Electromagnetic Waves. Radiation (Part - 1)

Q.189. An electromagnetic wave of frequency v = 3.0 MHz passes from vacuum into a non-magnetic medium with permittivity a = 4.0. Find the increment of its wavelength.

Ans. The velocity of light in a medium of relative permittivity ε is $\frac{c}{\sqrt{\varepsilon}}$. Thus the change in wavelength of light (from its value in vacuum to its value in the medium) is

$$\Delta\lambda = \frac{c/\sqrt{\varepsilon}}{\nu} - \frac{c}{\nu} = \frac{c}{\nu}\left(\frac{1}{\sqrt{\varepsilon}} - 1\right) = -50 \text{ m.}$$

Q.190. A plane electromagnetic wave falls at right angles to the surface of a plane-parallel plate of thickness l. The plate is made of non-magnetic substance whose permittivity decreases exponentially from a value al at the front surface down to a value a, at the rear one. How long does it take a given wave phase to travel across this plate?

Ans. From the data of the problem the relative permittivity of the medium varies as

$$\varepsilon(x) = \varepsilon_1 e^{-(x/l)\ln\frac{\varepsilon_1}{\varepsilon_2}}$$

Hence the local velocity of light

$$v(x) = \frac{c}{\sqrt{\varepsilon(x)}} = \frac{c}{\sqrt{\varepsilon_1}} e^{\frac{x}{2l}\ln\frac{\varepsilon_1}{\varepsilon_2}}$$

Thus the required time $t = \int_0^l \frac{dx}{v(x)} = \frac{\sqrt{\varepsilon_1}}{c}\int_0^l e^{-\frac{x}{2l}\ln\frac{\varepsilon_1}{\varepsilon_2}}$

$$dx = \frac{\sqrt{\varepsilon_1}}{c}\,\frac{-e^{-\frac{1}{2}\ln\frac{\varepsilon_1}{\varepsilon_2}}+1}{\frac{1}{2l}\ln\frac{\varepsilon_1}{\varepsilon_2}} = \frac{2l}{c}\,\frac{\sqrt{\varepsilon_1}-\sqrt{\varepsilon_2}}{\ln\frac{\varepsilon_1}{\varepsilon_2}}$$

Q.191. A plane electromagnetic wave of frequency v = 10 MHz propagates in a poorly conducting medium with conductivity σ = = 10 mS/m and permittivity ε = 9. Find the ratio of amplitudes of conduction and displacement current densities.

Ans. Conduction current density $= \sigma\vec{E}$

Displacement current density $= \frac{\partial \vec{D}}{\partial t} = \varepsilon \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = i \omega \varepsilon \varepsilon_0 \vec{E}$

Ratio of magnitudes... $= \frac{\sigma}{\omega \varepsilon \varepsilon_0} = \frac{j_c}{j_{dis}} = 2,$ on putting the values.

Q.192. A plane electromagnetic wave E = E_m cos (ωt— kr) propagates in vacuum. Assuming the vectors E_m and k to be known, find the vector H as a function of time t at the point with radius vector r = 0.

Ans.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$$

$$= \nabla \cos(\omega t - \vec{k} \cdot \vec{r}) \times \vec{E}_m = \vec{k} \times \vec{E}_m \sin(\omega t - \vec{k} \cdot \vec{r})$$

At $\vec{r} = 0$

$$\frac{\partial \vec{H}}{\partial t} = -\frac{\vec{k} \times \vec{E}_m}{\mu_0} \sin \omega t$$

So integrating (ignoring a constant) and using $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$$\vec{H} = \frac{\vec{k} \times \vec{E}_m}{\mu_0} \cos c k t \times \frac{1}{c k} = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{\vec{k} \times \vec{E}_m}{k} \cos c k t$$

Q.193. A plane electromagnetic wave E = E_m cos (ωt— kr), where E_m, = $E_m e_y$, k = $k e_x$, e_x, e_u are the unit vectors of the x, y axes, propagates in vacuum. Find the vector H at the point with radius vector r = $x e_x$ at the moment (a) t = 0, (b) t = t_o. Consider the case when E_m =1 60 V/m, k = 0.51 m^{-1}, x = 7.7 m, and t_o = = 33 ns.

Ans. As in the previous problem

$$\vec{H} = \frac{\vec{k} \times \vec{E}_m}{\mu_0 \omega} \cos(\omega t - \vec{k} \cdot \vec{r}) = \frac{E_m}{\mu_0 c} \hat{e}_z \cos(k x - \omega t)$$

$$= \sqrt{\frac{\varepsilon_0}{\mu_0}} E_m \hat{e}_z \cos(k x - \omega t)$$

Thus

(a) at $t = 0$ $\vec{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E_m\, \hat{e}_z \cos kx$

(b)at $t = t_0$, $\vec{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E_m\, \hat{e}_z \cos(kx - \omega t_0)$

Q.194. A plane electromagnetic wave E = E_m, cos (ωt— kx) propagating in vacuum induces the emf gin, in a square frame with side l. The orientation of the frame is shown in Fig. 4.37. Find the amplitude value ε_{ind}, if E_m = 0.50 mV/m, the frequency v =5.0 MHz and l = 50 cm.

