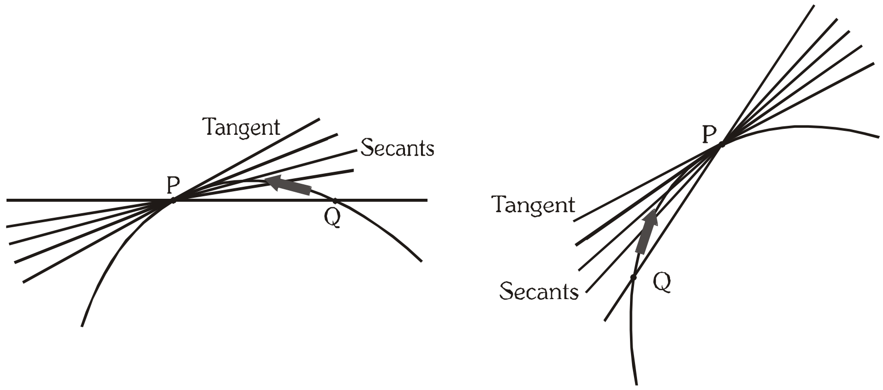
APPLICATION OF DERIVATIVES

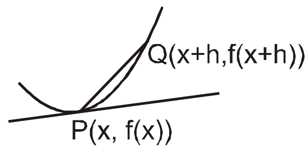
1. TANGENT & NORMAL :

The tangent to the curve at 'P' is the line through P whose slope is limit of the secant slopes as $Q \rightarrow P$ from either side. Line perpendicular to tangent & passingh through P is called normal at P.



1.1 Geometrical Meaning of $\frac{dy}{dx}$

As $Q \rightarrow P$, $h \rightarrow 0$ and slope of chord PQ tends to slope of tangent at P (see figure).



Slope of chord PQ = $\frac{f(x+h)-f(x)}{h}$

$\lim\limits_{Q\rightarrow P}$ slope of chord PQ = $\lim\limits_{h\rightarrow 0}\frac{f(x+h)-f(x)}{h}$

$\Rightarrow$ slope of tangent at P = $f'(x) = \frac{dy}{dx}$

1.2 Equation of tangent and normal

(a) The value of the derivative at $P(x_1, y_1)$ gives the slope of the tangent to the curve at P. Symbolically

$$f'(x_1) = \left.\frac{dy}{dx}\right]_{(x_1,\, y_1)}$$ = Slope of tangent at $P(x_1, y_1)$ = m(say).

(b) Equation of tangent at (x_1, y_1) is ; $$y - y_1 = \left.\frac{dy}{dx}\right]_{(x_1,\, y_1)}(x - x_1)$$

(c) Equation of normal at (x_1, y_1) is; $$y - y_1 = -\frac{1}{\left.\frac{dy}{dx}\right]_{(x_1, y_1)}}(x - x_1).$$

If $f'(x_1) = 0$, then tangent is the line $y = y_1$ and normal is the line $x = x_1$.

If $\lim\limits_{h\rightarrow 0}\frac{f(x_1+h)-f(x_1)}{h} = \infty$ or $-\infty$, then $x = x_1$ is tangent **(VERTICAL TANGENT)** and $y = y_1$ is normal.

Note :

(i) If the tangent at any point P on the curve is parallel to the axis of x then dy/dx = 0 at the point P.

(ii) If the tangent at any point on the curve is parallel to the axis of y, then dy/dx not defined or dx/dy = 0.

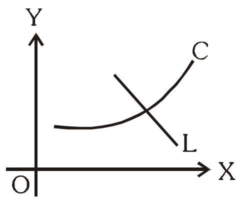
(iii) If the tangent at any point on the curve is equally inclined to both the axes then dy/dx = ±1.

(iv) If a curve passing through the origin be given by a rational integral algebraic equation, then the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation. e.g. If the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$

1.3 Myths About Tangent :

(a) **Myth :** A line meeting the curve only at one point is a tangent to the curve.

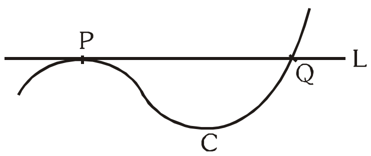
Explanation : A line meeting the curve in one point is not necessarily tangent to it.



Here L is not tangent to C

(b) **Myth :** A line meeting the curve at more than one point is not a tangent to the curve.

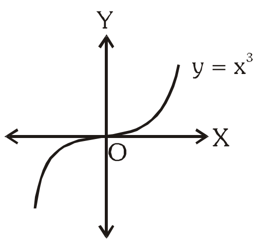
Explanation : A line may meet the curve at several points and may still be tangent to it at some point



Here L is tangent to C at P, and cutting it again at Q.

(c) **Myth :** Tangent at a point to the curve can not cross it at the same point.

Explanation : A line may be tangent to the curve and also cross it.



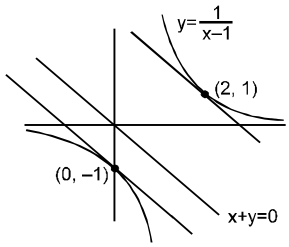
Here X-axis is tangent to $y = x^3$ at origin.

SOLVED EXAMPLE

Example 1 : Find the equation of all straight lines which are tangent to curve $y = \frac{1}{x-1}$ and which are parallel to the line x + y = 0.

Solution : Suppose the tangent is at (x_1, y_1) and it has slope – 1.

$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = -1.$



$\Rightarrow \quad -\frac{1}{(x_1-1)^2} = -1.$

$\Rightarrow \quad x_1 = 0 \quad$ or $\quad 2$

$\Rightarrow \quad y_1 = -1 \quad$ or $\quad 1$

Hence tangent at (0, – 1) and (2, 1) are the required lines (see figure) with equations

$-1(x-0) = (y+1) \quad$ and $\quad -1(x-2) = (y-1)$

$\Rightarrow \quad x + y + 1 = 0 \quad$ and $\quad y + x = 3$

Example # 2 Find equation of normal to the curve $y = |x^2 - |x||$ at x = – 2.

Solution : In the neighborhood of x = – 2, $y = x^2 + x$.

Hence the point of contact is (– 2, 2)

$\frac{dy}{dx} = 2x + 1 \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=-2} = -3.$

So the slope of normal at (– 2, 2) is $\frac{1}{3}$.

Hence equation of normal is

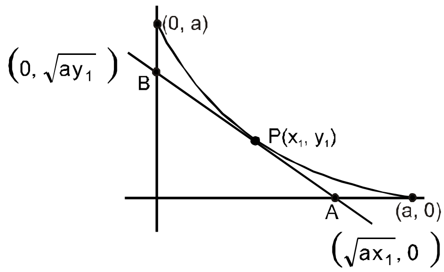
$\frac{1}{3}(x+2) = y - 2 \quad \Rightarrow \quad 3y = x + 8$

Example # 3 Prove that sum of intercepts of the tangent at any point to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ on the coordinate axis is constant.

Solution : Let $P(x_1, y_1)$ be a variable point on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$, as shown in figure.

$\Rightarrow$ equation of tangent at point P is

$-\frac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1) = (y - y_1)$



$\Rightarrow \quad -\frac{x}{\sqrt{x_1}} + \sqrt{x_1} = \frac{y}{\sqrt{y_1}} - \sqrt{y_1}$

$\Rightarrow \quad \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{x_1} + \sqrt{y_1} \Rightarrow \frac{x}{\sqrt{x_1}} + \frac{y}{\sqrt{y_1}} = \sqrt{a} \quad (\because \quad \sqrt{x_1} + \sqrt{y_1} = \sqrt{a})$

Hence point A is $(\sqrt{ax_1}, 0)$ and coordinates of point B is $(0, \sqrt{ay_1})$. Sum of intercepts

$= \sqrt{a}\,(\sqrt{x_1} + \sqrt{y_1}) = \sqrt{a}\,.\,\sqrt{a} = a \quad$ (which is constant)

Example # 4 : Find the equation of the tangent to the curve $y=(x^3-1)(x-2)$ at the points where the curve cuts the x-axis.

Solution : The equation of the curve is $y=(x^3-1)(x-2)$ .... (i)

It cuts x-axis at y = 0. So, putting y = 0 in (i), we get $(x^3-1)(x-2)=0$

$\Rightarrow (x-1)(x-2)(x^2+x+1)=0 \Rightarrow x-1=0,\ x-2=0 \qquad [\because x^2+x+1\neq 0]$

$\Rightarrow x=1,\ 2$.

Thus, the points of intersection of curve (i) with x-axis are (1, 0) and (2, 0). Now,

$y=(x^3-1)(x-2) \quad \Rightarrow \frac{dy}{dx}=3x^2(x-2)+(x^3-1) \Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)}=-3$ and $\left(\frac{dy}{dx}\right)_{(2,0)}=7$

The equations of the tangents at (1, 0) and (2, 0) are respectively

$y-0=-3(x-1)$ and $y-0=7(x-2) \Rightarrow y+3x-3=0$ and $7x-y-14=0$

Example # 5 : Find the equation of normal to the curve $x+y=x^y$, where it cuts x-axis.

Solution : Given curve is $x+y=x^y$ ..... (i)

at x-axis y = 0,

$\therefore\ x+0=x^0 \quad \Rightarrow \quad x=1$

$\therefore$ Point is A(1, 0)

Now to differentiate $x+y=x^y$ take log on both sides

$\Rightarrow \log(x+y)=y\log x \qquad \therefore\ \frac{1}{x+y}\left\{1+\frac{dy}{dx}\right\}=y.\frac{1}{x}+(\log x)\frac{dy}{dx}$

Putting $x=1, y=0$ $\quad \left\{1+\frac{dy}{dx}\right\}=0 \Rightarrow \left(\frac{dy}{dx}\right)_{(1,0)}=-1$

$\therefore$ slope of normal = 1

Equation of normal is, $\frac{y-0}{x-1}=1 \quad \Rightarrow \quad y=x-1$

<u>Problems for Self Practice</u>-1 :

(1) Find the slope of the normal to the curve $x = 1 - a\sin\theta,\ y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.

(2) Find the equation of the tangent and normal to the given curves at the given points.

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3) (ii) $y^2 = \frac{x^3}{4-x}$ at (2, – 2).

(3) Prove that area of the triangle formed by any tangent to the curve $xy = c^2$ and coordinate axes is constant.

(4) A curve is given by the equations $x = at^2$ & $y = at^3$. A variable pair of perpendicular lines through the origin 'O' meet the curve at P & Q. Show that the locus of the point of intersection of the tangents at P & Q is $4y^2 = 3ax - a^2$.

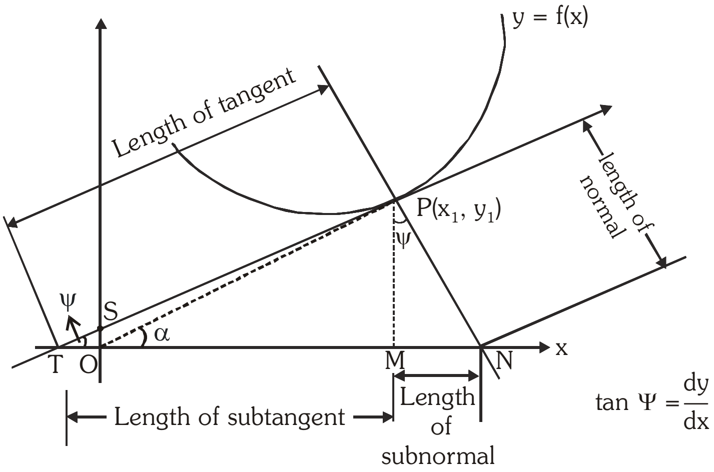
Answers :

(1) $-\frac{a}{2b}$

(2) (i) Tangent : y = 2x + 1, Normal : x + 2y = 7

(ii) Tangent : 2x + y = 2, Normal : x – 2y = 6

2. LENGTH OF TANGENT, SUBTANGENT, NORMAL & SUBNORMAL :



(a) Length of the tangent (PT) = $\left|\dfrac{y_1\sqrt{1+[f'(x_1)]^2}}{f'(x_1)}\right|$

(b) Length of Subtangent (MT) = $\left|\dfrac{y_1}{f'(x_1)}\right|$

(c) Length of Normal (PN) = $\left|y_1\sqrt{1+\left[\ f'(x_1)\ \right]^2}\right|$

(d) Length of Subnormal (MN) = $|y_1 f'(x_1)|$

(e) **Initial ordinate :** Y intercept of tangent at point P(x, y) = OS

(f) **Radius vector (Polar radius) :** Line segment joining origin to point P(x, y) = OP

(g) **Vectorial angle :** Angle made by radius vector with positive direction of x-axis in anticlock wise direction is called vectorial angle. In given figure α is vectorial angle.

