

DAY SIXTEEN

Definite Integrals

Learning & Revision for the Day

- Concept of Definite Integrals
- Leibnitz Theorem
- Walli's Formula
- Inequalities in Definite Integrals
- Definite Integration as the Limit of a sum

Concept of Definite Integrals

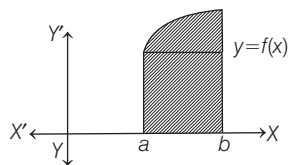
Let $\phi(x)$ be an anti-derivative of a function $f(x)$ defined on $[a, b]$ i.e. $\frac{d}{dx} [\phi(x)] = f(x)$. Then, definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x) dx$ and is defined as $[\phi(b) - \phi(a)]$ i.e. $\int_a^b f(x) dx = \phi(b) - \phi(a)$. The numbers a and b are called the limits of integration, where a is called **lower limit** and b is **upper limit**.

NOTE

- Every definite integral has a unique value.
- The above definition is nothing but the statement of second fundamental theorem of integral calculus.

Geometrical Interpretation of Definite Integral

In general, $\int_a^b f(x) dx$ represents an algebraic sum of areas of the region bounded by the curve $y = f(x)$, the X -axis, and the ordinates $x = a$ and $x = b$ as show in the following figure.



Evaluation of Definite Integrals by Substitution

When the variable in a definite integral is changed due to substitution, then the limits of the integral will accordingly be changed.

For example, to evaluate definite integral of the form

$$\int_a^b f[g(x)] \cdot g'(x) dx, \text{ we use the following steps}$$

Step I Substitute $g(x) = t$ so that $g'(x) dx = dt$

Step II Find the limits of integration in new system of variable. Here, the lower limit is $g(a)$, the upper limit is $g(b)$ and the integral is now $\int_{g(a)}^{g(b)} f(t) dt$.

Step III Evaluate the integral, so obtained by usual method.

Properties of Definite Integrals

$$(i) \int_a^a f(x) dx = 0$$

$$(ii) \int_a^b f(x) dx = \int_a^b f(t) dt$$

$$(iii) \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$(iv) \int_a^b f(x) dx = \int_a^{c_1} f(x) dx + \int_{c_1}^{c_2} f(x) dx + \dots + \int_{c_n}^b f(x) dx$$

where, $\alpha < c_1 < c_2 < \dots < c_n < \beta$

$$(v) (a) \int_a^\beta f(x) dx = \int_a^\beta f(\alpha + \beta - x) dx$$

$$(b) \int_0^\alpha f(x) dx = \int_0^\alpha f(\alpha - x) dx$$

$$(vi) \int_{-\alpha}^\alpha f(x) dx$$

$$= \begin{cases} 2 \int_0^\alpha f(x) dx, & \text{if } f(-x) = f(x) \\ & \text{i.e. } f(x) \text{ is an even function} \\ 0, & \text{if } f(-x) = -f(x) \\ & \text{i.e. } f(x) \text{ is an odd function} \end{cases}$$

$$(vii) \int_0^{2\alpha} f(x) dx = \int_0^\alpha f(x) dx + \int_0^\alpha f(2\alpha - x) dx$$

$$(viii) \int_0^{2\alpha} f(x) dx = \begin{cases} 2 \int_0^\alpha f(x) dx, & \text{if } f(2\alpha - x) = f(x) \\ 0, & \text{if } f(2\alpha - x) = -f(x) \end{cases}$$

$$(ix) \int_\alpha^\beta f(x) dx = (\beta - \alpha) \int_0^1 f[(\beta - \alpha)x + \alpha] dx$$

(x) If $f(x)$ is a periodic function with period T , then

$$(a) \int_\alpha^{\alpha + nT} f(x) dx = n \int_0^T f(x) dx, n \in I$$

$$(b) \int_{\alpha T}^{\beta T} f(x) dx = (\beta - \alpha) \int_0^T f(x) dx, \alpha, \beta \in I$$

$$(c) \int_{\alpha + nT}^{\beta + nT} f(x) dx = \int_\alpha^\beta f(x) dx, n \in I$$

(xi) Some important integrals, which can be obtained with the help of above properties.

$$(a) \int_0^{\pi/2} \log \sin x dx = \int_0^{\pi/2} \log \cos x dx = \frac{\pi}{2} \log \left(\frac{1}{2} \right) \cdot 3$$

$$(b) \int_0^{\pi/4} \log(1 + \tan x) dx = \frac{\pi}{8} \log 2.$$

(xii) If a function $f(x)$ is discontinuous at points x_1, x_2, \dots, x_n in (a, b) , then we can define sub-intervals

$(a, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n), (x_n, b)$ such that $f(x)$ is continuous in each of these sub-intervals and

$$\int_a^b f(x) dx = \int_a^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_{n-1}}^{x_n} f(x) dx + \int_{x_n}^b f(x) dx.$$

Leibnitz Theorem

If function $\phi(x)$ and $\psi(x)$ are defined on $[\alpha, \beta]$ and differentiable on $[\alpha, \beta]$ and $f(t)$ is continuous on $[\psi(\alpha), \phi(\beta)]$, then

$$\frac{d}{dx} \left[\int_{\phi(x)}^{\psi(x)} f(t) dt \right] = \left\{ \frac{d}{dt} \psi(x) \right\} f(\psi(x)) - \left\{ \frac{d}{dx} \phi(x) \right\} f(\phi(x))$$

Walli's Formula

This is a special type of integral formula whose limits from 0 to $\pi/2$ and integral is either integral power of $\cos x$ or $\sin x$ or $\cos x \sin x$.

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx$$

$$= \begin{cases} \frac{(n-1)(n-3)(n-5) \dots 5 \cdot 3 \cdot 1}{n(n-2)(n-4) \dots 6 \cdot 4 \cdot 2} \times \frac{\pi}{2}, & \text{if } n = 2m \text{ (even)} \\ \frac{(n-1)(n-3)(n-5) \dots 6 \cdot 4 \cdot 2}{n(n-2)(n-4) \dots 5 \cdot 3 \cdot 1}, & \text{if } n = 2m + 1 \text{ (odd)} \end{cases}$$

where, n is positive integer.

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x dx$$

$$= \begin{cases} \frac{(m-1)(m-3) \dots (2 \text{ or } 1) \cdot (n-1)(n-3) \dots (2 \text{ or } 1) \cdot \frac{\pi}{2}}{(m+n)(m+n-2) \dots (2 \text{ or } 1)}, & \text{when both } m \text{ and } n \text{ are even positive integers} \\ \frac{(m-1)(m-3) \dots (2 \text{ or } 1) \cdot (n-1)(n-3) \dots (2 \text{ or } 1)}{(m+n)(m+n-2) \dots (2 \text{ or } 1)}, & \text{when either } m \text{ or } n \text{ or both are odd positive integers} \end{cases}$$

Inequalities in Definite Integrals

$$(i) \text{ If } f(x) \geq g(x) \text{ on } [\alpha, \beta], \text{ then } \int_\alpha^\beta f(x) dx \geq \int_\alpha^\beta g(x) dx$$

$$(ii) \text{ If } f(x) \geq 0 \text{ in the interval } [\alpha, \beta], \text{ then } \int_\alpha^\beta f(x) dx \geq 0$$

$$(iii) \text{ If } f(x), g(x) \text{ and } h(x) \text{ are continuous on } [a, b] \text{ such that } g(x) \leq f(x) \leq h(x), \text{ then } \int_a^b g(x) dx \leq \int_a^b f(x) dx \leq \int_a^b h(x) dx$$

(iv) If f is continuous on $[\alpha, \beta]$ and $l \leq f(x) \leq M, \forall x \in [\alpha, \beta]$, then $l(\beta - \alpha) \leq \int_{\alpha}^{\beta} f(x) dx \leq M(\beta - \alpha)$

(v) If f is continuous on $[\alpha, \beta]$, then $\left| \int_{\alpha}^{\beta} f(x) dx \right| \leq \int_{\alpha}^{\beta} |f(x)| dx$

(vi) If f is continuous on $[\alpha, \beta]$ and $|f(x)| \leq k, \forall x \in [\alpha, \beta]$, then $\left| \int_{\alpha}^{\beta} f(x) dx \right| \leq k(\beta - \alpha)$

Definite Integration as the Limit of a Sum

Let $f(x)$ be a continuous function defined on the closed interval $[a, b]$, then $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} h \sum_{r=0}^{n-1} f(a + rh)$

where, $h = \frac{b-a}{n} \rightarrow 0$ as $n \rightarrow \infty$

The converse is also true, i.e. if we have an infinite series of the above form, it can be expressed as definite integral.