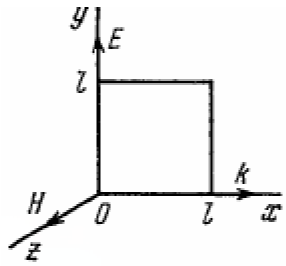


Fig. 4.37.

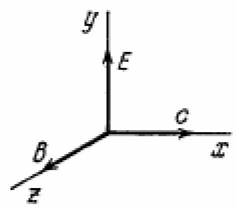


Fig. 4.38.

Ans.

$$\xi_{ind} = \oint \vec{E}\cdot d\vec{l} = E_m l(\cos\omega t - \cos(\omega t - kl))$$

$$= -2E_m l \sin\frac{\omega l}{2c}\sin\left(\omega t - \frac{\omega l}{2c}\right)$$

Putting the values

$$E_m = 50\text{ m } V/m,\ l = \frac{1}{2}\text{metre}$$

$$\frac{\omega l}{c} = \frac{2\pi\nu l}{c} = \frac{\pi\times 10^8}{3\times 10^8} = \frac{\pi}{3}$$

$$\xi_{ind} = 50\text{ m}V\left(-\sin\frac{\pi}{6}\right)\sin\left(\omega t - \frac{\pi}{6}\right)$$

$$= -25\sin\left(\omega t + \frac{\pi}{6} - \frac{\pi}{2}\right) = 25\cos\left(\omega t - \frac{\pi}{3}\right)\text{m}V$$

Q.195. Proceeding from Maxwell's equations show that in the case of a plane electromagnetic wave (Fig. 4.38) propagating in vacuum the following relations hold:

$$\frac{\partial E}{\partial t} = -c^2\frac{\partial B}{\partial x},\quad \frac{\partial B}{\partial t} = -\frac{\partial E}{\partial x}.$$

Ans.

$\vec{E} = \hat{j}\, E(t, x)$

$$\vec{B} = \hat{k}\, B(t_1 x)$$

and $\quad \text{Curl } \vec{E} = \hat{k}\, \frac{\partial E}{\partial x} = -\frac{\partial \vec{B}}{\partial t} = -\hat{k}\, \frac{\partial B}{\partial t}$

so $\quad -\frac{\partial E}{\partial x} = \frac{\partial B}{\partial t}$

Also $\quad \text{Curl } \vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$

and $\quad \text{Curl } \vec{B} = -\hat{j}\, \frac{\partial B}{\partial x}$ so $\frac{\partial B}{\partial x} = -\frac{1}{c^2} \frac{\partial E}{\partial t}.$

Q.196. Find the mean Pointing vector (8) of a plane electromagnetic wave E = E$_m$ cos (ωt— kr) if the wave propagates in vacuum.

Ans.

$\vec{E} = \vec{E}_m \cos(\omega t - \vec{k} \cdot \vec{r})$ then as before

$$\vec{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \frac{\vec{k} \times \vec{E}_m}{k} \cos(\omega t - \vec{k} \cdot \vec{r})$$

So

$$\vec{S} = \vec{E} \times \vec{H} = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \vec{E}_m \times (\vec{k} \times \vec{E}_m) \frac{1}{k} \cos^2(\omega t - \vec{k} \cdot \vec{r})$$
$$= \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \vec{E}_m^2\, \frac{\vec{k}}{k} \cos^2(\omega t - \vec{k} \cdot \vec{r})$$
$$\langle \vec{S} \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}}\, \vec{E}_m^2\, \frac{\vec{k}}{k}$$

Q.197.. A plane harmonic electromagnetic wave with plane polarization propagates in vacuum. The electric component of the wave has a strength amplitude E$_m$ = 50 mV/m, the frequency is v 100 MHz. Find:
(a) the efficient value of the displacement current density;
(b) the mean energy flow density averaged over an oscillation period.

Ans.

$E = E_m \cos(2\pi \nu t - kx)$

(a) $j_{dis} = \frac{\partial D}{\partial t} = -2\pi \varepsilon_0 \nu E_m \sin(\omega t - kx)$

Thus

$$(j_{dis})_{rms} = < j_{dis}^2 >^{1/2}$$
$$= \sqrt{2}\ \pi\varepsilon_0 \nu E_m = 0.20\ \text{mA/m}^2.$$

(b) $< S_x > = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\ E_m^2$ as in (196). Thus $< S_x > = 3.3\ \mu\,\text{W/m}^2$

Q.198. A ball of radius R = 50 cm is located in a non-magnetic medium with permittivity ε = 4.0. In that medium a plane electromagnetic wave propagates, the strength amplitude of whose electric component is equal to E_m = 200 Vim. What amount of energy reaches the ball during a time interval t = 1.0 min?

Ans. For the Pointing vector we can derive as in (196)

$< S > = \frac{1}{2}\sqrt{\frac{\varepsilon\varepsilon_0}{\mu_0}}\ E_m^2$ along the direction of propagation.

