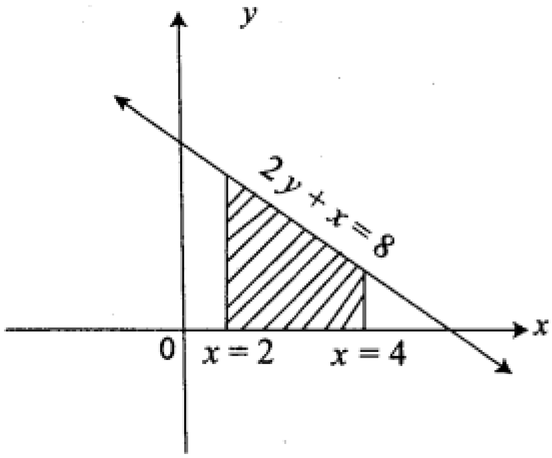
Chapter 3
Integral Calculus II

Ex 3.1

Question 1.
Using Integration, find the area of the region bounded the line 2y + x = 8, the x-axis and the lines x = 2, x = 4

Solution:
The given lines are 2y + x = 8, x-axis, x = 2, x = 4



$$\text{Required area} = \int_2^4 y\,dx$$

$$\text{Now } 2y + x = 8 \Rightarrow y = \frac{8-x}{2}$$

$$\text{So area} = \int_2^4 \left(\frac{8-x}{2}\right) dx$$

$$= \frac{1}{2}\left[8x - \frac{x^2}{x}\right]_2^4$$

$$= \frac{1}{2}[32 - 8 - 16 + 2] = 5 \text{ sq. units}$$

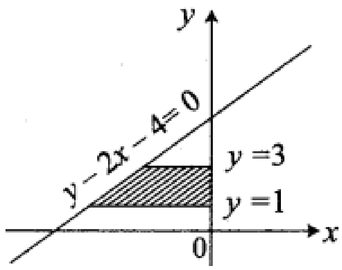
Question 2.
Find the area bounded by the lines y – 2x – 4 = 0, y = 1, y = 3 and the y-axis.

Solution:

Given lines are y – 2x – 4 = 0, y = 1, y = 3, y-axis

y – 2x = 4 = 0 ⇒ x = $\frac{y-4}{2}$

We observe that the required area lies to the left to the y-axis



$$\text{So Area} = \int_1^3 -x\,dy$$

$$= -\int_1^3 \frac{y-4}{2}\,dy = -\frac{1}{2}\left[\frac{y^2}{2} - 4y\right]_1^3$$

$$= -\frac{1}{2}\left[\frac{9}{2} - 12 - \frac{1}{2} + 4\right] = -\frac{1}{2}(-4) = 2 \text{ sq.units}$$

Question 3.

Calculate the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

Solution:

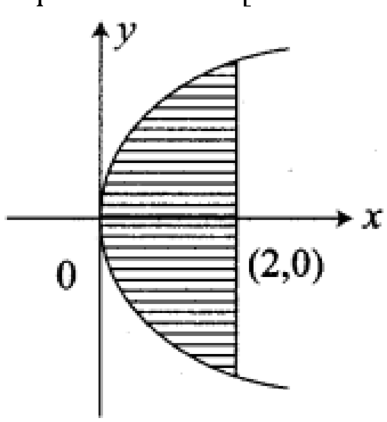
The latus rectum of the parabola is the line x = a through its focus (a, 0) perpendicular to the x-axis.

Equation of parabola

$y^2 = 4ax$

$y = \sqrt{4ax} \Rightarrow y = 2\sqrt{a}\,\sqrt{x}$

Required area = 2[Area in the first quadrant between the limits x = 0 and x = a]



$$= 2\int_0^a y\,dx$$

$$= 2\int_0^a \sqrt{4ax}\ dx$$

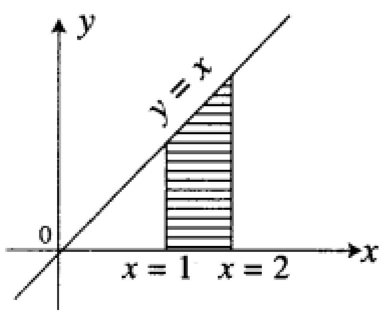
$$= 2\left(2\sqrt{a}\right)\int_0^a x^{\frac{1}{2}}\ dx$$

$$= 4\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$$

$$= \left(4\sqrt{a}\right)\frac{2}{3}a^{\frac{3}{2}} = \frac{8}{3}a^2 \text{ sq.units}$$

Question 4.
Find the area bounded by the line y = x, the x-axis and the ordinates x = 1, x = 2.

Solution:
Given lines are y = x, x-axis, x = 1, x = 2

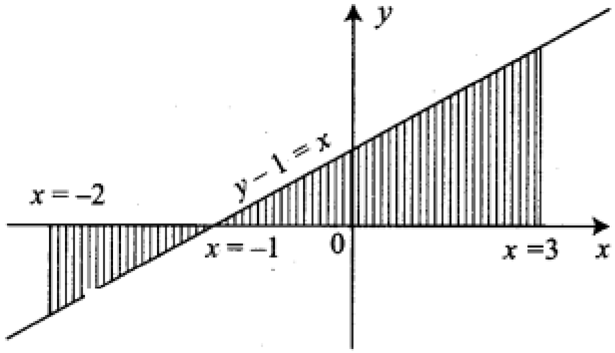


$$\textbf{Required area} = \int_1^2 y\,dx = \int_1^2 x\,dx = \frac{x^2}{2}\int_1^2 = 2 - \frac{1}{2}$$

$$= \frac{3}{2} \text{ sq.units}$$

Question 5.
Using integration, find the area of the region bounded by the line y – 1 = x, the x axis and the ordinates x = -2, x = 3
Solution:

Given lines are y – 1 = x, x-axis, x = -2, x = 3



$$\text{Required area} = \int_{-2}^{-1} -y\,dx + \int_{-1}^{3} y\,dx$$

$$= -\int_{-2}^{-1} (x+1)\,dx + \int_{-1}^{3} (x+1)\,dx$$

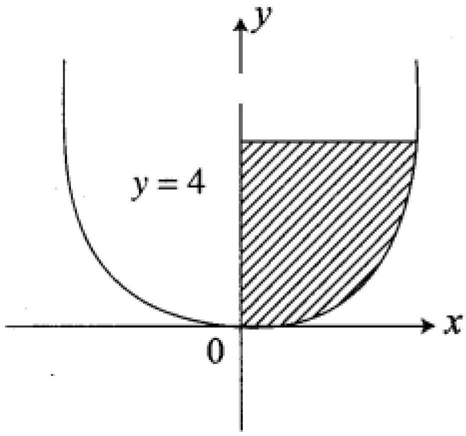
$$= -\left[\frac{(x+1)^2}{2}\right]_{2}^{-1} + \left[\frac{(x+1)^2}{2}\right]_{-1}^{3}$$

$$= -\frac{1}{2}[-1] + \frac{1}{2}[16-0]$$

$$= \frac{1}{2} + 8 = \frac{17}{2} \text{ sq.units}$$

Question 6.
Find the area of the region lying in the first quadrant bounded by the region $y = 4x^2$, $x = 0$, $y = 0$ and $y = 4$.

Solution:
The given parabola is $y = 4x^2$
$x^2 = \frac{y}{4}$
comparing with the standared form $x^2 = 4ay$
4a=14⇒a=116
The parabola is symmetric about y-axis
We require the area in the first quadrant.