Some Particular Cases

$$(i) \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} f\left(\frac{r}{n}\right) \text{ or } \lim_{n \rightarrow \infty} \sum_{r=1}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x) dx$$

$$(ii) \lim_{n \rightarrow \infty} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_{\alpha}^{\beta} f(x) dx$$

$$\text{where, } \alpha = \lim_{n \rightarrow \infty} \frac{r}{h} = 0 \quad (\because r = 1)$$

$$\text{and } \beta = \lim_{n \rightarrow \infty} \frac{r}{h} = p \quad (\because r = pn)$$

DAY PRACTICE SESSION 1

FOUNDATION QUESTIONS EXERCISE

1 $\int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$ is equal to

- (a) -2 (b) 2 (c) 4

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- (d) -1

2 If $f(x)$ is continuous function, then

(a) $\int_{-2}^2 f(x) dx = \int_0^2 [f(x) - f(-x)] dx$

(b) $\int_{-3}^5 2f(x) dx = \int_6^{10} f(x-1) dx$

(c) $\int_{-3}^5 f(x) dx = \int_{-4}^4 f(x-1) dx$

(d) $\int_{-3}^5 f(x) dx = \int_{-2}^6 f(x-1) dx$

3 $\int_0^{\pi/4} [\sqrt{\tan x} + \sqrt{\cot x}] dx$ is equal to

- (a) $\sqrt{2}\pi$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{\sqrt{2}}$ (d) 2π

4 $\int_0^1 \sin\left(2 \tan^{-1} \sqrt{\frac{1+x}{1-x}}\right) dx$ is equal to

- (a) $\pi/6$ (b) $\pi/4$ (c) $\pi/2$ (d) π

5 If $I_{1(n)} = \int_0^{\pi/2} \frac{\sin(2n-1)}{\sin x} dx$ and $I_{2(n)} = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx$,

$n \in N$, then

(a) $I_{2(n+1)} - I_{2(n)} = I_{1(n)}$ (b) $I_{2(n+1)} - I_{2(n)} = I_{1(n+1)}$

(c) $I_{2(n+1)} + I_{1(n)} = I_{2(n)}$ (d) $I_{2(n+1)} + I_{1(n+1)} = I_{2(n)}$

6 $\int_{-1}^1 \{x^2 + x - 3\} dx$, where $\{x\}$ denotes the fractional part of x , is equal to

- (a) $\frac{1}{3}(1 + 3\sqrt{5})$ (b) $\frac{1}{6}(1 + 3\sqrt{5})$

- (c) $\frac{1}{3}(3\sqrt{5} - 1)$ (d) $\frac{1}{6}(3\sqrt{5} - 1)$

7 $\int_0^2 [x^2] dx$ is equal to

- (a) $2 - \sqrt{2}$ (b) $2 + \sqrt{2}$
(c) $\sqrt{2} - 1$ (d) $-\sqrt{2} - \sqrt{3} + 5$

8 If $\int_{-2}^x |2t| dt = f(x)$, then for any $x \geq 0$, $f(x)$ is equal to

- (a) $4 + x^2$ (b) $4 - x^2$
(c) $\frac{1}{2}(4 + x^2)$ (d) $\frac{1}{4}(4 - x^2)$

9 $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^x} dx$ is equal to

- (a) 1 (b) 0
(c) -1 (d) None of these

10 Let a, b and c be non-zero real numbers such that

$$\int_0^3 (3ax^2 + 2bx + c) dx = \int_1^3 (3ax^2 + 2bx + c) dx, \text{ then}$$

- (a) $a + b + c = 3$ (b) $a + b + c = 1$
(c) $a + b + c = 0$ (d) $a + b + c = 2$

11 The value of $\int_1^a [x] f'(x) dx$, $a > 1$, where $[x]$ denotes the greatest integer not exceeding x , is

- (a) $[a]f(a) - \{f(1) + f(2) + \dots + f([a])\}$
(b) $[a]f([a]) - \{f(1) + f(2) + \dots + f(a)\}$
(c) $af([a]) - \{f(1) + f(2) + \dots + f(a)\}$
(d) $af(a) - \{f(1) + f(2) + \dots + f([a])\}$

12 The correct evaluation of $\int_0^{\pi/2} \left| \sin\left(x - \frac{\pi}{4}\right) \right| dx$ is

- (a) $2 + \sqrt{2}$ (b) $2 - \sqrt{2}$
(c) $-2 + \sqrt{2}$ (d) 0

13 $\int_0^{\lceil 3\pi/2 \rceil} \sin\left(\left\lceil \frac{2x}{\pi} \right\rceil\right) dx$, where $\lceil \cdot \rceil$ denotes the greatest

integer function is equal to

- (a) $\frac{\pi}{2}(\sin 1 + \cos 1)$ (b) $\frac{\pi}{2}(\sin 1 - \sin 2)$
 (c) $\frac{\pi}{2}(\sin 1 - \cos 1)$ (d) $\frac{\pi}{2}(\sin 1 + \sin 2)$

14 If $\lceil \cdot \rceil$ denotes the greatest integer function, then the value of $\int_0^{1.5} x \lceil x^2 \rceil dx$ is

- (a) $\frac{5}{4}$ (b) 0 (c) $\frac{3}{2}$ (d) $\frac{3}{4}$

15 The value of $\int_0^1 \frac{8 \log(1+x)}{1+x^2} dx$ is

- (a) $\frac{\pi}{8} \log 2$ (b) $\frac{\pi}{2} \log 2$ (c) $\log 2$ (d) $\pi \log 2$

16 $\int_0^{\pi/2} \sqrt{1 - \sin 2x}$ is equal to

- (a) $2\sqrt{2}$ (b) $2(\sqrt{2} + 1)$ (c) 2 (d) $2(\sqrt{2} - 1)$

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17 The value of $\int_{e^{-1}}^{e^2} \left| \frac{\log_e x}{x} \right| dx$ is

- (a) 3/2 (b) 5/2 (c) 3 (d) 5

18 The value of $\sum_{k=1}^n \int_0^1 f(k-1+x) dx$ is

- (a) $\int_0^1 f(x) dx$ (b) $\int_0^2 f(x) dx$ (c) $\int_0^n f(x) dx$ (d) $n \int_0^1 f(x) dx$

19 $f(x)$ is a continuous function for all real values of x and satisfies $\int_n^{n+1} f(x) dx = \frac{n^2}{2}$, $\forall n \in I$, then $\int_{-3}^5 f(|x|) dx$ is equal to

- (a) $\frac{19}{2}$ (b) $\frac{35}{2}$
 (c) $\frac{17}{2}$ (d) None of these

20 If $I_1 = \int_0^1 2^{x^2} dx$, $I_2 = \int_0^1 2^{x^3} dx$, $I_3 = \int_1^2 2^{x^2} dx$ and $I_4 = \int_1^2 2^{x^3} dx$, then

- (a) $I_3 > I_4$ (b) $I_3 = I_4$
 (c) $I_1 > I_2$ (d) $I_2 > I_1$

21 The value of $\int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x} dx$ is

- (a) $\log 2\sqrt{2}$ (b) $\frac{3}{2} \log 2$
 (c) $2 \log 2$ (d) $\log \sqrt{2}$

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22 $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$ is equal to

- (a) $\frac{1}{\sqrt{2}} \log(\sqrt{2} + 1)$ (b) $\frac{1}{2} \log(\sqrt{2} + 1)$
 (c) $-\log(\sqrt{2} + 1)$ (d) None of these

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23 The value of $\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \frac{1}{n+1} + \frac{2}{n+2} + \dots + \frac{3n}{4n} \right\}$ is

- (a) $5 - 2 \ln 2$ (b) $4 - 2 \ln 2$
 (c) $3 - 2 \ln 2$ (d) $2 - 2 \ln 2$

24 If f and g are continuous functions in $[0, 1]$ satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = a$, then

$\int_0^a f(x) \cdot g(x) dx$ is equal to

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- (a) $\frac{a}{2}$ (b) $\frac{a}{2} \int_0^a f(x) dx$
 (c) $\int_0^a f(x) dx$ (d) $a \int_0^a f(x) dx$

25 The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$ is

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- (a) π (b) $\frac{\pi}{2}$ (c) 4π (d) $\frac{\pi}{4}$

26 $\int_0^{\pi} \lceil \cot x \rceil dx$, where $\lceil \cdot \rceil$ denotes the greatest integer function, is equal to

- (a) $\frac{\pi}{2}$ (b) 1 (c) -1 (d) $-\frac{\pi}{2}$

27 The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)} dx$ is equal to

- (a) 2 (b) 4 (c) 1 (d) 6

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28 $\int_{-3\pi/2}^{-\pi/2} [(x + \pi)^3 + \cos^2(x + 3\pi)] dx$ is equal to

- (a) $\left(\frac{\pi^4}{32}\right) + \frac{\pi}{2}$ (b) $\frac{\pi}{2}$ (c) $\left(\frac{\pi}{4}\right) - 1$ (d) $\frac{\pi^4}{32}$

29 If $f: R \rightarrow R$ and $g: R \rightarrow R$ are one to one, real valued function, then the value of the integral

$\int_{-\pi}^{\pi} [f(x) + f(-x)][g(x) - g(-x)] dx$ is

- (a) 0 (b) π (c) 1 (d) None of these

30 The value of $\lim_{n \rightarrow \infty} \left(\frac{n}{(n+1)\sqrt{2n+1}} + \frac{n}{(n+2)\sqrt{2(2n+2)}} + \frac{n}{(n+3)\sqrt{3(2n+3)}} + \dots + \frac{1}{2n\sqrt{3}} \right)$ is

- (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$
 (c) $\frac{\pi}{4}$ (d) None of these

31 If $P = \int_0^{3\pi} f(\cos^2 x) dx$ and $Q = \int_0^{\pi} f(\cos^2 x) dx$, then

- (a) $P - Q = 0$ (b) $P - 2Q = 0$
 (c) $P - 3Q = 0$ (d) $P - 5Q = 0$

32 $\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx$ is equal to

- (a) $\frac{\pi^2}{4}$ (b) π^2 (c) 0 (d) $\frac{\pi}{2}$

33 If $f(x)$ is differentiable and $\int_0^{t^2} x f(x) dx = \frac{2}{5} t^5$, then

$f\left(\frac{4}{25}\right)$ is equal to

- (a) $\frac{2}{5}$ (b) $-\frac{5}{2}$ (c) 1 (d) $\frac{5}{2}$

34 If $f(x) = \frac{1}{x^2} \int_4^x [4t^2 - 2f'(t)] dt$, then $f'(4)$ is equal to

- (a) 32 (b) $\frac{32}{3}$
(c) $\frac{32}{9}$ (d) None of these

35 The value of $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sin \sqrt{t} dt}{x^3}$ is

- (a) 0 (b) $\frac{2}{9}$
(c) $\frac{1}{3}$ (d) $\frac{2}{3}$

36 If $I_1 = \int_1^2 \frac{dx}{\sqrt{1+x^2}}$ and $I_2 = \int_1^2 \frac{dx}{x}$, then

- (a) $I_1 > I_2$ (b) $I_2 > I_1$ (c) $I_1 = I_2$ (d) $I_1 > 2 I_2$

37 If $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$, then $\frac{d^2y}{dx^2}$ is equal to

- (a) y (b) $\sqrt{1+y^2}$
(c) $\frac{y}{\sqrt{1+y^2}}$ (d) y^2

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38 Let $f: R \rightarrow R$ be a differentiable function having $f(2) = 6$,

$f'(2) = \left(\frac{1}{48}\right)$. Then, $\lim_{x \rightarrow 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt$ is equal to

- (a) 18 (b) 12 (c) 36 (d) 24

39 $\lim_{n \rightarrow \infty} \frac{[1+2^4+3^4+\dots+n^4]}{n^5} - \frac{[1+2^3+3^3+\dots+n^3]}{n^5}$ is equal to

- (a) $\frac{1}{30}$ (b) 0 (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

40 If $I = \int_0^{\pi/2} \frac{dx}{\sqrt{1+\sin^3 x}}$, then

- (a) $0 < I < 1$ (b) $I > \frac{\pi}{2\sqrt{2}}$
(c) $I < \sqrt{2} \pi$ (d) $I < 2\pi$

41 If $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx$. Then, which one of the following is true?

- (a) $I > \frac{2}{3}$ and $J < 2$ (b) $I > \frac{2}{3}$ and $J > 2$
(c) $I < \frac{2}{3}$ and $J < 2$ (d) $I < \frac{2}{3}$ and $J > 2$

42 Statement I $\int_0^2 f(x) dx = \frac{4(\sqrt{2}-1)}{3}$,

where, $f(x) = \begin{cases} x^2, & \text{for } 0 \leq x < 1 \\ \sqrt{x}, & \text{for } 1 \leq x \leq 2 \end{cases}$

Statement II $f(x)$ is continuous in $[0, 2]$.