SOLVED EXAMPLE

Example # 6 : Find the length of the normal to the curve $x = a(\theta + \sin\theta),\ y = a(1-\cos\theta)$ at $\theta = \dfrac{\pi}{2}$.

Solution : $$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2} \Rightarrow \left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{2}} = \tan\left(\frac{\pi}{4}\right) = 1$$

Also at $\theta = \dfrac{\pi}{2},\ y = a\left(1-\cos\dfrac{\pi}{2}\right) = a$

$\therefore$ required length of normal $= y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = a\sqrt{1+1} = \sqrt{2}a$

Example # 7 : Find the length of the tangent to the curve $x = a\left(\cos t + \log\tan\frac{t}{2}\right),\ y = a\sin t$.

Solution : $\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right) = \frac{a\cos t}{a\left(-\sin t + \frac{1}{\sin t}\right)} = \tan t$

$\therefore$ length of the tangent $= y\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\left(\frac{dy}{dx}\right)} = a\sin t\frac{\sqrt{1+\tan^2 t}}{\tan t} = a\sin t\left(\frac{\sec t}{\tan t}\right) = a$

Example # 8 Find the length of tangent for the curve $y = x^3 + 3x^2 + 4x - 1$ at point $x = 0$.

Solution : Here, $m = \left.\frac{dy}{dx}\right|_{x=0}$

$\frac{dy}{dx} = 3x^2 + 6x + 4 \quad \Rightarrow \quad m = 4$

and, $k = y(0) \quad \Rightarrow \quad k = -1$

$\ell = |k|\sqrt{1 + \frac{1}{m^2}} \quad \Rightarrow \quad \ell = |(-1)|\sqrt{1+\frac{1}{16}} = \frac{\sqrt{17}}{4}$

Example # 9 Prove that for the curve $y = be^{x/a}$, the length of subtangent at any point is always constant.

Solution : $y = be^{x/a}$

Let the point be $(x_1, y_1) \quad \Rightarrow \quad m = \left.\frac{dy}{dx}\right|_{x=x_1} = \frac{b.e^{x_1/a}}{a} = \frac{y_1}{a}$

Now, length of subtangent $= \left|\frac{y_1}{m}\right| = \left|\frac{y_1}{y_1/a}\right| = |a|$; which is always constant.

<u>Problems for Self Practice</u> -2 :

(1) Prove that at any point of a curve, the product of the length of sub tangent and the length of sub normal is equal to square of the ordinates of point of contact.

(2) Find the length of subtangent to the curve $x^2 + y^2 + xy = 7$ at the point $(1, -3)$.

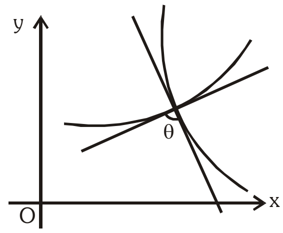
(3) For the curve $x^{m+n} = a^{m-n}y^{2n}$, where a is a positive constant and m, n are positive integers, prove that the m^{th} power of subtangent varies as n^{th} power of subnormal.

(4) Prove that the segment of the tangent to the curve $y = \frac{a}{2}\ \ell n\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}} - \sqrt{a^2-x^2}$ contained between the y-axis & the point of tangency has a constant length .

(5) Find the length of the subnormal to the curve $y^2 = x^3$ at the point (4, 8).

Answers : (2) 15 (5) 24

3. ANGLE OF INTERSECTION BETWEEN TWO CURVES :



Angle of intersection between two curves is defined as the angle between the two tangents drawn to the two curves at their point of intersection.

3.1 Orthogonal curves :

If the angle between two curves at each point of intersection is 90° then they are called **orthogonal curves.** For example, the curves $x^2 + y^2 = r^2$ & $y = mx$ are orthogonal curves.

SOLVED EXAMPLE

Example # 10 : Find the angle of intersection between the curve $x^2 = 32y$ and $y^2 = 4x$ at point (16, 8).

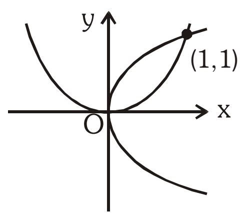
Solution : $x^2 = 32y \Rightarrow \frac{dy}{dx} = \frac{x}{16} \Rightarrow y^2 = 4x \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

$\therefore$ at $(16, 8), \left(\frac{dy}{dx}\right)_1 = 1, \left(\frac{dy}{dx}\right)_2 = \frac{1}{4}$

So required angle $= \tan^{-1}\left(\frac{1-\frac{1}{4}}{1+1\left(\frac{1}{4}\right)}\right) = \tan^{-1}\left(\frac{3}{5}\right)$

Example # 11 : Check the orthogonality of the curves $y^2 = x$ & $x^2 = y$.

Solution : Solving the curves simultaneously we get points of intersection as (1, 1) and (0, 0).



At (1,1) for first curve $2y\left(\frac{dy}{dx}\right)_1 = 1 \Rightarrow m_1 = \frac{1}{2}$

& for second curve $2x = \left(\frac{dy}{dx}\right)_2 \Rightarrow m_2 = 2$

$m_1m_2 \neq -1$ at (1,1).

But at (0, 0) clearly x-axis & y-axis are their respective tangents hence they are orthogonal at (0,0) but not at (1,1). Hence these curves are not said to be orthogonal.

Example # 12 : If curve $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then find the value of a.

Solution : $y = 1 - ax^2 \Rightarrow \frac{dy}{dx} = -2ax$ $\qquad$ $y = x^2 \Rightarrow \frac{dy}{dx} = 2x$

Two curves intersect orthogonally if $\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2 = -1$

$\Rightarrow (-2ax)(2x) = -1 \qquad \Rightarrow 4ax^2 = 1$ ..... (i)

Now eliminating y from the given equations we have $1 - ax^2 = x^2$

$\Rightarrow (1+a)x^2 = 1$ ..... (ii)

Eliminating x^2 from (i) and (ii) we get $\frac{4a}{1+a} = 1 \Rightarrow a = \frac{1}{3}$

Problems for Self Practice-3 :

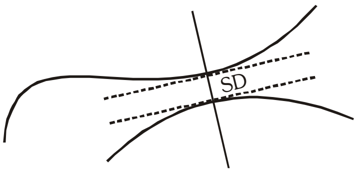
(1) If two curves $y = a^x$ and $y = b^x$ intersect at an angle α, then find the value of $\tan\alpha$.

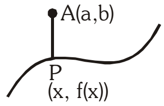
(2) Find the angle of intersection of curves $y = 4 - x^2$ and $y = x^2$.

Answers : **(1)** $\left|\dfrac{\ell na - \ell nb}{1 + \ell na\, \ell nb}\right|$ **(2)** $\tan^{-1}\left(\dfrac{4\sqrt{2}}{7}\right)$

4. LEAST/GREATEST DISTANCE BETWEEN TWO CURVES :

Least/Greatest distance between two non-intersecting curves usually lies along the common normal. (Wherever defined)





Note : Given a fixed point A(a, b) and a moving point P(x, f(x)) on the curve y = f(x). Then AP will be maximum or minimum if it is normal to the curve at P.

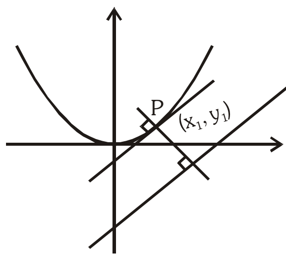
SOLVED EXAMPLE

Example # 13 : Find the co-ordinates of the point on the curve $x^2 = 4y$, which is at least distance from the line y = x – 4.

Solution : Let $P(x_1 y_1)$ be a point on the curve $x^2 = 4y$

at which normal is also a perpendicular to the line y = x – 4.

Slope of the tangent at (x_1, y_1) is $2x = 4\dfrac{dy}{dx} \Rightarrow \left.\dfrac{dy}{dx}\right|_{(x_1,y_1)} = \dfrac{x_1}{2}$



$\therefore \quad \dfrac{x_1}{2} = 1 \Rightarrow x_1 = 2$

$\because \quad x_1^2 = 4y_1 \Rightarrow y_1 = 1$

Hence required point is (2, 1)

Problems for Self Practice-4 :

(1) Find the coordinates of point on the curve $y^2 = 8x$, which is at minimum distance from the line x + y = –2.

Answers : **(1)** (2,–4)

5. RATE MEASUREMENT :

Whenever one quantity y varies with another quantity x, satisfying some rule y = f(x), then $\dfrac{dy}{dx}$ (or $f'(x)$) represents the instantanious rate of change of y with respect to x and $\left.\dfrac{dy}{dx}\right]_{x=a}$ (or $f'(a)$) represents the rate of change of y with respect to x at x = a.

SOLVED EXAMPLE

Example # 14 How fast the area of a circle increases when its radius is 5cm;
(i) with respect to radius (ii) with respect to diameter

Solution : (i) $A = \pi r^2$, $\frac{dA}{dr} = 2\pi r$

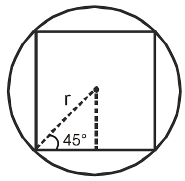
$\therefore$ $\left.\frac{dA}{dr}\right]_{r=5} = 10\pi$ cm²/cm.

(ii) $A = \frac{\pi}{4}D^2$, $\frac{dA}{dD} = \frac{\pi}{2}D$

$\therefore$ $\left.\frac{dA}{dD}\right]_{D=10} = \frac{\pi}{2}\,.\,10 = 5\pi$ cm²/cm.

Example # 15 If area of circle increases at a rate of 2cm²/sec, then find the rate at which area of the inscribed square increases.

Solution : Area of circle, $A_1 = \pi r^2$. Area of square, $A_2 = 2r^2$ (see figure)



$\frac{dA_1}{dt} = 2\pi r\,\frac{dr}{dt}$, $\frac{dA_2}{dt} = 4r\,.\,\frac{dr}{dt}$

$\because$ $2 = 2\pi r\,.\,\frac{dr}{dt}$ $\Rightarrow$ $r\,\frac{dr}{dt} = \frac{1}{\pi}$

$\therefore$ $\frac{dA_2}{dt} = 4\,.\,\frac{1}{\pi} = \frac{4}{\pi}$ cm²/sec

$\therefore$ Area of square increases at the rate $\frac{4}{\pi}$ cm²/sec.

Example # 16 The volume of a cube is increasing at a rate of 7 cm³/sec. How fast is the surface area increasing when the length of an edge is 4 cm?

Solution. Let at some time t, the length of edge is x cm.

$v = x^3$ $\Rightarrow$ $\frac{dv}{dt} = 3x^2\,\frac{dx}{dt}$ (but $\frac{dv}{dt} = 7$)

$\Rightarrow$ $\frac{dx}{dt} = \frac{7}{3x^2}$ cm/sec.

