Chapter 9

Binomial Theorem

Solutions

SECTION - A

Objective Type Questions (One option is correct)

1. If $(1-x^3)^n = \sum_{r=0}^{n} a_r x^r (1-x)^{3n-2r}$, then the value of a_r, where $n \in N$ is

(1) ${}^nC_r \cdot 3^r$ (2) ${}^nC_{3r}$ (3) ${}^nC_{r-1} 2^{r-1}$ (4) ${}^nC_r 2^r$

Sol. Answer (1)

$$\left\{(1-x)\left(1+x+x^2\right)\right\}^n = \sum_{r=0}^{n} a_r \cdot x^r \cdot (1-x)^{3n-2r}$$

$$\Rightarrow \left(1+x+x^2\right)^n = \sum_{r=0}^{n} a_r \cdot x^r (1+x^2-2x)^{n-r}$$

So, $\dfrac{a_r}{3^r} = {}^nC_r$

$$\Rightarrow \boxed{a_r = {}^nC_r \cdot 3^r}$$

2. Let $(1 + x^2)^2(1 + x)^n = \sum_{k=0}^{n+4} a_k \cdot x^k$. If a_1, a_2 and a_3 are in A.P., then the value of n is

(1) 2, 3, 4 (2) 5, 6, 7 (3) 8, 9, 10 (4) –1, 4, 6

Sol. Answer (1)

$(1 + x^2)^2 (1 + x)^n = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + \ldots a_{n+4}x^{n+4}$

$\Rightarrow (1 + x^4 + 2x^2)({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots {}^nC_nx^n) = a_0 + a_1x + a_2x^2 + \ldots$comparing the coefficients we find that

$a_1 = {}^nC_1 = n$ ...(i)

$$a_2 = {}^nC_2 + 2.\,{}^nC_0 = \frac{n(n-1)}{2} + 2$$

$$a_3 = {}^nC_3 + 2.{}^nC_1 = \frac{n(n-1)(n-2)}{6} + 2n$$

But $2a_2 = a_1 + a_3$

$\Rightarrow n(n-1)+4 = n + \dfrac{n(n-1)(n-2)}{6} + 2n$

$6n(n-1) + 24 = 6n + n(n-1)(n-2)+12n$

$6n^2 - 6n + 24 = 18n + n(n^2 - 3n + 2)$

$n^3 - 3n^2 + 2n + 24n - 6n^2 - 24 = 0$

$n^3 - 9n^2 + 26n - 24 = 0$

$\Rightarrow n^2(n-2) - 7n(n-2) + 12(n-2) = 0$

$(n-2)(n^2 - 7n + 12) = 0$

$(n-2)(n-3)(n-4) = 0$

$\Rightarrow n = 2, 3, 4$

3. $\sum_{r=0}^{n}(-1)^r . {}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\ldots m \text{ terms}\right]=$

(1) $\dfrac{2^{mn}-1}{2^{mn}(2^n-1)}$ (2) $\dfrac{2^{mn}-1}{2^n-1}$ (3) $\dfrac{2^{mn}+1}{2^n+1}$ (4) $\dfrac{2^{mn}+1}{2^n-1}$

Sol. Answer (1)

$$\sum_{r=0}^{n}(-1)^r\ {}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\ldots\text{upto } m \text{ terms}\right]$$

$$= \sum_{r=0}^{n}(-1)^r\ {}^nC_r\left[\left(\frac{1}{2}\right)^r+\left(\frac{3}{4}\right)^r+\left(\frac{7}{8}\right)^r+\ldots\text{upto } m \text{ terms}\right]$$

$$= \sum_{r=0}^{n}(-1)^r\ {}^nC_r\left(\frac{1}{2}\right)^r+\sum_{r=0}^{n}(-1)^r\ {}^nC_r\left(\frac{3}{4}\right)^r+\sum_{r=0}^{n}{}^nC_r(-1)^r\left(\frac{7}{8}\right)^r+\ldots\text{upto } m \text{ terms}$$

$$= \left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\ldots\text{upto } m \text{ terms}$$

$$= \left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n+\left(\frac{1}{8}\right)^n+\ldots\text{upto } m \text{ terms}$$

$$= \frac{\left(\frac{1}{2}\right)^n\left[\left(\frac{1}{2}\right)^{nm}-1\right]}{\left(\frac{1}{2}\right)^n-1} = \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

4. In the expansion of $(x + a)^n$, the sum of even terms is E, the sum of odd terms is O, then $O^2 - E^2$ is equal to

(1) $(x^2 + a^2)^n$ (2) $(x^2 - a^2)^n$ (3) $(x^2 - a^2)^{2n}$ (4) $(x^2 + a^2)^{2n}$

Sol. Answer (2)

$$(x+a)^n = {}^nC_0x^na^0 + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + ... + {}^nC_na^n$$

$$= T_1 + T_2 + T_3 + ...T_{n+1}$$

$$= (T_1 + T_3 + T_5 + ...) + (T_2 + T_4 + T_6 + ...)$$

$(x+a)^n = O + E$ ...(i)

Similarly $(x-a)^n = O - E$ ...(ii)

Multiplying (i) and (ii)

$(x^2-a^2)^n = O^2 - E^2$

5. For $4 \le r \le n$,

$\binom{n}{r}+4\binom{n}{r+1}+6\binom{n}{r+2}+4\binom{n}{r+3}+\binom{n}{r+4}$ equals

(1) $\binom{n+4}{r+4}$ (2) $\binom{n+4}{r}$ (3) $\binom{n+3}{r-1}$ (4) $\binom{n+4}{r+3}$

Sol. Answer (1)

$$\binom{n}{r}+4\binom{n}{r+1}+6\binom{n}{r+2}+4\binom{n}{r+3}+\binom{n}{r+4}$$

$$= {}^nC_r + 4\cdot {}^nC_{r+1} + 6\cdot {}^nC_{r+2} + 4\cdot {}^nC_{r+3} + {}^nC_{r+4}$$

$$= \left({}^nC_r + {}^nC_{r+1}\right) + 3\left({}^nC_{r+1} + {}^nC_{r+2}\right) + 3\left({}^nC_{r+2} + {}^nC_{r+3}\right) + \left({}^nC_{r+3} + {}^nC_{r+4}\right)$$

$$= \left({}^{n+1}C_{r+1}\right) + 3\left({}^{n+1}C_{r+2}\right) + 3\left({}^{n+1}C_{r+3}\right) + \left({}^{n+1}C_{r+4}\right)$$

$$= \left({}^{n+1}C_{r+1} + {}^{n+1}C_{r+2}\right) + 2\left({}^{n+1}C_{r+2} + {}^{n+1}C_{r+3}\right) + \left({}^{n+1}C_{r+3} + {}^{n+1}C_{r+4}\right)$$

$$= {}^{n+2}C_{r+2} + 2\left({}^{n+2}C_{r+3}\right) + {}^{n+2}C_{r+4}$$

$$= \left({}^{n+2}C_{r+2} + {}^{n+2}C_{r+3}\right) + \left({}^{n+2}C_{r+3} + {}^{n+2}C_{r+4}\right)$$

$$= {}^{n+3}C_{r+3} + {}^{n+3}C_{r+4}$$

$$= {}^{n+4}C_{r+4} = \binom{n+4}{r+4}$$

6. If $\sum_{k=0}^{n}\left(k^2+k+1\right)k! = (2007).\ 2007!$, then value of n is

(1) 2007 (2) 2006 (3) 2008 (4) 2005

Sol. Answer (2)

$$\sum_{k=0}^{n}(k^2+k+1)k! \equiv 2007\ .\ 2007!$$

But $\sum_{k=0}^{n}(k^2+k+1)k! = \sum_{k=0}^{n}(k^2+k+1)k!$

$= \sum_{k=0}^{n}\left(k^2.k! + (k+1)!\right) = \sum_{k=0}^{n}\left((k^2-1+1)k! + (k+1)!\right)$

$= \sum_{k=0}^{n}\left[((k-1)(k+1)+1)\,k! + (k+1)!\right] = \sum_{k=0}^{n}[(k-1)(k+1)! + k! + (k+1)!]$

$= \sum_{k=0}^{n}(k+1)!\,k + k! = \sum_{k=0}^{n}[(k+1)!\,(k+1-1) + k!]$

$= \sum_{k=0}^{n}\left([(k+1)!\,(k+1)] + [k! - (k+1)!]\right) = \sum_{k=0}^{n}[(k+2)! - (k+1)!] + [k! - (k+1)!]$

$= \sum_{k=0}^{n}[(k+2)! - (k+1)!] - [(k+1)! - k!]$

$= [(2! - 1!) - (1! - 0!)] + [(3! - 2!) - (2! - 1!)] + \ldots + [((n+2)! - (n+1)!) - ((n+1)! - n!)]$

$= [(2! - 1!) + (3! - 2!) + \ldots ((n+2)! - (n+1)!)] - [(1! - 0!) + (2! - 1!) + (3! - 2!) + \ldots ((n+1)! - n!)]$

$= [(n+2)! - 1!] - [(n+1)! - 0!]$

$= (n+2)! - (n+1)! = (n+1)!\,(n+1) = (2007)\,(2007!)$

Comparing $n = 2006$

7. If $x + y = 1$, then $\sum_{r=0}^{n} r^2\ {}^nC_r\, x^r\, y^{n-r}$ equals

(1) nxy (2) $nx(x + yn)$ (3) $nx(nx + y)$ (4) 1

Sol. Answer (3)

$S = \sum_{r=0}^{n} r^2 x^r y^{n-r}$

As $(y + x)^n = {}^nC_0 y^n x^0 + {}^nC_1 y^{n-1} x^1 + {}^nC_2 y^{n-2} x^2 + \ldots + {}^nC_n x^n$

Differentiating w.r.t. (x) keeping y as constant

$n(y + x)^{n-1} = {}^nC_1 y^{n-1} + 2\,{}^nC_2 y^{n-2} x + \ldots + n.{}^nC_n x^{n-1}$

$nx(y + x)^{n-1} = {}^nC_1 y^{n-1} x + 2\,{}^nC_2 y^{n-2} x^2 + \ldots + n.{}^nC_n x^n$

Again differentiating w.r.t. (x) keeping y constant

$n[x.(n-1)(y+x)^{n-2} + (y+x)^{n-1}]$

$= {}^nC_1 y^{n-1} + 2^2.{}^nC_2 y^{n-2} x + 3^2.{}^nC_3 y^{n-3} x^2 + \ldots + n^2.{}^nC_n x^{n-1}$

But $x + y = 1$

$\Rightarrow n[x\,(n-1) + 1] = S$

$\Rightarrow n[nx + 1 - x] = S$

$\Rightarrow S = n[nx + y]$

8. The value of $(C_0 + C_1)(C_1 + C_2)........(C_{n-1} + C_n)$ is

(1) $\dfrac{(n+1)^n}{n!}\cdot C_1C_2........C_n$

(2) $\dfrac{(n-1)^n}{n!}\cdot C_1C_2........C_n$

(3) $\dfrac{(n)^n}{(n+1)!}\cdot C_1C_2........C_n$

(4) $\dfrac{(n)^n}{n!}\cdot C_1C_2........C_n$

Sol. Answer (1)