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

43 Statement I The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ is $\frac{\pi}{6}$.

Statement II $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

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- (a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true; Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

44 Statement I If $\int_0^1 e^{\sin x} dx = \lambda$, then $\int_0^{200} e^{\sin x} dx = 200\lambda$

Statement II If $\int_0^{na} f(x) dx = n \int_0^a f(x) dx$, $n \in I$ and $f(a+x) = f(x)$

- (a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true; Statement II is true, Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

45 Statement I $\int_0^1 e^{-x} \cos^2 x dx < \int_0^1 e^{-x^2} \cos^2 x dx$

Statement II $\int_a^b f(x) dx < \int_a^b g(x) dx$, $\forall f(x) < g(x)$

- (a) Statement I is true; Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true; Statement II is true, Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

DAY PRACTICE SESSION 2

PROGRESSIVE QUESTIONS EXERCISE

- 1** If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all non-zero x , then $\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x) dx$ is equal to
 (a) $\sin\theta + \operatorname{cosec}\theta$ (b) $\sin^2\theta$
 (c) $\operatorname{cosec}^2\theta$ (d) None of these
- 2** If $\frac{d}{dx} f(x) = \frac{e^{\sin x}}{x}$; $x > 0$. If $\int_1^4 \frac{3}{x} e^{\sin x^3} dx = f(k) - f(1)$, then the possible value of k , is
 (a) 15 (b) 16 (c) 63 (d) 64
- 3** If $g(1) = g(2)$, then $\int_1^2 [f\{g(x)\}]^{-1} f'\{g(x)\} g'(x) dx$ is equal to
 (a) 1 (b) 2 (c) 0 (d) None of these
- 4** For $0 \leq x \leq \frac{\pi}{2}$, the value of $\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t}) dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t}) dt$ is **→ JEE Mains 2013**
 (a) $\frac{\pi}{4}$ (b) 0 (c) 1 (d) $-\frac{\pi}{4}$
- 5** If $F(x) = f(x) + f\left(\frac{1}{x}\right)$, where $f(x) = \int_1^x \frac{\log t}{1+t} dt$. Then, $F(e)$ is equal to
 (a) $1/2$ (b) 0 (c) 1 (d) 2
- 6** The expression $\frac{\int_0^n [x] dx}{\int_0^n \{x\} dx}$, where $[x]$ and $\{x\}$ are integral and fractional part of x and $n \in N$, is equal to
 (a) $\frac{1}{n-1}$ (b) $\frac{1}{n}$ (c) n (d) $n-1$
- 7** If $f(x) = \frac{e^x}{1+e^x}$, $I_1 = \int_{f(-a)}^{f(a)} xg[x(1-x)] dx$ and $I_2 = \int_{f(-a)}^{f(a)} g[x(1-x)] dx$, then the value of $\frac{I_2}{I_1}$ is
 (a) 2 (b) -3 (c) -1 (d) 1
- 8** The value of $\lim_{x \rightarrow \infty} \frac{\left\{ \int_0^x e^{t^2} dt \right\}^2}{\int_0^x e^{2t^2} dt}$ is
 (a) $1/3$ (b) $2/3$ (c) 1 (d) None of these
- 9** The integral $\int_0^{\pi} \sqrt{1 + 4 \sin^2 \frac{x}{2}} - 4 \sin \frac{x}{2} dx$ is equal to **→ JEE Mains 2014**
 (a) $\pi - 4$ (b) $\frac{2\pi}{3} - 4 - 4\sqrt{3}$
 (c) $4\sqrt{3} - 4$ (d) $4\sqrt{3} - 4 - \frac{\pi}{3}$
- 10** $\int_0^{\pi} xf(\sin x) dx$ is equal to
 (a) $\pi \int_0^{\pi} f(\sin x) dx$ (b) $\frac{\pi}{2} \int_0^{\pi/2} f(\sin x) dx$
 (c) $\pi \int_0^{\pi/2} f(\cos x) dx$ (d) $\pi \int_0^{\pi} f(\cos x) dx$
- 11** If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$, then $f(x)$ increases in
 (a) (2, 2) (b) no value of x
 (c) (0, ∞) (d) $(-\infty, 0)$
- 12** $\lim_{n \rightarrow \infty} \left(\frac{(n+1)(n+2)\dots 3n}{n^{2n}} \right)^{1/n}$ is equal to **→ JEE Mains 2016**
 (a) $\frac{18}{e^4}$ (b) $\frac{27}{e^2}$
 (c) $\frac{9}{e^2}$ (d) $3 \log 3 - 2$
- 13** The least value of the function $f(x) = \int_{5\pi/4}^x (3 \sin u + 4 \cos u) du$ on the interval $\left[\frac{5\pi}{4}, \frac{4\pi}{3} \right]$ is
 (a) $\sqrt{3} + \frac{3}{2}$ (b) $-2\sqrt{3} + \frac{3}{2} + \frac{1}{\sqrt{2}}$
 (c) $\frac{3}{2} + \frac{1}{\sqrt{2}}$ (d) None of these
- 14** If $g(x) = \int_0^x f(t) dt$, where f is such that, $\frac{1}{2} \leq f(t) \leq 1$, for $t \in [0, 1]$ and $0 \leq f(t) \leq \frac{1}{2}$, for $t \in [1, 2]$. Then, $g(2)$ satisfies the inequality
 (a) $-\frac{3}{2} \leq g(2) < \frac{1}{2}$ (b) $\frac{1}{2} \leq g(2) \leq \frac{3}{2}$
 (c) $\frac{3}{2} < g(2) \leq \frac{5}{2}$ (d) $2 < g(2) < 4$
- 15** Let $n \geq 1, n \in Z$. The real number $a \in (0, 1)$ that minimizes the integral $\int_0^1 |x^n - a^n| dx$ is
 (a) $\frac{1}{2}$ (b) 2
 (c) 1 (d) $\frac{1}{3}$
- 16** The value of $\lim_{n \rightarrow \infty} \left\{ \tan\left(\frac{\pi}{2n}\right) \tan\left(\frac{2\pi}{2n}\right) \tan\left(\frac{3\pi}{2n}\right) \dots \tan\left(\frac{n\pi}{2n}\right) \right\}^{1/n}$ is
 (a) 1 (b) 2
 (c) 3 (d) Not defined

- 17** If $f(x) = \frac{x-1}{x+1}$, $f^2(x) = f(x)$, ..., $f^{k+1}(x) = f\{f^k(x)\}$,
 $k = 1, 2, 3, \dots$ and $g(x) = f^{1998}(x)$, then $\int_{1/e}^1 g(x) dx$ is equal
to
(a) 0 (b) 1 (c) -1 (d) e

- 18** If $f(x)$ is a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and $g(x)$ is a function that satisfies $f(x) + g(x) = x^2$. Then, the value of $\int_0^1 f(x)g(x) dx$, is
(a) $e - \frac{e^2}{2} - \frac{5}{2}$ (b) $e + \frac{e^2}{2} - \frac{3}{2}$
(c) $e - \frac{e^2}{2} - \frac{3}{2}$ (d) $e + \frac{e^2}{2} + \frac{5}{2}$

- 19** If $n > 1$, then
Statement I $\int_0^\infty \frac{dx}{1+x^n} = \int_0^1 \frac{dx}{(1-x^n)^{1/n}}$
Statement II $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

- 20** Consider $\sin^6 x$ and $\cos^6 x$ is a periodic function with π .
Statement I $\int_0^{\pi/2} (\sin^6 x + \cos^6 x) dx$ lie in the interval $(\frac{\pi}{8}, \frac{\pi}{2})$.
Statement II $\sin^6 x + \cos^6 x$ is periodic with period $\pi/2$.
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true; Statement II is false
(d) Statement I is false; Statement II is true

ANSWERS

SESSION 1

1 (b)	2 (d)	3 (c)	4 (b)	5 (b)	6 (b)	7 (d)	8 (a)	9 (a)	10 (c)
11 (a)	12 (b)	13 (d)	14 (d)	15 (d)	16 (d)	17 (b)	18 (c)	19 (b)	20 (c)
21 (b)	22 (a)	23 (c)	24 (b)	25 (d)	26 (d)	27 (c)	28 (b)	29 (a)	30 (a)
31 (c)	32 (b)	33 (a)	34 (c)	35 (d)	36 (b)	37 (a)	38 (a)	39 (d)	40 (b)
41 (c)	42 (d)	43 (d)	44 (d)	45 (a)					

SESSION 2

1 (d)	2 (d)	3 (c)	4 (a)	5 (a)	6 (d)	7 (a)	8 (d)	9 (d)	10 (c)
11 (d)	12 (b)	13 (b)	14 (b)	15 (a)	16 (a)	17 (c)	18 (c)	19 (b)	20 (b)

Hints and Explanations

SESSION 1

1 Let $I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$
 $= \int_{\pi/4}^{3\pi/4} \frac{1 - \cos x}{1 - \cos^2 x} dx$
 $= \int_{\pi/4}^{3\pi/4} \frac{1 - \cos x}{\sin^2 x} dx$
 $= \int_{\pi/4}^{3\pi/4} (\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x) dx$
 $= [-\cot x + \operatorname{cosec} x]_{\pi/4}^{3\pi/4}$
 $= [(1 + \sqrt{2}) - (-1 + \sqrt{2})] = 2$

2 Since, f is continuous function.

Let $x = t - 1$

$\therefore dx = dt$

When x tends to -3 and 5 , then t tends to -2 , 6 .