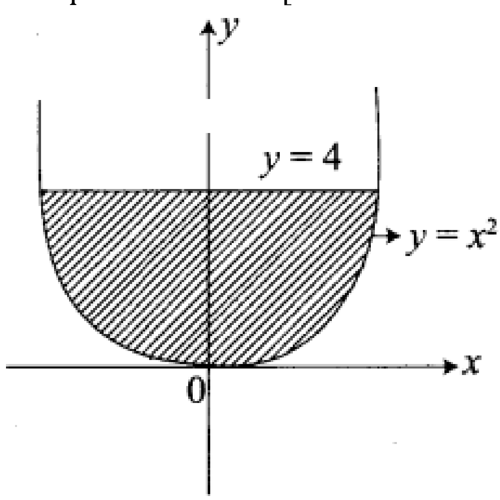


$$\text{Area} = \int_0^4 x\,dy = \int_0^4 \sqrt{\frac{y}{4}}\,dy = \frac{1}{2}\int_0^4 \sqrt{y}\,dy$$

$$= \frac{1}{2}\left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 = \frac{1}{3}(4)^{\frac{3}{2}} = \frac{8}{3} \text{ sq.units}$$

Question 7.
Find the area bounded by the curve $y = x^2$ and the line $y = 4$

Solution:
Given the parabola is $y = x^2$ and line $y = 4$
The parabola is symmetrical about the y-axis.
So required area = 2 [Area in the first quadrant between limits y = 0 and y = 4]



$$= \quad 2\int_0^4 x\,dy = 2\int_0^4 \sqrt{y}\,dy$$

$$= \quad 2\left[\frac{2}{3}y^{\frac{3}{2}}\right]_0^4 = \frac{4}{3}.8 = \frac{32}{3}\text{ sq.units}$$

Ex 3.2

Question 1.
The cost of an overhaul of an engine is ₹ 10,000 The operating cost per hour is at the rate of 2x – 240 where the engine has run x km. Find out the total cost if the engine runs for 300 hours after overhaul.

Solution:
M.C = 2x – 240
Total cost C = $\int$(M.C)dx
= $\int$(2x – 240)dx
C = $2(\frac{x^2}{2}) - 240x + c$
Where C = 10000
C = $x^2 - 240x + 10000$
Where x = 300
C = $(300)^2 - 240(300) + 10,000$
= 90000 – 72000 + 10,000
= 100000 – 72000
C = Rs 28,000

Question 2.
Elasticity of a function $\frac{E_y}{E_x}$ is given by $\frac{E_y}{E_x} = \frac{-7x}{(1-2x)(2+3x)}$ Find the function when x = 2, y = $\frac{3}{8}$

Solution:

$$\text{Given } \eta = \frac{E_y}{E_x} = \frac{-7x}{(1-2x)(2+3x)} \Rightarrow \frac{x}{y}\frac{dy}{dx} = \frac{-7x}{(1-2x)(2+3x)}$$

$$\frac{dy}{y} = \frac{-7x}{(1-2x)(2+3x)}\ \frac{dx}{x}$$

$$\int\frac{dy}{y} = 7\int\frac{dx}{(2x-1)(3x+2)} \qquad .....(1)$$

$$\frac{1}{(2x-1)(3x+2)} = \frac{A}{(2x-1)} + \frac{B}{(3x+2)}$$

$$1 = A(3x+2) + B(2x-1)$$

$$\text{Let } x = \frac{1}{2} \Rightarrow 1 = A\left(\frac{3}{2}+2\right)$$

$$1 = A\left(\frac{7}{2}\right) \Rightarrow A = \frac{2}{7}$$

$$\text{Let } x = \frac{-2}{3} \Rightarrow 1 = B\left[2\left(\frac{-2}{3}\right) - 1\right]$$

$$1 = B\left(\frac{-4}{3} - 1\right)$$

$$1 = B\left(\frac{-7}{3}\right) \Rightarrow B = \frac{-3}{7}$$

Using these values in (1) we get

$$\int \frac{dy}{y} = 7\int \frac{\frac{2}{7}}{2x-1}dx - 7\int \frac{\frac{3}{7}}{3x+2}dx$$

$$\int \frac{dy}{y} = \int \frac{2dx}{2x-1} - \int \frac{3dx}{3x+2}$$

$$\log y = \log(2x-1) - \log(3x+2) + \log k$$

$$y = \left(\frac{2x-1}{3x+2}\right)k$$

$$\text{when } x = 2, y = \frac{3}{8} \Rightarrow \frac{3}{8} = \frac{3}{8}k$$

$$\Rightarrow k = 1$$

Hence the function is $y = \dfrac{2x-1}{3x+2}$

Question 3.

The elasticity of demand with respect to price for a commodity is given by $\frac{(4-x)}{x}$ where p is the price when demand is x. Find the demand function when the price is 4 and the demand is 2. Also, find the revenue function.

Solution:

$$\text{Given} \quad \eta_d = \frac{4-x}{x}$$

$$(i.e) \quad \frac{-p}{x}\frac{dx}{dp} = \frac{4-x}{x}$$

$$\Rightarrow \quad \frac{-dx}{4-x} = \frac{dp}{p}$$

$$\int \frac{dx}{x-4} = \int \frac{dp}{p}$$

$$\log (x-4) = \log p + \log k$$

$$x-4 = pk$$

When $p = 4, x = 2$

gives $\quad 2 - 4 = 4k$

$$\Rightarrow \quad k = -\frac{1}{2}$$

Hence $\quad p = \dfrac{x-4}{\left(-\frac{1}{2}\right)}$

$p = 8 - 2x$ is the demand function.

The Revenue function $\quad R = px$

So $\quad R = 8x - 2x^2$

Question 4.
A company receives a shipment of 500 scooters every 30 days. From experience, it is known that the inventory on hand is related to the number of days x. Since the shipment, $I(x) = 500 - 0.03x^2$, the daily holding cost per scooter is ₹0.3. Determine the total cost for maintaining inventory for 30 days.

Solution:
Here inventory $I(x) = 500 - 0.03x^2$
Unit holding cost C_1 = ₹0.3
T = 30 days
So total inventory carrying cost

$$= C_1 \int_0^T I(x)\, dx$$

$$= 0.3 \int_0^{30} \left(500 - 0.03x^2\right) dx$$

$$= 0.3 \left(500x - \frac{0.03x^3}{3}\right)_0^{30}$$

$$= 0.3 \left[500(30) - \frac{0.03}{3}(30)^3\right]$$

$$= 0.3\,[15000 - 270]$$

$$= 4419$$

Hence the total cost for maintaining inventory for 30 days is ₹ 4,419.

Question 5.
An account fetches interest at the rate of 5% per annum compounded continuously. An individual deposits ₹ 1,000 each year in his account. How much will be in the account after 5 years. ($e^{0.25} = 1.284$)

Solution:
p = 1000, N = 5, r = 5% = 0.05

$$\text{Annuity} = \int_0^5 1000 e^{0.05t}\,dt$$
$$= \frac{1000}{0.05}\left(e^{0.05t}\right)_0^5$$
$$= 20000\,[\,e^{0.25} - e^0]$$
$$= 20000\,(1.284 - 1]$$
$$= 5680$$

After 5 years ₹ 5680 will be in the account

Question 6.
The marginal cost function of a product is given by $\frac{dc}{dx} = 100 - 10x + 0.1x^2$ where x is the output. Obtain the total and the average cost function of the firm under the assumption, that its fixed cost is ₹ 500.

Solution:
Given MC $= \frac{dc}{dx} = 100 - 10x + 0.1x^2$
$C = \int MC\, dx + k$
$C = \int (100 - 10x + 0.1x^2)\, dx + k$
$C = 100x - 5x^2 + \frac{0.1x^3}{3} + k$
The fixed cost is 500 ⇒ k = 500
Hence total cost function $= 100x - 5x^2 + \frac{0.1x^3}{3} + 500$
Average cost function AC $= \frac{c}{x}$
$= 100 - 5x + \frac{x^2}{30} + \frac{500}{x}$

Question 7.
The marginal cost function is MC = $300\,x^{\frac{2}{5}}$ and fixed cost is zero. Find out the total cost and average cost functions.

Solution:

$$\text{Given} \qquad MC = 300x^{\frac{2}{5}}$$

$$C = \int 300x^{\frac{2}{5}}\,dx + k = 300\frac{x^{\frac{7}{5}}}{\frac{7}{5}} + k$$

Since fixed cost = 0, $k = 0$

$$\text{So} \qquad C = \frac{1500}{7}x^{\frac{7}{5}}$$

$$\text{Average cost} = \frac{C}{x} = \frac{1500}{7}x^{\frac{2}{5}}$$

Question 8.