$(C_0+C_1)(C_1+C_2).....(C_{n-1}+C_n)$

${}^{n+1}C_1\cdot{}^{n+1}C_2.....{}^{n+1}C_n$

$$=\frac{1\cdot{}^{n+1}C_1\cdot 2\cdot{}^{n+1}C_2\cdot 3\cdot{}^{n+1}C_3.....n\cdot{}^{n+1}C_n}{1\cdot 2\cdot 3.....n}$$

$$=\frac{(n+1)^n\cdot{}^nC_0\cdot{}^nC_1.....{}^nC_n}{n!}$$

$$=\frac{(n+1)^n}{n!}\cdot C_1\cdot C_2\cdot C_3.....C_n$$

9. The coefficient of x^n in the polynomial

$(x + {}^nC_0)\ (x + 3.{}^nC_1)\ (x + 5.{}^nC_2)\ (x + 7.{}^nC_3)\ .....\ (x + (2n+1)\ {}^nC_n)$ is

(1) $n.2^{n-1}$ (2) $n.2^n$ (3) $n.2^{n+1}$ (4) $(n + 1)\ 2^n$

Sol. Answer (4)

$(x+{}^nC_0)(x+3\ {}^nC_1)(x+5\ {}^nC_2)....(x+(2n+1),\ {}^nC_n)$

$= x^{n+1}+({}^nC_0+3\ {}^nC_1+5\ {}^nC_2+....(2n+1)\ {}^nC_n)x^n+....$

Coefficient of x^n = ${}^nC_0+3\ {}^nC_1+5\ {}^nC_2+....(2n+1)\ {}^nC_n$

$= {}^nC_0+(1+2).{}^nC_1+(1+4).{}^nC_2+...+(1+2n).{}^nC_n$

$= ({}^nC_0+{}^nC_1+....+{}^nC_n)+2(1.{}^nC_1+2.{}^nC_2+....+n.{}^nC_n)$

$= 2^n+2n.2^{n-1}$

$= 2^n+n.2^n$

$= (n+1).2^n$

10. $C_0{}^{2n}C_n - C_1{}^{2n-2}C_n + C_2{}^{2n-4}C_n$ …… equals to

(1) 2^n (2) $2^n(n + 1)$ (3) 2^{n-1} (4) $\dfrac{(n+1)2^{n-1}}{2}$

Sol. Answer (1)

$C_0 + {}^{2n}C_n - C_1{}^{2n-2}C_n + C_2{}^{.2n-4}C_n$ ……

= Coeff. of x^n in $\{(1 + x)^2 - 1\}^n$

$= 2^n$

11. If S be such that $S = {}^nC_1 - 3\,{}^nC_3 + 3^2\cdot{}^nC_5 - 3^3\cdot{}^nC_7$ .... ; then S is equal to

(1) $(-1)^n\cdot\dfrac{2^n}{\sqrt{3}}\sin\dfrac{2n\pi}{3}$

(2) $(-1)^{n+1}\cdot\dfrac{2^n}{\sqrt{3}}\sin\dfrac{2n\pi}{3}$

(3) $(-1)^{n+1}\cdot\dfrac{2^n}{\sqrt{3}}\cos\dfrac{2n\pi}{3}$

(4) $(-1)^n\cdot\dfrac{2^n}{\sqrt{3}}\cos\dfrac{2n\pi}{3}$

Sol. Answer (2)

$$\omega^n = \left(-\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^n = \left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^n = \cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}$$

Now $\left(-\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}\right)^n = \dfrac{(-1)^n}{2^n}(1-i\sqrt{3})^n$

$$=\frac{(-1)^n}{2^n}\left[1+{}^nC_1\left(-i\sqrt{3}\right)+{}^nC_2\left(-i\sqrt{3}\right)^2+{}^nC_3\left(-i\sqrt{3}\right)^3........\right]$$

$$=\frac{(-1)^n}{2^n}\left[\left(1-3\,{}^nC_2........\right)-i\sqrt{3}\left({}^nC_1-3\,{}^nC_3+3^2\,{}^nC_5-3^3\,{}^nC_7........\right)\right]$$

Equating the real and imaginary parts, we get

$$S=(-1)^{n+1}\frac{2^n}{\sqrt{3}}\sin\frac{2n\pi}{3}$$

12. Let ${}^nC_r = C_r$, then value of $\sum\limits_{0\le r<s\le n} r\cdot s\cdot C_r\cdot C_s =$

(1) $\dfrac{n^2}{2}\left(2^{2n-2}-{}^{2n-2}C_{n-1}\right)$

(2) $\dfrac{n^2}{2}\left(2^{n-1}-{}^{2n-1}C_{n-1}\right)$

(3) $\dfrac{n^2}{2}\left(2^{2n-1}-{}^{2n}C_n\right)$

(4) $\dfrac{n^2}{2}\left(2^{n+1}-{}^{2n-2}C_{n-1}\right)$

Sol. Answer (1)

$\because\ C_0 + C_1x + C_2x^2 + ... = (1 + x)^n$

On differentiating and replacing $x = 1$, we get

$$\sum_{r=1}^{n} r\cdot C_r = n\cdot 2^{n-1}$$

Now, $(1\cdot C_1+2\cdot C_2x+3\cdot C_3x^2+...)\times\left(1\cdot C_1+2\cdot\dfrac{C_2}{x}+3\cdot\dfrac{C_3}{x^2}+...\right) = n(1+x)^{n-1}\cdot n\left(1+\dfrac{1}{x}\right)^{n-1}$

On comparing the constant terms from both sides

$$\sum_{r=1}^{n} r^2C_r^2 = n^2\cdot{}^{2n-2}C_{n-1}$$

Now, $\sum\limits_{1\le r<s\le n} r\cdot s\cdot C_r\cdot C_s = \dfrac{1}{2}\left[\left(\sum\limits_{r=1}^{n} r\,C_r\right)^2-\sum\limits_{r=1}^{n} r^2C_r^2\right] = \dfrac{n^2}{2}-\left[2^{2n-2}-{}^{2n-2}C_{n-1}\right]$

13. The remainder when $2^{2019} + 2020$ is divided by 17 is

(1) 13 (2) 9 (3) 5 (4) 2

Sol. Answer (3)

$2^{2019} = 8 \,.\, 16^{504} = 8 \,.\, (17 - 1)^{504} = 8 + \text{multiple of } 17$

and 2020 = 14 + multiple of 17

$\therefore\ 2^{2019} + 2020 = 22 + \text{multiple of } 17 = 5 + \text{multiple of } 17.$

14. The largest integer k such that 2^k divides $3^{2^n} - 1$ is

(1) $n + 2$ (2) n (3) $n - 1$ (4) $n + 1$

Sol. Answer (1)

$$3^{2^n} - 1 = (4-1)^{2^n} - 1 = 4^{2^n} - {}^{2^n}C_1 4^{2^n - 1} + \ldots - {}^{2^n}C_{n-1} \cdot 4$$

which is divisible by 2^{n+2}

$\therefore\ k = n + 2$

15. Let $f(x) = \dfrac{\sum_{r=0}^{100} {}^{100}C_r \sin 2rx}{\sum_{r=0}^{100} {}^{100}C_r \cos 2r}$ then $f\left(\dfrac{\pi}{800}\right)$ is equal to

(1) $\sqrt{2}+1$ (2) $\sqrt{2}-2$ (3) $\sqrt{2}-1$ (4) $2\sqrt{2}-2$

Sol. Answer (3)

$$\because\ \sum_{r=0}^{100} {}^{100}C_r \sin(2rx) = 2^{101}\cos^{100} x \cdot \sin 100x$$

$$\text{and } \sum_{r=0}^{100} {}^{100}C_r \cos(2rx) = 2^{101}\cos^{100} x \cdot \cos 100x$$

$$\therefore\ f(x) = \tan 100x$$

$$\therefore\ f\left(\frac{\pi}{800}\right) = \tan\frac{\pi}{8} = \frac{2\sin\frac{\pi}{8}\cdot\cos\frac{\pi}{8}}{2\cos^2\frac{\pi}{8}} = \frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}} = \frac{1}{\sqrt{2}+1} = \sqrt{2}-1$$

SECTION - B

Objective Type Questions (More than one options are correct)

1. For a positive integer n, if the expansion of $\left(\dfrac{5}{x^2} + x^4\right)^n$ has a term independent of x, then n can be

(1) 18 (2) 21 (3) 27 (4) 99

Sol. Answer (1, 2, 3, 4)

$$T_{r+1} = {}^nC_r\left(\frac{5}{x^2}\right)^{n-r}\left(x^4\right)^r = {}^nC_r\, 5^{n-r} x^{-2n+2r} x^{4r} = {}^nC_r\, 5^{n-r} x^{-2n+6r}$$

For term independent of 'x'

$-2n + 6r = 0$

$\Rightarrow n = 3r = 0, 3, 6, 9, \ldots$

2. The positive value of a so that the coefficients of x^5 is equal to that of x^{15} in the expansion of $\left(x^2 + \frac{a}{x^3}\right)^{10}$ is

(1) $\frac{1}{2\sqrt{3}}$ (2) $\frac{1}{\sqrt{3}}$ (3) $\frac{\sqrt{3}}{6}$ (4) $\frac{1}{3}$

Sol. Answer (1, 3)

$\left(x^2 + \frac{a}{x^3}\right)^{10}$

$T_{r+1} = {}^{10}C_r (x^2)^{10-r} \left(\frac{a}{x^3}\right)^r$

$= {}^{10}C_r a^r x^{20-2r} x^{-3r}$

$= {}^{10}C_r\ a^r x^{20-5r}$

If $20 - 5r = 5 \Rightarrow r = 3$

$\Rightarrow$ Coefficient of $x^5 = {}^{10}C_3 a^3$

If $20 - 5r = 15 \Rightarrow r = 1$

$\Rightarrow$ Coefficient of $x^{15} = {}^{10}C_1 a^1 = {}^{10}C_1 a$

Given that ${}^{10}C_3 a^3 = {}^{10}C_1 a^1$

$\Rightarrow \frac{10 \times 9 \times 8}{3 \times 2} a^3 = 10 \times a$

$\Rightarrow a^2 = \frac{1}{12}$

$\Rightarrow a = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$

Positive value of $a = \frac{1}{2\sqrt{3}}$ or $\frac{\sqrt{3}}{6}$

3. The sum of the coefficients of all the even powers of x in the expansion of $(2x^2 - 3x + 1)^{11}$ is

(1) 3.6^{10} (2) 6^{11} (3) $2^{10}.3^{11}$ (4) $2^{11}.3^{10}$

Sol. Answer (1, 3)

$(2x^2 - 3x + 1)^{11} = A_0 + A_1 x + A_2 x^2 + \ldots + A_{22} x^{22}$

Putting $x = 1$, we get

$0 = A_0 + A_1 + A_2 + \ldots + A_{22}$ …(i)

Putting $x = -1$, we get

$6^{11} = A_0 - A_1 + A_2 - A_3 + \ldots + A_{22}$ …(ii)

Adding equation (i) and (ii)

$6^{11} = 2[A_0 + A_2 + A_4 + \ldots]$

Sum of coefficient of even powers of x

$= A_0 + A_2 + A_4 + \ldots.$

$= \dfrac{6^{11}}{2} = \dfrac{6.6^{10}}{2}$

$= 3.6^{10} = 3.2^{10}.3^{10} = 2^{10}.3^{11}$

4. The term independent of x in the expansion of $\left(x-\dfrac{1}{x}\right)^4\left(x+\dfrac{1}{x}\right)^3$ is

(1) –3 (2) 0 (3) ${}^nP_k$, where $k > n$ (4) ${}^nC_r$, where $r > n$

Sol. Answer (2, 3, 4)

$\left(x-\dfrac{1}{x}\right)^4\left(x+\dfrac{1}{x}\right)^3$

$= \left(x-\dfrac{1}{x}\right)\left(x^2-\dfrac{1}{x^2}\right)^3$

$= \left(x-\dfrac{1}{x}\right)\left(x^6+\dfrac{1}{x^6}-3\left(x^2-\dfrac{1}{x^2}\right)\right)$

$= \left(x-\dfrac{1}{x}\right)\left(x^6+\dfrac{1}{x^6}-3x^2+\dfrac{3}{x^2}\right)$

Clearly no term is independent of x

Hence coefficient = 0

Also coefficient = ${}^nC_k$ if $k > n$

= ${}^nP_k$, if $k > n$

Hence, options, 2, 3, 4 are correct.