Hence in time t (which is much longer than the time period T of the wave), the eneigy reaching the ball is

$$\pi R^2 \times \frac{1}{2}\sqrt{\frac{\varepsilon\varepsilon_0}{\mu_0}}\ E_m^2 \times t = 5\ \text{kJ}.$$

Q.199. A standing electromagnetic wave with electric component E = E_m cos kx•cos ωt is sustained along the x axis in vacuum. Find the magnetic component of the wave B (x, t). Draw the approximate distribution pattern of the wave's electric and magnetic components (E and B) at the moments t = 0 and t = T/4, where T is the oscillation period.

Ans.

Here $\vec{E} = \vec{E}_m \cos kx \cos \omega t$
From div $\vec{E} = 0$ we get $E_{mx} = 0$ so $\vec{E}_m$ is in the $y-z$ plane.

Also

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} = -\nabla \cos kx \times \vec{E}_m \cos \omega t$$
$$= \vec{k}\times\vec{E}_m \sin kx \cos \omega t$$

so $$\vec{B} = \frac{\vec{k}\times\vec{E}_m}{\omega}\sin kx \sin\omega t = \vec{B}_m \sin kx \sin\omega t$$

Where $\left|\vec{B}_m\right| = \frac{E_m}{c}$ and $\vec{B}_m \perp \vec{E}_m$ in the $y-z$ plane.

At $t = 0,\ \vec{B} = 0,\ E = E_m \cos kx$

At $t = T/4\ \ \vec{E} = 0,\ B = B_m \sin kx$

Q.200. A standing electromagnetic wave E = E_m cos kx• cos ωt is sustained along the x axis in vacuum. Find the projection of the Pointing vector on the x axis s_x (x, t) and the mean value of that projection averaged over an oscillation period.

Ans.

$$\vec{E} = \vec{E}_m \cos kx\,\omega t$$

$$\vec{H} = \frac{\vec{k}\times\vec{E}_m}{\mu_0\omega}\sin kx \sin\omega t \quad \text{(exactly as in 199)}$$

$$\vec{S} = \vec{E}\times\vec{H} = \frac{\vec{E}_m\times(\vec{k}\times\vec{E}_m)}{\mu_0\omega}\frac{1}{4}\sin 2kx\ \sin 2\omega t$$

Thus

$$S_x = \frac{1}{4}\varepsilon_0 c E_m^2 \sin 2kx \sin 2\omega t \quad \left(\text{as } \frac{1}{\mu_0 c} = \varepsilon_0 c\right)$$

$$\langle S_x \rangle = 0$$

Q.201. A parallel-plate air capacitor whose electrodes are shaped as discs of radius R = 6.0 cm is connected to a source of an alternating sinusoidal voltage with frequency ω = 1000 s^{-1}. Find the ratio of peak values of magnetic and electric energies within the capacitor.

Ans. Inside the condenser the peak electrical energy $W_e = \frac{1}{2} C V_m^2$

$$= \frac{1}{2} V_m^2 \frac{\varepsilon_0 \pi R^2}{d}$$

(d = separation between the plates, πR^2 = area of each plate.).

$V = V_m \sin\omega t,\ V_m$ is the maximum voltage

Changing electric field causes a displacement current

$$j_{dis} = \frac{\partial D}{\partial t} = \varepsilon_0 E_m \omega \cos \omega t$$

$$= \frac{\varepsilon_0 \omega V_m}{d} \cos \omega t$$

This gives rise to a magnetic field B (r) (at a radial distance r from the centre of the plate)

$$B(r) \cdot 2\pi r = \mu_0 \pi r^2 j_{dis} = \mu_0 \pi r^2 \frac{\varepsilon_0 \omega V_m}{d} \cos \omega t$$

$$B = \frac{1}{2} \varepsilon_0 \mu_0 \omega \frac{r}{d} V_m \cos \omega t$$

Energy associated with this field is

$$= \int d^3 r \frac{B^2}{2\mu_0} = \frac{1}{8} \varepsilon_0^2 \mu_0 \frac{\omega^2}{d^2} 2\pi \int_0^R r^2 r\, dr \times d \times V_m^2 \cos^2 \omega t$$

$$= \frac{1}{16} \pi \varepsilon_0^2 \mu_0 \frac{\omega^2 R^4}{d} V_m^2 \cos^2 \omega t$$

Thus the maximum magnetic energy

$$W_m = \frac{\varepsilon_0^2 \mu_0}{16} (\omega R)^2 \frac{\pi R^2}{d} V_m^2$$

Hence

$$\frac{W_m}{W_e} = \frac{1}{8} \varepsilon_0 \mu_0 (\omega R)^2 = \frac{1}{8} \left(\frac{\omega R}{c}\right)^2 = 5 \times 10^{-15}$$

The approximation are valid only if $\omega R << c$.

Q.202. An alternating sinusoidal current of frequency ω = = 1000 s^{-1} f lows in the winding of a straight solenoid whose cross-sectional radius is equal to R = 6.0 cm. Find the ratio of peak values of electric and magnetic energies within the solenoid.