If the marginal cost function of x units of output is $\frac{a}{\sqrt{ax+b}}$ and if the cost of output is zero. Find the total cost as a function of x.

Solution:

$$\text{Given} \qquad MC = \frac{a}{\sqrt{ax+b}}$$

$$\text{Total cost} = \int \frac{a}{\sqrt{ax+b}}\,dx + k$$

$$C = 2\sqrt{ax+b} + k$$

The cost of output is zero $\Rightarrow x = 0, C = 0$

$$0 = 2\sqrt{b} + k \Rightarrow k = -2\sqrt{b}$$

So total cost function is $2\sqrt{ax+b} - 2\sqrt{b}$

Question 9.

Determine the cost of producing 200 air conditioners if the marginal cost (is per unit) is C'(x) = $\left(\frac{x^2}{200} + 4\right)$

Solution:

$$\text{Given } MC = C'(x) = \left(\frac{x^2}{200} + 4\right)$$

$$\text{Total cost} \qquad C = \int \left(\frac{x^2}{200} + 4\right) dx + k$$

$$C = \frac{x^3}{600} + 4x + k$$

When $x = 0$, $c = 0 \Rightarrow k = 0$

So $$C = \frac{x^3}{600} + 4x$$

When $x = 200$, $$C = \frac{(200)^3}{600} + 4(200) = \frac{8,000,000}{600} + 800$$

$$C = 14133.33$$

So the cost of producing 200 air conditioners is ₹ 14133.33

Question 10.
The marginal revenue (in thousands of Rupees) functions for a particular commodity is $5 + 3e^{-0.03x}$ where x denotes the number of units sold. Determine the total revenue from the sale of 100 units. (Given $e^{-3} = 0.05$ approximately)

Solution:
Given, marginal Revenue $R'(x) = 5 + 3e^{-0.03x}$
Total revenue from the sale of 100 units is

$$R = \int_0^{100} \left(5 + 3e^{-0.03x}\right) dx$$

$$R = \left[5x + \frac{3e^{-0.03x}}{-0.03}\right]_0^{100}$$

$$R = \left(500 + \frac{3e^{-0.03(100)}}{-0.03}\right) - \left(0 - \frac{3}{0.03}\right)$$

$$R = 500 - 100\,e^{-3} + 100$$

$$R = 600 - 100\,(0.05) = 595$$

Total revenue = 595 × 1000 = ₹ 5,95,000

Question 11.
If the marginal revenue function for a commodity is $MR = 9 - 4x^2$. Find the demand function.

Solution:
Given, marginal Revenue function $MR = 9 - 4x^2$

Revenue function, R = $\int$(MR) dx + k

$$R = \int (9 - 4x^2)\,dx + k$$

$$R = 9x - \frac{4}{3}x^3 + k$$

Since R = 0 when $x = 0$, $k = 0$

$$R = 9x - \frac{4}{3}x^3$$

Demand function $\quad P = \frac{R}{x}$

$$P = 9 - \frac{4}{3}x^2$$

Question 12.

Given the marginal revenue function $\frac{4}{(2x+3)^2} - 1$, show that the average revenue function is P = $\frac{4}{6x+9} - 1$

Solution:

Given $\quad MR = \frac{4}{(2x+3)^2} - 1$

$$R = \int \frac{4}{(2x+3)^2}\,dx - \int dx$$

$$R = \frac{4}{-(2x+3)^2} - x + k$$

Since R = 0 when $x = 0$

$$0 = \frac{2}{-3} + k \Rightarrow k = \frac{2}{3}$$

So $\quad R = \frac{-2}{2x+3} - x + \frac{2}{3}$

Average revenue function $\quad P = \frac{R}{x}$

$$P = \frac{-2}{x(2x+3)} - 1 + \frac{2}{3x}$$

$$= \frac{2}{x}\left[\frac{1}{3} - \frac{1}{2x+3}\right] - 1$$

$$= \frac{2}{x}\left[\frac{2x+3-3}{3(2x+3)}\right]-1$$

$$= \frac{2}{x}\left(\frac{2x}{3(2x+3)}\right)-1$$

$$P = \frac{4}{6x+9}-1$$

which is the required answer.

Question 13.

A firm's marginal revenue function is MR = $20\, e^{\frac{-x}{10}} \left(1 - \frac{x}{10}\right)$. Find the corresponding demand function.

Solution:

$$MR = 20e^{\frac{-x}{10}}\left(1-\frac{x}{10}\right)$$

$$R = \int 20e^{\frac{-x}{10}}\left(1-\frac{x}{10}\right) dx + k$$

$$R = 20\int\left(e^{\frac{-x}{10}} - \frac{x}{10}e^{\frac{-x}{10}}\right) dx + k$$

$$R = 20\int d\left(xe^{\frac{-x}{10}}\right) + k$$

$$R = 20xe^{\frac{-x}{10}} + k$$

When $x = 0$, $R = 0$, so $k = 0$

$$R = 20xe^{\frac{-x}{10}}$$

The demand function $\quad P = \frac{R}{x} = 20e^{\frac{-x}{10}}$

Question 14.

The marginal cost of production of a firm is given by C'(x) = 5 + 0.13x, the marginal revenue is given by R'(x) = 18 and the fixed cost is ₹ 120. Find the profit function.

Solution:

C(x) = ∫C '(x) dx = ∫(5 + 0.13x) dx

$= 5x + 0.13(\frac{x^2}{2}) + k_1$

$C = 5x + 0.065\ x^2 + 120$ …….. (1)

Total Revenue function

$R(x) = \int R'(x)\ dx = \int 18\ dx$

$R = 18x + k_2$

when $x = 0$; $R = 0 \Rightarrow k_2 = 0$

$R = 18x$ …….. (2)

Profit function $p = R – C$

$= 18x – [5x + 0.065\ x^2 + 120]$

$= 18x – 5x – 0.065x^2 – 120$

$\therefore p = 13x – 0.065x^2 – 120$

Question 15.

If the marginal revenue function is $R'(x) = 1500 – 4x – 3x^2$. Find the revenue function and average revenue function.

Solution:

The marginal Revenue function

$R'(x) = 1500 – 4x – 3x^2$

Revenue function

$R = \int R'(x)\ dx$

$= \int (1500 – 4x – 3x^2)\ dx$

$R = 1500\ x – 4(\frac{x^2}{2}) – 3(\frac{x^3}{3}) + k$

$R = 1500\ x – 2x^2 – x^3 + k$

when $x = 0$; $R = 0 \Rightarrow k = 0$

$\therefore$ Revenue function

$R = 1500\ x – 2x^2 – x^3$

Average Revenue function A.R $= \frac{R(x)}{x}$

$\frac{1500x - 2x^2 - x^3}{x}$

A.R $= 1500 – 2x – x^2$

Question 16.

Find the revenue function and the demand function if the marginal revenue for x units is $MR = 10 + 3x – x^2$

Solution:

Given $\quad \text{MR} = 10 + 3x - x^2$

Revenue function $\quad R(x) = \int (\text{MR})\, dx + k$

$$R = \int (10 + 3x - x^2)\, dx + k$$

$$= 10x + \frac{3}{2}x^2 - \frac{x^3}{3} + k$$

When $x = 0, R = 0, \Rightarrow k = 0$

$$R = 10x + \frac{3}{2}x^2 - \frac{x^3}{3}$$

Demand function $\quad P = \frac{R}{x} = 10 + \frac{3}{2}x - \frac{x^2}{3}$

Question 17.

The marginal cost function of a commodity is given by MC = $\frac{14000}{\sqrt{7x+4}}$ and the fixed cost is ₹ 18,000. Find the total cost and average cost.