Therefore, $\int_{-3}^5 f(x) dx = \int_{-2}^6 f(t - 1) dt$
 $= \int_{-2}^6 f(x - 1) dx$

3 Let $I = \int_0^{\pi/4} [\sqrt{\tan x} + \sqrt{\cot x}] dx$
 $= \int_0^{\pi/4} \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$
 $= \sqrt{2} \int_0^{\pi/4} \frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} dx$

Put $\sin x - \cos x = t$

$\Rightarrow (\cos x + \sin x) dx = dt$

$\therefore I = \sqrt{2} \int_{-1}^0 \frac{dt}{\sqrt{1 - t^2}} \Rightarrow I = \sqrt{2} [\sin^{-1} t]_{-1}^0$

$= \sqrt{2} [0 - (-\pi/2)] = \frac{\pi}{\sqrt{2}}$

4 $\int_0^1 \sin \left[2 \tan^{-1} \frac{1+x}{1-x} \right] dx$

Put $x = \cos \theta$, then

$\sin \left[2 \tan^{-1} \frac{1 + \cos \theta}{1 - \cos \theta} \right]$

$= \sin \left[2 \tan^{-1} \left(\cot \frac{\theta}{2} \right) \right]$

$= \sin \left[2 \tan^{-1} \left\{ \tan \left(\frac{\pi}{2} - \frac{\theta}{2} \right) \right\} \right]$

$= \sin \left[2 \left(\frac{\pi}{2} - \frac{\theta}{2} \right) \right]$

$= \sin (\pi - \theta) = \sin \theta$

$= \sqrt{1 - \cos^2 \theta} = \sqrt{1 - x^2}$

$\therefore \int_0^1 \sin \left[2 \tan^{-1} \frac{1+x}{1-x} \right] dx$

$= \int_0^1 \sqrt{1 - x^2} dx$

$= \left[\frac{1}{2} x \sqrt{1 - x^2} \right]_0^1 + \frac{1}{2} [\sin^{-1} x]_0^1 = \frac{\pi}{4}$

5 $I_{2(n)} - I_{2(n-1)}$
 $= \int_0^{\pi/2} \frac{[\sin^2 nx - \sin^2(n-1)x]}{\sin^2 x} dx$

$= \int_0^{\pi/2} \frac{\sin(2n-1)x \cdot \sin x}{\sin^2 x} dx$

$= \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx = I_{1(n)}$

$\Rightarrow I_{2(n+1)} - I_{2(n)} = I_{1(n+1)}$

6 $\int_{-1}^1 \{x^2 + x - 3\} dx = \int_{-1}^1 \{x^2 + x\} dx$
 $= \int_{-1}^1 (x^2 + x - [x^2 + x]) dx$

$= \left(\frac{x^3}{3} + \frac{x^2}{2} \right)_{-1}^1 - \int_{-1}^0 [x^2 + x] dx$

$- \int_0^{\frac{\sqrt{5}-1}{2}} [x^2 + x] dx$

$- \int_{\frac{1}{\sqrt{5}-1}}^1 [x^2 + x] dx$

$= \frac{2}{3} + 1 - 0 - 1 \left(1 - \frac{\sqrt{5}-1}{2} \right)$

$= \frac{2}{3} + \frac{\sqrt{5}-1}{2} = \frac{1}{6} (1 + 3\sqrt{5})$

7 $\int_0^2 [x^2] dx = \int_0^1 [x^2] dx$

$+ \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2] dx$

$+ \int_{\sqrt{3}}^2 [x^2] dx$

$= \int_0^1 0 dx + \int_1^{\sqrt{2}} 1 dx$

$+ \int_{\sqrt{2}}^{\sqrt{3}} 2 dx + \int_{\sqrt{3}}^2 3 dx$

$= [x]_1^{\sqrt{2}} + [2x]_{\sqrt{2}}^{\sqrt{3}} + [3x]_{\sqrt{3}}^2$

$= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3}$

$= 5 - \sqrt{3} - \sqrt{2}$

8 $\int_{-2}^x |2t| dt = f(x)$

$= \int_{-2}^0 |2t| dt + \int_0^x 2t dt$

$= -2 \left[\frac{t^2}{2} \right]_{-2}^0 + 2 \left[\frac{t^2}{2} \right]_0^x$

$= -2 [0 - 2] + 2 \left[\frac{x^2}{2} - 0 \right] = 4 + x^2$

9 Let $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1 + e^x} dx$

$= \int_{-\pi/2}^0 \frac{\cos x}{1 + e^x} dx$

$+ \int_0^{\pi/2} \frac{\cos x}{1 + e^x} dx \dots (i)$

On putting $x = -x$ in 1st integral, we get

$\int_{-\pi/2}^0 \frac{\cos x}{1 + e^x} dx = \int_0^{\pi/2} \frac{\cos x}{1 + e^{-x}} dx$

$\therefore I = \int_0^{\pi/2} \frac{e^x \cos x}{1 + e^x} dx + \int_0^{\pi/2} \frac{\cos x}{1 + e^x} dx$

$= \int_0^{\pi/2} \frac{(1 + e^x) \cos x}{(1 + e^x)} dx$

$= \int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = 1$

10 $\int_0^3 (3ax^2 + 2bx + c) dx$

$= \int_1^3 (3ax^2 + 2bx + c) dx$

$\Rightarrow \int_0^1 (3ax^2 + 2bx + c) dx$

$+ \int_1^3 (3ax^2 + 2bx + c) dx$

$= \int_1^3 (3ax^2 + 2bx + c) dx$

$\Rightarrow \int_0^1 (3ax^2 + 2bx + c) dx = 0$

$\Rightarrow \left[\frac{3ax^3}{3} + \frac{2bx^2}{2} + cx \right]_0^1 = 0$

$\therefore a + b + c = 0$

11 Since, $\int_1^a [x] f'(x) dx = \int_1^a f'(x) dx$

$+ \int_1^3 2f'(x) dx + \dots + \int_{[a]}^a [a] f'(x) dx$

$= [f(x)]_1^2 + 2[f(x)]_2^3 + \dots + [a][f(x)]_{[a]}^a$

$= f(2) - f(1) + 2f(3) - 2f(2) + \dots$

$+ [a]f(a) - [a]f([a])$

$= [a]f(a) - \{f(1) + f(2) + \dots + f([a])\}$

12 Let $I = \int_0^{\pi/2} \left| \sin \left(x - \frac{\pi}{4} \right) \right| dx$

$= - \int_0^{\pi/4} \sin \left(x - \frac{\pi}{4} \right) dx$

$+ \int_{\pi/4}^{\pi/2} \sin \left(x - \frac{\pi}{4} \right) dx$

$= \left[\cos \left(x - \frac{\pi}{4} \right) \right]_0^{\pi/4} - \left[\cos \left(x - \frac{\pi}{4} \right) \right]_{\pi/4}^{\pi/2}$

$= 1 - \frac{1}{\sqrt{2}} - \left(\frac{1}{\sqrt{2}} - 1 \right) = 2 - \sqrt{2}$

13 $\int_0^{3\pi/2} \sin \left[\frac{2x}{\pi} \right] dx = \int_0^{\pi/2} \sin \left[\frac{2x}{\pi} \right] dx$

$+ \int_{\pi/2}^{\pi} \sin \left[\frac{2x}{\pi} \right] dx + \int_{\pi}^{3\pi/2} \sin \left[\frac{2x}{\pi} \right] dx$

$= 0 + \sin 1 \int_{\pi/2}^{\pi} dx + \sin 2 \int_{\pi}^{3\pi/2} dx$

$= \frac{\pi}{2} (\sin 1 + \sin 2)$

14 Here, $\int_0^{1.5} x[x^2] dx$

$I = \int_0^1 x \cdot 0 dx + \int_1^{\sqrt{2}} x \cdot 1 dx + \int_{\sqrt{2}}^{1.5} x \cdot 2 dx$

$$\begin{aligned}
&= 0 + \left[\frac{x^2}{2} \right]_1^{\sqrt{2}} + [x^2]_{\sqrt{2}}^{1.5} \\
&= \frac{1}{2} \{2 - 1\} + \{(1.5)^2 - 2\} \\
&= \frac{1}{2} + 2.25 - 2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}
\end{aligned}$$

15 Let $I = \int_0^1 \frac{8 \log(1+x)}{(1+x^2)} dx$

Put $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$

When $x = 0 \Rightarrow \tan \theta = 0$

$\therefore \theta = 0$

When $x = 1 = \tan \theta \Rightarrow \theta = \frac{\pi}{4}$

$$\therefore I = \int_0^{\pi/4} \frac{8 \log(1 + \tan \theta)}{1 + \tan^2 \theta} \cdot \sec^2 \theta d\theta$$

$$= 8 \int_0^{\pi/4} \log(1 + \tan \theta) d\theta$$

$$I = 8 \int_0^{\pi/4} \log(1 + \tan \theta) d\theta \quad \dots(i)$$

$$= 8 \cdot \left(\frac{\pi}{2} \cdot \log 2 \right)$$

$$\left[\because \int_0^{\pi/4} \log(1 + \tan \theta) d\theta = \frac{\pi}{8} \log 2 \right]$$

$$= \pi \log 2$$

16 Let $I = \int_0^{\pi/2} \sqrt{1 - \sin 2x} dx$

$$\begin{aligned}
&= \int_0^{\pi/4} \sqrt{(\cos x - \sin x)^2} dx \\
&\quad + \int_{\pi/4}^{\pi/2} \sqrt{(\sin x - \cos x)^2} dx
\end{aligned}$$