5. If the second, third and fourth terms in the expansion of $(x + y)^n$ be 135, 30 and 10/3 respectively, then

(1) $x = 3$ (2) $y = \dfrac{1}{3}$ (3) $n = 5$ (4) $n = 7$

Sol. Answer (1, 2, 3)

$T_2 = {}^nC_1 x^{n-1} y = 135$ …(i)

$T_3 = {}^nC_2 x^{n-2} y^2 = 30$ …(ii)

$T_4 = {}^nC_3\ {}^nC_3 x^{n-3} y^3 = \dfrac{10}{3}$ …(iii)

If (ii) is divided by (i)

Then, we get $\left(\dfrac{n-1}{2}\right)\left(\dfrac{y}{x}\right) = \dfrac{30}{135}$ …(iv)

By dividing equation (iii) by (ii)

$\left(\dfrac{n-2}{3}\right)\left(\dfrac{y}{x}\right) = \dfrac{10}{3\times 30} = \dfrac{1}{9}$ …(v)

$$\frac{\left(\frac{n-1}{2}\right)}{\left(\frac{n-2}{3}\right)} = \frac{\left(\frac{30}{135}\right)}{\left(\frac{1}{9}\right)} \qquad \text{by (iv) and (v)}$$

$$\frac{3(n-1)}{2(n-2)} = \frac{9 \times 30}{135}$$

$$\frac{3(n-1)}{2(n-2)} = \frac{30}{15} = 2$$

$3n - 3 = 4n - 8$

$n = 5$

By (iv) $\left(\frac{5-1}{2}\right)\left(\frac{y}{x}\right) = \frac{30}{135} = \frac{2}{9}$

$$2\left(\frac{y}{x}\right) = \frac{2}{9}$$

$\frac{y}{x} = \frac{1}{9}$ – (iv) $\Rightarrow$ y = $\frac{x}{9}$

By eqn. (i) ${}^5C_1\, x^4 \left(\frac{x}{9}\right) = 135$

$\Rightarrow x^5 = 3^5 \Rightarrow x = 3$ and $y = \frac{1}{3}$

6. $\sum_{r=0}^{6} (-1)^r \binom{16}{r}$ is divisible by

(1) 5 (2) 7 (3) 11 (4) 13

Sol. Answer (1, 2, 3, 4)

$$\sum_{r=0}^{16} (-1)^r\ {}^{16}C_r = {}^{16}C_0 - {}^{16}C_1 + {}^{16}C_2 - {}^{16}C_3 + \ldots\ {}^{16}C_{16} = 0$$

Hence, the number is divisible by all 5, 7, 11, 13

Hence, all options are correct.

7. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + a_3x^3 + .. + a_{20}x^{20}$, then

(1) $a_1 = 20$ (2) $a_2 = 210$ (3) $a_4 = 8085$ (4) $a_{20} = 2^2.3^7.7$

Sol. Answer (1, 2, 3)

$(1 + 2x + 3x^2)^{10} = 1 + {}^{10}C_1(2x+3x^2) + {}^{10}C_2(2x+3x^2)^2 + {}^{10}C_3(2x+3x^2)^3 + {}^{10}C_4(2x+3x^2)^4 + \ldots$

$= 1 + x(2.{}^{10}C_1) + x^2(3.{}^{10}C_1 + 4.{}^{10}C_2) + x^3(12.{}^{10}C_2 + 8) + x^4(9.{}^{10}C_2 + 3.2.3.2.{}^{10}C_3 + {}^{10}C_4 \cdot 2^4)$

Comparing the coefficients

$a_0 = 1,\ a_1 = 2 \times 10 = 20$

$a_2 = 30 + \dfrac{4\times10\times9}{2} = 30 + 180 = 210$

$a_4 = 9.{}^{10}C_2 + 36.{}^{10}C_3 + {}^{10}C_4.16$

$= 9\cdot\dfrac{10\times9}{2} + 36\times\dfrac{10\times9\times8}{3\times2} + \dfrac{10\times9\times8\times7}{4\times3\times2\times1}\cdot16$

= 8085

8. The maximum value of ${}^nC_r$ is obtained when r is equal to

(1) $\dfrac{n}{3}$ (2) $\dfrac{n}{4}$ (3) $\dfrac{n-1}{2}$ or $\dfrac{n+1}{2}$ (4) $\dfrac{n}{2}$

Sol. Answer (3, 4)

$${}^nC_r \text{ is maximum if } r = \begin{cases} \dfrac{n}{2} & ;\text{ if } n \text{ is even} \\ \dfrac{n-1}{2} \text{ or } \dfrac{n+1}{2} & ;\text{ if } n \text{ is odd} \end{cases}$$

Hence options (3), (4) are correct.

9. Given that the 4th term in the expansion of $\left(2+\dfrac{3x}{8}\right)^{10}$ has the maximum numerical value, then x lies in the interval

(1) $\left(2, \dfrac{64}{21}\right)$ (2) $\left(-\dfrac{60}{23}, -2\right)$ (3) $\left(-\dfrac{64}{21}, -2\right)$ (4) $\left(2, -\dfrac{60}{23}\right)$

Sol. Answer (1, 3)

If 4th term has maximum numerical value then

$|T_5| < |T_4| > |T_3|$

(i) Using $|T_4| > |T_3|$

$$\left|\frac{T_4}{T_3}\right| > 1$$

$$\Rightarrow \left|\frac{{}^{10}C_3(2)^7\left(\frac{3x}{8}\right)^3}{{}^{10}C_2(2)^8\left(\frac{3x}{8}\right)^2}\right| > 1 \quad \Rightarrow \left(\frac{10-3+1}{3}\right)\left|\frac{3x}{8}\times\frac{1}{2}\right| > 1$$

$$\Rightarrow \left|\frac{8}{3}\right|\left|\frac{3}{8}\cdot\frac{x}{2}\right| > 1$$

$\Rightarrow |\,x\,| > 2$ $\Rightarrow x \in (-\infty, -2) \cup (2, \infty)$ … (i)

(ii) Using $|T_4| > |T_5|$

$|T_5| < |T_4|$

$$\left|\frac{T_5}{T_4}\right| < 1$$

$$\Rightarrow \left|\left(\frac{10-4+1}{4}\right)\frac{\left(\frac{3x}{8}\right)}{2}\right| < 1 \quad \Rightarrow \left|\frac{7}{4}.\frac{3x}{8}\times\frac{1}{2}\right| < 1$$

$$\Rightarrow \ |x| < \frac{64}{21}$$

$$\Rightarrow -\frac{64}{21} < x < \frac{64}{21} \qquad \text{... (ii)}$$

By equation (i) and (ii)

$$x \in \left(2, \frac{64}{21}\right) \text{ or } \left(-\frac{64}{21}, -2\right)$$

10. The value of the expression $C_0^2 - C_1^2 + C_2^2 - .... + (-1)^n C_n^2$ is

(1) 0 if n is odd

(2) $(-1)^n$ if n is odd

(3) $(-1)^{n/2}\, {}^nC_{n/2}$ if n is even

(4) $(-1)^{n-1}\, {}^nC_{n-1}$ if n is even

Sol. Answer (1, 3)

$$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_{n-1}x + C_n \qquad \text{... (i)}$$

$$(1 \ \ x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + ...(-1)^n.C_nx^n \qquad \text{... (ii)}$$

Multiply in equation (i) and (ii)

$$(1 \ \ x^2)^n = (C_0^2 - C_1^2 + C_2^2 - C_3^2 + ...(-1)^n.C_n^2)x^n + ...$$

Hence $C_0^2 - C_1^2 + C_2^2 - C_3^2 + ...(-1)^n.C_n^2$ = Coefficient of x^n in $(1 - x^2)^n$

$$= \begin{cases} 0 & ;\text{if } n \text{ is odd} \\ (-1)^{\frac{n}{2}}\ {}^nC_{\frac{n}{2}} & ;\text{if } n \text{ is even} \end{cases}$$

11. The number $101^{100} - 1$ is divisible by

(1) 10^2

(2) 10^3

(3) 10^4

(4) 10^5

Sol. Answer (1, 2, 3)

$$\left((101)^{100} - 1\right) = (1+100)^{100} - 1$$

$$= \left(1 + {}^{100}C_1 100 + {}^{100}C_2(100)^2 + ... + {}^{100}C_{100}(100)^{100}\right) - 1$$

$= 10^4\ (1 + {}^{10}C_4 + ...)$

$= 10^4 \times$ integer

Hence, $(101)^{100} - 1$ is divided by 10^4 and hence divided by 10^3 and 10^2 also.