Solution:

Given $\quad \text{MC} = \frac{14000}{\sqrt{7x+4}} \quad$ fixed cost = ₹18,000

$$\text{Total cost} = \int (\text{MC})\, dx + k$$

$$= \int \frac{14000}{\sqrt{7x+4}}\, dx + k$$

$$= 14000\left(\frac{2}{7}\sqrt{7x+4}\right) + k$$

$$= 4000\sqrt{7x+4} + k$$

Since the fixed cost is ₹18,000, when $x = 0$, $k = 18,000$

$\Rightarrow \quad$ Total cost $\quad C = 4000\sqrt{7x+4} + 18000$

$$\text{Average cost A.C} = \frac{C}{x}$$

$$= \frac{4000}{x}\sqrt{7x+4} + \frac{18000}{x}$$

Question 18.
If the marginal cost (MC) of production of the company is directly proportional to the number of units (x) produced, then find the total cost function, when the fixed cost is ₹ 5,000 and the cost of producing 50 units is ₹ 5,625.

Solution:
Given that the marginal cost MC is directly proportional to the number of units x.
That is, MC ∝ x
MC = kx, where k is the constant of proportionality
Total cost C = $\int$(MC) dx + c_1

$$= \int (kx)\, dx + c_1$$

$$C = \frac{kx^2}{2} + c_1$$

The fixed cost is given as 5000. So $c_1 = 5000$

$$C = \frac{kx^2}{2} + 5000$$

When $x = 50$, C = 5625

So $$5625 = \frac{k}{2}(50)^2 + 5000$$

$$625 = \frac{2500}{2}k$$

$$k = \frac{1}{2}$$

Thus total cost function $C = \frac{1}{2}\left(\frac{x^2}{2}\right) + 5000$

$$C = \frac{x^2}{4} + 5000$$

Question 19.
If MR = 20 – 5x + 3x^2

Solution:

$$MR = 20 - 5x + 3x^2$$
$$R = \int (MR)\, dx + k$$
$$= \int (20 - 5x + 3x^2)\, dx + k$$
$$R = 20x - \frac{5x^2}{2} + x^3 + k$$
Since $R = 0$, when $x = 0$, $k = 0$
$$\Rightarrow \quad R = 20x - \frac{5x^2}{2} + x^3 \text{ is the total revenue function}$$

Question 20.
If MR = 14 – 6x + 9x², find the demand function.
Solution:
$MR = 14 - 6x + 9x^2$
Total Revenue function

$R = \int (MR)\, dx = \int (14 - 6x + 9x^2)\, dx$

$R = 14x - 6\left(\frac{x^2}{2}\right) + 9\left(\frac{x^3}{3}\right) + k$

$R = 14x - 3x^2 + 3x^3 + k$

when $x = 0$; $R = 0 \Rightarrow k =$

$R = 14x - 3x^2 + 3x^3 + k$

$\Rightarrow px = 14x - 3x^2 + 3x^3$

$p = \frac{14x - 3x^2 + 3x^3}{x}$

∴ The demand function

$p = 14 - 3x + 3x^2$

Ex 3.3

Question 1.
Calculate consumer's surplus if the demand function $p = 50 - 2x$ and $x = 20$

Solution:
Given demand function $p = 50 - 2x$, $x_0 = 20$

$$CS = \int_0^{x_0} p(x)\,dx - x_0 p_0$$

$$\text{When } x = 20,\ p_0 = 50 - 2(20) = 10$$

$$CS = \int_0^{20} (50 - 2x)\,dx - (20)(10)$$

$$= \left[50x - x^2\right]_0^{20} - 200$$

$$= [1000 - 400] - 200 = 400$$

Hence the consumer's surplus is 400 units.

Question 2.
Calculate consumer's surplus if the demand function $p = 122 - 5x - 2x^2$, and $x = 6$

Solution:
Demand function $p = 122 - 5x - 2x^2$ and $x = 6$
when $x = x_0 = 6$
$p_0 = 122 - 5(6) - 2(36)$
$= 122 - 30 - 72$
$= 20$

$$CS = \int_0^{6} (122 - 5x - 2x^2)\,dx - (20)(6)$$

$$= \left[122x - \frac{5x^2}{2} - \frac{2x^3}{3}\right]_0^6 - 120$$

$$= (122)(6) - \frac{5}{2}(36) - \frac{2}{3}(216) - 120$$

$$= 732 - 90 - 144 - 120 = 378$$

Hence the consumer's surplus is 378 units

Question 3.
The demand function $p = 85 - 5x$ and supply function $p = 3x - 35$. Calculate the equilibrium price and quantity demanded. Also, calculate consumer's surplus.

Solution:
Given $p_d = 85 - 5x$ and $p_s = 3x - 35$
At equilibrium prices $p_d = p_s$
$85 - 5x = 3x - 35$
$\Rightarrow 8x = 120$
$\Rightarrow x = 15$
$p_0 = 85 - 5(15) = 85 - 75 = 10$

$$p_0 = 85 - 5(15) = 85 - 75 = 10$$

$$CS = \int_0^{x_0} p\,dx - x_0 p_0,\ x_0 = 15$$

$$CS = \int_0^{15} (85 - 5x)\,dx - (15)(10)$$

$$= \left(85x - \frac{5x^2}{2}\right)_0^{15} - 150$$

$$= 85(15) - \frac{5(225)}{2} - 150$$

$$= 562.5$$

The equilibrium price is ₹10, the quantity demanded is 15. The consumer surplus is 562.50 units.

Question 4.
The demand function for a commodity is $p = e^{-x}$ .Find the consumer's surplus when p = 0.5.

Solution:
Given demand function $p = e^{-x}$
At p = 0.5, (i.e) $p_0 = 0.5$,
we have $p_0 = e{-}x0$
$\Rightarrow 0.5 = e{-}x0$
Taking $\log_e$ on both sides
$\log_e(0.5) = -x_0$

$$\log_e\left(\frac{1}{2}\right) = -x_0$$

$$-\log_e 2 = -x_0$$

$$\Rightarrow x_0 = \log_e 2$$

$$CS = \int_0^{\log_e 2} e^{-x}\,dx - (\log_e 2)(0.5)$$

$$= \left[-e^{-x}\right]_0^{\log_e 2} - \frac{\log_e 2}{2}$$

$$= \frac{-1}{2} + 1 - \frac{\log_e 2}{2} = \frac{1}{2} - \frac{\log_e 2}{2}$$

$$CS = \frac{1}{2}[1 - \log_e 2] \text{ units}$$

Question 5.
Calculate the producer's surplus at x = 5 for the supply function p = 7 + x.

Solution:
Given supply function is p = 7 + x, $x_0 = 5$
$p_0 = 7 + x_0 = 7 + 5 = 12$
Producer's surplus

$$PS = x_0 p_0 - \int_0^{x_0} p(x)\, dx$$

$$= 5(12) - \int_0^5 (7 + x) dx$$

$$= 60 - \left(7x + \frac{x^2}{2}\right)_0^5 = 60 - 35 - \frac{25}{2} = \frac{25}{2}$$

Hence the producer's surplus is 252 units

Question 6.
If the supply function for a product is $p = 3x + 5x^2$. Find the producer's surplus when x = 4.

Solution:
Given the supply function $p_s = 3x + 5x^2$
when x = 4, (i.e) $x_0 = 4$,
$p_0 = 3(4) + 5(4)^2 = 12 + 80 = 92$

$$PS = x_0 p_0 - \int_0^{x_0} p_s(x) dx$$

$$= 4(92) - \int_0^4 \left(3x + 5x^2\right) dx$$

$$= 368 - \left[\frac{3x^2}{2} + \frac{5x^3}{3}\right]_0^4$$

$$= 368 - \left[\frac{48}{2} + \frac{5}{3}(64)\right] = 368 - 24 - 106.67$$

$$= 237.33$$

Hence the producer's surplus is 237.3 units.

Question 7.

The demand function for a commodity is $p = \frac{36}{x+4}$. Find the consumer's surplus when the prevailing market price is ₹ 6.

Solution:

Given $p = \frac{36}{x+4}$

The marker price is ₹6 (i.e) $p_0 = 6$

$$p_o = \frac{36}{x_0 + 4} \Rightarrow 6 = \frac{36}{x_0 + 4} \Rightarrow x_o = 2$$

$$CS = \int_0^2 \left(\frac{36}{x+4}\right) dx - p_0 x_0$$

$$= 36\int_0^2 \left(\frac{1}{x+4}\right) dx - (6)(2)$$

$$= 36[\log(x+4)]_0^2 - 12$$

$$= 36 [\log 6 - \log 4] - 12$$

$$= 36 \log\frac{3}{2} - 12$$

So the consumer's surplus when the prevailing market price is ₹ 6 is $(36 \log\frac{3}{2} - 12)$ units.