Now $S = 6x^2$

$\frac{dS}{dt} = 12x\,\frac{dx}{dt}$ $\Rightarrow$ $\frac{dS}{dt} = 12x.\,\frac{7}{3x^2} = \frac{28}{x}$

when x = 4 cm, $\frac{dS}{dt} = 7$ cm²/sec.

Example # 17 Sand is pouring from pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one - sixth of radius of base. How fast is the height of the sand cone increasing when height is 4 cm?

Solution. $V = \frac{1}{3}\pi r^2 h$

but $h = \frac{r}{6}$

$\Rightarrow$ $V = \frac{1}{3}\pi (6h)^2 h$

$\Rightarrow$ $V = 12\pi h^3$

$\frac{dV}{dt} = 36\pi h^2 . \frac{dh}{dt}$

when, $\frac{dV}{dt} = 12$ cm^3/s and $h = 4$ cm

$\frac{dh}{dt} = \frac{12}{36\pi.(4)^2} = \frac{1}{48\pi}$ cm/sec.

Example # 18 : x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of the second square with respect to the first square.

Solution : Given x and y are sides of two squares. Thus the area of two squares are x^2 and y^2

We have to obtain $\frac{d(y^2)}{d(x^2)} = \frac{2y\frac{dy}{dx}}{2x} = \frac{y}{x}.\frac{dy}{dx}$ .... (i)

where the given curve is, $y = x - x^2$ $\Rightarrow$ $\frac{dy}{dx} = 1 - 2x$ .... (ii)

Thus, $\frac{d(y^2)}{d(x^2)} = \frac{y}{x}(1-2x)$ [from (i) and (ii)]

or $\frac{d(y^2)}{d(x^2)} = \frac{(x-x^2)(1-2x)}{x}$ $\Rightarrow$ $\frac{d(y^2)}{d(x^2)} = (2x^2 - 3x + 1)$

The rate of change of the area of second square with respect to first square is $(2x^2 - 3x + 1)$

Problems for Self Practice-5 :

(1) Radius of a circle is increasing at rate of 3 cm/sec. Find the rate at which the area of circle is increasing at the instant when radius is 10 cm.

(2) A ladder of length 5 m is leaning against a wall. The bottom of ladder is being pulled along the ground away from wall at rate of 2cm/sec. How fast is the top part of ladder sliding on the wall when foot of ladder is 4 m away from wall.

(3) Water is dripping out of a conical funnel of semi-vertical angle 45° at rate of 2cm^3/s. Find the rate at which slant height of water is decreasing when the height of water is $\sqrt{2}$ cm.

(4) A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the lift-off point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment.

(5) A stone is dropped into a quiet lake and waves move in circles at the speed of 5cm/s. At the instant when the radius of the circular wave is 8cm, how fast is the enclosed area increasing?

Answers : (1) 60π cm^2/sec (2) $\frac{8}{3}$ cm/sec (3) $\frac{1}{\sqrt{2}\pi}$ cm/sec. (4) 140 ft/min.

(5) $80\ \pi$cm^2/s

6. APPROXIMATION USING DIFFERENTIALS :

In order to calculate the approximate value of a function, differentials may be used where the differential of a function is equal to its derivative multiplied by the differential of the independent variable.

In general dy = f'(x)dx or df(x) = f'(x)dx

Note :

(i) For the independent variable 'x', increment Δ x and differential dx are equal but this is not the case with the dependent variable 'y' i.e. $\Delta y \neq dy$.

$\therefore$ Approximate value of y when increment Δx is given to independent variable x in y = f(x) is

$$y + \Delta y = f(x + \Delta x) = f(x) + \frac{dy}{dx}.\Delta x$$

(ii) The relation dy = f'(x) dx can be written as $\frac{dy}{dx} = f'(x)$; thus the quotient of the differentials of 'y' and 'x' is equal to the derivative of 'y' w.r.t. 'x'.

SOLVED EXAMPLE

Example # 19 : Find the approximate value of square root of 25.2.

Solution : Let f(x) = $\sqrt{x}$

Now, $f(x + \Delta x) - f(x) = f'(x) . \Delta x = \frac{\Delta x}{2\sqrt{x}}$

we may write, 25.2 = 25 + 0.2

Taking x = 25 and Δx = 0.2, we have

$$f(25.2) - f(25) = \frac{0.2}{2\sqrt{25}}$$

or $f(25.2) - \sqrt{25} = \frac{0.2}{10} = 0.02 \Rightarrow f(25.2) = 5.02$

or $\sqrt{(25.2)} = 5.02$

<u>Problems for Self Practice</u>-6 :

(1) Find the approximate value of $(0.009)^{1/3}$.

Answers : **(1)** 0.208

7. MONOTONICITY OF A FUNCTION :

Let f be a real valued function having domain D(D $\subset$ R) and S be a subset of D. f is said to be monotonically increasing in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$. f is said to be monotonically decreasing in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

f is said to be strictly increasing in S if for $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$. Similarly, f is said to be strictly decreasing in S if for every $x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

Notes : (i) f is strictly increasing $\Rightarrow$ f is monotonically increasing. But converse need not be true.

(ii) f is strictly decreasing $\Rightarrow$ f is monotonically decreasing. Again, converse need not be true.

(iii) If f(x) = constant in S, then f is increasing as well as decreasing in S

(iv) A function f is said to be an increasing function if it is increasing in the domain. Similarly, if f is decreasing in the domain, we say that f is monotonically decreasing

(v) f is said to be a monotonic function if either it is monotonically increasing or monotonically decreasing

(vi) If f is increasing in a subset of S and decreasing in another subset of S, then f is non monotonic in S.

7.1 Application of differentiation for detecting monotonicity :

Let I be an interval (open or closed or semi open and semi closed)

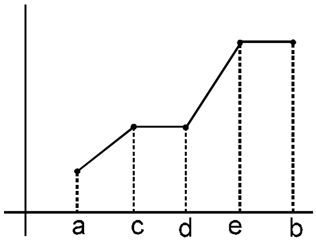
(i) If $f'(x) > 0\ \forall\ x \in I$, then f is strictly increasing in I

(ii) If $f'(x) < 0\ \forall\ x \in I$, then f is strictly decreasing in I

Note : Let I be an interval (or ray) which is a subset of domain of f. If $f'(x) > 0$, $\forall\ x \in I$, except for countably many points where $f'(x) = 0$, then f(x) is strictly increasing in I.

{$f'(x) = 0$ at countably many points $\Rightarrow f'(x) = 0$ does not occur on an interval which is a subset of I }

Let us consider another function whose graph is shown below for $x \in (a, b)$.



Here also $f'(x) \geq 0$ for all $x \in (a, b)$. But, note that in this case, $f'(x) = 0$ holds for all $x \in (c, d)$ and (e,b). Thus the given function is increasing (monotonically increasing) in (a, b), but not strictly increasing.

SOLVED EXAMPLE

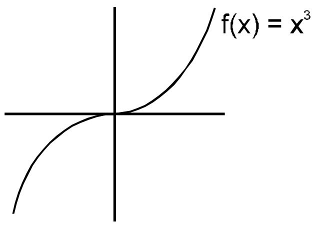
Example # 20 : Let $f(x) = x^3$. Find the intervals of monotonicity.

Solution : $f'(x) = 3x^2$

$f'(x) > 0$ everywhere except at $x = 0$.

Hence f(x) will be strictly increasing function

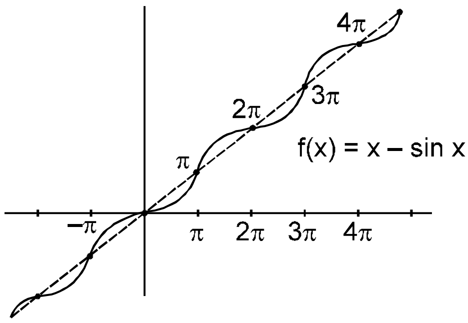
for $x \in R$ {see figure}



Example # 21 : Let $f(x) = x - \sin x$. Find the intervals of monotonicity.

Solution : $f'(x) = 1 - \cos x$

Now, $f'(x) > 0$ every where, except at $x = 0, \pm 2\pi, \pm 4\pi$ etc. But all these points are discrete (countable) and do not form an interval. Hence we can conclude that f(x) is strictly increasing in R. In fact we can also see it graphically.



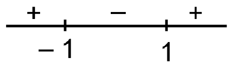
Example # 22 : Find the intervals in which $f(x) = x^3 - 3x + 2$ is increasing.

Solution : $f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

$f'(x) = 3(x - 1)(x + 1)$

for M.I. $f'(x) \geq 0 \quad \Rightarrow \quad 3(x - 1)(x + 1) \geq 0$



$\Rightarrow \quad x \in (-\infty, -1] \cup [1, \infty)$, thus f is increasing in $(-\infty, -1]$ and also in $[1, \infty)$

Example # 23 : Find the intervals of monotonicity of the following functions.

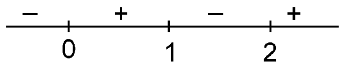
(i) $f(x) = x^2 (x - 2)^2$

(ii) $f(x) = x\, \ell n\, x$

(iii) $f(x) = \sin x + \cos x \ ; \ x \in [0, 2\pi]$

Solution : (i) $f(x) = x^2 (x - 2)^2 \Rightarrow \quad f'(x) = 4x (x - 1)(x - 2)$

observing the sign change of $f'(x)$



Hence increasing in $[0, 1]$ and in $[2, \infty)$

and decreasing for $x \in (-\infty, 0]$ and $[1, 2]$

(ii) $f(x) = x\, \ell n\, x$

$f'(x) = 1 + \ell n\, x$

$f'(x) \geq 0 \quad \Rightarrow \quad \ell n\, x \geq -1 \quad \Rightarrow \quad x \geq \dfrac{1}{e}$

$\Rightarrow$ increasing for $x \in \left[\dfrac{1}{e}, \infty\right)$ and decreasing for $x \in \left(0, \dfrac{1}{e}\right]$.

(iii) $f(x) = \sin x + \cos x$

$f'(x) = \cos x - \sin x$

for increasing $f'(x) \geq 0 \quad \Rightarrow \quad \cos x \geq \sin x$

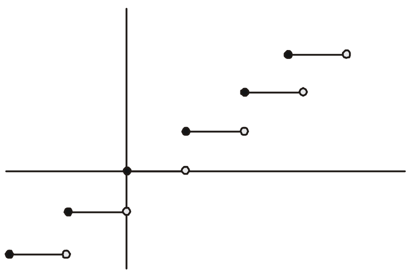
$\Rightarrow$ f is increasing in $\left[0, \dfrac{\pi}{4}\right]$ and $\left[\dfrac{5\pi}{4}, 2\pi\right]$

f is decreasing in $\left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

Note : If a function f(x) is increasing in (a, b) and f(x) is continuous in [a, b], then f(x) is increasing on [a, b]

Example # 24 : $f(x) = [x]$ is a step up function. Is it a strictly increasing function for $x \in R$.

Solution : No, $f(x) = [x]$ is increasing (monotonically increasing) (non-decreasing), but not strictly increasing function as illustrated by its graph.



Example # 25 : If $f(x) = \sin^4x + \cos^4x + bx + c$, then find possible values of b and c such that f(x) is monotonic for all $x \in R$

Solution : $f(x) = \sin^4x + \cos^4x + bx + c$

$f'(x) = 4\sin^3x\cos x - 4\cos^3x\sin x + b = -\sin 4x + b.$

Case - (i) : for M.I. $f'(x) \geq 0$ for all $x \in R$

$\Leftrightarrow$ $b \geq \sin 4x$ for all $x \in R$ $\Leftrightarrow$ $b \geq 1$

Case - (ii) : for M.D. $f'(x) \leq 0$ for all $x \in R$

$\Leftrightarrow$ $b \leq \sin 4x$ for all $x \in R$ $\Leftrightarrow$ $b \leq -1$

Hence for f(x) to be monotonic $b \in (-\infty, -1] \cup [1, \infty)$ and $c \in R$.