12. If n is a positive integer and $(3\sqrt{3}+5)^{2n+1} = I + f$ where I is an integer and $0 < f < 1$, then

(1) I is an even integer

(2) $(I + f)f$ is divisible by 2^{2n+1}

(3) The integer just less than $(3\sqrt{3}+5)^{2n+1}$ is divisible by 3

(4) I is divisible by 10

Sol. Answer (1, 2, 4)

$\left(3\sqrt{3}+5\right)^{2n+1} = I + f$ where I is an integer and $0 \; f < 1$

Let us take $f' = \left(3\sqrt{3}-5\right)^{2n+1}$ where $0 < f' < 1$

Here for solutions we expand both $I + f$ and f'

$(I + f) = \left(3\sqrt{3}+5\right)^{2n+1}$

$$= {}^{2n+1}C_0\left(3\sqrt{3}\right)^{2n+1} + {}^{2n+1}C_1\left(3\sqrt{3}\right)^{2n}(5) + \ldots + {}^{2n+1}C_{2n+1}5^{2n+1} \quad \ldots\text{(i)}$$

$f' = \left(3\sqrt{3}-5\right)^{2n+1}$

$$= {}^{2n+1}C_0\left(3\sqrt{3}\right)^{2n+1} - {}^{2n+1}C_1\left(3\sqrt{3}\right)^{2n}(5) + \ldots + {}^{2n+1}C_{2n+1}5^{2n+1} \quad \ldots\text{(ii)}$$

By (i) and (ii)

$$I + f - f' = 2\left[{}^{2n+1}C_1\left(3\sqrt{3}\right)^{2n}(5) + {}^{2n+1}C_3\left(3\sqrt{3}\right)^{2n-2}(5)^3 + \ldots + 5^{2n+1}\right] \quad \ldots\text{(iii)}$$

$= 2 \times$ an integer = even integer

$\Rightarrow I + f - f'$ = an integer

$\Rightarrow f - f'$ is an integer

Again $0 < f < 1$ ...(iv)

$0 < f' < 1$

$\Rightarrow -1 < f' < 0$ ...(v)

Adding (iv) and (v)

$-1 < f - f' < 1$, but $f - f'$ is an integer hence

$f - f' = 0 \qquad \Rightarrow f = f'$

As $f - f' = 0$ and $I + f - f'$ = even integer

$\Rightarrow I$ is an even integer

By (iii) $I = 2 \times 5 \times$ Integer

$I = 10 \times$ integer, hence I is divisible by 10

If an integer just less than $\left(3\sqrt{3}+5\right)^{2n+1}$ is I which is an even integer hence it may or may not be divisible by 3.

$(I + f)f = \left(3\sqrt{3}+5\right)^{2n+1}\left(3\sqrt{3}-5\right)^{2n+1}$

$= (27 - 25)^{2n+1}$

$= 2^{2n+1}$

Hence $(I + f)f$ is divisible by 2^{2n+1}

13. If n is a positive integer and if $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then

(1) $a_r = a_{2n-r}$, for $0 \le r \le 2n$

(2) $a_0 + a_1 + \ldots\ldots + a_{n-1} = \frac{1}{2}(3^n - a_n)$

(3) $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots\ldots + a_{2n}^2 = a_n$

(4) $a_0 + a_2 + \ldots\ldots + a_{2n} = \frac{1}{2}(3^n + 1)$

Sol. Answer (1, 2, 3, 4)

We have

$$\sum_{r=0}^{2n} a_r \cdot x^r = (1+x+x^2)^n$$

Replacing x by $\frac{1}{x}$ on both sides,

$$\sum_{r=0}^{2n} a_r \cdot \left(\frac{1}{x}\right)^r = \left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n$$

$$= \left(\frac{x^2+x+1}{x^2}\right)^n = (x^2+x+1)^n \cdot \frac{1}{x^{2n}}$$

$$\Rightarrow \sum_{r=0}^{2n} a_r \cdot x^{2n-r} = (x^2+x+1)^n = \sum_{r=0}^{2n} a_r x^r$$

$$= \sum_{r=0}^{2n} a_{2n-r} x^{2n-r} \quad \ldots(i)$$

Comparing both sides we get,

$a_r = a_{2n-r}$, for $0 \le r \le 2n$

Again we have,

$a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_{2n} x^{2n} = (1 + x + x^2)^n$

Putting $x = 1$ both sides we get,

$a_0 + a_1 + a_2 + \ldots\ldots + a_{2n} = 3^n$.

But $a_r = a_{2n-r}$

$2(a_0 + a_1 + a_2 + \ldots\ldots + a_{n-1}) + a_n = 3^n$

$\Rightarrow a_0 + a_1 + a_2 + \ldots\ldots + a_{n-1} = \frac{1}{2}[3^n - a_n]$

Again,

$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_{2n}x^{2n}$ ...(ii)

Replacing x by $-\frac{1}{x}$, we get

$$\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 - \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \ldots\ldots + \frac{a_{2n}}{x^{2n}} \quad \ldots(iii)$$

Multiplying (ii) and (iii), comparing constant term we get,

$$a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots\ldots + a_{2n}^2 = a_n$$

$$a_0 - a_1 + a_2 - a_3 + \ldots\ldots + a_{2n} = 1$$

$$a_0 + 0\,a_1 + a_2 - a_3 + \ldots\ldots + a_{2n} = 3^n$$

$$a_0 + a_2 + \ldots\ldots\ldots + a_{2n} = \frac{1}{2}(3^n + 1)$$

Hence (1), (2), (3), (4) are correct.

14. Which of the following is/are correct?

(1) $(101^{50} - 99^{50}) > 100^{50}$

(2) $(101)^{50} - 100^{50} > 99^{50}$

(3) $(1000)^{1000} > (1001)^{999}$

(4) $(1001)^{999} > (1000)^{1000}$

Sol. Answer (1, 2, 3)

$101^{50} - 99^{50} = (100 + 1)^{50} - (100 - 1)^{50}$

$= [\,{}^{500}C_0 \cdot (100)^{50} \cdot 1^{50} + {}^{500}C_1 \cdot (100)^{49} \cdot 1^1 + {}^{500}C_2 \cdot (100)^{48} \cdot 1^2 + \ldots\ldots\,]$

$- [\,{}^{500}C_0 \cdot (100)^{50} \cdot (-1)^0 - {}^{500}C_1 \cdot 100^{49} \cdot 1^1 + {}^{500}C_0 \cdot 100^{48} \cdot 1^2 + \ldots\ldots\,]$

$= (100)^{50} + 2[\,{}^{50}C_3(100)^{47} + {}^{50}C_5(100)^{45} + \ldots\ldots\,] > (100)^{50}$

$\Rightarrow (101)^{50} - (100)^{50} > 99^{50}$

Also, $\left(\frac{1005}{1000}\right)^{999} = \left(1+\frac{1}{1000}\right)^{999} = 1 + {}^{999}C_1 \cdot \frac{1}{1000} + {}^{999}C_2 \cdot \left(\frac{1}{1000}\right)^2 + \ldots\ldots < 1 + 1 + \ldots\ldots + 1 = 1000$

$\therefore \left(\frac{1001}{1000}\right)^{999} < 1000$

$\Rightarrow (1001)^{999} < (1000)^{1000}$

Hence option (1), (2), (3) are correct.

15. If C_r stands for ${}^nC_r = \frac{n!}{r!\ n-r!}$ and $\sum_{r=1}^{n} r.C_r^{\,2} = \lambda$ for $n \geq 2$, then λ is divisible by

(1) $3(n - 1)$

(2) $n + 1$

(3) $n(2n - 1)$

(4) $n^2 + 1$

Sol. Answer (3)

$(x + 1)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + \ldots {}^nC_n1^n$ ... (i)

or $(x + 1)^n = C_0x^n + C_1x^{n-1} + \ldots + C_n$ ...(ii) where $C_r = {}^nC_r$

Again $(1 + x)^n = C_0 + C_1x + C_2x^2 + ...C_nx^n$

Differentiating

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + ...nC_nx^{n-1}$ …(iii)

Multiplying (ii) and (iii), we get

$n(1 + x)^{2n-1} = (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ...C_n)(C_1 + 2C_2x + 3C_3x^2 + ... n.C_nx^{n-1})$

Comparing the coefficient of x^{n-1}

$n.\ {}^{2n-1}C_{n-1} = C_1^2 + 2C_2^2 + 3C_3^2 + ... + n.C_n^2$

$\Rightarrow C_1^2 + 2C_2^2 + 3C_3^2 + ... + n.C_n^2 = n.\frac{(2n-1)!}{(n-1)!\,n!}$

which is clearly divisible by $n(2n - 1)$

SECTION - C

Linked Comprehension Type Questions

Comprehension-I

If C_0, C_1, C_2, C_3, ..., C_n be binomial coefficients in the expansion of $(1 + x)^n$, then

1. The value of the expression $C_0 + 2C_1 + 3C_2 + ... + (n + 1)C_n$ is equal to

   (1) $2^{n-1}(n + 1)$ (2) $2^{n-1}(n + 2)$ (3) $2^n(n + 2)$ (4) None of these

2. The value of the expression $C_0 - 2C_1 + 3C_2 - .... + (-1)^n(n+1)C_n$ is equal to

   (1) 0 (2) $2^n(n + 3)$ (3) $2^{n-1}(n - 2)$ (4) None of these

3. The value of $C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + ... + (C_0 + C_1 + C_2 + ... + C_{n-1})$ equal

   (1) $n\,2^n$ (2) $n\,2^{n-1}$ (3) $(n + 2)\,2^n$ (4) None of these

Solution of Comprehension-I

$(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

$x(1 + x)^n = C_0x + C_1x^2 + C_2x^3 + ... + C_nx^{n+1}$

Differentiating

$x.n(1 + x)^{n-1} + (1 + x)^n = C_0 + 2C_1x + 3C_2x^2 + ... + n.C_nx^{n-1}$ …(i)

By putting $x = 1$, in (i) we get

$C_0 + 2C_1 + 3C_2 + 4C_3 + ... = 2^{n-1}(n + 2)$

Putting $x = -1$, in (i)

$0 = C_0 - 2C_1 + 3C_2 + ... (-1)^n(n + 1)C_n$

1. Answer (2)

   $(1 + x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$

   $\Rightarrow x(1 + x)^n = C_0x + C_1x^2 + C_2x^3 + ... + C_nx^{n+1}$ …(1)

   $\Rightarrow$ differentiating w.r.t. 'x' & putting $x = 1$,

   $C_0 + 2C_1 + 3C_2 + ... + (n + 1)\,C_n = 2^{n-1}(n + 2)$

2. Answer (1)

putting $x = -1$ in last questions equation (1)

$C_0 - 2C_1 + 3C_2 \ldots\ldots + (-1)^n.(n + 1)C_n = 0$

3. Answer (2)

$y = C_0 + (C_0 + C_1) + (C_0 + C_1 + C_2) + \ldots + (C_0 + C_1 + C_2 + \ldots C_{n-1})$

Adding the terms equidistant from the ends we get

$y = (C_0 + C_0 + C_1 + C_2 + \ldots + C_{n-1}) + (C_0 + C_1 + C_0 + C_1 + \ldots + C_{n-2}) + \ldots$

Putting $C_0 = C_n$ in first bracket

Putting $C_0 = C_n$ and $C_1 = C_{n-1}$ in second bracket

And so on

In this way

$y = (C_0 + C_1 + C_2 \ldots C_n) + (C_0 + C_1 + C_2 \ldots C_n) + \ldots \frac{n}{2}$ times