Ans. Here $I = I_m \cos \omega t$, then the peak magnetic eneigy is

$$W_m = \frac{1}{2} L I_m^2 = \frac{1}{2} \mu_0 n^2 I_m^2 \pi R^2 d$$

Changing magnetic field induces an electric field which by Faraday's law is given by

$$E \cdot 2\pi r = -\frac{d}{dt}\int \vec{B}\cdot d\vec{S} = \pi r^2 \mu_0 n I_m \omega \sin \omega t$$

$$E = \frac{1}{2} r \mu_0 n I_m \omega \sin \omega t$$

The associated peak electric eneigy is

$$W_e = \int \frac{1}{2}\varepsilon_0 E^2 d^3 r = \frac{1}{8}\varepsilon_0 \mu_0^2 n^2 I_m^2 \omega^2 \sin^2 \omega t \times \frac{\pi R^4}{2} d$$

Hence $$\frac{W_e}{W_m} = \frac{1}{8}\varepsilon_0 \mu_0 (\omega R)^2 = \frac{1}{8}\left(\frac{\omega R}{c}\right)^2$$

Again we expect the results to be valid if and only if

$$\left(\frac{\omega R}{c}\right) << 1$$

Q.203. A parallel-plate capacity whose electrodes are shaped as round discs is charged slowly. Demonstrate that the flux of the Pointing vector across the capacitor's lateral surface is equal to the increment of the capacitor's energy per unit time. The dissipation of field at the edge is to be neglected in calculations.

Ans. If the chaige on the capacitor is Q, the rate of increase of the capacitor’s energy

$$= \frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C}\right) = \frac{Q\dot{Q}}{C} = \frac{d}{\varepsilon_0 \pi R^2} Q\dot{Q}$$

Now electric field between the plates (inside it) is, $E = \frac{Q}{\pi R^2 \varepsilon_0}$.

So displacement current $= \frac{\partial D}{\partial t} = \frac{\dot{Q}}{\pi R^2}$

This will lead to a magnetic field, (circuital) inside the plates. At a radial distance r

$$2\pi r H_\theta (r) = \pi r^2 \frac{\dot{Q}}{\pi R^2} \quad \text{or} \quad H_\theta = \frac{\dot{Q} r}{2\pi R^2}$$

Hence $H_\theta (R) = \frac{\dot{Q}}{2\pi R}$ at the edge.

Thus inward Pointing vector $= S = \frac{\dot{Q}}{2\pi R} \times \frac{Q}{\pi R^2 \varepsilon_0}$

Total flow = $2\pi R d \times S = \frac{Q Q d}{\pi R^2 \varepsilon_0}$ Proved

Q.204. A current I flows along a straight conductor with round cross-section. Find the flux of the Pointing vector across the lateral surface of the conductor's segment with resistance R.

Ans. Suppose the radius of the conductor is R_0. Then the conduction current density is

$$j_c = \frac{I}{\pi R_0^2} = \sigma E \quad \text{or} \quad E = \frac{I}{\pi R_0^2 \sigma} = \frac{\rho I}{\pi R_0^2}$$

where $: \rho = \frac{1}{\sigma}$ is the resistivity.

Inside the conductor there is a magnetic field given by

$$H \cdot 2\pi R_0 = I \quad \text{or} \quad H = \frac{I}{2\pi R_0} \text{ at the edge}$$

∴ Energy flowing in per second in a section of length l is

$$E H \times 2\pi R_0 l = \frac{\rho I^2 l}{\pi R_0^2}$$

But the resistance $R = \frac{\rho l}{\pi R_0^2}$

Thus the energy flowing into the conductor = $I^2 R$.

Q.205. Non-relativistic protons accelerated by a potential difference U form a round beam with current I. Find the magnitude and direction of the Pointing vector outside the beam at a distance r from its axis.

Ans. Here $n e v = I/\pi R^2$ where R = radius of cross section of the conductor and n = chaige density (per unit volume)

Also

$$\frac{1}{2} m v^2 = e U \quad \text{or} \quad v = \sqrt{\frac{2 e U}{m}}.$$

Thus, the moving protons have a charge per unit length

$$= n e \pi R^2 = I\sqrt{\frac{m}{2eU}}.$$

This gives rise to an electric field at a distance r given by

$$E = \frac{I}{\varepsilon_0}\sqrt{\frac{m}{2eU}} \Big/ 2\pi r$$

The magnetic field is $H = \frac{I}{2\pi r}$ (for $r > R$)

Thus

$S = \frac{I^2}{\varepsilon_0 4\pi^2 r^2}\sqrt{\frac{m}{2eU}}$ Radially outward from the axis

This is the Pointing vector.

Q.206. A current flowing in the winding of a long straight solenoid is increased at a sufficiently slow rate. Demonstrate that the rate at which the energy of the magnetic field in the solenoid increases is equal to the flux of the Pointing vector across the lateral surface of the solenoid.

Ans. Within the solenoid $B = \mu_0 n I$ and the rate of change of magnetic energy

$$= \dot{W}_m = \frac{d}{dt}\left(\frac{1}{2}\mu_0 n^2 I^2 \pi R^2 l\right) = \mu_0 n^2 \pi R^2 l I \dot{I}$$

Where R = radius of cross section of the solenoid l = length.

Also $H = B/\mu_0 = nI$ along the axis within the solenoid.