Question 8.

The demand and supply functions under perfect competition are $p_d = 1600 - x^2$ and $p_s = 2x^2 + 400$ respectively. Find the producer's surplus.

Solution:

Given demand function $p_d = 1600 - x^2$ and

Supply function $p_s = 2x^2 + 400$

Perfect competition means there is equilibrium between supply and demand

$p_s = p_d$

$\Rightarrow 1600 - x^2 = 2x^2 + 400$

$\Rightarrow 3x^2 = 1200$

$\Rightarrow x^2 = 400$

$\Rightarrow x = \pm 20$

The value of x cannot be negative. So x = 20 we take $x_0 = 20$.
$p_0 = 1600 - (20)^2 = 1600 - 400 = 1200$

$$\text{PS} = x_0 p_0 - \int_0^{x_0} p_s \, dx$$

$$= (20)(1200) - \int_0^{20} \left(2x^2 + 400\right) dx$$

$$= 24000 - \left[\frac{2x^3}{3} + 400x\right]_0^{20}$$

$$= 24000 - \left[\frac{16000}{3} + 8000\right]$$

$$= 16000 - \frac{16000}{3} = \frac{32000}{3}$$

Hence the producer's surplus is $\frac{32000}{3}$ units.

Question 9.
Under perfect competition for a commodity the demand and supply laws are $p_d = \frac{8}{x+1} - 2$ and $p_s = \frac{x+3}{2}$ respectively. Find the consumer's and producer's surplus.

Solution:
Given $p_d = \frac{8}{x+1} - 2$ and $p_s = \frac{x+3}{2}$
Here, since there is perfect competition, there is equilibrium, that is $p_d = p_s$

$$\frac{8}{x+1} - 2 = \frac{x+3}{2}$$

$$\frac{8 - 2x - 2}{x+1} = \frac{x+3}{2}$$

$$\frac{6 - 2x}{x+1} = \frac{x+3}{2}$$

$$(x+1)(x+3) = 12 - 4x$$

$$x^2 + 4x + 3 = 12 - 4x$$

$$x^2 + 8x - 9 = 0$$

$$(x+9)(x-1) = 0$$

$$x = -9, 1$$

Since the value of x cannot be negative, x = 1 we take this value as x_0

$$p_0 = \frac{8}{x_0+1} - 2 \quad = \quad \frac{8}{2} - 2 = 2$$

$$\text{CS} \quad = \quad \int_0^1 p_d dx - x_0 p_0$$

$$= \quad \int_0^1 \left(\frac{8}{x+1} - 2 \right) dx - (1)(2)$$

$$= \quad \left[8 \log(x+1) - 2x \right]_0^1 - 2$$

$$= \quad 8 \log 2 - 8 \log 1 - 2 - 0 - 2$$

$$= \quad 8 \log 2 - 4$$

$$\text{PS} \quad = \quad x_0 p_0 - \int_0^{x_0} \text{p}_s \, dx$$

$$= \quad 2 - \int_0^1 \frac{x+3}{2} dx \; = \; 2 - \frac{1}{2} \left(\frac{x^2}{2} + 3x \right)_0^1$$

$$= \; 2 - \frac{1}{2} \left(\frac{1}{2} + 3 \right) = 2 - \frac{7}{2} = \frac{1}{4}$$

Hence under perfect competition,
(i) The consumer's surplus is (8 log 2-4) units
(ii) The producer's surplus is 14 units.

Question 10.
The demand equation for a products is $x = \sqrt{100 - p}$ and the supply equation is $x = \frac{p}{2} - 10$.
Determine the consumer's surplus and producer's surplus, under market equilibrium.

Solution:
Given demand equation is $x = \sqrt{100 - p}$ and supply equation is $x = \frac{p}{2} - 10$
So the demand law is $x^2 = 100 - p$
$\Rightarrow p_d = 100 - x^2$
Supply law is given by $x + 10 = \frac{p}{2}$
$\Rightarrow p_s = 2(x + 10)$
Under equilibrium $p_d = p_s$
$\Rightarrow 100 - x^2 = 2(x + 10)$
$\Rightarrow 100 - x^2 = 2x + 20$
$\Rightarrow x^2 + 2x - 80 = 0$
$\Rightarrow (x + 10)(x - 8) = 0$
$\Rightarrow x = -10, 8$
The value of x cannot be negative, So x = 8

When $x_0 = 8$, $p_0 = 100 - 8^2 = 100 - 64 = 36$

$$\text{CS} = \int_0^8 \left(100 - x^2\right) dx - (8)(36)$$

$$= \left(100x - \frac{x^3}{3}\right)_0^8 - 288$$

$$= 800 - \frac{512}{3} - 288 = \frac{1024}{3}$$

$$\text{So consumer surplus} = \frac{1024}{3} \text{ units}$$

$$\text{PS} = 8\,(36) - \int_0^8 2(x+10)\,dx$$

$$= 288 - 2\left[\frac{x^2}{2} + 10x\right]_0^8$$

$$= 288 - 2\left[\frac{64}{2} + 80\right]$$

= 288 – 2(112)
= 64
So the producer's surplus is 64 units.

Question 11.
Find the consumer's surplus and producer's surplus for the demand function $p_d = 25 - 3x$ and supply function $p_s = 5 + 2x$.

Solution:
Given $p_d = 25 - 3x$ and $p_s = 5 + 2x$
At market equilibrium, $p_d = ps_s$
⇒ $25 - 3x = 5 + 2x$
⇒ $5x = 20$
⇒ $x = 4$
When $x_0 = 4$, $p_0 = 25 - 12 = 13$

$$\text{CS} = \int_0^4 (25 - 3x)\,dx - 13(4)$$

$$= \left(25x - \frac{3x^2}{2}\right)_0^4 - 52$$

$$= 100 - \frac{3}{2}(16) - 52 = 24$$

So the consumer’s surplus is 24 units.

$$\text{PS} = 13(4) - \int_0^4 (2x+5)dx$$
$$= 52 - (x^2 + 5x)_0^4 = 52 - 16 - 20 = 16$$

So the producer’s surplus is 16 units.

Ex 3.4

Choose the correct answer form the given alternatives.

Question 1.
Area bounded by the curve y = x(4 – x) between the limits 0 and 4 with x-axis is _______
(a) $\frac{30}{3}$ sq.units
(b) $\frac{31}{2}$ sq.units
(c) $\frac{32}{3}$ sq.units
(d) $\frac{15}{2}$ sq.units

Answer
(c) $\frac{32}{3}$sq.units
Hint:

$$\text{Area} = \int_0^4 x(4-x)dx = \int_0^4 (4x-x^2)dx$$
$$= \left(2x^2 - \frac{x^3}{3}\right)_0^4 = 32 - \frac{64}{3} = \frac{32}{3}$$

Question 2.
Area bounded by the curve y = e^{-2x} between the limits $0 \leq x \leq \infty$ is _____
(a) 1 sq.units
(b) $\frac{1}{2}$ sq.unit
(c) 5 sq.units
(d) 2 sq.units

Answer:
(b) $\frac{1}{2}$ sq.unit
Hint:

$$\text{Area} = \int_0^\infty e^{-2x}dx = \left(\frac{e^{-2x}}{-2}\right)_0^\infty = 0 + \frac{1}{2} = \frac{1}{2}$$

Question 3.
Area bounded by the curve y = $\frac{1}{x}$ between the limits 1 and 2 is _______
(a) log 2 sq.units

(b) log 5 sq.units
(c) log 3 sq.units
(d) log 4 sq.units
Answer:
(a) log 2 sq.units
Hint:

Area A = $\int_1^2 y dx = \int_1^2 \frac{1}{x} dx = [\log x]_1^2$
= log 2 – log 1 = log (2/1) = log (2)