$y = \frac{n}{2}.2^n = n \,.\, 2^{n-1}$

Comprehension-II

Let n be a positive integer and

$(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_rx^r + \ldots + C_{n-1}x^{n-1} + C_nx^n$

where C_r stands for ${}^nC_r$, then

1. The value of $\sum_{r=0}^{n} \sum_{s=0}^{n} (C_r + C_s)$ is

(1) $(n + 1)2^{n+1}$ (2) $n.2^n$ (3) $(n + 1)2^n$ (4) $n.2^{n-1}$

Sol. Answer (1)

$$\sum_{r=0}^{n}\sum_{s=0}^{n}(C_r + C_s) = \{(C_0 + C_0) + (C_0 + C_1) + \ldots(C_0 + C_n)\} + \{(C_1 + C_0) + (C_1 + C_1) + (C_1 + C_2) + \ldots(C_1 + C_n)\} + \ldots + \{(C_n + C_0) + (C_n + C_1) + \ldots(C_n + C_n)\}$$

$= 2(n + 1)(C_0 + C_1 + C_2 + \ldots C_n)$

$= (n + 1) \cdot 2 \cdot 2^n = (n + 1)2^{n+1}$

2. The value of $\sum\sum_{0 \le r < s \le n} (C_r \pm C_s)^2$ is

(1) $(n \mp 1)\,{}^{2n}C_n \pm 2^{2n}$ (2) $(n+1)\,{}^{2n}C_n + 2^{2n-1}$ (3) $(n \pm 1)\,{}^{2n}C_n \pm 2^{2n}$ (4) $(n+1)\,{}^{2n}C_n - 2^{2n-1}$

Sol. Answer (1)

$$\sum_{0 \le r < s \le n} (C_r \pm C_s)^2$$

$$= \sum_{0 \le r < s \le n} (C_r^2 + C_s^2 \pm 2C_rC_s)$$

$= \sum_{0\le r<s\le n} (C_r^2 + C_s^2) \pm \sum_{0\le r<s\le n} \sum C_r C_s$

$= n(C_0^2 + C_1^2 + ... + C_n^2) \pm (2^{2n} - {}^{2n}C_n)$

$= (C_0 + C_1 + ...C_n)^2 = C_0^2 + C_1^2 + ...C_n^2 + 2\sum_{0\le r<s\le n} \sum C_r C_s$

$\Rightarrow \sum_{0\le r<s\le n} \sum (C_r \pm C_s)^2 = n.{}^{2n}C_n \pm (2^{2n} - {}^{2n}C_n)$

$= (n \mp 1)\,{}^{2n}C_n \pm 2^{2n}$

Comprehension-III

Let n is a rational number and x is a real number such that $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)}{3!}\cdot x^3 + ...$$

This can be used to find the sum of different series.

1. Sum of infinite series

$1 + \frac{2}{3}\cdot\frac{1}{2} + \frac{2}{3}\cdot\frac{5}{6}\cdot\frac{1}{2^2} + \frac{2}{3}\cdot\frac{5}{6}\cdot\frac{8}{9}\cdot\frac{1}{2^3} + ...\infty$ is

(1) $2^{1/3}$ (2) $4^{1/3}$ (3) $8^{1/3}$ (4) $4^{2/3}$

Sol. Answer (2)

We have, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + ...$

Here, $nx = \frac{2}{3}\cdot\frac{1}{2}$ ...(i)

$\frac{n(n-1)}{2}x^2 = \frac{2\cdot 5}{3\cdot 6}\cdot\frac{1}{2^2}$ ...(ii)

Equation (ii) divided by Equation (i)

$\frac{n(n-1)x^2}{2\times nx} = \frac{2\cdot 5\cdot 1}{3\cdot 6\cdot 4}\cdot\frac{3\cdot 2}{2\cdot 1}$

$\Rightarrow \frac{(n-1)x}{2} = \frac{5}{12}$

$\Rightarrow nx - x = \frac{5}{6}$

$\Rightarrow x = nx - \frac{5}{6} = \frac{1}{3} - \frac{5}{6} = \frac{2-5}{6} = -\frac{1}{2}$

$\therefore n\cdot\frac{-1}{2} = \frac{1}{3} \Rightarrow n = -\frac{2}{3}$

$\therefore$ The require sum $= \left(\frac{1}{2}\right)^{-\frac{2}{3}} = \frac{1}{\left(\frac{1}{2}\right)^{2/3}} = \frac{1}{\left(\frac{1}{4}\right)^{1/3}} = \frac{1}{\frac{1}{4^{1/3}}} = 4^{1/3}$

2. The sum of the series $1+\frac{1}{3^2}+\frac{1\cdot4}{1\cdot2}\cdot\frac{1}{3^4}+\frac{1\cdot4\cdot7}{1\cdot2\cdot3}\cdot\frac{1}{3^6}+........$ is

(1) $\sqrt{\frac{3}{2}}$ (2) $\left(\frac{3}{2}\right)^{1/3}$ (3) $\sqrt{\frac{1}{3}}$ (4) $\left(\frac{2}{3}\right)^{1/3}$

Sol. Answer (2)

In this problem

$$nx=\frac{1}{3^2}$$

$$\frac{n(n-1)}{2}x^2=\frac{1\cdot4}{1\cdot2}\cdot\frac{1}{3^4}$$

Solving above two we get,

$$n=-\frac{1}{3},\text{ and } x=-\frac{1}{3}$$

$\therefore$ Required sum $=\left(1-\frac{1}{3}\right)^{-\frac{1}{3}}=\left(\frac{2}{3}\right)^{-\frac{1}{3}}=\frac{1}{\left(\frac{2}{3}\right)^{1/3}}=\left(\frac{3}{2}\right)^{1/3}$

SECTION - D

Matrix-Match Type Questions

Let n be a positive integer and $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_nx^n$.

1. Match the following:

Column-I	**Column-II**
(A) If the second term of the expansion $\left(x^{\frac{1}{13}}+\frac{x}{\sqrt{x^{-1}}}\right)^n$ is $14x^{5/2}$, then the value of $\frac{^nC_3}{^nC_2}$ is	(p) 5
(B) In the binomial $\left(2^{\frac{1}{3}}+3^{-\frac{1}{3}}\right)^n$, if the ratio of the seventh term from beginning of the expansion to the seventh term from its end is 1/6, then n is equal to	(q) 7
(C) Given that the term of the expansion $(x^{\frac{1}{3}}-x^{-\frac{1}{2}})^{15}$ which does not contain x is 5m, where $m \in N$, then $\frac{m}{143}$ is equal to	(r) 4
(D) If the coefficients of x^7 and x^8 in the expansion of $\left(2+\frac{x}{3}\right)^n$ are equal, then the value of $\frac{n}{11}$ is	(s) 9

Sol. Answer A(r); B(s); C(q); D(p)

(A) $T_2 = {}^nC_1\left(x^{\frac{1}{13}}\right)^{n-1}\left(\dfrac{x}{x^{-\frac{1}{2}}}\right)^1 = 14x^{\frac{5}{2}}$

$\Rightarrow {}^nC_1 x^{\frac{n-1}{13}+\frac{3}{2}} = 14x^{\frac{5}{2}}$

$\Rightarrow n = 14$

Hence, $\dfrac{{}^nC_3}{{}^nC_2} = \dfrac{{}^{14}C_3}{{}^{14}C_2} = \dfrac{14-3+1}{3} = 4$

(B) $\dfrac{T_7 \text{ from the beginning}}{T_7 \text{ from the end}} = \dfrac{{}^nC_6\left(2^{\frac{1}{3}}\right)^{n-6}\left(3^{-\frac{1}{3}}\right)^6}{{}^nC_{n-6}\left(-3^{-\frac{1}{3}}\right)^{n-6}\left(2^{\frac{1}{3}}\right)^6} = \dfrac{1}{6}$

$\Rightarrow \left(2^{\frac{1}{3}}\right)^{n-6-6}\left(3^{-\frac{1}{3}}\right)^{6-n+6} = \dfrac{1}{6}$

$\Rightarrow 2^{\frac{n-12}{3}}\cdot 3^{-\frac{12-n}{6}} = 2^{-1}3^{-1}$

$\Rightarrow n - 12 = -3$ or $\dfrac{12-n}{6} = 1$

$\Rightarrow n = 9$

(C) $T_{r+1} = {}^{15}C_r\left(x^{\frac{1}{3}}\right)^{15-r}\left(-x^{-\frac{1}{2}}\right)^r$

$= {}^{15}C_r x^{\frac{15-r}{3}}(-1)^{\frac{r}{2}} x^{-\frac{r}{2}}$

$\Rightarrow \dfrac{15-r}{3} - \dfrac{r}{2} = 0$

$\Rightarrow 30 - 2r - 3r = 0 \Rightarrow r = 6$

Coefficient of term = ${}^{15}C_6 = 5m$

$\Rightarrow \dfrac{15\times14\times13\times12\times11}{6\times5\times4\times3\times2\times1} = 5$

$\Rightarrow m = 7 \times 143$

$\dfrac{m}{143} = 7$

(D) Coefficient of x^7 = Coefficient of x^8

$${}^nC_7 2^{n-7}\left(\frac{1}{3}\right)^7 = {}^nC_8 2^{n-8}\left(\frac{1}{3}\right)^8$$

$$\Rightarrow\ 2\cdot {}^nC_7 = {}^nC_8\cdot\left(\frac{1}{3}\right)$$

$$\Rightarrow\ 6\cdot {}^nC_7 = {}^nC_8$$

$$\frac{n-8+1}{8} = 6\ \Rightarrow\ \frac{n-7}{8} = 6$$

$\Rightarrow$ n = 55

$\frac{n}{11}$ = 5

2. Match the following:

Column-I	**Column-II**
(A) The remainder when 2^{2007} is divided by 17 is	(p) 8
(B) The greatest value of the term independent of x in the expansion of $(x \sin p + x^{-1} \cos p)^{10}$, $p \in R$ is equal to k, then $\frac{8k}{9}$ is equal to	(q) 9
(C) In the expansion of $(7^{\frac{1}{3}} + 11^{\frac{1}{9}})^{2007}$, the number of rational terms is r then $\frac{r}{14}$ is equal to	(r) 7
(D) If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_{2n}x^{2n}$ then $8\,[a_0 - a_2 + a_4 - a_6 + ...]^2$, if $n = 4p + 2$ is equal to	(s) 16

Sol. Answer A(q); B(r); C(s); D(p)

(A) $y = 2^{2007} = 2^3(2^{2004}) = 2^3(16)^{501}$

$= 2^3(17 - 1)^{501} = -2^3(1 - 17)^{501}$

$= -8(1 - {}^{501}C_1(17) + {}^{50}C_2(-17)^2 + \dots {}^{501}C_{501}(-17)^{501})$

$= -8 + 17 \times I$, where I is an integer.