By Faraday's law, the induced electric field is

$$E_\theta\, 2\pi r = \pi r^2 \dot{B} = \pi r^2 \mu_0 n \dot{I}$$

Or $E_\theta = \frac{1}{2}\mu_0 n \dot{I} r$

so at the edge $E_\theta(R) = \frac{1}{2}\mu_0 n \dot{I} R$ (circuital)

Then $S_r = E_\theta H_z$ (radially inward)

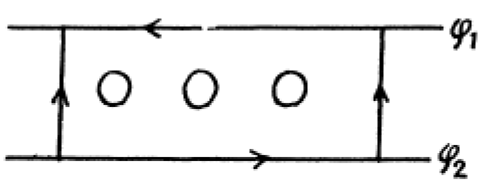
and $\dot{W}_m = \frac{1}{2}\mu_0 n^2 I \dot{I} R \times 2\pi R l = \mu_0 n^2 \pi R^2 l I \dot{I}$ as before.

Q.207. Fig. 4.39 illustrates a segment of a double line carrying direct current whose direction is indicated by the arrows. Taking into account that the potential $\varphi_2 > \varphi_1$, and making use of the Pointing vector, establish on which side (left or right) the source of the current is located.

Fig. 4.39.

Ans. Given $\varphi_2 > \varphi_1$
The electric field is as shown by the dashed lines (— →).

The magnetic field is as shown (⊙) emerging out of the paper. $\vec{S} = \vec{E} \times \vec{H}$ is parallel to the wires and towards right.



Hence source must be on the left.

Q.208. The energy is transferred from a source of constant voltage V to a consumer by means of a long straight coaxial cable with negligible active resistance. The consumed current is I. Find the energy flux across the cross-section of the cable. The conductive sheath is supposed to be thin.

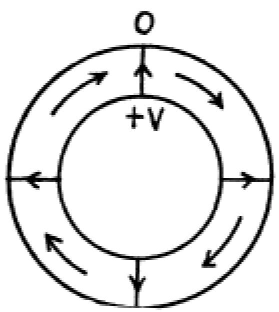
Ans. The electric field (— →) and the magnetic field (H→) are as shown. The electric field by Gauss's theorem is like

$$E_r = \frac{A}{r}$$

Integrating $$\varphi = A \ln \frac{r_2}{r}$$

so $$A = \frac{V}{\ln \frac{r_2}{r_1}} \quad (r_2 > r_1)$$

Then $$E = \frac{V}{r \ln \frac{r_2}{r_1}}$$



$$E = \frac{V}{r \ln \frac{r_2}{r_1}}$$
Then

Magnetic field is $H_\theta = \frac{I}{2\pi r}$

The Poynting vector $\bar{S}$ is along the Z axis and non zero between the two wires $(r_1 < r < r_2)$. The total power flux is

$$= \int_{r_1}^{r_2} \frac{IV}{2\pi r^2 \ln \frac{r_2}{r_1}} \cdot 2\pi r \, dr = IV$$

Q.209. A source of ac voltage V = V_o cos ωt delivers energy to a consumer by means of a long straight coaxial cable with negligible active resistance. The current in the circuit varies as I = = I_o cos ωt — φ). Find the time-averaged energy flux through the cross-section of the cable. The sheath is thin.

Ans. As in the previous problem

$$E_r = \frac{V_0 \cos \omega t}{r \ln \frac{r_2}{r_1}} \quad \text{and} \quad H_\theta = \frac{I_0 \cos(\omega t - \varphi)}{2 \pi r}$$

Hence time averaged power flux (along the z axis) = $\frac{1}{2} V_0 I_0 \cos \varphi$

On using $< \cos \omega t \cos(\omega t - \varphi) > = \frac{1}{2} \cos \varphi$.

Q.210. Demonstrate that at the boundary between two media the normal components of the Pointing vector are continuous, i.e. $S_{1n} = S_{2n}$.

Ans. Let $\vec{n}$ be along the z axis. Then

$$S_{1n} = E_{1x} H_{1y} - E_{1y} H_{1x}$$

and

$$S_{2n} = E_{2x} H_{2y} - E_{2y} H_{2x}$$

Using the boundary condition $E_{1t} = E_{2t}$, $H_{1t} = H_{2t}$ at the boundary ($t = x$ or y) we see

$$S_{1n} = S_{2n}$$

Electromagnetic Waves. Radiation (Part - 2)

Q.211. Demonstrate that a closed system of charged non-relativistic particles with identical specific charges emits no dipole radiation.

Ans.