$[\because \cos x - \sin x > 0$ when $x \in (0, \frac{\pi}{4})$ and

$\cos x - \sin x < 0$ when $x \in (\frac{\pi}{4}, \frac{\pi}{2})]$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx$$

$$+ \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \left[\left(\sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right) - (\sin 0 + \cos 0) \right]$$

$$+ \left[\left(-\cos \frac{\pi}{2} - \sin \frac{\pi}{2} \right) - \left(-\cos \frac{\pi}{4} - \sin \frac{\pi}{4} \right) \right]$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \right) \right] + \left[-1 - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \right]$$

$$= \left[\frac{2}{\sqrt{2}} - 1 \right] + [-1 + \sqrt{2}]$$

$$= (\sqrt{2} - 1) + (-1 + \sqrt{2}) = 2(\sqrt{2} - 1)$$

17 $\int_{e^{-1}}^{e^2} \left| \frac{\log_e x}{x} \right| dx = \int_{e^{-1}}^1 \left| \frac{\log_e x}{x} \right| dx$

$$+ \int_1^{e^2} \left| \frac{\log_e x}{x} \right| dx$$

$$= \int_{e^{-1}}^1 \frac{-\log_e x}{x} dx + \int_1^{e^2} \frac{\log_e x}{x} dx$$

$$= \int_{-1}^0 -z dz + \int_0^2 z dz$$

[put $\log_e x = z \Rightarrow (1/x) dx = dz$]

$$\therefore I = \left[-\frac{z^2}{2} \right]_{-1}^0 + \left[\frac{z^2}{2} \right]_0^2 = \frac{1}{2} + 2 = \frac{5}{2}$$

18 Let $I = \int_0^1 f(k-1+x) dx$

$$\Rightarrow I = \int_{k-1}^k f(t) dt, \text{ where}$$

$$t = k - 1 + x$$

$$\Rightarrow dt = dx$$

$$\Rightarrow I = \int_{k-1}^k f(x) dx$$

$$\therefore \sum_{k=1}^n \int_{k-1}^k f(x) dx$$

$$= \int_0^1 f(x) dx + \int_1^2 f(x) dx + \dots + \int_{n-1}^n f(x) dx$$

$$= \int_0^n f(x) dx$$

19 $\int_{-3}^5 f(|x|) dx = \int_{-3}^0 f(|x|) dx + \int_0^5 f(|x|) dx$

$$= 2 \int_0^3 f(x) dx + \int_3^5 f(x) dx$$

$$= 2 \left(\int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^3 f(x) dx \right)$$

$$+ \left(\int_3^4 f(x) dx + \int_4^5 f(x) dx \right)$$

$$= 2 \left(0 + \frac{1}{2} + \frac{2^2}{2} \right) + \left(\frac{9}{2} + \frac{16}{2} \right) = \frac{35}{2}$$

20 Given that, $I_1 = \int_0^1 2^{x^2} dx$

$$I_2 = \int_0^1 2^{x^3} dx,$$

$$I_3 = \int_1^2 2^{x^2} dx$$

and $I_4 = \int_1^2 2^{x^3} dx$

Since, $2^{x^3} < 2^{x^2}$ for $0 < x < 1$

and $2^{x^3} > 2^{x^2}$ for $x > 1$

$$\therefore \int_0^1 2^{x^3} dx < \int_0^1 2^{x^2} dx$$

and $\int_1^2 2^{x^3} dx > \int_1^2 2^{x^2} dx$

$$\Rightarrow I_2 < I_1 \text{ and } I_4 > I_3$$

21 $I = \int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x} dx$

$$= \int_{7\pi/4}^{2\pi} |\tan x| dx + \int_{2\pi}^{7\pi/3} |\tan x| dx$$

$$= - \int_{7\pi/4}^{2\pi} \tan x dx + \int_{2\pi}^{7\pi/3} \tan x dx$$

$$= - [\log \sec x]_{7\pi/4}^{2\pi} + [\log \sec x]_{2\pi}^{7\pi/3}$$

$$= - \left[\log \sec 2\pi - \log \sec \frac{7\pi}{4} \right]$$

$$+ \left[\log \sec \frac{7\pi}{3} - \log \sec 2\pi \right]$$

$$= - \left[\log 1 - \log \sec \frac{\pi}{4} \right]$$

$$+ \left[\log \sec \frac{\pi}{3} - \log 1 \right]$$

$$= \log \sqrt{2} + \log 2 = \frac{1}{2} \log 2 + \log 2$$

$$= \frac{3}{2} \log 2$$

22 We have, $I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx$

$$= \int_0^{\pi/2} \frac{\sin^2 \left(\frac{\pi}{2} - x \right)}{\sin \left(\frac{\pi}{2} - x \right) + \cos \left(\frac{\pi}{2} - x \right)} dx$$

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x + \cos x} dx$$

$$\text{Thus, } 2I = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos \left(x - \frac{\pi}{4} \right)}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sec \left(x - \frac{\pi}{4} \right) dx$$

$$= \frac{1}{\sqrt{2}} \left[\log \left\{ \sec \left(x - \frac{\pi}{4} \right) + \tan \left(x - \frac{\pi}{4} \right) \right\} \right]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} \left[\log \left(\sec \frac{\pi}{4} + \tan \frac{\pi}{4} \right) - \log \left\{ \sec \left(-\frac{\pi}{4} \right) + \tan \left(-\frac{\pi}{4} \right) \right\} \right]$$

$$= \frac{1}{\sqrt{2}} [\log(\sqrt{2}+1) - \log(\sqrt{2}-1)]$$

$$= \frac{1}{\sqrt{2}} \log \frac{|\sqrt{2}+1|}{|\sqrt{2}-1|} = \frac{1}{\sqrt{2}} \log \left\{ \frac{(\sqrt{2}+1)^2}{1} \right\}$$

$$= \frac{2}{\sqrt{2}} \log(\sqrt{2}+1)$$

$$\therefore I = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$$

23 $\lim_{n \rightarrow \infty} \frac{1}{n} \left\{ \frac{1}{n+1} + \frac{2}{n+2} + \dots + \frac{3n}{n+3n} \right\}$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^{3n} \frac{1}{n} \left(\frac{r}{n+r} \right)$$

$$= \int_0^3 \frac{x}{1+x} dx$$

$$= \int_0^3 \left(1 - \frac{1}{1+x} \right) dx$$

$$= [x - \ln(1+x)]_0^3 = 3 - \ln 4$$

$$= 3 - 2 \ln 2$$

24 $\therefore I = \int_0^a f(x) \cdot g(x) dx$

$$= \int_0^a f(a-x) g(a-x) dx$$

$$= \int_0^a f(x) \{a-g(x)\} dx$$

$$= a \int_0^a f(x) dx - \int_0^a f(x) \cdot g(x) dx$$

$$= a \int_0^a f(x) dx - I$$

$$\therefore I = \frac{a}{2} \int_0^a f(x) dx$$

25 Let $I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 \left(\frac{-\pi}{2} + \frac{\pi}{2} - x \right)}{1 + 2^{\frac{-\pi}{2} + \frac{\pi}{2} - x}} dx$$

$$\left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

$$\begin{aligned} \Rightarrow I &= \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^{-x}} dx \\ \Rightarrow I &= \int_{-\pi/2}^{\pi/2} \frac{2^x \cdot \sin^2 x}{1+2^x} dx \\ I &= \int_{-\pi/2}^{\pi/2} \frac{2^x \cdot \sin^2 x}{1+2^x} dx \\ \Rightarrow 2I &= \int_{-\pi/2}^{\pi/2} \sin^2 x dx = \int_0^{\pi/2} 2\sin^2 x dx \\ &= \int_0^{\pi/2} (1 - \cos 2x) dx \\ \Rightarrow 2I &= \left[x - \frac{\sin 2x}{2} \right]_0^{\pi/2} = \frac{\pi}{2} \\ \therefore I &= \frac{\pi}{4} \end{aligned}$$

26 Let $I = \int_0^\pi [\cot x] dx \quad \dots(i)$

$$\begin{aligned} \Rightarrow I &= \int_0^\pi [\cot(\pi - x)] dx \\ &= \int_0^\pi [-\cot x] dx \quad \dots(ii) \end{aligned}$$

On adding Eqs. (i) and (ii), we get

$$\begin{aligned} 2I &= \int_0^\pi [\cot x] dx + \int_0^\pi [-\cot x] dx \\ &= \int_0^\pi (-1) dx \\ &= \int_0^\pi [-x]_0^\pi = -\pi \\ \therefore I &= -\frac{\pi}{2} \end{aligned}$$

27 $I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)} dx$

$$\begin{aligned} &= \int_2^4 \frac{2 \log x}{2 \log x + \log(6-x)^2} dx \\ &= \int_2^4 \frac{2 \log x dx}{2[\log x + \log(6-x)]} \end{aligned}$$

$$\Rightarrow I = \int_2^4 \frac{\log x dx}{[\log x + \log(6-x)]} \quad \dots(i)$$

$$\Rightarrow I = \int_2^4 \frac{\log(6-x)}{\log(6-x) + \log x} dx \quad \dots(ii)$$

$$\left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

On adding Eqs.(i) and (ii), we get

$$\begin{aligned} 2I &= \int_2^4 \frac{\log x + \log(6-x)}{\log x + \log(6-x)} dx \\ \Rightarrow 2I &= \int_2^4 dx = [x]_2^4 \\ \Rightarrow 2I &= 2 \Rightarrow I = 1 \end{aligned}$$