$\Rightarrow$ When p is divided by 17, then remainder = 17 – 8 = 9

(B) $T_{r+1} = {}^{10}C_r(x \sin p)^{10-r}\,(x^{-1} \cos p)^{rb}$

$$= {}^{10}C_r\, x^{10-r}\, x^{-r}\,(\sin p)^{10-r}\,(\cos p)^r$$

For independent term

$10 - r - r = 0 \Rightarrow\ \ r = 5$

$\Rightarrow$ The coefficient of independent term

$= (\sin p)^5 (\cos p)^5 \times {}^{10}C_5$

$= \frac{1}{32}(2\sin p \cos p)^5 \times {}^{10}C_5$

$= \frac{1}{32}(\sin 2p)^5 \times {}^{10}C_5$

Maximum value of coefficient

$= \frac{1}{32} \times 1 \times \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1}$

$K = \frac{63}{8}$

$\Rightarrow \frac{8k}{9} = \frac{8}{9} \times \frac{63}{8} = 7$

(C) $\left(7^{\frac{1}{3}} + 11^{\frac{1}{9}}\right)^{2007}$

Let general term is T_{r+1}

$T_{r+1} = {}^{2007}C_r \left(7^{\frac{1}{3}}\right)^{2007-r} (11)^{\frac{r}{9}} = {}^{2007}C_r 7^{\frac{2007-r}{3}} (11)^{\frac{r}{9}}$

For rational term both $\frac{2007-r}{3}$ and $\frac{r}{9}$ must be integer

If $\frac{r}{9}$ is integer then r = 9, 18, 27, 36, …. 2007

At these values of r $\frac{2007-r}{3}$ is integer

Hence total term are $\frac{2007}{9} + 1 = 224$

$r = 224$

$\frac{r}{14} = \frac{224}{14} = 16$

(D) $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots. a_{2n}x^{2n}$

Putting $x = i = \sqrt{-1}$

$(1 + i + i^2)^n = a_0 + a_1i + a_2i^2 + a_3i^3 + \ldots. a_{2n}(i)^{2n}$

$\Rightarrow (i)^{2n} = (a_0 - a_2 + a_4 - a_6 + \ldots) + i(a_1 - a_3 + a_5 - a_7 + \ldots)$

$n = 4p + 2$

$\Rightarrow (i)^{8p+4} = (a_1 - a_2 + a_4 - a_6 + \ldots) + i\,(a_1 - a_3 + a_5 - a_7 + \ldots)$

Comparing $a_0 - a_2 + a_4 - a_6 + \ldots = 1$

$8[a_0 - a_2 + a_4 - a_6 + \ldots] = 8$

3. Match the following:

Column-I | **Column-II**

(A) $(3 + x^{10} + x^{11})^{2009} = a_0 + a_1x + a_2x^2 + a_3x^3$.....,

then $a_0 - \frac{a_1}{2} - \frac{a_2}{2} + a_3 - \frac{a_4}{2} - \frac{a_5}{2} + a_6$......

will be equal to

(B) $(3 + x^2 + x^3)^{2008} = a_0 + a_1x + a_2x^2 +$........, then $a_0 + a_1$ is equal to

(C) $(3 - x + x^2)^{2009} = a_0 + a_1x + a_2x^2 +$......., then $a_0 - a_1$ is equal to

(D) In expansion of $\left(x + \frac{1}{x}\right)^{2009}$ the coefficient of middle term may be

(p) 2^{2009}

(q) 3^{2008}

(r) ${}^{2009}C_{1004}$

(s) ${}^{2009}C_{1005}$

(t) 2012.3^{2008}

Sol. Answer A(p); B(q); C(t); D(r, s)

(A) $(3 + x^{10} + x^{11})^{2009} = a_0 + a_1x + a_2x^2 + a_3x^3 +$..........

Put $x = \omega$ and ω^2

$(3 + \omega + \omega^2)^{2009} = a_0 + a_1\omega + a_2\omega^2 + a_3\omega^3 +$..........

$(3 + \omega^2 + \omega)^{2009} = a_0 + a_1\omega^2 + a_2\omega + a_3 + a_4\omega^2 +$ .....+ $= 2^{2009}$

On adding

$2^{2009} + 2^{2009} = 2a_0 - a_1 - a_2 + 2a_3 - a_4 - a_5 + 2a_6 +$.........$= 2^{2009}$

$\Rightarrow a_0 - \frac{a_1}{2} - \frac{a_2}{2} + a_3 - \frac{a_4}{2} - \frac{a_5}{2} + 96$.........$= 2^{2009}$

(B) $(3 + x^2 + x^3)^{2008} = a_0 + a_1x + a_2x^2$ ....

Put $x = 0$

$a_0 = 3^{2008}$, on differentiating and putting $x = 1$

$2008(3 + x^2 + x^3)^{2007}(2x + 3x^2) = a_1 + 2a_2x +$.........

Put $x = 0 \Rightarrow a_1 = 0$

$a_0 + a_1 = 3^{2008}$

(C) $(3 - x + x^2)^{2009} = a_0 + a_1x + a_2x^2 +$ ....

Put $x = 0 \Rightarrow a_0 = 3^{2009}$

On differentiating

$2009(3 - x + x^2)^{2008}(-1 + 2x) = a_1 + 2a_2x +$...........

Put $x = 0$

$a_1 = 2009(3)^{2008}(-1)$

$a_1 = -2009(3)^{2008}$

$a_0 - a_1 = 3^{2009} + 2009 \,.\, 3^{2008}$

$= 3^{2008}(3 + 2009)$

$= 2012 \,.\, 3^{2008}$

(D) Coefficient of middle term $= {}^{2009}C_{1005} = {}^{2009}C_{1004}$

4. Match the entries of Column-I with those of Column-II.

Column-I	Column-II
(A) If the fourth term in the expression of $\left[\sqrt{x^{\left(\frac{1}{\log_{10}x+1}\right)}}+x^{\frac{1}{12}}\right]^6$ equal to 200 and $x > 1$, then x equals	(p) 1
(B) The number of non-zero terms in the expansion of $\left[\left(1+3\sqrt{2}x\right)^9+(1-3\sqrt{2}x)^9\right]$ is	(q) 5
(C) If $\frac{T_2}{T_3}$ in the expansion of $(a + b)^n$ and $\frac{T_3}{T_4}$ in the expansion of $(a + b)^{n+3}$ are equal, then n is equal to	(r) 10
(D) The digit of unit's place in the number $17^{1995} + 11^{1995} - 7^{1995}$ is	(s) $^5C_2$
	(t) $^5C_1$

Sol. Answer A(r, s); B(q, t); C(q, t); D(p)

(A) $T_4 = 200$

$$\Rightarrow {}^6C_3\left[x^{\frac{3}{2(\log x+1)}}\cdot x^{\frac{1}{4}}\right]=200$$

$$\Rightarrow x^{\frac{3}{2(\log x+1)}+\frac{1}{4}}=10$$

$$\Rightarrow \frac{3}{2(\log x+1)}+\frac{1}{4}=\frac{1}{\log_{10}x}$$

$$\Rightarrow \frac{6+\log x+1}{4(\log x+1)}=\frac{1}{\log_{10}x}$$

Now let $\log_{10}x = y$

$$\Rightarrow \frac{7+y}{4(y+1)}=\frac{1}{y}$$

$\Rightarrow 4y + y^2 = 4y + 4$

$\Rightarrow y^2 + 3y - 4 = 0$

$\Rightarrow y^2 + 4y - y - 4 = 0$

$\Rightarrow y = 1, -4$

$\Rightarrow x = 10^1, 10^{-4}$

$\because x > 1$

$\therefore x = 10$

(B) Required expansion = $2[T_1 + T_3 + T_5 + T_7 + T_9]$

$\therefore$ Number of non-zero terms = 5

(C) Since $\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\left(\frac{b}{a}\right)$

$\therefore\ \frac{T_3}{T_2}=\frac{n-1}{2}\left(\frac{b}{a}\right)$

$\frac{T_4}{T_3}=\frac{n+1}{3}\left(\frac{b}{a}\right)$

Now, $\frac{n-1}{2}=\frac{n+1}{3}$

$\Rightarrow 3n - 3 = 2n + 2$

$\Rightarrow n = 5$

(D) $17^{1995} + 11^{1995} - 7^{1995}$

$= (7 + 10)^{1995} + (10 + 1)^{1995} - (7)^{1995}$

$= ({}^{1995}C_{1}\cdot 7^{1994}\cdot 10^1 + \ldots\ldots + 10^{1995}) + ({}^{1995}C_{1}\cdot 10^1 + \ldots\ldots\ {}^{1995}C_{1995} + 10^{1995}) + 1$

= (a multiple of 10) + 1

Hence the units place digit = 1.

SECTION - E

Assertion-Reason Type Questions

1. STATEMENT-1 : The number of distinct term in the expansion of $(1 + px)^{20} + (1 - px)^{20}$ is 42.

and

STATEMENT-2 : Number of term in the expansion of $(1 + x)^n$ is $(n + 1)$.

Sol. Answer (4)

$(1 + px)^{20} = {}^{20}C_0 + {}^{20}C_1(px) + {}^{20}C_2(px)^2 + \ldots + {}^{20}C_{20}(px)^{20}$ ...(i)

$(1 - px)^{20} = {}^{20}C_0 - {}^{20}C_1(px) + {}^{20}C_2(px)^2 - \ldots + {}^{20}C_{20}(px)^{20}$ ...(ii)

Adding the equations

$(1 + px)^{20} + (1 - px)^{20} = 2[{}^{20}C_0 + {}^{20}C_2(px)^2 + {}^{20}C_4(px)^4 + \ldots + {}^{20}C_{20}(px)^{20}]$

Hence total terms is 11.

Hence statement-1 is false but statement-2 is true.

2. STATEMENT-1 : The coefficient of $a^3b^4c^3$ in the expansion of $(a - b + c)^{10}$ is $\frac{10!}{3!\,4!\,3!}$.

and

STATEMENT-2 : The coefficient of $x^py^qz^r$ in the expansion of $(x + y + z)^n$ is $\frac{n!}{p!\,q!\,r!}$ for all integer n.

Sol. Answer (3)

Statement-1 is correct because coefficient is

$(-1)^4\frac{10!}{3!\,4!\,3!}=\frac{10!}{3!\,4!\,3!}$

But statement-2 is false because no conditions is given on p, q, r

If $p + q + r \neq n$, then coefficient will be zero.

3. STATEMENT-1 : If $\sum_{r=1}^{n} r^3 \left(\frac{{}^nC_r}{{}^nC_{r-1}}\right)^2 = 196$, then the sum of the coefficients of power of x in the expansion of the polynomial $(x - 3x^2 + x^3)^n$ is –1.

and

STATEMENT-2 : $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$ $\forall\ n \in N$ and $r \in W$.