$P \cdot \alpha\ |\ddot{\vec{p}}|^2$ when

$$\vec{p} = \sum e_i \vec{r_i} = \sum \frac{e_i}{m_i} m_i \vec{r_i} = \frac{e}{m} \sum m_i \vec{r_i}$$

if $$\frac{e_i}{m_i} = \frac{e}{m} = \text{fixed}$$

But $$\frac{d^2}{dt^2} \sum m_i \vec{r_i} = 0 \quad \text{for a closed system}$$

Hence P = 0

Q.212. Find the mean radiation power of an electron performing harmonic oscillations with amplitude a = 0.10 nm and frequent cy ω = 6.5.10^{14}S^{-1}

Ans.

$$P = \frac{1}{4\pi\varepsilon_0} \frac{2(\ddot{\vec{p}})^2}{3c^3}$$

$$|\ddot{\vec{p}}|^2 = (e\omega^2 a)^2 \cos^2 \omega t$$

Thus $$\langle P \rangle = \frac{1}{4\pi\varepsilon_0} \frac{2}{3c^3} (e\omega^2 a)^2 \times \frac{1}{2} = \frac{e^2 \omega^4 a^2}{12\pi\varepsilon_0 c^3} = 5.1 \times 10^{-15} \text{ W.}$$

Q.213. Find the radiation power developed by a non-relativistic particle with charge e and mass m, moving along a circular orbit of radius R in the field of a stationary point charge q.

Ans. Here

$$\ddot{\vec{p}} = \frac{e}{m} \times \text{force} = \frac{e^2 q}{mR^2} \frac{1}{4\pi\varepsilon_0}.$$

Thus $P = \frac{1}{(4\pi\varepsilon_0)^3}\left(\frac{e^2 q}{mR^2}\right)^2 \frac{2}{3c^3}$

Q.214. A particle with charge e and mass m. flies with non-relativistic velocity v at a distance b past a stationary particle with charge q. Neglecting the bending of the trajectory of the moving particle, find the energy lost by this particle due to radiation during the total flight time

Ans. Most of the radiation occurs when the moving particle is closest to the stationary particle. In that region, we can write

$R^2 = b^2 + v^2 t^2$

And apply the previous problem's formula

Thus $\Delta W = \frac{1}{(4\pi\varepsilon_0)^3}\frac{2}{3c^3}\int_{-\infty}^{\infty}\left(\frac{qe^2}{m}\right)^2 \frac{dt}{(b^2+v^2t^2)^2}$

(The integral can be taken between $\pm\infty$ with little error.)

Now $\int_{-\infty}^{\infty}\frac{dt}{(b^2+v^2t^2)^2} = \frac{1}{v}\int_{-\infty}^{\infty}\frac{dx}{(b^2+x^2)^2} = \frac{\pi}{2vb^3}.$

Hence, $\Delta W = \frac{1}{(4\pi\varepsilon_0)^3}\frac{\pi q^2 e^4}{3c^3 m^2 v b^3}.$

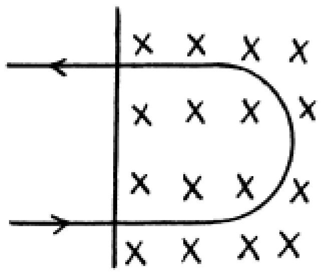
Q.215. A non-relativistic proton enters a half-space along the normal to the transverse uniform magnetic field whose induction equals B = 1.0 T. Find the ratio of the energy lost by the proton due to radiation during its motion in the field to its initial kinetic energy.

Ans. For the semicircular path on the right

$$\frac{mv^2}{R} = Bev \quad \text{or} \quad v = \frac{BeR}{m}.$$

Thus K.E. $= T = \frac{1}{2}mv^2 = \frac{B^2e^2R^2}{2m}.$

Power radiated $= \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{ev^2}{R}\right)^2$



Hence energy radiated $= \Delta W$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{B^2e^3R}{m^2}\right)^2\cdot\frac{\pi R}{BeR}m = \frac{B^3e^5R^2}{6\varepsilon_0 m^3 c^3}$$

So $$\frac{\Delta W}{T} = \frac{Be^3}{3\varepsilon_0 c^3 m^2} = 2.06 \times 10^{-18}.$$

(Neglecting the change in v due to radiation, correct if $\Delta W/T \ll 1$).

Q.216. A non-relativistic charged particle moves in a transverse uniform magnetic field with induction B. Find the time dependence of the particle's kinetic energy diminishing due to radiation. How soon will its kinetic energy decrease e-fold? Calculate this time interval for the case
(a) of an electron,
(b) of a proton.

Ans.

$$R = \frac{mv}{eB}.$$

Then

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{ev^2}{R}\right)^2 = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{e^2Bv}{m}\right)^2$$
$$= \frac{1}{3\pi\varepsilon_0 c^3}\left(\frac{B^2e^4}{m^3}\right)T$$

This is the radiated power so

$$\frac{dT}{dt} = -\frac{B^2e^4}{3\pi\varepsilon_0 m^3 c^3}T$$

Integrating, $T = T_0\, e^{-t/\tau}$

$$\tau = \frac{3\pi\varepsilon_0 m^3 c^3}{B^2 e^4}$$

τ is $(1836)^3 \approx 10^{10}$ Times less for an electron than for a proton so electrons radiate away their energy much faster in a magnetic field.

Q.217. A charged particle moves along the y axis according to the law y = a cos ωt, and the point of observation P is located on the axis at a distance 1 from the particle (l ≫ a). Find the ratio of electromagnetic radiation flow densities S_1/S_2 at the point P at the moments when the coordinate of the particle $y_1 = 0$ and $y_2 = a$. Calculate that ratio if co = 3.3.10^6 s^{-1} and l = 190 m.