Example # 26: Find possible values of 'a' such that $f(x) = e^{2x} - (a + 1)e^x + 2x$ is monotonically increasing for $x \in R$

Solution : $f(x) = e^{2x} - (a + 1)e^x + 2x$

$f'(x) = 2e^{2x} - (a + 1)e^x + 2$

Now, $2e^{2x} - (a + 1)e^x + 2 \geq 0$ for all $x \in R$

$\Rightarrow$ $2\left(e^x + \frac{1}{e^x}\right) - (a + 1) \geq 0$ for all $x \in R$

$(a + 1) \leq 2\left(e^x + \frac{1}{e^x}\right)$ for all $x \in R$

$\Rightarrow$ $a + 1 \leq 4$ $\left(\because \ e^x + \frac{1}{e^x} \text{ has minimum value } 2\right)$

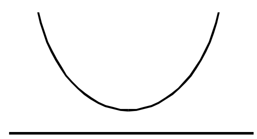
$\Rightarrow$ $a \leq 3$

Aliter (Using graph)

$2e^{2x} - (a + 1)e^x + 2 \geq 0$ for all $x \in R$

putting $e^x = t$; $t \in (0, \infty)$

$2t^2 - (a + 1)t + 2 \geq 0$ for all $t \in (0, \infty)$



Case - (i) : $D \leq 0$

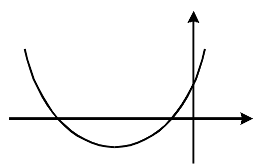
$\Rightarrow$ $(a + 1)^2 - 4 \leq 0$

$\Rightarrow$ $(a + 5)(a - 3) \leq 0$

$\Rightarrow$ $a \in [-5, 3]$

or

Case - (ii) : both roots are non positive



$D \geq 0$ & $-\frac{b}{2a} < 0$ & $f(0) \geq 0$

$\Rightarrow$ $a \in (-\infty, -5] \cup [3, \infty)$ & $\frac{a+1}{4} < 0$ & $2 \geq 0$

$\Rightarrow$ $a \in (-\infty, -5] \cup [3, \infty)$ & $a < -1$ & $a \in R$

$\Rightarrow$ $a \in (-\infty, -5]$

Taking union of (i) and (ii), we get $a \in (-\infty, 3]$.

Problems for Self Practice-7 :

(1) Find the intervals of monotonicity of the following functions.

(i) $f(x) = -x^3 + 6x^2 - 9x - 2$ (ii) $f(x) = x + \dfrac{1}{x+1}$

(iii) $f(x) = x \,.\, e^{x-x^2}$ (iv) $f(x) = x - \cos x$

(2) Let $f(x) = x - \tan^{-1}x$. Prove that f(x) is monotonically increasing for $x \in R$.

(3) If $f(x) = 2e^x - ae^{-x} + (2a + 1)\,x - 3$ monotonically increases for $\forall\, x \in R$, then find range of values of a

(4) Let $f(x) = e^{2x} - ae^x + 1$. Prove that f(x) cannot be monotonically decreasing for $\forall\, x \in R$ for any value of 'a'.

(5) The values of 'a' for which function $f(x) = (a + 2)\,x^3 - ax^2 + 9ax - 1$ monotonically decreasing for $\forall$ $x \in R$.

Answers : (1) (i) I in $[1, 3]$; D in $(-\infty, 1] \cup (3, \infty)$

(ii) I in $(-\infty, -2] \cup [0, \infty)$; D in $[-2, -1) \cup (-1, 0]$

(iii) I in $\left[-\dfrac{1}{2},\ 1\right]$; D in $\left(-\infty, -\dfrac{1}{2}\right] \cup [1, \infty)$

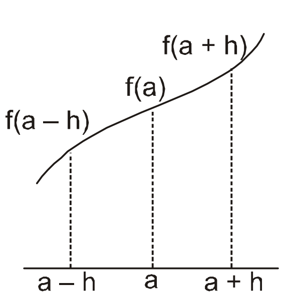
(iv) I for $x \in R$

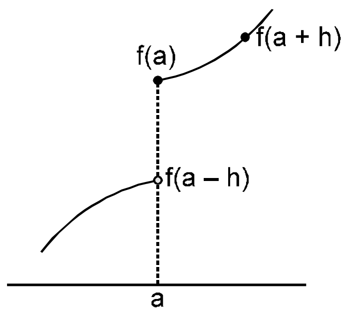
(3) $a \geq 0$

(5) $-\infty < a \leq -3$

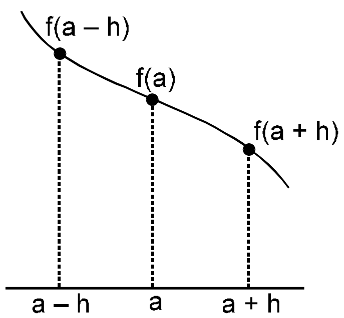
8. MONOTONICITY OF FUNCTION ABOUT A POINT :

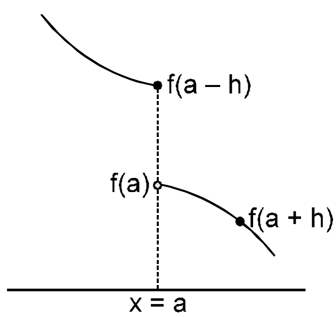
1. A function f(x) is called as a strictly increasing function about a point (or at a point) $a \in D_f$ if it is strictly increasing in an open interval containing a (as shown in figure).



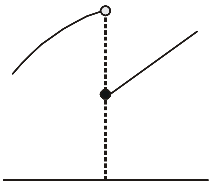


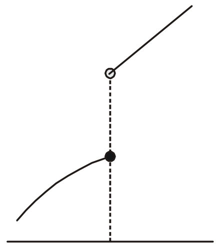
2. A function f(x) is called a strictly decreasing function about a point x = a, if it is strictly decreasing in an open interval containing a (as shown in figure).

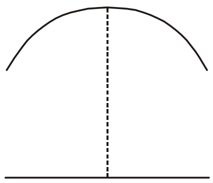


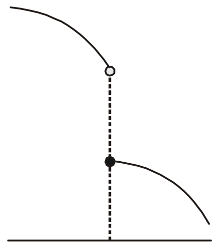


e.g. : Which of the following functions (as shown in figure) is increasing, decreasing or neither increasing nor decreasing at x = a.

(i)

x = a
neither increasing nor decreasing at x = a

(ii)

x = a
Increasing at x = a

(iii)

x = a
neither increasing nor decreasing at x = a

(iv)

x = a
decreasing at x = a

8.1 Test for increasing and decreasing for differentiable functions about a point

Let f(x) be differentiable.

(1) If $f'(a) > 0$ then f(x) is increasing at x = a.

(2) If $f'(a) < 0$ then f(x) is decreasing at x = a.

(3) If $f'(a) = 0$ then examine the sign of $f'(x)$ on the left neighbourhood and the right neighbourhood of a.

(i) If $f'(x)$ is positive on both the neighbourhoods, then f is increasing at x = a.

(ii) If $f'(x)$ is negative on both the neighbourhoods, then f is decreasing at x = a.

(iii) If $f'(x)$ have opposite signs on these neighbourhoods, then f is non-monotonic at x = a.

SOLVED EXAMPLE

Example # 27: Let $f(x) = x^3 - 3x + 2$. Examine the monotonicity of function at points x = 0, 1, 2.

Solution : $f(x) = x^3 - 3x + 2$

$f'(x) = 3(x^2 - 1)$

(i) $f'(0) = -3 \Rightarrow$ decreasing at x = 0

(ii) $f'(1) = 0$

also, $f'(x)$ is positive on left neighbourhood and $f'(x)$ is negative in right neighbourhood.

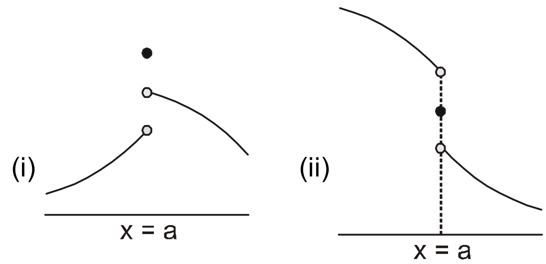
$\Rightarrow$ neither increasing nor decreasing at x = 1.

(iii) $f'(2) = 9 \Rightarrow$ increasing at x = 2

Note : Above method is applicable only for functions those are continuous at x = a.

Problems for Self Practice-8 :

(1) For each of the following graph comment on monotonicity of f(x) at x = a.



(2) Let $f(x) = x^3 - 3x^2 + 3x + 4$, comment on the monotonic behaviour of f(x) at (i) x = 0 (ii) x = 1.

(3) Draw the graph of function $f(x) = \begin{cases} x & 0 \le x \le 1 \\ [x] & 1 \le x \le 2 \end{cases}$. Graphically comment on the monotonic behaviour of f(x) at x = 1. Is f(x) M.I. for $x \in [0, 2]$?

Answers :
(1) (i) neither M.I. nor M.D. (ii) M.D.
(2) M.I. both at x = 0 and x = 1.
(3) Neither M.I. nor M.D. at x = 1. No, f(x) is not M.I. for $x \in [0, 2]$.

9. USE OF MONOTONICITY FOR PROVING INEQUALITIES

Comparison of two functions f(x) and g(x) can be done by analysing the monotonic behaviour of h(x) = f(x) – g(x)

SOLVED EXAMPLE

Example # 28 : For $x \in \left(0, \frac{\pi}{2}\right)$ prove that sin x < x < tan x

Solution : Let $f(x) = x - \sin x \Rightarrow f'(x) = 1 - \cos x$

$f'(x) > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ f(x) is M.I. $\Rightarrow f(x) > f(0)$

$\Rightarrow x - \sin x > 0 \Rightarrow x > \sin x$

Similarly consider another function $g(x) = x - \tan x \Rightarrow g'(x) = 1 - \sec^2 x$

$g'(x) < 0$ for $x \in \left(0, \frac{\pi}{2}\right) \Rightarrow$ g(x) is M.D.

Hence $g(x) < g(0)$

$x - \tan x < 0 \Rightarrow x < \tan x$

sin x < x < tan x Hence proved

Example # 29 : For $x \in (0, 1)$ prove that $x - \frac{x^3}{3} < \tan^{-1} x < x - \frac{x^3}{6}$ hence or otherwise find $\lim_{x \to 0} \left[\frac{\tan^{-1} x}{x}\right]$

Solution : Let $f(x) = x - \frac{x^3}{3} - \tan^{-1}x$

$f(x) = 1 - x^2 - \frac{1}{1+x^2}$

$f'(x) = -\frac{x^4}{1+x^2}$

$f'(x) < 0$ for $x \in (0, 1)$ $\Rightarrow$ $f(x)$ is M.D.

$\Rightarrow$ $f(x) < f(0)$ $\Rightarrow$ $x - \frac{x^3}{3} - \tan^{-1}x < 0$

$\Rightarrow$ $x - \frac{x^3}{3} < \tan^{-1}x$ ....(i)

Similarly $g(x) = x - \frac{x^3}{6} - \tan^{-1}x$

$g'(x) = 1 - \frac{x^2}{2} - \frac{1}{1+x^2}$

$g'(x) = \frac{x^2(1-x^2)}{2(1+x^2)}$

$g'(x) > 0$ for $x \in (0, 1)$ $\Rightarrow$ $g(x)$ is M.I.