28 Let $I = \int_{-\pi/2}^{\pi/2} [(x + \pi)^3 + \cos^2 x] dx$

$$\begin{aligned} \Rightarrow I &= \int_{-\pi/2}^{\pi/2} \left[-\frac{\pi}{2} - \frac{3\pi}{2} - x + \pi \right]^3 \\ &\quad + \cos^2 \left(-\frac{\pi}{2} - \frac{3\pi}{2} - x \right) dx \\ \left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right] \end{aligned}$$

$$\begin{aligned} \Rightarrow I &= \int_{-\pi/2}^{\pi/2} [-(x + \pi)^3 + \cos^2 x] dx \\ \Rightarrow 2I &= \int_{-\pi/2}^{\pi/2} 2\cos^2 x dx \\ &= \int_{-\pi/2}^{\pi/2} (1 + \cos 2x) dx \\ &= \left[x + \frac{\sin 2x}{2} \right]_{-\pi/2}^{\pi/2} \\ &= \left[\frac{\pi}{2} + \frac{\sin(-\pi)}{2} \right. \\ &\quad \left. - \left(-\frac{3\pi}{2} + \frac{\sin(-3\pi)}{2} \right) \right] = \pi \\ \therefore I &= \frac{\pi}{2} \end{aligned}$$

29 Let $\phi(x) = [f(x) + f(-x)] [g(x) - g(-x)]$

$$\begin{aligned} \therefore \phi(-x) &= [f(-x) + f(x)] [g(-x) \\ &\quad - g(x)] = -\phi(x) \\ \therefore \int_{-\pi}^{\pi} \phi(x) dx &= 0 \end{aligned}$$

($\because \phi(x)$ is an odd function)

30 $S = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{(n+r)\sqrt{r(2n+r)}}$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n \left(1 + \frac{r}{n} \right) \sqrt{\frac{r}{n} \left(2 + \frac{r}{n} \right)}} \\ S &= \int_0^1 \frac{dx}{(1+x)\sqrt{2x+x^2}} \\ &= \int_0^1 \frac{dx}{(1+x)\sqrt{(1+x)^2 - 1}} \end{aligned}$$

$$\begin{aligned} S &= [\sec^{-1}(1+x)]_0^1 \\ &= \sec^{-1} 2 - \sec^{-1} 1 = \frac{\pi}{3} \end{aligned}$$

31 $P = \int_0^{3\pi} f(\cos^2 x) dx$ and $Q = \int_0^\pi f(\cos^2 x) dx$

Also, $P = 3 \int_0^\pi f(\cos^2 x) dx = 3Q$

$$\therefore P - 3Q = 0$$

32 Let $I = \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx$

$$\begin{aligned} &= \int_{-\pi}^{\pi} \frac{2x}{1 + \cos^2 x} dx \\ &\quad + \int_{-\pi}^{\pi} \frac{2x \sin x}{1 + \cos^2 x} dx \\ \Rightarrow I &= 0 + 4 \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx \quad \dots(i) \end{aligned}$$

$$\left[\because \frac{2x}{1 + \cos^2 x} \text{ is an odd function} \right.$$

$$\left. \text{and } \frac{2x \sin x}{1 + \cos^2 x} \text{ is an even function} \right]$$

$$\Rightarrow I = 4 \int_0^\pi \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$$

$$\begin{aligned} \Rightarrow I &= 4 \int_0^\pi \frac{\pi \sin x}{1 + \cos^2 x} dx \\ &\quad - 4 \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx \\ \Rightarrow I &= 4\pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx - I \\ &\quad \text{[from Eq. (i)]} \\ \Rightarrow I &= 2\pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx \\ \text{Put } \cos x &= t \\ \Rightarrow -\sin x dx &= dt \\ \therefore I &= -2\pi \int_1^{-1} \frac{1}{1+t^2} dt \\ &= 2\pi [\tan^{-1} t]_{-1}^1 = 2\pi \left[\frac{\pi}{4} + \frac{\pi}{4} \right] \\ &= \pi^2 \end{aligned}$$

33 Using Newton-Leibnitz's formula, we get

$$\begin{aligned} t^2 \{f(t^2)\} \left\{ \frac{d}{dt}(t^2) \right\} - 0 \\ \cdot f(0) \left\{ \frac{d}{dt}(0) \right\} = 2t^4 \\ \Rightarrow t^2 \{f(t^2)\} 2t = 2t^4 \\ \Rightarrow f(t^2) = t \\ \Rightarrow f\left(\frac{4}{25}\right) = \pm \frac{2}{5} \quad \left[\text{put } t = \pm \frac{2}{5} \right] \\ \therefore f\left(\frac{4}{25}\right) = \frac{2}{5} \end{aligned}$$

[neglecting negative sign]

34 We have,

$$\begin{aligned} f(x) &= \frac{1}{x^2} \int_4^x [4t^2 - 2f'(t)] dt \\ \therefore f'(x) &= \frac{1}{x^2} [4x^2 - 2f'(x)] \\ &\quad - \frac{2}{x^3} \int_4^x [4t^2 - 2f'(t)] dt \\ \Rightarrow f'(4) &= \frac{1}{16} [64 - 2f'(4)] - 0 \\ \therefore f'(4) &= \frac{32}{9} \end{aligned}$$

35 $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sin \sqrt{t} dt}{x^3} \left[\text{form } \frac{0}{0} \right]$

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{\sin x \cdot 2x}{3x^2} \\ &= \frac{2}{3} \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= \frac{2}{3} \cdot 1 = \frac{2}{3} \end{aligned}$$

36 We have, $(1+x^2) > x^2, \forall x$

$$\begin{aligned} \Rightarrow \sqrt{1+x^2} &> x, \forall x \in (1, 2) \\ \Rightarrow \frac{1}{\sqrt{1+x^2}} &< \frac{1}{x}, \forall x \in (1, 2) \\ \therefore \int_1^2 \frac{dx}{\sqrt{1+x^2}} &< \int_1^2 \frac{dx}{x} \Rightarrow I_1 < I_2 \\ \Rightarrow I_2 &> I_1 \end{aligned}$$

37 We have, $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$

By Leibnitz rule, we get

$$1 = \frac{1}{\sqrt{1+y^2}} \cdot \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{1+y^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{y}{\sqrt{1+y^2}} \cdot \frac{dy}{dx}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{y}{\sqrt{1+y^2}} \cdot \sqrt{1+y^2} = y$$

38 $\lim_{x \rightarrow 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt = \lim_{x \rightarrow 2} \frac{\int_6^{f(x)} 4t^3 dt}{(x-2)}$ [form $\frac{0}{0}$]

[by Leibnitz's rule]

$$= \lim_{x \rightarrow 2} \frac{4\{f(x)\}^3}{1} f'(x) = 4\{f(2)\}^3 f'(2)$$

$$= 4 \times (6)^3 \times \frac{1}{48}$$

$$\left[\because f(2) = 6 \text{ and } f'(2) = \frac{1}{48}, \text{ given} \right]$$

$$= 18$$

39 $\lim_{n \rightarrow \infty} \left[\frac{1+2^4+3^4}{n^5} - \lim_{n \rightarrow \infty} \left[\frac{1+2^3+3^3}{n^5} \right] \right]$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n} \right)^4 - \lim_{n \rightarrow \infty} \frac{1}{n} \times \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n} \right)^3$$

$$= \int_0^1 x^4 dx - \lim_{n \rightarrow \infty} \frac{1}{n} \times \int_0^1 x^3 dx$$

$$= \left[\frac{x^5}{5} \right]_0^1 - 0 = \frac{1}{5}$$

40 Since, $x \in \left[0, \frac{\pi}{2} \right] \Rightarrow 1 \leq 1 + \sin^3 x \leq 2$

$$\Rightarrow \frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{1 + \sin^3 x}} \leq 1$$

$$\Rightarrow \int_0^{\pi/2} \frac{1}{\sqrt{2}} dx \leq \int_0^{\pi/2} \frac{dx}{\sqrt{1 + \sin^3 x}} \leq \int_0^{\pi/2} dx$$

$$\therefore \frac{\pi}{2\sqrt{2}} \leq I \leq \frac{\pi}{2}$$

41 Since, $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \frac{x}{\sqrt{x}} dx$,

because in $x \in (0, 1)$, $x > \sin x$

$$\Rightarrow I < \int_0^1 \sqrt{x} dx = \frac{2}{3} [x^{3/2}]_0^1$$

$$\Rightarrow I < \frac{2}{3}$$

and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 \frac{1}{\sqrt{x}}$

$$= \int_0^1 x^{-1/2} dx = 2 [x^{1/2}]_0^1 = 2$$

$$\therefore J < 2$$

42 Since, $f(x)$ is continuous in $[0, 2]$.

$$\therefore \int_0^2 f(x) dx = \int_0^1 f(x) dx + \int_1^2 f(x) dx$$

$$= \int_0^1 x^2 dx + \int_1^2 \sqrt{x} dx$$

$$= \left[\frac{x^3}{3} \right]_0^1 + \left[\frac{2x^{3/2}}{3} \right]_1^2$$

$$= \frac{1}{3} + \frac{2}{3} (2^{3/2} - 1)$$

$$= \frac{1}{3} + \frac{4\sqrt{2}}{3} - \frac{2}{3} = \left(\frac{4\sqrt{2} - 1}{3} \right)$$

43 Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$... (i)

$$\therefore I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan\left(\frac{\pi}{2} - x\right)}}$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x} dx}{1 + \sqrt{\tan x}} \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} dx$$

$$\Rightarrow 2I = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{6}$$

$$\Rightarrow I = \frac{1}{2} \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi}{12}$$

Hence, Statement I is false but $\int_a^b f(x) dx$

$= \int_a^b f(a+b-x) dx$ is a true statement by property of definite integrals.