Ans. P is a fixed point at a distance l from the equilibrium position of the particle. Because / > a, to first order in $\frac{a}{l}$ the distance between P and the instantaneous position of the particle is still l For the first case y = 0 so t = T/4

The corresponding retarded time is $t' = \frac{T}{4} - \frac{l}{c}$

Now $\ddot{y}(t') = -\omega^2 a \cos\omega\left(\frac{T}{4} - \frac{l}{c}\right) = -\omega^2 a \sin\frac{\omega l}{c}$

For the second case y = a at t = 0 so at the retarded time : $t' = -\frac{\omega l}{c}$

Thus $\ddot{y}(t') = -\omega^2 a \cos\frac{\omega l}{c}$

The radiation fluxes in the two cases are proportional to $(\ddot{y}(t'))^2$

$\frac{S_1}{S_2} = \tan^2\frac{\omega l}{c} = 3.06$ On substitution.

Note: The radiation received at P at time t depends on the acceleration of the charge at the retarded time.

Q.218. A charged particle moves uniformly with velocity v along a circle of radius R in the plane xy (Fig. 4.40). An observer is located

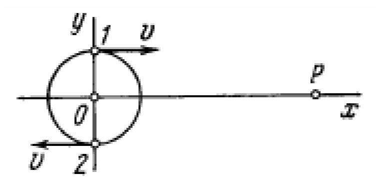


Fig. 4.40.

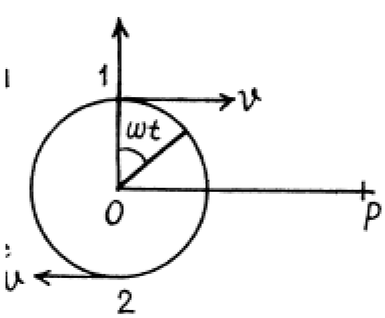
on the x axis at a point P which is removed from the centre of the circle by a distance much exceeding R. Find:

(a) the relationship between the observed values of the y projection of the particle's acceleration and the y coordinate of the particle;

(b) the ratio of electromagnetic radiation flow densities S_1/S_2 at the point P at the moments of time when the particle moves, from the standpoint of the observer P, toward him and away from him, as shown in the figure.

Ans. Along the circle x = sin ωt, y = cos ωt

Where $\omega = \frac{v}{R}$ If t is the parameter in x (r), y (r) and



t ' is the observer time then

$$t' = t + \frac{l - x(t)}{c}$$

Where we have neglected the effect of the y--coordinate which is of second order. The observed coordinate are

$$x'(t') = x(t),\ y'(t') = y(t)$$

Then

$$\frac{dy'}{dt'} = \frac{dy}{dt'} = \frac{dt}{dt'}\frac{dy}{dt} = \frac{-\omega R \sin \omega t}{1 - \frac{\omega R}{c}\cos \omega t} = \frac{-\omega x}{1 - \frac{\omega y}{c}} = \frac{-v\,x/R}{1 - \frac{v\,y}{c\,R}}$$

And

$$\frac{d^2 y'}{dt'^2} = \frac{dt}{dt'}\frac{d}{dt}\left(\frac{-v\,x/R}{1 - \frac{v\,y}{c\,R}}\right)$$

$$= \frac{1}{1 - \frac{v\,y}{c\,R}}\left\{\frac{-\frac{v^2}{R^2}y}{1 - \frac{v\,y}{c\,R}} + \frac{\frac{v\,x}{R}\left(\frac{v^2}{c\,R^2}x\right)}{\left(1 - \frac{v\,y}{c\,R}\right)^2}\right\} = \frac{\frac{v^2}{R}\left(\frac{v}{c} - \frac{y}{R}\right)}{\left(1 - \frac{v\,y}{c\,R}\right)^3}.$$

This is the observed acceleration.

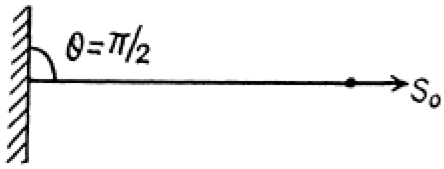
(b) Energy flow density of JEM radiation S is proportional to the square of the y-projection of the observed acceleration of the particle $\left(\text{i.e. } \frac{d^2 y'}{dt'^2}\right)$.

Thus $$\frac{S_1}{S_2} = \left[\frac{\left(\frac{v}{c} - 1\right)}{\left(1 - \frac{v}{c}\right)^3} \Bigg/ \frac{\left(\frac{v}{c} + 1\right)}{\left(1 + \frac{v}{c}\right)^3}\right]^2 = \frac{\left(1 + \frac{v}{c}\right)^4}{\left(1 - \frac{v}{c}\right)^4}.$$

Q.219. An electromagnetic wave emitted by an elementary dipole propagates in vacuum so that in the far field zone the mean value of the energy flow density is equal to S_0 at the point removed from the dipole by a distance r along the perpendicular drawn to the dipole's axis. Find the mean radiation power of the dipole. pole's axis. Find the mean radiation power of the dipole.