$\Rightarrow$ $g(x) > g(0)$

$x - \frac{x^3}{6} - \tan^{-1}x > 0$

$x - \frac{x^3}{6} > \tan^{-1}x$ ....(ii)

from (i) and (ii), we get

$x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6}$ Hence Proved

Also, $1 - \frac{x^2}{3} < \frac{\tan^{-1}x}{x} < 1 - \frac{x^2}{6}$, for $x > 0$

Hence by sandwich theorem we can prove that $\lim_{x\to 0} \frac{\tan^{-1}x}{x} = 1$ but it must also be noted that as $x \to 0$, value of $\frac{\tan^{-1}x}{x} \longrightarrow 1$ from left hand side i.e. $\frac{\tan^{-1}x}{x} < 1$

$\Rightarrow$ $\lim_{x\to 0} \left[\frac{\tan^{-1}x}{x}\right] = 0$

NOTE : In proving inequalities, we must always check when does the equality takes place because the point of equality is very important in this method. Normally point of equality occur at end point of the interval or will be easily predicted by hit and trial.

Example # 30 : For $x \in \left(0, \frac{\pi}{2}\right)$, prove that $\sin x > x - \frac{x^3}{6}$

Solution : Let $f(x) = \sin x - x + \frac{x^3}{6}$

$f'(x) = \cos x - 1 + \frac{x^2}{2}$

we cannot decide at this point whether $f'(x)$ is positive or negative, hence let us check for monotonic nature of $f'(x)$

$f''(x) = x - \sin x$

Since $f''(x) > 0 \Rightarrow f'(x)$ is M.I. for $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f'(x) > f'(0) \Rightarrow f'(x) > 0$

$\Rightarrow f(x)$ is M.I. $\Rightarrow f(x) > f(0)$

$\Rightarrow \sin x - x + \frac{x^3}{6} > 0 \Rightarrow \sin x > x - \frac{x^3}{6}$. Hence proved

Example # 31 : Examine which is greater : sin x tan x or x^2. Hence evaluate $\lim_{x\to 0}\left[\frac{\sin x \tan x}{x^2}\right]$, where $x \in \left(0, \frac{\pi}{2}\right)$

Solution : Let $f(x) = \sin x \tan x - x^2$

$f'(x) = \cos x \,.\, \tan x + \sin x \,.\, \sec^2 x - 2x$

$\Rightarrow f'(x) = \sin x + \sin x \sec^2 x - 2x$

$\Rightarrow f''(x) = \cos x + \cos x \sec^2 x + 2\sec^2 x \sin x \tan x - 2$

$\Rightarrow f''(x) = (\cos x + \sec x - 2) + 2\sec^2 x \sin x \tan x$

Now $\cos x + \sec x - 2 = \left(\sqrt{\cos x} - \sqrt{\sec x}\right)^2$ and $2\sec^2 x \tan x \,.\, \sin x > 0$ because $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow f''(x) > 0 \Rightarrow f'(x)$ is M.I.

Hence $f'(x) > f'(0)$

$\Rightarrow f'(x) > 0 \Rightarrow f(x)$ is M.I.

$\Rightarrow f(x) > 0 \Rightarrow \sin x \tan x - x^2 > 0$

Hence $\sin x \tan x > x^2$

$\Rightarrow \frac{\sin x \tan x}{x^2} > 1 \Rightarrow \lim_{x\to 0}\left[\frac{\sin x \tan x}{x^2}\right] = 1.$

Example # 32 : Prove that $f(x) = \left(1+\frac{1}{x}\right)^x$ is monotonically increasing in its domain. Hence or otherwise draw graph of f(x) and find its range

Solution : $f(x) = \left(1+\frac{1}{x}\right)^x$, for Domain of f(x), $1 + \frac{1}{x} > 0$

$\Rightarrow \frac{x+1}{x} > 0 \Rightarrow (-\infty, -1) \cup (0, \infty)$

Consider $f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) + \frac{x}{1+\frac{1}{x}}\ \frac{-1}{x^2}\right]$

$\Rightarrow \quad f'(x) = \left(1+\frac{1}{x}\right)^x \left[\ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\right]$

Now $\left(1+\frac{1}{x}\right)^x$ is always positive, hence the sign of $f'(x)$ depends on sign of $\ell n\left(1+\frac{1}{x}\right) - \frac{1}{1+x}$

i.e. we have to compare $\ell n\left(1+\frac{1}{x}\right)$ and $\frac{1}{1+x}$

So lets assume $g(x) = \ell n\left(1+\frac{1}{x}\right) - \frac{1}{x+1}$

$g'(x) = \frac{1}{1+\frac{1}{x}}\ \frac{-1}{x^2} + \frac{1}{(x+1)^2} \quad \Rightarrow \quad g'(x) = \frac{-1}{x(x+1)^2}$

(i) for $x \in (0, \infty)$, $g'(x) < 0$ $\Rightarrow$ $g(x)$ is M.D. for $x \in (0, \infty)$

$g(x) > \lim\limits_{x\to\infty} g(x)$

$g(x) > 0.$ and since $g(x) > 0$ $\Rightarrow$ $f'(x) > 0$

(ii) for $x \in (-\infty, -1)$, $g'(x) > 0$ $\Rightarrow$ $g(x)$ is M.I. for $x \in (-\infty, -1)$

$\Rightarrow$ $g(x) > \lim\limits_{x\to-\infty} g(x)$ $\Rightarrow$ $g(x) > 0$ $\Rightarrow$ $f'(x) > 0$

Hence from (i) and (ii) we get $f'(x) > 0$ for all $x \in (-\infty, -1) \cup (0, \infty)$

$\Rightarrow$ f(x) is M.I. in its Domain

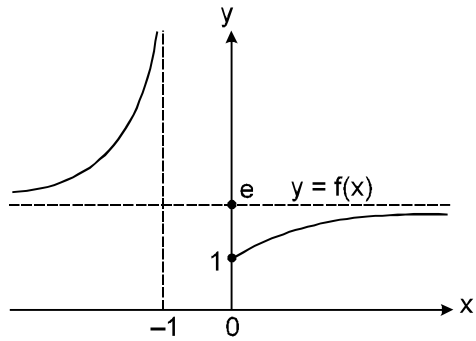
For drawing the graph of f(x), its important to find the value of f(x) at boundary points

i.e. $\pm\infty, 0, -1$

$\lim\limits_{x\to\pm\infty} \left(1+\frac{1}{x}\right)^x = e$

$\lim\limits_{x\to 0^+} \left(1+\frac{1}{x}\right)^x = 1$ and $\lim\limits_{x\to -1} \left(1+\frac{1}{x}\right)^x = \infty$

so the graph of f(x) is



Range is $y \in (1, \infty) - \{e\}$

Example # 33 : Compare which of the two is greater $(100)^{1/100}$ or $(101)^{1/101}$.

Solution : Assume $f(x) = x^{1/x}$ and let us examine monotonic nature of f(x)

$f'(x) = x^{1/x} . \left(\frac{1-\ell nx}{x^2}\right)$

$f'(x) > 0 \quad \Rightarrow \quad x \in (0,e)$

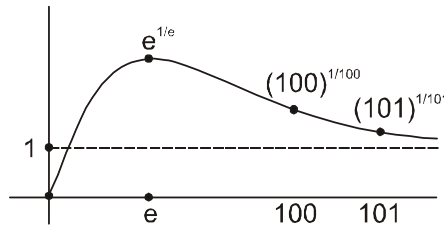
and $f'(x) < 0 \quad \Rightarrow \quad x \in (e,\infty)$

Hence f(x) is M.D. for $x \geq e$

and since 100 < 101

$\Rightarrow$ $f(100) > f(101)$

$\Rightarrow$ $(100)^{1/100} > (101)^{1/101}$



Problems for Self Practice-9 :

(1) Prove the following inequalities

(i) $x < -\ell n(1-x)$ for $x \in (0, 1)$

(ii) $x > \tan^{-1}(x)$ for $x \in (0, \infty)$

(iii) $e^x > x + 1$ for $x \in (0, \infty)$

(iv) $\frac{x}{1+x} \le \ell n(1+x) \le x$ for $x \in (0, \infty)$

(v) $\frac{2}{\pi} < \frac{\sin x}{x} < 1$ for $x \in \left(0, \frac{\pi}{2}\right)$

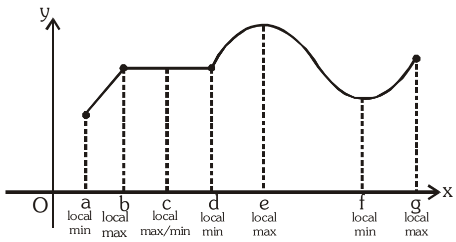
10. MAXIMA & MINIMA :

(a) Local Maxima/Relative maxima :

A function $f(x)$ is said to have a local maxima at x = a

if $f(a) \geq f(x)\ \forall\ x \in (a-h, a+h) \cap D_{f(x)}$

Where h is some positive real number.



(b) Local Minima/Relative minima :

A function $f(x)$ is said to have a local minima at x = a if $f(a) \le f(x)\ \forall\ x \in (a-h, a+h) \cap D_{f(x)}$

Where h is some positive real number.

(c) Absolute maxima (Global maxima) :

A function f has an absolute maxima (or global maxima) at c if $f(c) \ge f(x)$ for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D.

(d) Absolute minima (Global minima) :

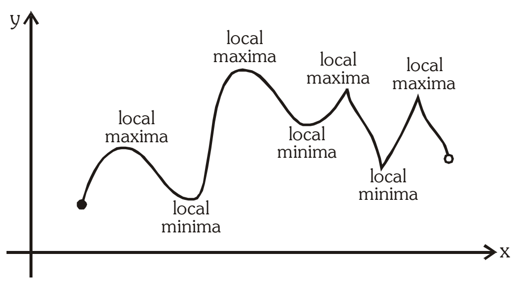
A function f has an absolute minima at c if $f(c) \le f(x)$ for all x in D and the number f(c) is called the minimum value of f on D.

Note :

(i) The term 'extrema' is used for both maxima or minima.

(ii) A local maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iii) A function can have several extreme values such that local minimum value may be greater than a local maximum value.



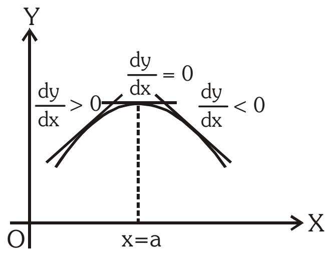
11. DERIVATIVE TEST FOR ASCERTAINING MAXIMA/MINIMA :

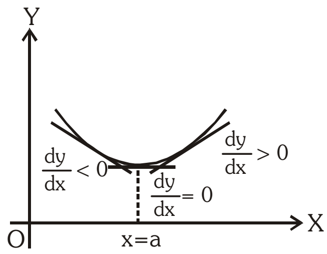
11.1 First derivative test :

If $f'(x) = 0$ at a point (say x = a) and

(i) If f'(x) changes sign from positive to negative in the neighbourhood of x = a then x = a is said to be a point **local maxima.**

(ii) If f'(x) changes sign from negative to positive in the neighbourhood of x = a then x = a is said to be a point **local minima.**





Note : If f '(x) does not change sign i.e. has the same sign in a certain complete neighbourhood of a, then f(x) is either increasing or decreasing throughout this neighbourhood implying that x=a is not a point of extremum of f.

SOLVED EXAMPLE

Example # 34 : Let $f(x) = x + \frac{1}{x}$; $x \neq 0$. Discuss the local maximum and local minimum values of f(x).