Sol. Answer (4)

$$\sum_{r=1}^{n} r^3 \left(\frac{n-r+1}{r}\right)^2 = \sum_{r=1}^{n} r(n-r+1)^2$$

$$= \sum_{r=1}^{n} r\,\{(n+1)^2 - 2(n+1)r + r^2\}$$

$$= (n+1)^2 \sum_{r=1}^{n} r - 2(n+1)\sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r^3$$

$$= (n+1)^2 \cdot \frac{n(n+1)}{2} - \frac{2(n+1)\cdot n(n+1)(2n+1)}{6} + \left(\frac{n(n+1)}{2}\right)^2$$

$$\Rightarrow \frac{(n+1)^2 \cdot n(n+2)}{12} = 14^2$$

$$\Rightarrow \boxed{n = 6}$$

$\therefore$ Sum of coefficients = $(1 - 3 + 1)^6 = 1$

4. STATEMENT-1 : The number of terms in the expansion of $\left(x + \frac{1}{x} + 1\right)^n$ is $2n + 1$.

and

STATEMENT-2 : The number of terms in the expansion of $(a_1 + a_2 + \text{........ } a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

Sol. Answer (2)

Statement-1:

Given expression is = $\left(1 + \left(x + \frac{1}{x}\right)\right)^n$

$$= {}^nC_0 \cdot 1^n \cdot \left(x + \frac{1}{x}\right)^0 + {}^nC_1 \cdot 1^{n-1}\left(x + \frac{1}{x}\right)^1 + {}^nC_2 \cdot 1^{n-2}\left(x + \frac{1}{x}\right)^2 + \text{........} + \left(x + \frac{1}{x}\right)^n$$

This will be of the form $= a_0 + a_1x + a_2x^2 + \text{........} + a_nx^n + \frac{b_1}{x} + \frac{b_2}{x^2} + \text{........} + \frac{b_n}{x^n}$

Hence, number of term $= n + n + 1 = 2n + 1$

Statement-2, obviously the number of terms in the expansion of

$(a_1 + a_2 + \text{........} + a_{-n})^n$ is ${}^{n+m-1}C_{m-1}$

Hence Statement-1 and Statement-2 both are true but (2) is not the correct explanation of (1).

5. STATEMENT-1 : Sum of the coefficients of last 30 terms in the expansion of $(1 + x)^{49}$, when expanded in ascending powers of x, is 2^{48}.

and

STATEMENT-2 : P^{th} term from the end in the expansion of $(x + y)^n$ is $(n – P + 2)^{th}$ term from the beginning.

Sol. Answer (4)

The sum of last 25 terms in the expansion of $(1 + x)^{49}$ will be 2^{48}. Hence, sum of 30 terms will be obviously greater than 2^{48}.

Hence, Statement-1 is false but Statement-2 is true.

6. STATEMENT-1 : In the expansion of $(\sqrt{5}+3^{1/5})^{10}$, sum of integral terms is 3134.

and

STATEMENT-2 : $(x+y)^n=\sum_{r=0}^{n} {}^nC_r\cdot x^{n-r}y^r$.

Sol. Answer (1)

Sum of integral term

$= {}^{10}C_0\cdot(5^{1/2})^0(3^{1/5})^{10}+{}^{10}C_{10}\cdot(5^{1/2})^{10}\cdot(3^{1/5})^0$

$= 3^2 + 5^5$

$= 9 + 3125 = 3134$

Hence, Statement-1 is true and Statement-2 is also true and Statement-2 is the correct explanation of Statement-1.

SECTION - F

Integer Answer Type Questions

1. If the coefficient of x^3 is 1140 in the expansion of $(1 + 2x + kx^2)^{10}$, then the value of k is ________.

Sol. Answer (1)

General term in the expansion of

$(1+2x+kx^2)^{10}$ is $=\frac{10!}{p!\,q!\,r!}\cdot 1^p\cdot(2x)^q\cdot(kx^2)^r \;=\frac{10!}{p!\,q!\,r!}\cdot 2^q\cdot k^r\cdot x^{q+2r}$

Under the condition

$p + q + r = 10$

$q + 2r = 3$

p	q	r
7	3	0
8	1	1

Now, the coefficient of x^3 in $(1 + 2x + kx^2)^{10}$

$= \frac{10!}{7!3!0!}\cdot 2^3\cdot k^0+\frac{10}{8!1!1!}\cdot 2^1\cdot k^1$

$= 960 + 180k$

Now, $960 + 180k = 1140$

$\Rightarrow 180k = 180$

$\Rightarrow k = 1$

2. The sum $\sum_{i=0}^{3m}\binom{10}{i}\binom{20}{3m-i}$, where $\binom{p}{q}=0$ if $p < q$ is maximum then m is ________.

Sol. Answer (5)

$$\sum_{i=0}^{3m}\binom{10}{i}\binom{20}{3m-i}=\sum_{i=0}^{3m} {}^{10}C_i \cdot {}^{20}C_{3m-i}$$

= Coefficient of x^{3m} in the expansion of $(1 + x)^{10}(1 + x)^{20} = {}^{30}C_{3m}$

It is maximum when $3m=\frac{30}{2}$

$\Rightarrow m=\frac{15}{3}=5$

3. If $(1 + x)^{15} = C_0 + C_1x + C_2x^2 + C_3x^3 + ... + C_{15}x^{15}$ and $(k = C_2 + 2C_3 + 3C_4 + ... + 14C_{15})$ then the value of $\frac{k-993}{53000}$ is equal to ....

Sol. Answer (4)

$(1 + x)^{15} = C_0+C_1x+C_2x^2+....+C_{15}x^{15}$

$$\frac{(1+x)^{15}}{x} = \frac{C_0}{x}+C_1+C_2x+C_3x^2+....+C_{15}x^{14}$$

Differentiating with respect to x

$$\frac{(1+x)^{15}-15(1+x)^{14}x}{x^2} = -\frac{C_0}{x^2}+0+C_2x+2C_3x+3C_4x^3+....14.C_{15}x^{13}$$

Putting $x = 1$, we get

$$\frac{2^{15}-15.2^{14}}{1}=-1+k$$

$\Rightarrow k = 2^{15} - 15.2^{14} + 1$

$\Rightarrow k - 993 = 2^{15} - 15.2^{14} - 992$

$$\Rightarrow \frac{k-993}{53000}=\frac{2^{15}-15.2^{14}-992}{53000} = \frac{212}{53}=4$$

4. If C_r stands for ${}^nC_r$, then the sum of the first $(n + 1)$ terms of the series $aC_0 - (a + d) C_1 + (a + 2d) C_2 - (a + 3d) C_3 +$ ...., is equal to ....

Sol. Answer (0)

$aC_0-(a+d)C_1+(a+2d)C_2-(a+3d)C_3+....$

$=a(C_0-C_1+C_2-C_3+....)-d(C_1-2C_2+3C_3-4C_4+....)$

$= a \times 0 - d \times 0 = 0$

5. In the expansion of $(1 + x)^{10} = {}^{10}C_0 + {}^{10}C_1x + {}^{10}C_2x^2 + ... + {}^{10}C_{10}x^{10}$, the value of

$$528\left[\frac{{}^{10}C_0}{2} - \frac{{}^{10}C_1}{3} + \frac{{}^{10}C_2}{4} - \frac{{}^{10}C_3}{5} + ... + \frac{{}^{10}C_{10}}{12}\right]$$ is equal to ....

Sol. Answer (4)

$$(1 + x)^{10} = {}^{10}C_0 + {}^{10}C_1x + {}^{10}C_2x^2 + ... + {}^{10}C_{10}x^{10}$$

$$x(1 + x)^{10} = {}^{10}C_0x + {}^{10}C_1x^2 + {}^{10}C_2x^3 + ... + {}^{10}C_{10}x^{11}$$

$$\int x(1+x)^{10}dx = \int({}^{10}C_0 + {}^{10}C_1x + {}^{10}C_2x^2 + ...)xdx$$

$$\int x(1+x)^{10}dx = \int({}^{10}C_0x + {}^{10}C_1x^2 + ...{}^{10}C_{10}x^{11})dx$$

$$\int x(1+x)^{10}dx = {}^{10}C_0\frac{x^2}{2} + {}^{10}C_1\frac{x^3}{3} + {}^{10}C_2\frac{x^4}{4} + ... + {}^{10}C_{10}\frac{x^{12}}{12} + k$$

$$= \frac{x(1+x)^{11}}{11} - \int\frac{(1+x)^{11}}{11}dx = {}^{10}C_0\frac{x^2}{2} + \frac{{}^{10}C_1x^3}{3} + ... + \frac{{}^{10}C_{10}x^{12}}{12} + k$$

$$\Rightarrow \frac{x(1+x)^{11}}{11} - \frac{(1+x)^{12}}{132} = {}^{10}C_0\frac{x^2}{2} + \frac{{}^{10}C_1x^3}{3} + ... + \frac{{}^{10}C_{10}x^{12}}{12} + k$$

Put $x = 0$

$$-\frac{1}{132} = k$$

$$\Rightarrow \frac{x(1+x)^{11}}{11} - \frac{(1+x)^{12}}{132} = {}^{10}C_0\frac{x^2}{2} + \frac{{}^{10}C_1x^3}{3} + ... + \frac{{}^{10}C_{10}x^{12}}{12} - \frac{1}{132}$$

At $x = -1$

$$\Rightarrow 0 = \frac{{}^{10}C_0}{2} - \frac{{}^{10}C_1}{3} + \frac{{}^{10}C_2}{4} - \frac{{}^{10}C_3}{5} + ... + \frac{{}^{10}C_{10}}{12} - \frac{1}{132}$$

$$\Rightarrow \frac{{}^{10}C_0}{2} - \frac{{}^{10}C_1}{3} + \frac{{}^{10}C_2}{4} - \frac{{}^{10}C_3}{5} + ... = \frac{1}{132}$$

$$528\left[\frac{{}^{10}C_0}{2} - \frac{{}^{10}C_1}{3} + \frac{{}^{10}C_2}{4} - \frac{{}^{10}C_3}{5}...\right] = 528 \times \frac{1}{132} = 4$$

6. Let $\sum_{r=0}^{2010} a_r x^r = (1 + x + x^2 + x^3 + x^4 + x^5)^{402}$ and $\sum_{r=0}^{2010} a_r = a$, then the value of $\left(\frac{\sum_{r=0}^{2010} r.a_r}{\sum_{r=0}^{2010} a_r}\right) - 1000$ is ....

Sol. Answer (5)

$$(1 + x + x^2 + x^3 + x^4 + x^5)^{402} = \sum_{r=0}^{2010} a_r x^r \quad ...(i)$$

$$\text{At } x = 1,\ 6^{402} = \sum_{r=0}^{2010} a_r \quad ...(ii)$$

Differentiating equation (i) with respect to x

$\Rightarrow 402(1 + x + x^2 + x^3 + x^4 + x^5)^{401}(1 + 2x + 3x^2 + 4x^3 + 5x^4)$

$$= \sum_{r=0}^{2010} r.a_r.x^{r-1}$$

At $x = 1$, we get

$$402\times 6^{401}\left(\frac{5\times 6}{2}\right) = \sum_{r=0}^{2010} r.a_r \qquad \ldots\text{(iii)}$$

Dividing equation (iii) by (ii)

$$\frac{\sum_{r=0}^{2010} r.a_r}{\sum_{r=0}^{2010} a_r} = \frac{(402)\times 6^{401}\times 15}{6^{402}} = \frac{402}{6}\times 15 = 67\times 15 = 1005$$

$$\left(\frac{\sum_{r=0}^{2010} r.a_r}{\sum_{r=0}^{2010} a_r}\right) - 1000 = 1005 - 1000 = 5$$

7. The digit at tenth place of $(81)^{100}(121)^{100} - 1$ is

Sol. Answer (0)

$(81)^{100}(121)^{100} - 1$

$= (81 \times 121)^{100} - 1$

$= (99)^{200} - 1 = (100 - 1)^{200} - 1$

Clearly the digits at units and tens place is 0.