Ans. We know that

$$S_0(r) \propto \frac{1}{r^2}$$



At other angles $S(r, \theta) \propto \sin^2\theta$

Thus $S(r, \theta) = S_0(r)\sin^2\theta = S_0 \sin^2\theta$

Average power radiated

$$= S_0 \times 4\pi r^2 \times \frac{2}{3} = \frac{8\pi}{3} S_0 r^2$$

$$\left(\text{Average of } \sin^2\theta \text{ over whole sphere is } \frac{2}{3}\right)$$

Q.220. The mean power radiated by an elementary dipole is equal to P_o. Find the mean space density of energy of the electromagnetic field in vacuum in the far field zone at the point removed from the dipole by a distance r along the perpendicular drawn to the dipole's axis.

Ans. From the previous problem.

$$P_0 = \frac{8\pi S_0 r^2}{3}$$

or $S_0 = \frac{3P_0}{8\pi r^2}$

Thus $\langle w \rangle = \frac{S_0}{c} = \frac{3P_0}{8\pi c r^2}$

(Pointing flux vector is the energy contained is a box of unit cross section and length c).

Q.221. An electric dipole whose modulus is constant and whose moment is equal to p rotates with constant angular velocity w about the axis drawn at right angles to the axis of the dipole and passing through its midpoint. Find the power radiated by such a dipole.

Ans. The rotating dispole has moments

$$p_x = p\cos\omega t,\ p_y = p\sin\omega t$$

Thus

$$P = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\omega^4 p^2 = \frac{p^2\omega^4}{6\pi\varepsilon_0 c^3}.$$

Q.222. A free electron is located in the field of a plane electromagnetic wave. Neglecting the magnetic component of the wave disturbing its motion, find the ratio of the mean energy radiated by the oscillating electron per unit time to the mean value of the energy flow density of the incident wave.

Ans. If the electric field of the wave is

$$\vec{E} = \vec{E_0}\cos\omega t$$

Then this induces a dipole moment whose second derivative is

$$\ddot{\vec{p}} = \frac{e^2\vec{E_0}}{m}\cos\omega t$$

Hence radiated mean power $$<P> = \frac{1}{4\pi\varepsilon_0}\frac{2}{3c^3}\left(\frac{e^2 E_0}{m}\right)^2\times\frac{1}{2}$$

On the other hand the mean Pointing flux of the incident radiation is

$$<S_{inc}> = \sqrt{\frac{\varepsilon_0}{\mu_0}}\times\frac{1}{2}E_0^2$$

Thus

$$\frac{P}{<S_{inc}>} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3}(\varepsilon_0\mu_0)^{3/2}\left(\frac{e^2}{m}\right)^2\times\sqrt{\frac{\mu_0}{\varepsilon_0}}$$
$$= \frac{\mu_0^2}{6\pi}\left(\frac{e^2}{m}\right)^2$$

Q.223. A plane electromagnetic wave with frequency ω falls upon an elastically bonded electron whose natural frequency equals ω_0. Neglecting the damping of oscillations, find the ratio of the mean energy dissipated by the electron per unit time to the mean value of the energy flow density of the incident wave.

Ans. For the elastically bound electron

$$m \ddot{\vec{r}} + m \omega_0^2 \vec{r} = e \vec{E_0} \cos \omega t$$

This equation has the particular integral (i.e. neglecting the part which does not have the frequency of the impressed force)

$$\vec{r} = \frac{e \vec{E_0}}{m} \frac{\cos \omega t}{\omega_0^2 - \omega^2} \quad \text{so and} \quad \ddot{\vec{p}} = -\frac{e^2 \vec{E_0} \omega^2}{(\omega_0^2 - \omega^2) m} \cos \omega t$$

Hence P = mean radiated power

$$= \frac{1}{4 \pi \varepsilon_0} \frac{2}{3 c^3} \left(\frac{e^2 \omega^2}{m (\omega_0^2 - \omega^2)} \right)^2 \frac{1}{2} E_0^2$$

The mean incident poynting flux is

$$< S_{inc} > = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{1}{2} E_0^2$$

Thus

$$\frac{P}{< S_{inc} >} = \frac{\mu_0^2}{6 \pi} \left(\frac{e^2}{m} \right)^2 \cdot \frac{\omega^4}{(\omega_0^2 - \omega^2)^2}.$$

Q.224. Assuming a particle to have the form of a ball and to absorb all incident light, find the radius of a particle for which its gravitational attraction to the Sun is counterbalanced by the force that light exerts on it. The power of light radiated by the Sun equals P = 4.10^{26} W, and the density of the particle is p = 1.0 g/cm^3

Ans. Let r = radius of the ball R = distance between the ball & the Sun (r « R).

M = mass of the Sun

y = gravitational constant

Then

$$\frac{\gamma M}{R^2} \frac{4 \pi}{3} r^3 \rho = \frac{P}{4 \pi R^2} \pi r^2 \cdot \frac{1}{c}$$

(The factor $\frac{1}{c}$ converts the energy received on the right into momentum received. Then the right hand side is the momentum received per unit time and must equal tLe negative of the impressed force for equilibrium).

Thus $r = \frac{3P}{16\pi\gamma M c \rho} = 0.606\ \mu\text{m}.$