Solution : Here, $f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}$

Using number line rule, f(x) will have local maximum at x = –1 and local minimum at x = 1

$\therefore$ local maximum value of f(x) = –2 at x = –1

and local minimum value of f(x) = 2 at x = 1

Example # 35 : If $f(x) = \begin{cases} 3x^2+12x-1, & -1 \le x \le 2 \\ 37-x, & 2 < x \le 3, \end{cases}$, then

(A) f(x) is increasing on [–1, 2)
(B) f(x) is continuous on [–1, 3]
(C) f'(x) does not exist at x = 2
(D) f(x) has the maximum value at x = 2

Solution : Given,

$$f(x) = \begin{cases} 3x^2+12x-1, & -1 \le x \le 2 \\ 37-x, & 2 < x \le 3 \end{cases}$$

$\Rightarrow$

$$f'(x) = \begin{cases} 6x+12, & -1 \le x \le 2 \\ -1, & 2 < x \le 3 \end{cases}$$

(A) which shows $f'(x) > 0$ for $x \in [-1, 2)$
So, $f(x)$ is increasing on $[-1, 2)$
Hence, (A) is correct.

(B) for continuity of $f(x)$. (check at $x = 2$)
RHL = 35, LHL = 35 and $f(2) = 35$
So, (B) is correct

(C) $Rf'(2) = -1$ and $Lf'(2) = 24$
so, not differentiable at $x = 2$.
Hence, (C) is correct.

(D) we know $f(x)$ is increasing on $[-1, 2)$ and decreasing on $(2, 3]$,
Thus maximum at $x = 2$,
Hence, (D) is correct.

$\therefore$ (A), (B), (C), (D) all are correct.

Example # 36 : Let $f(x) = x^3 + 3(a - 7)x^2 + 3(a^2 - 9)x - 1$. If $f(x)$ has positive point of maxima, then find possible values of 'a'.

Solution. $f'(x) = 3[x^2 + 2(a - 7)x + (a^2 - 9)]$
Let α, β be roots of $f'(x) = 0$ and let α be the smaller root. Examining sign change of $f'(x)$.

$$\underset{\qquad}{+}\;\underset{\alpha}{|}\;\;-\;\;\underset{\beta}{|}\;\underset{\qquad}{+}$$

Maxima occurs at smaller root α which has to be positive. This basically implies that both roots of $f'(x) = 0$ must be positive and distinct.

(i) $D > 0 \Rightarrow a < \dfrac{29}{7}$

(ii) $-\dfrac{b}{2a} > 0 \Rightarrow a < 7$

(iii) $f'(0) > 0 \Rightarrow a \in (-\infty, -3) \cup (3, \infty)$

from (i), (ii) and (iii) $\Rightarrow a \in (-\infty, -3) \cup \left(3, \dfrac{29}{7}\right)$

Problems for Self Practice-10 :

(1) Find local maxima and local minima for the function $f(x) = x^3 - 3x$.

(2) If function $f(x) = x^3 - 62x^2 + ax + 9$ has local maxima at $x = 1$, then find the value of a.

(3) Let $f(x) = \dfrac{x}{2} + \dfrac{2}{x}$. Find local maximum and local minimum value of $f(x)$. Can you explain this discrepancy of locally minimum value being greater than locally maximum value.

(4) If $f(x) = \begin{cases} (x+\lambda)^2 & x < 0 \\ \cos x & x \ge 0 \end{cases}$, find possible values of λ such that $f(x)$ has local maxima at $x = 0$.

Answers :
(1) local max. at $x = -1$, local min. at $x = 1$
(2) 121
(3) Local maxima at $x = -2$, $f(-2) = -2$; Local minima at $x = 2$, $f(2) = 2$.
(4) $\lambda \in [-1, 1)$

11.2 Maxima, Minima by higher order derivatives :

11.2.1 Second derivative test :

Let f(x) have derivatives up to second order

Step - I. Find $f'(x)$

Step - II. Solve $f'(x) = 0$. Let $x = c$ be a solution

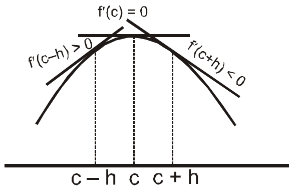
Step - III. Find $f''(c)$

Step - IV.

(i) If $f''(c) = 0$ then further investigation is required

(ii) If $f''(c) > 0$ then $x = c$ is a point of minima.

(iii) If $f''(c) < 0$ then $x = c$ is a point of maxima.



For maxima $f'(x)$ changes from positive to negative (as shown in figure).

$\Rightarrow$ $f'(x)$ is decreasing hence $f''(c) < 0$

SOLVED EXAMPLE

Example # 37 : Find the points of local maxima or minima for $f(x) = \sin 2x - x$, $x \in (0, \pi)$.

Solution. $f(x) = \sin 2x - x$

$f'(x) = 2\cos 2x - 1$

$f'(x) = 0 \quad \Rightarrow \quad \cos 2x = \frac{1}{2} \quad \Rightarrow \quad x = \frac{\pi}{6}, \frac{5\pi}{6}$

$f''(x) = -4 \sin 2x$

$f''\left(\frac{\pi}{6}\right) < 0 \quad \Rightarrow \quad$ Maxima at $x = \frac{\pi}{6}$

$f''\left(\frac{5\pi}{6}\right) > 0 \quad \Rightarrow \quad$ Minima at $x = \frac{5\pi}{6}$

11.2.2 n^{th} Derivative test :

Let f(x) have derivatives up to nth order

If $f'(c) = f''(c) = \ldots\ldots\ldots = f^{n-1}(c) = 0$ and

$f^n(c) \neq 0$ then we have following possibilities

(i) n is even, $f^{(n)}(c) < 0 \Rightarrow x = c$ is point of maxima

(ii) n is even, $f^{(n)}(c) > 0 \Rightarrow x = c$ is point of minima.

(iii) n is odd, $f^{(n)}(c) < 0 \Rightarrow f(x)$ is decreasing about $x = c$

(iv) n is odd, $f^{(n)} > 0 \Rightarrow f(x)$ is increasing about $x = c$.

SOLVED EXAMPLE

Example # 38 : Find points of local maxima or minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Solution. $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^2 (x - 1)(x - 3)$

$f'(x) = 0 \quad \Rightarrow \quad x = 0, 1, 3$

$f''(x) = 10x (2x^2 - 6x + 3)$

Now, $f''(1) < 0$ $\Rightarrow$ Maxima at x = 1

$f''(3) > 0$ $\Rightarrow$ Minima at x = 3

and, $f''(0) = 0$ $\Rightarrow$ II[nd] derivative test fails

so, $f'''(x) = 30\,(2x^2 - 4x + 1)$

$f'''(0) = 30$

$\Rightarrow$ Neither maxima nor minima at x = 0.

Note : It was very convenient to check maxima/minima at first step by examining the sign change of $f'(x)$ no sign change of $f'(x)$ at x = 0

$f'(x) = 5x^2\,(x - 1)\,(x - 3)$

$$\begin{array}{ccccccc} + & & + & & - & & + \\ \hline & 0 & & 1 & & 3 & \\ & & & \text{max.} & & \text{min.} & \end{array}$$

Example # 39 : If $f(x) = 2x^3 - 3x^2 - 36x + 6$ has local maximum and minimum at x = a and x = b respectively, then find ordered pair (a, b)

Solution : $f(x) = 2x^3 - 3x^2 - 36x + 6$

$f'(x) = 6x^2 - 6x - 36$ & $f''(x) = 12x - 6$

Now $f'(x) = 0 \Rightarrow 6(x^2 - x - 6) = 0 \Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = -2, 3$

$f''(-2) = -30$

$\therefore$ x = –2 is a point of local maximum

$f''(3) = 30$

$\therefore$ x = 3 is a point of local minimum

Hence, (–2, 3) is the required ordered pair.

Example # 40 : Find the point of local maxima of $f(x) = \sin x\,(1+\cos x)$ in $x \in (0, \pi/2)$.

Solution : Let $f(x) = \sin x\,(1 + \cos x) = \sin x + \dfrac{1}{2}\sin 2x$

$\Rightarrow$ $f'(x) = \cos x + \cos 2x$

$f''(x) = -\sin x - 2\sin 2x$

Now $f'(x) = 0 \Rightarrow \cos x + \cos 2x = 0$

$\Rightarrow$ $\cos 2x = \cos(\pi - x) \Rightarrow x = \pi/3$

Also $f''(\pi/3) = -\sqrt{3}/2 - \sqrt{3} < 0$ $\therefore$ f(x) has a maxima at $x = \pi/3$ **Ans.**

<u>Problems for Self Practice</u>-11 :

(1) Find local maximum value of function $f(x) = \dfrac{\ell nx}{x}$

(2) If $f(x) = x^2e^{-2x}$ (x > 0), then find the local maximum value of f(x).

(3) Identify the point of local maxima/minima in $f(x) = (x - 3)^{10}$.

(4) Find the points of local maxima or minima of $f(x) = \sin 2x - x$

(5) Let $f(x) = \sin x\,(1 + \cos x)$; $x \in (0, 2\pi)$. Find the number of critical points of f(x). Also identify which of these critical points are points of Maxima/Minima.

Answers : (1) 1/e (2) $1/e^2$ (3) local minima at x = 3

(4) Maxima at $x = n\pi + \dfrac{\pi}{6}$; Minima at $x = n\pi - \dfrac{\pi}{6}$

(5) Three

$x = \dfrac{\pi}{3}$ is point of maxima.

$x = \pi$ is not a point of extrema.

$x = \dfrac{5\pi}{3}$ is point of minima.

11.3 Global extrema for continuous functions :

(i) Function defined on closed interval

Let f(x), $x \in [a, b]$ be a continuous function

Step - I : Find critical points. Let it be $c_1, c_2, \ldots, c_n$

Step - II : Find $f(a), f(c_1), \ldots, f(c_n), f(b)$

Let $M = \max\cdot\{f(a), f(c_1), \ldots, f(c_n), f(b)\}$

$m = \min\cdot\{f(a), f(c_1), \ldots f(c_n), f(b)\}$

Step - III : M is global maximum.

m is global minimum.

(ii) Function defined on open interval.

Let f(x), $x \in (a, b)$ be continuous function.

Step - I Find critical points . Let it be $c_1, c_2, \ldots c_n$

Step - II Find $f(c_1), f(c_2), \ldots, f(c_n)$

Let $M = \max\cdot\{f(c_1), \ldots f(c_n)\}$

$m = \min\cdot\{f(c_1), \ldots, f(c_n)\}$

Step - III $\lim\limits_{x\to a^+} f(x) = \ell_1$ (say), $\lim\limits_{x\to b^-} f(x) = \ell_2$ (say).

Let $\ell = \min.\{\ell_1, \ell_2\}$, $L = \max.\{\ell_1, \ell_2\}$

Step - IV

(i) If $m \le \ell$ then m is global minimum

(ii) If $m > \ell$ then f(x) has no global minimum

(iii) If $M \ge L$ then M is global maximum

(iv) If $M < L$, then f(x) has no global maximum

SOLVED EXAMPLE

Example # 41 : Find the greatest and least values of $f(x) = x^3 - 12x$ $x \in [-1, 3]$

Solution. The possible points of maxima/minima are critical points and the boundary points.

for $x \in [-1, 3]$ and $f(x) = x^3 - 12x$

x = 2 is the only critical point.

Examining the value of f(x) at points x = –1, 2, 3. We can find greatest and least values.

x	f(x)
–1	11
2	–16
3	–9

$\therefore$ Minimum f(x) = – 16 & Maximum f(x) = 11.

Example # 42 : Find the global maximum and global minimum of $f(x) = \dfrac{e^x + e^{-x}}{2}$ in $[-\log_e 2, \log_e 7]$.

Solution : $f(x) = \dfrac{e^x + e^{-x}}{2}$ is differentiable at all x in its domain.