8. If $(1+x)^n = \sum_{r=0}^{n} a_r x^r$ and $b_0 = 1 + \frac{a_r}{a_{r-1}}$ and $\prod_{r=1}^{n} b_r = \frac{10^9}{9!}$, then n is ________.

Sol. Answer (9)

We have, $(1+x)^n = \sum_{r=0}^{n} a_r x^r$

$\therefore \quad a_r = {}^nC_r$

Now, $b_r = 1 + \frac{{}^nC_r}{{}^nC_{r-1}}$

$$= 1 + \frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(n-r+1)!(r-1)!}}$$

$$= 1 + \frac{(n-r+1)!\,(r-1)!}{r!\cdot(n-r)!} = 1 + \frac{n-r+1}{r}$$

$$= \frac{r+n-r+1}{r} = \frac{n+1}{r}$$

$$\therefore \prod_{r=1}^{n} b_r = \prod_{r=1}^{n} \frac{n+1}{r} = \frac{(n+1)^n}{n!}$$

Now, $\frac{(n+1)^n}{n!} = \frac{10^9}{9!}$

$\Rightarrow$ $n = 9$

9. If $(1 + x + x^2)^{3n+1} = a_0 + a_1x + a_2x^2 + \ldots a_{6n+2}x^{6n+2}$, then find the value of $\sum_{r=0}^{2n} a_{3r} - \left(\frac{a_{3r+1}+a_{3r+2}}{2}\right)$ is

Sol. Answer (0)

$(1 + x + x^2)^{3n+1} = a_0 + a_1x + a_2x^2 + \ldots a_{6n}x^{6n} + a_{6n+1}x^{6n+1} + a_{6n+2}x^{6n+2}$

putting $x = \omega$ and ω^2, we get

$(1 + \omega + \omega^2)^{3n+1} = 0 = a_0 + a_1\omega + a_2\omega^2 + a_3 + \ldots + a_{6n} + a_{6n+1}\omega + a_{6n+2}\omega^2\ldots$ ...(1)

$(1 + \omega^2 + \omega^4)^{3n+1} = 0 = a_0 + a_1\omega^2 + a_2\omega + a_3 + \ldots + a_{6n+1}\omega^2 + a_{6n+2}\omega\ldots$ ...(2)

Adding (1) and (2), we get

$$\sum_{r=0}^{2n} (2a_{3r} - (a_{3r+1} + a_{3r+2})) = 0.$$

10. Find the value of

${}^nC_0 \cdot {}^{2n}C_n - {}^nC_1 \cdot {}^{2n-1}C_n + {}^nC_2 \cdot {}^{2n-2}C_n - {}^nC_3 \cdot {}^{2n-3}C_n + \ldots + (-1)^n \cdot {}^nC_n \cdot {}^nC_n$ (where $n \in N$)

Sol. Answer (1)

Value of given expression will be coefficient of x^n in

$[{}^nC_0(1 + x)^{2n} - {}^nC_1(1 + x)^{2n-1} + {}^nC_2(1 + x)^{2n-2} + \ldots + (-1)^n \cdot {}^nC_n(1 + x)^n]$

Coefficient of x^n in

$(1 + x)^n[{}^nC_0(1 + x)^n - {}^nC_1(1 + x)^{n-1} + {}^nC_2(1 + x)^{n-2} + \ldots + (-1)^n \cdot {}^nC_n]$

Coefficient of x^n in

$(1 + x)^n[(1 + x) - 1]^n = 1$

11. If P_r is the coefficient of x^r in the expansion of $(1+x)^2\left(1+\frac{x}{2}\right)^2\left(1+\frac{x}{2^2}\right)^2\ldots$, if $P_r = \frac{\lambda}{2^r-1}(P_{r-1}+P_{r-2})$ then the value of λ.

Sol. Answer (4)

Let $(1+x)^2\left(1+\frac{x}{2}\right)^2\left(1+\frac{x}{2^2}\right)^2\ldots = P_0 + P_1x + P_2x^2 + \ldots + P_rx^r + \ldots$ ...(1)

If x is replaced by $\frac{x}{2}$, then

$$\left(1+\frac{x}{2}\right)^2\left(1+\frac{x}{2^2}\right)^2+\left(1+\frac{x}{2^3}\right)^2\ldots\left(1+\frac{x}{2^n}\right)^2 = P_0 + P_1\left(\frac{x}{2}\right) + P_2\left(\frac{x}{2}\right)^2 + \ldots + P_r\left(\frac{x}{2}\right)^r$$

Now, equation (1) becomes

$$(1+x)^2\left[P_0+P_1\left(\frac{x}{2}\right)+P_2\left(\frac{x}{2}\right)^2+...+P_r\left(\frac{x}{2}\right)^r+...\right]=P_0+P_1x+P_2x^2+...P_rx^rx+...$$

Equating the coefficient of x^r, we get

$$\frac{P_r}{2^r}+2\frac{P_{r-1}}{2^{r-1}}+\frac{P_{r-2}}{2^{r-2}}=P_r \Rightarrow P_r+4P_{r-1}+4P_{r-2}=2^rP_r$$

$$\Rightarrow P_r=\frac{2^2(P_{r-1}+P_{r-2})}{2^r-1}$$

12. If ${}^{100}C_0\cdot{}^{100}C_2+{}^{100}C_2\cdot{}^{100}C_4+{}^{100}C_4\cdot{}^{100}C_6+...+{}^{100}C_{98}\cdot{}^{100}C_{100}=\frac{1}{\lambda}\left[{}^{200}C_{98}-{}^{100}C_{49}\right]$ then find the value of λ.

Sol. Answer (2)

$(1+x)^{100}={}^{100}C_0x^{100}+{}^{100}C_1x^{99}+...+{}^{100}C_{100}$

$(1+x)^{100}={}^{100}C_0+{}^{100}C_1x+{}^{100}C_2x^2+...+{}^{100}C_{100}x^{100}$

$\Rightarrow\ (1+x)^{200}=({}^{100}C_0x^{100}+{}^{100}C_1x^{99}+{}^{100}C_2x^{98}+...+{}^{100}C_{100})$

$({}^{100}C_0+{}^{100}C_1x+{}^{100}C_2x^2+...+{}^{100}C_{100}x^{100})$

$\Rightarrow\ ({}^{100}C_0\cdot{}^{100}C_2+{}^{100}C_1\cdot{}^{100}C_3+{}^{100}C_2\cdot{}^{100}C_4+...+{}^{100}C_{98}\cdot{}^{100}C_{100})={}^{200}C_{102}$

$\Rightarrow\ x+y={}^{200}C_{102}$

Where $x={}^{100}C_0\cdot{}^{100}C_2+{}^{100}C_2\cdot{}^{100}C_4+...+{}^{100}C_{98}\cdot{}^{100}C_{100}$

and $y={}^{100}C_1\cdot{}^{100}C_3+{}^{100}C_3\cdot{}^{100}C_5+...+{}^{100}C_{97}\cdot{}^{100}C_{99}$

Also $(1+x)^{100}(1-x)^{100}=({}^{100}C_0x^{100}+{}^{100}C_1x^{99}+...+{}^{100}C_{100})$

$({}^{100}C_0-{}^{100}C_1x+{}^{100}C_2x^2-...+{}^{100}C_{100}x^{100})$

${}^{100}C_0\cdot{}^{100}C_2-{}^{100}C_1\cdot{}^{100}C_3+{}^{100}C_2\cdot{}^{100}C_4-...+{}^{100}C_{98}\cdot{}^{100}C_{100}=x-y=-{}^{100}C_{49}$

$\Rightarrow\ 2x={}^{200}C_{98}-{}^{100}C_{49}$

$$x=\frac{1}{2}\left[{}^{200}C_{98}-{}^{100}C_{49}\right]$$

Hence $\lambda=2$.

13. If ${}^9C_n+3\cdot{}^9C_{n+1}+3\cdot{}^9C_{n+2}+{}^9C_{n+3}>{}^{12}C_{n+2}$ then find the greatest possible value of n.

Sol. Answer (3)

We have

${}^9C_n+3\cdot{}^9C_{n+1}+3\cdot{}^9C_{n+2}+{}^9C_{n+3}=({}^9C_n+{}^9C_{n+1})+2({}^9C_{n+1}+{}^9C_{n+2})+{}^9C_{n+2}+{}^9C_{n+3}$

$={}^{10}C_{n+1}+2\cdot{}^{10}C_{n+2}+{}^{10}C_{n+3}={}^{10}C_{n+1}+{}^{10}C_{n+2}+{}^{10}C_{n+2}+{}^{10}C_{n+3}$

$={}^{11}C_{n+2}+{}^{11}C_{n+3}={}^{12}C_{n+3}$.

Hence the given inequality becomes ${}^{12}C_{n+3}>{}^{12}C_{n+2}$

$$\Rightarrow\frac{(12)!}{(n+3)!(9-n)!}>\frac{(12)!}{(n+2)!(10-n)!}$$

$\Rightarrow 10-n>n+3\Rightarrow n<\frac{7}{2}$, but n is an integer.

Hence greatest value of $n=3$.

14. Find the unit digit in the expansion of $(101)^{202} + (202)^{101} - (107)^{99}$.

Sol. Answer (0)

$(101)^{202} + (202)^{101} - (107)^{99}$

$(101)^{202} = (100 + 1)^{202}$

$= {}^{202}C_0 100^{202} + \dots {}^{202}C_{201} 100 + 1$ $\Rightarrow$ Unit digit is 1

$(202)^{101} = (200 + 2)^{101}$

$= {}^{101}C_0 200^{101} + \dots {}^{101}C_{100}(200)2^{100} + 2^{101}$

$M(100) + 2^{101}$

$2^{101} = 2(2^4)^{25} = 2[(2^4)(2^4) \dots (2^4)25 \text{ times}]. 2[6.6\dots6]$ 25 times

$\Rightarrow$ Units digit is 2

$\therefore$ Units digit in $2(2^4)^{25}$ is 2.

Also $(107)^{99} = (100 + 7)^{99}$

$= {}^{99}C_0 100^{99} + \dots {}^{99}C_{98}(100)7^{98} + 7^{99}$

Now $7^{99} = 7^3(7^4)^{24}$.

$= 7^3(7^4 \cdot 7^4 \dots 24 \text{ times})$

$\Rightarrow$ Unit digit is 3.

$\Rightarrow$ Unit digit in $(101)^{202} + (202)^{101} - (107)^{99}$

$\Rightarrow$ $1 + 2 - 3 = 0$

15. Find the digit at the unit place of the number $(1997)! + 3^{1997}$.

Sol. Answer (3)

Clearly the digit at the unit place of 1997! is 0.

Also the digit at unit place of 3^4 is 1.

Now $3^{1997} = (3^4)^{499} \cdot 3$

$\Rightarrow$ The digit at unit place of 3^{1997} is 3.

$\Rightarrow$ The digit at unit place of $(1997)! + 3^{1997}$ is 3

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