44 Since, period of $e^{\sin x}$ is 2π .

$$\therefore \int_0^{200} e^{\sin x} dx \neq 200\lambda$$

45 For $0 < x < 1, x > x^2$

$$\Rightarrow -x < -x^2 \Rightarrow e^{-x} < e^{-x^2}$$

$$\Rightarrow \int_0^1 e^{-x} \cos^2 x dx < \int_0^1 e^{-x^2} \cos^2 x dx$$

If $f(x) \geq g(x)$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$

SESSION 2

1 We have, $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$

$$\Rightarrow f(x) = -\frac{1}{x^2} f\left(\frac{1}{x}\right)$$

$$I = \int_{\sin\theta}^{\csc\theta} f(x) dx$$

$$= \int_{\sin\theta}^{\csc\theta} \left\{ -\frac{1}{x^2} f\left(\frac{1}{x}\right) \right\} dx$$

$$= \int_{\csc\theta}^{\sin\theta} f(t) dt, \text{ where } t = \frac{1}{x}$$

$$\Rightarrow I = -\int_{\sin\theta}^{\csc\theta} f(t) dt = -I$$

$$\Rightarrow 2I = 0 \Rightarrow I = 0$$

2 $\frac{d}{dx} f(x) = \frac{e^{\sin x}}{x}$

$$\therefore \int_1^4 \frac{3}{x} e^{\sin x^3} dx = \int_1^4 \frac{3x^2}{x^3} e^{\sin x^3} dx$$

Put $x^3 = t$

$$\Rightarrow 3x^2 dx = dt$$

$$\therefore f(t) = \int_1^{64} \frac{e^{\sin t}}{t} dt$$

$$= [f(t)]_1^{64}$$

$$= f(64) - f(1)$$

On comparing, we get

$$k = 64$$

3 Let $I = \int_1^2 [f\{g(x)\}]^{-1} f'\{g(x)\} \{g'(x)\} dx$

Put $f\{g(x)\} = z$

$$\Rightarrow f'\{g(x)\} g'(x) dx = dz$$

When $x = 1$, then $z = f\{g(1)\}$

When $x = 2$, then $z = f\{g(2)\}$

$$\therefore I = \int_{f\{g(1)\}}^{f\{g(2)\}} \frac{1}{z} dz = [\log z]_{f\{g(1)\}}^{f\{g(2)\}}$$

$$\Rightarrow I = \log f\{g(2)\} - \log f\{g(1)\} = 0$$

$$[\because g(2) = g(1)]$$

4 Put $t = \sin^2 z$ in 1st integral and

$t = \cos^2 u$ in 2nd integral, we get

$$dt = 2\sin z \cos z dz \text{ and}$$

$$dt = -2\cos u \sin u du$$

$$\therefore I = \int_0^x z(2\sin z \cos z) dz$$

$$+ \int_{\pi/2}^x -u(2\cos u \sin u) du$$

$$= \int_0^x z \sin 2z dz - \int_{\pi/2}^x u \sin 2u du$$

$$= \left[-z \cdot \frac{\cos 2z}{2} + \frac{\sin 2z}{4} \right]_0^x$$

$$- \left[\frac{-u \cos 2u}{2} + \frac{\sin 2u}{4} \right]_{\pi/2}^x$$

$$= \left[-x \cdot \frac{\cos 2x}{2} + \frac{\sin 2x}{4} - \{0 + 0\} \right]$$

$$- \left[-x \cdot \frac{\cos 2x}{2} + \frac{\sin 2x}{4} - \left(\frac{\pi}{4} + 0 \right) \right]$$

$$= \frac{\pi}{4}$$

5 Since, $f(x) = \int_1^x \frac{\log t}{1+t} dt$

and $f(x) = f(x) + f\left(\frac{1}{x}\right)$

$$\therefore F(e) = f(e) + f\left(\frac{1}{e}\right)$$

$$\Rightarrow F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^{1/e} \frac{\log t}{1+t} dt$$

Put $t = \frac{1}{t}$ in second integration

$$\therefore F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^e \frac{-\log t}{1+\frac{1}{t}} d\left(\frac{1}{t}\right)$$

$$\begin{aligned}
&= \int_1^e \frac{\log t}{1+t} dt - \int_1^e \frac{t \log t}{(1+t)} \times \left(\frac{-dt}{t^2} \right) \\
&= \int_1^e \frac{\log t}{1+t} dt + \int_1^e \frac{\log t}{t(1+t)} dt \\
&= \int_1^e \frac{\log t}{1+t} dt + \int_1^e \frac{\log t}{t} dt - \int_1^e \frac{\log t}{(1+t)} dt \\
&= \int_1^e \frac{\log t}{t} dt \quad \left[\because \frac{1}{t(1+t)} = \frac{1}{t} - \frac{1}{t+1} \right] \\
&= \left[\frac{(\log t)^2}{2} \right]_1^e \\
&= \frac{1}{2} [(\log e)^2 - (\log 1)^2] \\
&= \frac{1}{2}
\end{aligned}$$

6 We have, $\int_0^n [x] dx$

$$\begin{aligned}
&= \int_0^1 0 dx + \int_1^2 1 dx + \int_2^3 2 dx + \dots \\
&\quad + \int_{n-1}^n (n-1) dx \\
&= 1(2-1) + 2(3-2) + 3(4-3) + \dots + (n-1)\{n-(n-1)\} \\
&= 1 + 2 + 3 + \dots + (n-1) \\
&= \frac{n(n-1)}{2}
\end{aligned}$$

and $\int_0^n \{x\} dx = \int_0^n (x - [x]) dx = \frac{n}{2}$

$$\therefore \frac{\int_0^n [x] dx}{\int_0^n \{x\} dx} = n - 1$$

7 Given that, $f(x) = \frac{e^x}{1+e^x}$

$$\therefore f(a) = \frac{e^a}{1+e^a} \quad \dots(i)$$

and $f(-a) = \frac{e^{-a}}{1+e^{-a}} = \frac{1}{1+e^a} \quad \dots(ii)$

On adding Eqs. (i) and (ii), we get

$$\begin{aligned}
f(a) + f(-a) &= 1 \\
\Rightarrow f(a) &= 1 - f(-a)
\end{aligned}$$

Let $f(-a) = t$

$$\Rightarrow f(a) = 1 - t$$

Now, $I_1 = \int_t^{1-t} xg[x(1-x)] dx \quad \dots(iii)$

$$\left[\because I = \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

$$\Rightarrow I_1 = \int_t^{1-t} (1-x)g[x(1-x)] dx \quad \dots(iv)$$

On adding Eqs. (iii) and (iv), we get

$$2I_1 = \int_t^{1-t} g[x(1-x)] dx = I_2 \quad [\text{given}]$$

$$\therefore \frac{I_2}{I_1} = 2$$

8 $\lim_{x \rightarrow \infty} \frac{\left(\int_0^x e^{t^2} dt \right)^2}{\int_0^x e^{2t^2} dt}$

$$\begin{aligned}
&= \lim_{x \rightarrow \infty} \frac{\left(2 \int_0^x e^{t^2} dt \right) (e^{x^2})}{e^{2x^2}} \quad \left[\text{form } \frac{0}{0} \right] \\
&= \lim_{x \rightarrow \infty} \frac{2 \int_0^x e^{t^2} dt}{e^{x^2}} \left[\text{form } \frac{0}{0} \right] \\
&= \lim_{x \rightarrow \infty} \frac{2e^{x^2}}{2xe^{x^2}} = \lim_{x \rightarrow \infty} \frac{1}{x} = 0
\end{aligned}$$

9 Use the formula, $|x-a| = \begin{cases} x-a, & x \geq a \\ -(x-a), & x < a \end{cases}$ to break given integral in two parts and then integrate separately.

$$\begin{aligned}
&\int_0^\pi \sqrt{1-2\sin \frac{x}{2}} dx = \int_0^\pi |1-2\sin \frac{x}{2}| dx \\
&= \int_0^{\frac{\pi}{3}} \left(1-2\sin \frac{x}{2} \right) dx - \int_{\frac{\pi}{3}}^\pi \left(1-2\sin \frac{x}{2} \right) dx \\
&= \left(x + 4\cos \frac{x}{2} \right)_0^{\pi/3} - \left(x + 4\cos \frac{x}{2} \right)_{\frac{\pi}{3}}^\pi \\
&= 4\sqrt{3} - 4 - \frac{\pi}{3}
\end{aligned}$$

10 Let $I = \int_0^\pi x f(\sin x) dx \quad \dots(i)$

$$\Rightarrow I = \int_0^\pi (\pi - x) f[\sin(\pi - x)] dx$$

$$\Rightarrow I = \int_0^\pi (\pi - x) f(\sin x) dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^\pi \pi f(\sin x) dx$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^\pi f(\sin x) dx \quad \dots(iii)$$

$$\left[\because \int_0^{2a} f(x) dx = \right.$$

$$\left. \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases} \right]$$

$$\Rightarrow I = \pi \int_0^{\pi/2} f(\sin x) dx$$

Put $\frac{\pi}{2} - x = t \Rightarrow x = \frac{\pi}{2} - t$

Put $dx = -dt$ in Eq. (iii), we get

$$I = \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} f(\cos t) dt$$

$$= \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} f(\cos x) dx$$

$$= \pi \int_0^{\pi/2} f(\cos x) dx$$

[$\because f(\cos x)$ is an even function]

11 On differentiate the given interval by using Newton-Leibnitz formula,

we get $f'(x) = e^{-(x^2+1)^2} \cdot \left\{ \frac{d}{dx}(x^2+1) \right\} - e^{-(x^2)^2} \cdot \left\{ \frac{d}{dx}(x^2) \right\}$

$$\begin{aligned}
&= e^{-(x^2+1)^2} \cdot 2x - e^{-(x^2)^2} \cdot 2x \\
&= 2xe^{-(x^2+2x^2+1)}(1 - e^{2x^2+1})
\end{aligned}$$

For $f'(x) > 0$,

then $2x(1 - e^{2x^2+1}) > 0$

$$\Rightarrow 2x < 0$$

$$\Rightarrow x < 0$$

12 Let $l = \lim_{n \rightarrow \infty} \left(\frac{(n+1) \cdot (n+2) \dots (3n)}{n^{2n}} \right)^{\frac{1}{n}}$

$$= \lim_{n \rightarrow \infty} \left(\frac{(n+1) \cdot (n+2) \dots (n+2n)}{n^{2n}} \right)^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \left(\left(\frac{n+1}{n} \right) \left(\frac{n+2}{n} \right) \dots \left(\frac{n+2n}{n} \right) \right)^{\frac{1}{n}}$$