Then $f'(x) = \dfrac{e^x - e^{-x}}{2}$, $f''(x) = \dfrac{e^x + e^{-x}}{2}$

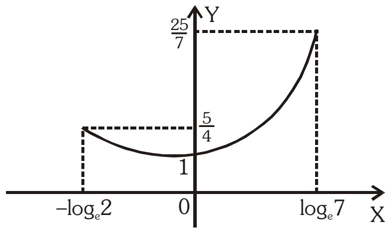
$f'(x) = 0 \Rightarrow \dfrac{e^x - e^{-x}}{2} = 0 \Rightarrow e^{2x} = 1 \Rightarrow x = 0$

$f''(0) = 1$ $\therefore$ x = 0 is a point of local minimum

Points $x = -\log_e 2$ and $x = \log_e 7$ are extreme points.

Now, check the value of f(x) at all these three points $x = -\log_e 2, 0, \log_e 7$

$\Rightarrow$ $f(-\log_e 2) = \dfrac{e^{-\log_e 2} + e^{+\log_e 2}}{2} = \dfrac{5}{4}$



$f(0) = \dfrac{e^{0} + e^{-0}}{2} = 1$

$f(\log_e 7) = \dfrac{e^{\log_e 7} + e^{-\log_e 7}}{2} = \dfrac{25}{7}$

$\therefore$ x = 0 is absolute minima & $x = \log_e 7$ is absolute maxima

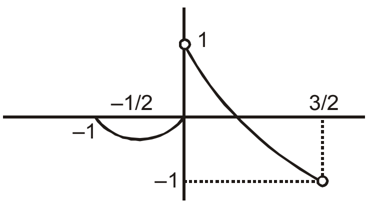
Hence, absolute/global minimum value of f(x) is 1 at x = 0

and absolute/global maximum value of f(x) is $\dfrac{25}{7}$ at $x = \log_e 7$ **Ans.**

Example # 43 : Let $f(x) = \begin{cases} x^2 + x & ; \ -1 \le x < 0 \\ \lambda & ; \ x = 0 \\ \log_{1/2}\left(x + \frac{1}{2}\right) & ; \ 0 < x < \frac{3}{2} \end{cases}$

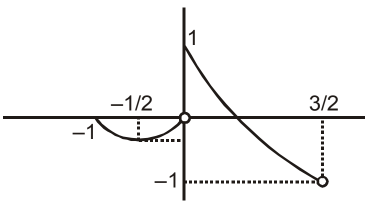
Discuss global maxima, minima for $\lambda = 0$ and $\lambda = 1$. For what values of λ does f(x) has global maxima

Solution : Graph of y = f(x) for $\lambda = 0$



No global maxima, minima

Graph of y = f(x) for $\lambda = 1$



Global maxima is 1, which occurs at x = 0

Global minima does not exists

$\lim\limits_{x \to 0^-} f(x) = 0$, $\lim\limits_{x \to 0^+} f(x) = 1$, $f(0) = \lambda$

For global maxima to exists

$f(0) \ge 1 \qquad \Rightarrow \qquad \lambda \ge 1.$

Example # 44 : Find extrema of $f(x) = 3x^4 + 8x^3 - 18x^2 + 60$. Draw graph of $g(x) = \dfrac{40}{f(x)}$ and comment on its local and global extrema.

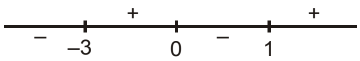
Solution : $f'(x) = 0$

$\Rightarrow$ $12x(x^2 + 2x - 3) = 0$

$\Rightarrow$ $12x(x - 1)(x + 3) = 0$

$\Rightarrow$ $x = -3, 0, 1$

$f'(x) = 12(x + 3)\,x(x - 1)$



local minima occurs at x = –3, 1

local maxima occurs at x = 0

f(–3) = – 75, f(1) = 53 are local minima

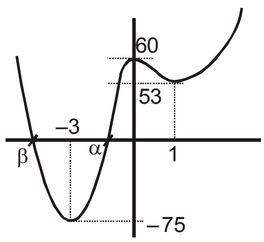
f(0) = 60 is local maxima

$\lim\limits_{x\to\infty} f(x) = \infty$, $\lim\limits_{x\to-\infty} f(x) = \infty$

Hence global maxima does not exists : Global minima is – 75

$g'(x) = \dfrac{-40}{(f(x))^2}\, f'(x)$ $\Rightarrow$ g(x) has same critical points as that of f(x).

A rough sketch of y = f(x) is



Let zeros of f(x) be α, β

$g(\alpha)$, $g(\beta)$ are undefined,

$\lim\limits_{x\to\beta^-} g(x) = \infty$, $\lim\limits_{x\to\beta^+} g(x) = -\infty$, $\lim\limits_{x\to\alpha^-} g(x) = -\infty$, $\lim\limits_{x\to\alpha^+} g(x) = \infty$

$x = \alpha$, $x = \beta$ are asymptotes of y = g(x).

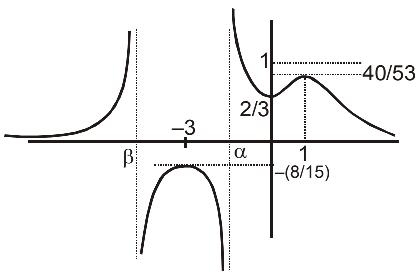
$\lim\limits_{x\to\infty} g(x) = 0$, $\lim\limits_{x\to-\infty} g(x) = 0$

$\Rightarrow$ y = 0 is also an asymptote. $\because$ x = –3, x = 1 are local minima of y = f(x) $\Rightarrow$ x = – 3, x = 1 are local maxima of y = g(x)

similarly, x = 0 is local minima of y = g(x)

Global extrema of g(x) does not exists.

A rough sketch of y = g(x) is



Problems for Self Practice-12 :

(1) Let $f(x) = 2x^3 - 9x^2 + 12x + 6$

(i) Find the possible points of Maxima/Minima of f(x) for $x \in R$.

(ii) Find the number of critical points of f(x) for $x \in [0, 2]$.

(iii) Discuss absolute (global) maxima/minima value of f(x) for $x \in [0, 2]$

(iv) Prove that for $x \in (1, 3)$, the function does not has a Global maximum.

Answers :

(1) (i) x = 1, 2 (ii) one

(iii) f(0) = 6 is the global minimum, f(1) = 11 is global maximum

12. USEFUL FORMULAE OF MENSURATION TO REMEMBER :

1. Volume of a cuboid = ℓbh.

2. Surface area of cuboid = $2(\ell b + bh + h\ell)$.

3. Volume of cube = a^3

4. Surface area of cube = $6a^2$

5. Volume of a cone = $\frac{1}{3}\pi r^2 h$.

6. Curved surface area of cone = $\pi r\ell$ (ℓ = slant height)

7. Curved surface area of a cylinder = $2\pi rh$.

8. Total surface area of a cylinder = $2\pi rh + 2\pi r^2$.

9. Volume of a sphere = $\frac{4}{3}\pi r^3$.

10. Surface area of a sphere = $4\pi r^2$.

11. Area of a circular sector = $\frac{1}{2} r^2 \theta$, when θ is in radians.

12. Volume of a prism = (area of the base) × (height).

13. Lateral surface area of a prism = (perimeter of the base) × (height).

14. Total surface area of a prism = (lateral surface area) + 2 (area of the base)
(Note that lateral surfaces of a prism are all rectangle).

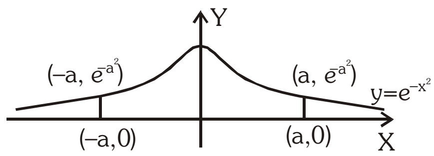
15. Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).

16. Curved surface area of a pyramid = $\frac{1}{2}$ (perimeter of the base) × (slant height).
(Note that slant surfaces of a pyramid are triangles).

SOLVED EXAMPLE

Example # 45 : Determine the largest area of the rectangle whose base is on the x-axis and two of its vertices lie on the curve y = e^{-x^2}.

Solution : Area of the rectangle will be A = 2a. e^{-a^2}



For max. area, $\frac{dA}{da} = \frac{d}{da}(2ae^{-a^2}) = e^{-a^2}[2 - 4a^2]$

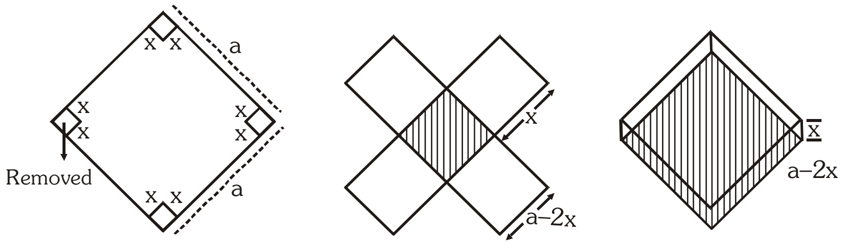
$$\frac{dA}{da} = 0 \Rightarrow a = \pm\frac{1}{\sqrt{2}}$$

& sign of $\frac{dA}{da}$ changes from positive to negative at $a = +\frac{1}{\sqrt{2}}$

$\Rightarrow \quad x = \frac{1}{\sqrt{2}}$ are points of maxima $\Rightarrow A_{max} = \frac{2}{\sqrt{2}} \cdot e^{-\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{\sqrt{2}}{e^{1/2}}$ sq units. **Ans.**

Example # 46 : A box of maximum volume with top open is to be made by cutting out four equal squares from four corners of a square tin sheet of side length a ft, and then folding up the flaps. Find the side of the square base cut off.

Solution : Volume of the box is, $V = x(a - 2x)^2$ i.e., squares of side x are cut out then we will get a box with a square base of side (a – 2x) and height x.



$\therefore \quad \frac{dV}{dx} = (a - 2x)^2 + x \cdot 2(a - 2x)(-2)$

$\frac{dV}{dx} = (a - 2x)(a - 6x)$

For V to be extremum $\frac{dV}{dx} = 0 \quad \Rightarrow \quad x = a/2, a/6$

But when x = a/2; V = 0 (minimum) and we know minimum and maximum occurs alternately in a continuous function.

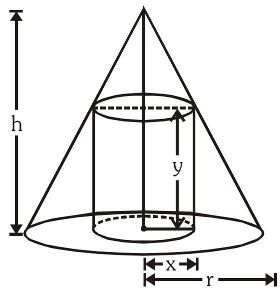
Hence, V is maximum when x = a/6. **Ans.**

Example #47 : If a right circular cylinder is inscribed in a given cone. Find the dimension of the cylinder such that its volume is maximum.

Solution : Let x be the radius of cylinder and y be its height

$V = \pi x^2 y$

x, y can be related by using similar triangles



$\frac{y}{r - x} = \frac{h}{r} \quad \Rightarrow \quad y = \frac{h}{r}(r - x)$

$\Rightarrow \quad V(x) = \pi x^2 \frac{h}{r}(r - x) \quad \Rightarrow \quad V(x) = \frac{\pi h}{r}(rx^2 - x^3)$

$V'(x) = \frac{\pi h}{r}(2rx - 3x^2)$

$V'(x) = 0 \Rightarrow \quad x = 0, \frac{2r}{3}$

$V''(x) = \frac{\pi h}{r}(2r - 6x)$

$V''(0) = 2\pi h \Rightarrow x = 0$ is point of minima

$V''\left(\frac{2r}{3}\right) = -2\pi h \Rightarrow x = \frac{2r}{3}$ is point of maxima

Thus volume is maximum at $x = \left(\frac{2r}{3}\right)$ and $y = \frac{h}{3}$.

Example # 48 : Find two positive numbers x and y such that x + y = 60 and xy^3 is maximum.