Question 4.
If the marginal revenue function of a firm is MR = e−x10, then revenue is ______
(a) $-10e^{\frac{-x}{10}}$
(b) $1 - e^{\frac{-x}{10}}$
(c) $10\left(1 - e^{\frac{-x}{10}}\right)$
(d) $e^{\frac{-x}{10}} + 10$

Answer:
(c) $10\left(1 - e^{\frac{-x}{10}}\right)$

Hint:

$$R = \int e^{\frac{-x}{10}} + k = \frac{e^{\frac{-x}{10}}}{\frac{-1}{10}} + k$$

When $R = 0, x = 0$

$$\text{So } 0 = \frac{1}{\frac{-1}{10}} + k \Rightarrow k = 10$$

$$\Rightarrow R = -10e^{\frac{-x}{10}} + 10 = 10\left(1 - e^{\frac{-x}{10}}\right)$$

Question 5.
If MR and MC denotes the marginal revenue and marginal cost functions, then the profit functions is ________
(a) P = ∫(MR – MC) dx + k
(b) P = ∫(MR + MC) dx + k
(c) P = ∫(MR) (MC) dx + k
(d) P = ∫(R – C) dx + k

Answer:
(a) $P = \int(MR - MC)\, dx + k$
Hint:
Profit = Revenue – Cost

Question 6.
The demand and supply functions are given by $D(x) = 16 - x^2$ and $S(x) = 2x^2 + 4$ are under perfect competition, then the equilibrium price x is ________
(a) 2
(b) 3
(c)4
(d) 5

Answer:
(a) 2
Hint:
$D(x) = 16 - x^2$, $S(x) = 2x^2 + 4$
Under equilibriuim, $D(x) = S(x)$
$\Rightarrow 16 - x^2 = 2x^2 + 4$
$\Rightarrow 3x^2 = 12$
$\Rightarrow x = \pm 2$.
Since x cannot be negative x = 2.

Question 7.
The marginal revenue and marginal cost functions of a company are $MR = 30 - 6x$ and $MC = -24 + 3x$ where x is the product, then the profit function is _______
(a) $9x^2 + 54x$
(b) $9x^2 - 54x$
(c) $54x - \frac{9x^2}{2}$
(d) $54x - \frac{9x^2}{2} + k$

Answer:
(d) $54x - \frac{9x^2}{2} + k$

Hint:
Profit $P = \int(MR - MC)\, dx$
$= \int[(30 - 6x) - (-24 + 3x)]\, dx$
$= \int(30 - 6x + 24 - 3x)\, dx$
$= \int(54 - 9x)\, dx = 54x - 9x22 + k$

Question 8.
The given demand and supply function are given by $D(x) = 20 - 5x$ and $S(x) = 4x + 8$ if they are under perfect competition then the equilibrium demand is _______

(a) 40
(b) $\frac{41}{2}$
(c) $\frac{40}{3}$
(d) $\frac{41}{5}$

Answer:
(c) $\frac{40}{3}$
Hint:
D(x) = S(x) in equilibrium
20 – 5x = 4x + 8
9x = 12
$x = \frac{4}{3} = x_0$
p_0 = 20 – 5($\frac{4}{3}$ [/ltex])=20–[latex]$\frac{20}{3}$
$= \frac{40}{3}$

Question 9.
If the marginal revenue MR = 35 + 7x – 3x², then the average revenue AR is _______
(a) $35x + \frac{7x^2}{2} - x^3$
(b) $35x + \frac{7x}{2} - x^2$
(c) $35 + \frac{7x}{2} + x^2$
(d) $35 + 7x + x^2$

Answer:
(b) $35x + \frac{7x}{2} - x^2$
Hint:

$$\begin{aligned} R &= \int (MR)\, dx + k \\ &= \int (35+7x-3x^2)\, dx + k \\ &= 35x + \frac{7x^2}{2} - x^3 + k \end{aligned}$$

When $R = 0$, $x = 0 \Rightarrow k = 0$

So revenue function R $= 35x + \frac{7x^2}{2} - x^3$

$$AR = \frac{R}{x} = 35 + \frac{7}{2}x - x^2$$

Question 10.
The profit of a function p(x) is maximum when _______
(a) MC – MR = 0
(b) MC = 0
(c) MR = 0
(d) MC + MR = 0

Answer:
(a) MC – MR = 0
Hint:
P = Revenue – Cost
P is maximum when $\frac{dp}{dx} = 0$
$\frac{dp}{dx} = R'(x) - C'(x) = MR - MC = 0$

Question 11.
For the demand function p(x), the elasticity of demand with respect to price is unity then ________
(a) revenue is constant
(b) cost function is constant
(c) profit is constant
(d) none of these
Answer:
(a) revenue is constant
Hint:

$$\begin{aligned} \eta_d &= 1 \\ \Rightarrow \frac{-p}{x}\frac{dx}{dp} &= 1 \\ \frac{dx}{x} &= \frac{-dp}{p} \\ \log x &= -\log p + \log k \\ \Rightarrow p\,x &= k, \text{ constant} \\ \text{But } p\,x &= \text{R (revenue), which is a constant.} \end{aligned}$$

Question 12.
The demand function for the marginal function MR = 100 – $9x^2$ is ______
(a) 100 – $3x^2$
(b) 100x – $3x^2$
(c) 100x – $9x^2$
(d) 100 + $9x^2$
Answer:
(a) 100 – $3x^2$
Hint:

$R = \int MR\, dx = \int (100 - 9x^2)\, dx$

$R = 100x - \frac{9x^3}{3} + k$

when x = 0, R = 0, then k = 0

$\therefore R = 100x - 3x^3$

Demand function

$p = \frac{R}{x} = \frac{100x - 3x^3}{x} = 100 - 3x^2$

Question 13.
When x_0 = 5 and p_0 = 3 the consumer's surplus for the demand function $p_d = 28 - x^2$ is ______

(a) 250 units
(b) $\frac{250}{3}$ units
(c) $\frac{251}{2}$ units
(d) $\frac{251}{3}$ units

Answer:
(b) $\frac{250}{3}$ units
Hint:

$$CS = \int_0^5 (28 - x^2)\, dx - (5)(3)$$

$$= \left(28x - \frac{x^3}{3}\right)_0^5 - 15$$

$$= 140 - \frac{125}{3} - 15 = \frac{250}{3} \text{ units}$$

Question 14.

When $x_0 = 2$ and $P_0 = 12$ the producer's surplus for the supply function $P_s = 2x^2 + 4$ is ______

(a) $\frac{31}{5}$ units
(b) $\frac{31}{2}$ units
(c) $\frac{32}{3}$ units
(d) $\frac{30}{7}$ units

Answer:
(c) $\frac{32}{3}$ units
Hint:

$$\begin{aligned} \text{PS} &= 2(12) - \int_0^2 \left(2x^2 + 4\right) dx \\ &= 24 - \left[\frac{2}{3}x^3 + 4x\right]_0^2 \\ &= 24 - \frac{16}{3} - 8 = \frac{32}{3} \text{ units} \end{aligned}$$

Question 15.
Area bounded by $y = x$ between the lines $y = 1$, $y = 2$ with y-axis is ______
(a) $\frac{1}{2}$ sq.units
(b) $\frac{5}{2}$ sq.units
(c) $\frac{3}{2}$ sq.units
(d) 1 sq.unit

Answer:
(c) $\frac{3}{2}$ sq.units
Hint:

$$\text{Area} = \int_1^2 y\,dy = \left(\frac{y^2}{2}\right)_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

Question 16.
The producer's surplus when the supply function for a commodity is $P = 3 + x$ and $x_0 = 3$ is ______

(a) $\frac{5}{2}$

(b) $\frac{9}{2}$
(c) $\frac{3}{2}$
(d) $\frac{7}{2}$

Answer:
(b) $\frac{9}{2}$
Hint:
$x_0 = 3$, $p = 3 + x \Rightarrow p_0 = 3 + 3 = 6$

$$\begin{aligned} \text{PS} &= (3)(6) - \int_0^3 (x+3)\,dx \\ &= 18 - \left[\frac{x^2}{2} + 3x\right]_0^3 = 18 - \frac{9}{2} - 9 = \frac{9}{2} \end{aligned}$$