On taking log on both sides, we get

$$\log l = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left[\left(1 + \frac{1}{n} \right) \left(1 + \frac{2}{n} \right) \dots \left(1 + \frac{2n}{n} \right) \right]$$

$$\begin{aligned}
\Rightarrow \log l &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log \left(1 + \frac{1}{n} \right) + \log \left(1 + \frac{2}{n} \right) + \dots \right. \\
&\quad \left. + \log \left(1 + \frac{2n}{n} \right) \right]
\end{aligned}$$

$$\Rightarrow \log l = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \log \left(1 + \frac{r}{n} \right)$$

$$\Rightarrow \log l = \int_0^2 \log(1+x) dx$$

$$\Rightarrow \log l = \left[\log(1+x) \right.$$

$$\left. \cdot x - \int \frac{1}{1+x} \cdot x dx \right]_0^2$$

$$\begin{aligned}
\Rightarrow \log l &= [\log(1+x) \cdot x]_0^2 \\
&\quad - \int_0^2 \frac{x+1-1}{1+x} dx
\end{aligned}$$

$$\Rightarrow \log l = 2 \cdot \log 3 - \int_0^2 \left(1 - \frac{1}{1+x} \right) dx$$

$$\Rightarrow \log l = 2 \cdot \log 3 - [x - \log|1+x|]_0^2$$

$$\Rightarrow \log l = 2 \cdot \log 3 - [2 - \log 3]$$

$$\Rightarrow \log l = 3 \cdot \log 3 - 2$$

$$\Rightarrow \log l = \log 27 - 2$$

$$\Rightarrow l = e^{\log 27 - 2} = 27 \cdot e^{-2} = \frac{27}{e^2}$$

13 We have, $f'(x) = 3 \sin x + 4 \cos x$

Since, in $\left[\frac{5\pi}{4}, \frac{4\pi}{3} \right]$, $f'(x) < 0$, so assume

the least value at the point $x = \frac{4\pi}{3}$.

Thus,

$$f \left[\frac{4\pi}{3} \right] = \int_{5\pi/4}^{4\pi/3} (3 \sin u + 4 \cos u) du$$

$$= [-3\cos u + 4\sin u]_{\frac{4\pi/3}{5\pi/4}}$$

$$= \frac{3}{2} - 2\sqrt{3} + \frac{1}{\sqrt{2}}$$

14 We have, $g(x) = \int_0^x f(t) dt$

$$\Rightarrow g(2) = \int_0^2 f(t) dt$$

$$\Rightarrow g(2) = \int_0^1 f(t) dt + \int_1^2 f(t) dt$$

We know that, $m \leq f(x) \leq M$ for $x \in [a, b]$

$$\Rightarrow m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

$$\therefore \frac{1}{2} \leq f(t) \leq 1, \text{ for } t \in [0, 1]$$

$$\text{and } 0 \leq f(t) \leq \frac{1}{2}, \text{ for } t \in [1, 2]$$

$$\Rightarrow \frac{1}{2}(1-0) \leq \int_0^1 f(t) dt \leq 1(1-0)$$

$$\text{and } 0(2-1) \leq \int_1^2 f(t) dt \leq \frac{1}{2}(2-1)$$

$$\Rightarrow \frac{1}{2} \leq \int_0^1 f(t) dt \leq 1 \text{ and } 0 \leq \int_1^2 f(t) dt \leq \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} \leq \int_0^1 f(t) dt + \int_1^2 f(t) dt \leq \frac{3}{2}$$

$$\therefore \frac{1}{2} \leq g(2) \leq \frac{3}{2}$$

15 Let $f(a) = \int_0^1 |x^n - a^n| dx$

$$= \int_0^a (a^n - x^n) dx + \int_a^1 (x^n - a^n) dx$$

$$= \left[a^n x - \frac{x^{n+1}}{n+1} \right]_0^a + \left[\frac{x^{n+1}}{n+1} - a^n \cdot x \right]_a^1$$

$$= \left[a^{n+1} - \frac{a^{n+1}}{n+1} \right] +$$

$$\left[\frac{1}{n+1} - a^n - \frac{a^{n+1}}{n+1} + a^{n+1} \right]$$

$$= 2a^{n+1} - \frac{2a^{n+1}}{n+1} - a^n + \frac{1}{n+1}$$

$$= 2a^{n+1} \left[\frac{n}{n+1} \right] - a^n + \frac{1}{n+1}$$

$$\Rightarrow f'(a) = n(2a-1)a^{n-1}$$

Thus, only critical point in $(0, 1)$ is

$$a = 1/2$$

$$\text{Also, } f'(a) < 0 \text{ for } a \in \left(0, \frac{1}{2}\right)$$

$$\text{and } f'(a) > 0 \text{ for } a \in \left(\frac{1}{2}, 1\right).$$

$$\therefore f(a) \text{ is minimum for } a = \frac{1}{2}$$

16 Let $P = \lim_{n \rightarrow \infty} \left\{ \prod_{r=1}^n \tan \left(\frac{r\pi}{2n} \right) \right\}^{1/n}$

$$\therefore \ln P = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \ln \tan \left(\frac{r\pi}{2n} \right)$$

$$= \int_0^1 \ln \tan \left(\frac{\pi x}{2} \right) dx$$

$$\Rightarrow \ln P = \frac{2}{\pi} \int_0^{\pi/2} \ln \tan x dx \quad \dots(i)$$

$$\text{and } \ln P = \frac{2}{\pi} \int_0^{\pi/2} \ln \cot x dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2 \ln P = \frac{2}{\pi} \int_0^{\pi/2} (\ln \tan x + \ln \cot x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \ln 1 dx = 0$$

$$\Rightarrow \ln P = 0$$

$$\therefore P = 1$$

17 We have, $f(x) = \frac{x-1}{x+1}$

$$\Rightarrow f^2(x) = f[f(x)] = f\left(\frac{x-1}{x+1}\right)$$

$$= \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1} = -\frac{1}{x}$$

$$\Rightarrow f^4(x) = f^2[f^2(x)] = f^2\left(-\frac{1}{x}\right)$$

$$= \frac{-1}{-\frac{1}{x}} = x$$

$$\therefore g(x) = f^{1998}(x) = f^2 \circ f^{1996}(x)$$

$$\Rightarrow g(x) = f^2[f^{1996}(x)]$$

$$\Rightarrow g(x) = f^2(x) \quad [\because f^{1996}(x) = \underbrace{\{(f^4 \circ f^4 \circ f^4 \circ \dots \circ f^4)\}}_{499 \text{ times}}(x) = x]$$

$$\Rightarrow g(x) = -\frac{1}{x}$$

$$\therefore \int_{1/e}^1 g(x) dx = \int_{1/e}^1 \left(-\frac{1}{x}\right) dx$$

$$= -[\log_e x]_{1/e}^1$$

$$\Rightarrow \int_{1/e}^1 g(x) dx = -\left[\log_e 1 - \log_e \frac{1}{e}\right]$$

$$= -[0 + 1] = -1$$

18 Given, $f'(x) = f(x)$

$$\text{and } f(0) = 1$$

$$\text{Let } f(x) = e^x \quad \dots(i)$$

$$\text{Also, } f(x) + g(x) = x^2$$

$$\Rightarrow g(x) = x^2 - e^x \quad \dots(ii)$$

$$\text{Now, } \int_0^1 f(x)g(x) dx$$

$$= \int_0^1 e^x(x^2 - e^x) dx$$

[from Eqs. (i) and (ii)]

$$= \int_0^1 x^2 e^x dx - \int_0^1 e^{2x} dx$$

$$= [x^2 e^x - \int 2x e^x dx]_0^1 - \frac{1}{2} [e^{2x}]_0^1$$

$$= [x^2 e^x - 2x e^x + 2e^x]_0^1 - \frac{1}{2} (e^2 - 1)$$

$$= [(x^2 - 2x + 2)e^x]_0^1 - \frac{1}{2} e^2 + \frac{1}{2}$$

$$= [(1 - 2 + 2)e^1 - (0 - 0 + 2)e^0] - \frac{1}{2} e^2 + \frac{1}{2}$$

$$= e - 2 - \frac{e^2}{2} + \frac{1}{2} = e - \frac{e^2}{2} - \frac{3}{2}$$

19 In LHS, put $x^n = \tan^2 \theta$

$$\Rightarrow nx^{n-1} dx = 2 \tan \theta \sec^2 \theta d\theta$$

$$\therefore \int_0^\infty \frac{dx}{1+x^n} = \frac{2}{n} \int_0^{\pi/2} \tan^{1-2+2/n} \theta d\theta$$

$$= \frac{2}{n} \int_0^{\pi/2} \tan^{(2/n)-1} \theta d\theta$$

In RHS, put $x^n = \sin^2 \theta$

$$\Rightarrow nx^{n-1} dx = 2 \sin \theta \cos \theta d\theta$$

$$\therefore \int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{2}{n} \int_0^{\pi/2} \frac{1}{\cos^{2/n} \theta}$$

$$\sin^{2-1} \theta \cos \theta d\theta = \frac{2}{n} \int_0^{\pi/2} \tan^{(2/n)-1} \theta d\theta$$

20 $\sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3$

$$= (\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x (\sin^2 x + \cos^2 x)$$

$$= 1 - 3 \sin^2 x \cos^2 x$$

$$= 1 - \frac{3}{4} \sin^2 2x \quad \left[\because \text{period} = \frac{\pi}{2} \right]$$

So, the least and greatest value of

$$\sin^6 x + \cos^6 x \text{ are } \frac{1}{4} \text{ and } 1.$$

$$\text{Hence, } \left(\frac{\pi}{2} - 0\right) \times \frac{1}{4}$$

$$< \int_0^{\pi/2} (\sin^6 x + \cos^6 x) dx < \left(\frac{\pi}{2} - 0\right) \times 1$$

$$\therefore \frac{\pi}{8} < \int_0^{\pi/2} (\sin^6 x + \cos^6 x) dx < \frac{\pi}{2}$$