Question 17.
The marginal cost function is MC = 100√x. find AC given that TC = 0 when the output is zero is ______
(a) $\frac{200}{3} x^{\frac{1}{2}}$
(b) $\frac{200}{3} x^{\frac{3}{2}}$
(c) $\frac{200}{3x^{\frac{3}{2}}}$
(d) $\frac{200}{3x^{\frac{1}{2}}}$

Answer:
(a) $\frac{200}{3} x^{\frac{1}{2}}$

Hint:

$$TC = \int (MC)\, dx + k$$
$$= \int 100\sqrt{x}\; dx + k$$
$$TC = 100\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + k$$

when $TC = 0, x = 0 \Rightarrow k = 0$

$$TC = \frac{200}{3}x^{\frac{3}{2}}$$
$$AC = \frac{TC}{x} = \frac{200}{3}x^{\frac{1}{2}}$$

Question 18.
The demand and supply function of a commodity are $P(x) = (x - 5)^2$ and $S(x) = x^2 + x + 3$ then the equilibrium quantity x_0 is ________
(a) 5
(b) 2
(c) 3
(d) 19

Answer:
(b) 2
Hint:
$P(x) = S(x)$
$(x - 5)^2 = x^2 + x + 3$
$x^2 - 10x + 25 = x^2 + x + 3$
$\Rightarrow 25 - 3 = x + 10x$
$\Rightarrow 11x = 22$
$\Rightarrow x = 2$

Question 19.
The demand and supply function of a commodity are $D(x) = 25 - 2x$ and $S(x) = \frac{10+x}{4}$ then the equilibrium price p_0 is ________
(a) 5
(b) 2
(c) 3
(d) 10

Answer:

(a) 5
Hint:
At market equilibrium, D(x) = S(x)
$25x - 2x = \frac{10+x}{4} \Rightarrow 4(25 - 2x) = 10 + x$
$100 - 8x = 10 + x \Rightarrow 100 - 10 = x + 8x$
$9x = 90 \Rightarrow x = 10$
when x = 10 P = 25 – 2 (10) = 25 – 20 = 5

Question 20.
If MR and MC denote the marginal revenue and marginal cost and $MR - MC = 36x - 3x^2 - 81$, then the maximum profit at x is equal to ______
(a) 3
(b) 6
(c) 9
(d) 5

Answer:
(c) 9
Hint:
The maximum profit occurs when MR – MC = 0
$\Rightarrow 36x - 3x^2 - 81 = 0$
$\Rightarrow x^2 - 12x + 27 = 0$
$\Rightarrow (x - 9)(x - 3) = 0$
$\Rightarrow x = 9, 3$
Now $\frac{dp}{dx} = 36x - 3x^2 - 81 \Rightarrow \frac{d^2p}{dx^2} = 36 - 9x$
At x = 9, $\frac{d^2p}{dx^2} = 36 - 81 < 0$
At x = 3, $\frac{d^2p}{dx^2} = 36 - 27 > 0$
Therefore profit is maximum when x = 9.

Question 21.
If the marginal revenue of a firm is constant, then the demand function is ______
(a) MR
(b) MC
(c) C(x)
(d) AC

Answer:
(a) MR
Hint:
MR = k (constant)
Revenue function $R = \int (MR)\, dx + c_1$
$= \int k dx + c_1$
$= kx + c_1$

When R = 0, x = 0, ⇒ $c_1 = 0$
R = kx
Demand function $p = \frac{R}{x} = \frac{kx}{x}$ = k constant
⇒ p = MR

Question 22.
For a demand function p, if $\int \frac{dp}{p} = k \int \frac{dx}{x}$ then k is equal to ______

(a) η_d
(b) $-\eta_d$
(c) $-1\eta d$
(d) $1\eta d$

Answer:
(c) $-1\eta d$
Hint:

$$\int \frac{dp}{p} = k\int \frac{dx}{x}$$

Comparing with $\int \frac{dp}{p} = \frac{-1}{\eta_d} \int \frac{dx}{x} \Rightarrow k = \frac{-1}{\eta_d}$

Question 23.
Area bounded by $y = e^x$ between the limits 0 to 1 is ________
(a) (e – 1) sq.units
(b) (e + 1) sq.units
(c) $(1 - \frac{1}{E})$ sq.units
(d) $(1 + \frac{1}{E})$ sq.units

Answer:
(a) (e – 1) sq.units
Hint:

$$\text{Area} = \int_0^1 e^x dx = \left(e^x\right)_0^1 = e^1 - e^0 = e - 1$$

Question 24.
The area bounded by the parabola $y^2 = 4x$ bounded by its latus rectum is ______
(a) $\frac{16}{3}$ sq.units
(b) $\frac{8}{3}$ sq.units

(c) $\frac{72}{3}$ sq.units

(d) $\frac{1}{3}$ sq.units

Answer:

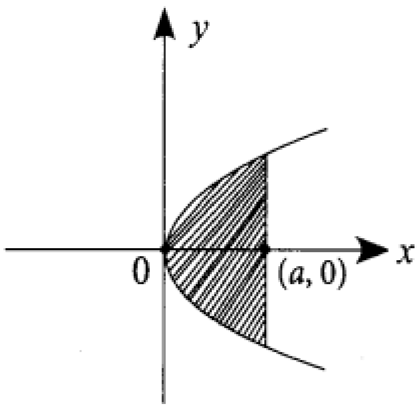
(b) $\frac{8}{3}$ sq.units

Hint:

$y^2 = 4x$

Comparing with $y^2 = 4ax$ gives a = 1

Since parabola is symmetric about x – axis,



$$\text{Area} = 2\int_0^a y\,dx = 2\int_0^1 2\sqrt{x}\,dx$$

$$= 4\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \frac{8}{3} \text{ sq.units}$$

Question 25.

Area bounded by y = |x| between the limits 0 and 2 is _______

(a) 1 sq.units

(b) 3 sq.units

(c) 2 sq.units

(d) 4 sq.units

Answer:

(c) 2 sq.units

Hint:

When x lies between 0 and 2

|x| = x

So area $= \int_0^2 x dx = \left(\frac{x^2}{2}\right)_0^2 = \frac{4}{2} = 2$

Area = 2 sq.units

Miscellaneous Problems

Question 1.
A manufacture's marginal revenue function is given by MR = 275 – x – 0.3x². Find the increase in the manufactures total revenue if the production is increased from 10 to 20 units.

Solution:
Given MR = 275 – x – 0.3x^2
R = $\int$(MR) dx + k
But given that production is increased from 10 to 20 units. so,

$$R = \int_{10}^{20}(MR)dx = \int_{10}^{20}\left(275 - x - 0.3x^2\right)dx$$

$$= \left(275x - \frac{x^2}{2} - 0.3\frac{x^3}{3}\right)_{10}^{20}$$

$$= \left[5500 - \frac{400}{2} - \frac{0.3}{3} - (8000)\right] - \left[2750 - \frac{100}{2} - \frac{0.3}{3} - (1000)\right]$$

$$= (5500 - 200 - 800) - (2750 - 50 - 100)$$

$$= 1900$$

Thus the total revenue is increased by ₹ 1900

Question 2.
A company has determined that marginal cost function for x product of a particular commodity is given by MC = 125 + 10x – $\frac{x^2}{9}$. Where C is the cost of producing x units of the commodity. If the fixed cost is ₹ 250 what is the cost of producing 15 units.

Solution:

Given $\quad MC = 125 + 10x - \frac{x^2}{9}$

$$\text{Then } C = \int (MC)\, dx + k$$

$$C = \int (125 + 10x - \frac{x^2}{9})\, dx + k$$

$$C = 125x + 5x^2 - \frac{x^3}{27} + k$$

The fixed cost is given as ₹ 250. So, $k = 250$

$$\Rightarrow C = 125x + 5x^2 - \frac{x^3}{27} + 250$$

When $x = 15$ units

$$C = 125(15) + 5(15)^2 - \frac{(15)^3}{27} + 250$$

$$C = 1875 + 1125 - 125 + 250$$

$$C = 3125$$

Thus the cost of producing 15 units is ₹ 3125

Question 3.

The marginal revenue function for a firm is given by MR = $\frac{2}{x+3} - \frac{2x}{(x+3)^2} + 5$. Show that the demand function is P = $\frac{2}{x+3} + 5$

Solution:

Given $$MR = \frac{2}{x+3} - \frac{2x}{(x+3)^2} + 5$$

$$MR = \frac{2}{x+3}\left[1 - \frac{x}{x+3}\right] + 5$$

$$= \frac{2}{x+3}\left[\frac{x+3-x}{x+3}\right] + 5 = \frac{6}{(x+3)^2} + 5$$

Revenue function $$R(x) = \int (MR)\, dx + k$$

$$= \int \frac{6}{(x+3)^2}\, dx + \int 5dx + k$$

$$= \frac{-6}{(x+3)} + 5x + k$$

$$R(x) = 2 - \frac{6}{(x+3)} - 2 + 5x + k = \frac{2x+6-6}{x+3} + 5x + k_1$$

$$R(x) = \frac{2x}{x+3} + 5x + k_1$$

When $x = 0$, $R = 0 \Rightarrow k_1 = 0$

So $$R(x) = \frac{2x}{x+3} + 5x$$

Demand function $$P = \frac{R}{x} = \frac{2}{x+3} + 5$$

Question 4.
For the marginal revenue function MR = $6 - 3x^2 - x^3$, Find the revenue function and demand function.

Solution:
MR = $6 - 3x^2 - x^3$
Revenue function R = $\int(6 - 3x^2 - x^3)\,dx + k$

$$R = 6x - x^3 - \frac{x^4}{4} + k$$

When R = 0, $x = 0 \Rightarrow k = 0$

So $$R = 6x - x^3 - \frac{x^4}{4}$$

Demand function $$P = \frac{R}{x} = 6 - x^2 - \frac{x^3}{4}$$

Question 5.
The marginal cost of production of a firm is given by $C'(x) = 20 + \frac{x}{20}$ the marginal revenue is given by $R'(x) = 30$ and the fixed cost is ₹ 100. Find the profit function.

Solution:

$$C'(x) = 20 + \frac{x}{20}$$

$$C'(x) = \int\left(20 + \frac{x}{20}\right) dx + k$$

$$= 20x + \frac{x^2}{40} + k$$

The fixed cost is given by ₹ 100

So cost function $$C = 20x + \frac{x^2}{40} + 100$$

The marginal revenue $R'(x) = 30$

$\Rightarrow$ Revenue function $R = \int 30\,dx + k = 30x + k$

When $x = 0$, $R = 0 \Rightarrow k = 0$

So $R = 30x$

The profit function is R – C

$$\Rightarrow \quad P = 30x - \left(20x + \frac{x^2}{40} + 100\right)$$

$$P = 10x - \frac{x^2}{40} - 100$$

Question 6.
The demand equation for a product is $p_d = 20 - 5x$ and the supply equation is $p_s = 4x + 8$. Determine the consumer's surplus and producer's surplus under market equilibrium.

Solution:
Given $p_d = 20 - 5x$ and $p_s = 4x + 8$
Under market equilibrium, $p_d = p_s$
$20 - 5x = 4x + 8$

$$9x = 12 \Rightarrow x = \frac{4}{3} = x_0$$

$$p_0 = 20 - 5x_0 = 20 - 5\left(\frac{4}{3}\right) = 20 - \frac{20}{3} = \frac{40}{3}$$

Consumer's surplus,

$$\text{CS} = \int_0^{\frac{4}{3}} (20 - 5x)\,dx - \left(\frac{4}{3}\right)\left(\frac{40}{3}\right)$$

$$= \left(20x - \frac{5x^2}{2}\right)_0^{\frac{4}{3}} - \frac{160}{9}$$

$$= \frac{80}{3} - \frac{80}{18} - \frac{160}{9} = \frac{80}{3} - \frac{40}{9} - \frac{160}{9} = \frac{40}{9}$$

Producers surplus,

$$\text{PS} = \left(\frac{4}{3}\right)\left(\frac{40}{3}\right) - \int_0^{\frac{4}{3}} (4x + 8)\,dx$$

$$= \frac{160}{9} - \left[2x^2 + 8x\right]_0^{\frac{4}{3}}$$

$$= \frac{160}{9} - \left[\frac{32}{9} + \frac{32}{3}\right] = \frac{160}{9} - \frac{128}{9} = \frac{32}{9}$$

Hence at market equilibrium,
(i) the consumer's surplus is $\frac{40}{9}$ units
(ii) the producer's surplus is $\frac{32}{9}$ units

Question 7.
A company requires f(x) number of hours to produce 500 units. It is represented by $f(x) = 1800x^{-0.4}$. Find out the number of hours required to produce additional 400 units.
$[(900)^{0.6} = 59.22, (500)^{0.6} = 41.63]$

Solution:
Given that for producing 500 units, the company requires f(x) = 1800 $x^{-0.4}$ hours.
Now it has to produce an additional 400 units.
So totally 900 units.
No. of hours needed

$$= \int_{500}^{900} f(x)\,dx$$

$$= \int_{500}^{900} 1800x^{-0.4}\,dx$$

$$= 1800\left[\frac{x^{0.6}}{0.6}\right]_{500}^{900} = 1800\left[\frac{(900)^{0.6}-(500)^{0.6}}{0.6}\right]$$

$$= \frac{1800}{0.6}[59.22-41.63] = 52,770$$

Question 8.
The price elasticity of demand for a commodity is $\frac{p}{X^3}$. Find the demand function if the quantity of demand is 3 when the price is ₹ 2.

Solution:

Given $\quad \eta_d = \frac{p}{x^3}$

(i.e) $\quad \frac{-p}{x}\frac{dx}{dp} = \frac{p}{x^3}$

$$\frac{-dx}{dp} = \frac{1}{x^2}$$

$$-\int x^2 dx = \int dp$$

$$\frac{-x^3}{3} = p + k$$

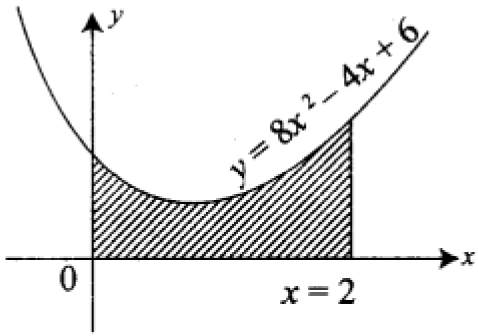
When $p = 2, x = 3$ So $\frac{-27}{3} = 2 + k$ gives $k = -11$

Hence demand function $\quad P = 11 - \frac{x^3}{3}$

Question 9.
Find the area of the region bounded by the curve between the parabola y = $8x^2 - 4x + 6$ the

y-axis and the ordinate at x = 2.

Solution:



The shaded region is the area required.

$$\text{Area} = \int_0^2 y\,dx = \int_0^2 \left(8x^2 - 4x + 6\right) dx$$

$$\text{Area} = \left(\frac{8x^3}{3} - 2x^2 + 6x\right)_0^2$$

$$= \frac{64}{3} - 8 + 12 = \frac{64}{3} + 4 = \frac{76}{3} \text{ sq.units}$$

Question 10.
Find the area of the region bounded by the curve $y^2 = 27x^3$ and the lines x = 0, y = 1 and y = 2.

Solution:
The given curve is $y^2 = 27x^3$
The lines are x = 0, y = 1 and y = 2

$$\text{Required area} = \int_1^2 x\,dy$$

$$\text{Now } x^3 = \frac{y^2}{27} \Rightarrow x = \frac{y^{\frac{2}{3}}}{3}$$

$$\text{So area} = \int_1^2 \frac{y^{\frac{2}{3}}}{3}\,dy$$

$$= \frac{1}{3}\left[\frac{3}{5}y^{\frac{5}{3}}\right]_1^2 = \frac{1}{5}\left[(2)^{\frac{5}{3}} - 1\right] \text{sq.units}$$