Solution. $x + y = 60$

$\Rightarrow \quad x = 60 - y \quad \Rightarrow \quad xy^3 = (60 - y)y^3$

Let $f(y) = (60 - y)\, y^3 \quad ; \quad y \in (0, 60)$

for maximizing f(y) let us find critical points

$f'(y) = 3y^2 (60 - y) - y^3 = 0$

$f'(y) = y^2 (180 - 4y) = 0$

$\Rightarrow \quad y = 45$

$f'(45^+) < 0$ and $f'(45^-) > 0$. Hence local maxima at y = 45.

So x = 15 and y = 45.

Example # 49 : Rectangles are inscribed inside a semicircle of radius r. Find the rectangle with maximum area.

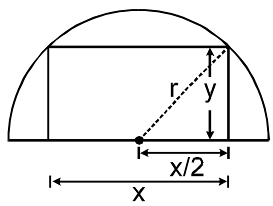
Solution. Let sides of rectangle be x and y (as shown in figure).

$\Rightarrow \quad A = xy.$

Here x and y are not independent variables and are related by Pythogorus theorem with r.

$\frac{x^2}{4} + y^2 = r^2 \quad \Rightarrow \quad y = \sqrt{r^2 - \frac{x^2}{4}}$

$\Rightarrow \quad A(x) = x\sqrt{r^2 - \frac{x^2}{4}}$



$\Rightarrow \quad A(x) = \sqrt{x^2r^2 - \frac{x^4}{4}}$

Let $f(x) = r^2x^2 - \frac{x^4}{4} \quad ; \quad x \in (0, r)$

A(x) is maximum when f(x) is maximum

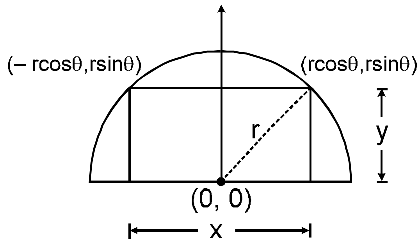
Hence $f'(x) = x(2r^2 - x^2) = 0 \quad \Rightarrow \quad x = r\sqrt{2}$

also $f'(r\sqrt{2^+}) < 0$ and $f'(r\sqrt{2^-}) > 0$

confirming at f(x) is maximum when $x = r\sqrt{2}$ & $y = \frac{r}{\sqrt{2}}$.

Aliter Let us choose coordinate system with origin as centre of circle (as shown in figure).

A = xy



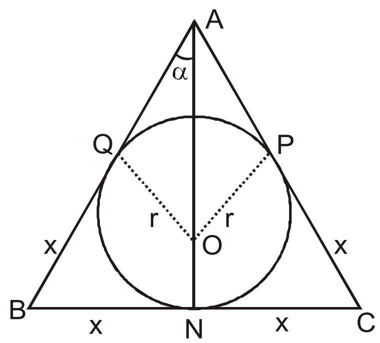
$\Rightarrow$ A = 2 (rcosθ) (rsinθ) $\Rightarrow$ A = r^2 sin2θ, $\theta \in \left(0, \frac{\pi}{2}\right)$

Clearly A is maximum when $\theta = \frac{\pi}{4}$ $\Rightarrow$ $x = r\sqrt{2}$ and $y = \frac{r}{\sqrt{2}}$.

Example # 50. Show that the least perimeter of an isosceles triangle circumscribed about a circle of radius 'r' is $6\sqrt{3}$ r .

Sol. AQ = r cot α = AP

AO = r cosecα



$\frac{x}{AO + ON} = \tan\alpha$

x = (r cosecα + r) tanα

x = r(secα + tanα)

Perimeter = p = 4x + 2AQ

p = 4r(secα + tanα) + 2rcotα

p = r(4secα + 4tanα + 2cotα)

$\frac{dp}{d\alpha} = r[4\sec\alpha\tan\alpha + 4\sec^2\alpha - 2\text{cosec}^2\alpha]$

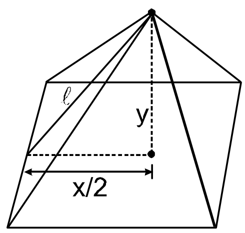
for max or min $\frac{dp}{d\alpha} = 0$ $\Rightarrow$ $2\sin^3\alpha + 3\sin^2\alpha - 1 = 0$ $\Rightarrow$ $(\sin\alpha + 1)(2\sin^2\alpha + \sin\alpha - 1) = 0$

$(\sin\alpha + 1)^2 (2\sin\alpha - 1) = 0$ $\Rightarrow$ sinα = 1/2 $\Rightarrow$ α = 30° = π/6

$$p_{least} = r\left[\frac{4.2}{\sqrt{3}} + \frac{4}{\sqrt{3}} + 2\sqrt{3}\right] = r\left[\frac{8+4+6}{\sqrt{3}}\right] = r\,\frac{\left(6\sqrt{3}\sqrt{3}\right)}{\sqrt{3}} = 6\sqrt{3}\ r$$

Example # 51 : Among all regular square pyramids of volume $36\sqrt{2}$ cm³. Find dimensions of the pyramid having least lateral surface area.

Solution. Let the length of a side of base be x cm and y be the perpendicular height of the pyramid (see figure).



$V = \frac{1}{3} \times$ area of base x height

$\Rightarrow$ $V = \frac{1}{3}x^2y = 36\sqrt{2}$ $\Rightarrow$ $y = \frac{108\sqrt{2}}{x^2}$

and $S = \frac{1}{2} \times$ perimeter of base x slant height

$= \frac{1}{2}(4x).\,\ell$

but $\ell = \sqrt{\frac{x^2}{4} + y^2}$ $\Rightarrow$ $S = 2x\sqrt{\frac{x^2}{4} + y^2} = \sqrt{x^4 + 4x^2y^2}$

$\Rightarrow$ $S = \sqrt{x^4 + 4x^2\left(\frac{108\sqrt{2}}{x^2}\right)^2}$

$S(x) = \sqrt{x^4 + \frac{8.(108)^2}{x^2}}$

Let $f(x) = x^4 + \frac{8.(108)^2}{x^2}$ for minimizing f(x)

$f'(x) = 4x^3 - \frac{16(108)^2}{x^3} = 0$ $\Rightarrow$ $f'(x) = 4\,\frac{(x^6 - 6^6)}{x^3} = 0$

$\Rightarrow$ x = 6, which a point of minima

Hence x = 6 cm and y = $3\sqrt{2}$.

Important note :

(i) If the sum of two real numbers x and y is constant then their product is maximum if they are equal.

i.e. $xy = \frac{1}{4}[(x + y)^2 - (x - y)^2]$

(ii) If the product of two positive numbers is constant then their sum is least if they are equal.

i.e. $(x + y)^2 = (x - y)^2 + 4xy$

Problems for Self Practice-13 :

(1) Find the two positive numbers x and y whose sum is 35 and the product $x^2 y^5$ maximum.

(2) A square piece of tin of side 18 cm is to be made into a box without top by cutting a square from each corner and folding up the slops to form a box. What should be the side of the square to be cut off such that volume of the box is maximum possible.

(3) Prove that a right circular cylinder of given surface area and maximum volume is such that the height is equal to the diameter of the base.

(4) A normal is drawn to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$. Find the maximum distance of this normal from the centre.

(5) A line is drawn passing through point P(1, 2) to cut positive coordinate axes at A and B. Find minimum area of ΔPAB.

(6) Two towns A and B are situated on the same side of a straight road at distances a and b respectively perpendiculars drawn from A and B meet the road at point C and D respectively. The distance between C and D is c. A hospital is to be built at a point P on the road such that the distance APB is minimum. Find position of P.

Answers : (1) x = 25, y = 10. (2) 3 cm (4) 1 unit

(5) 4 units (6) P is at distance of $\frac{ac}{a+b}$ from C.

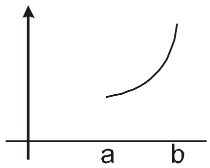
13. CONCAVITY AND POINT OF INFLECTION

A function f(x) is concave upward in (a, b) if tangent drawn at every point (x_0, (f(x_0)), for $x_0 \in$ (a, b) lie below the curve. f(x) is concave downward in (a,b) if tangent drawn at each point (x_0, f(x_0)), $x_0 \in$ (a, b) lie above the curve.

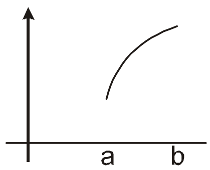
A point (c, f(c)) of the graph y = f(x) is said to be a point of inflection of the graph, if f(x) is concave upward in (c – δ, c) and concave downward in (c, c + δ) (or vice verse), for some $\delta \in R^+$.

Results :

1. If $f''(x) > 0 \ \forall \ x \in (a, b)$, then the curve y = f(x) is concave upward in (a, b)



2. If $f''(x) < 0 \ \forall \ x \in (a, b)$ then the curve y = f(x) is concave downward in (a, b)



3. If f is continuous at x = c and $f''(x)$ has opposite signs on either sides of c, then the point (c, f(c)) is a point of inflection of the curve

4. If $f''(c) = 0$ and $f'''(c) \neq 0$, then the point (c, f(c)) is a point of inflection

13.1 Proving Inequalities using curvature :

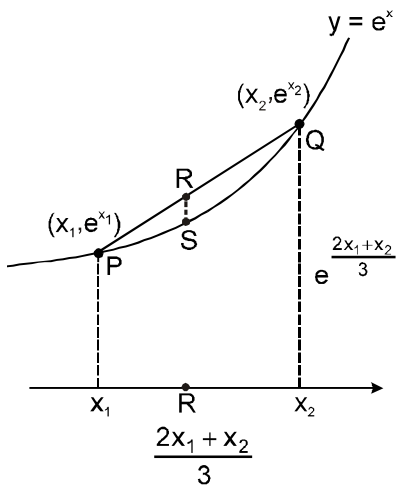
Generally these inequalities involve comparison between values of two functions at some particular points.

SOLVED EXAMPLE

Example # 52 : Prove that for any two numbers x_1 & x_2, $\dfrac{2e^{x_1} + e^{x_2}}{3} > e^{\frac{2x_1+x_2}{3}}$

Solution : Assume f(x) = e^x and let x_1 & x_2 be two points on the curve y = e^x.

Let R be another point which divides $\overline{PQ}$ in ratio 1 : 2.



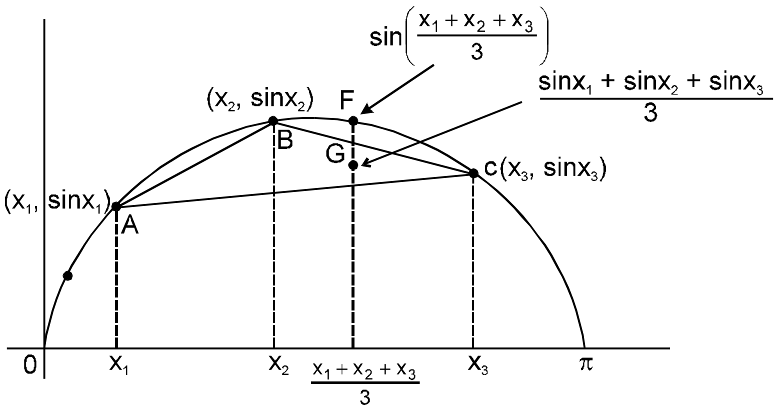
y coordinate of point R is $\frac{2e^{x_1}+e^{x_2}}{3}$ and y coordinate of point S is $e^{\frac{2x_1+x_2}{3}}$. Since f(x) = e^x is concave up, the point R will always be above the point S.

$\Rightarrow \quad \frac{2e^{x_1}+e^{x_2}}{3} > e^{\frac{2x_1+x_2}{3}}$

Alternate : Above inequality could also be easily proved using AM and GM.

Example # 53 : If $0 < x_1 < x_2 < x_3 < \pi$ then prove that $\sin\left(\frac{x_1+x_2+x_3}{3}\right) > \frac{\sin x_1+\sin x_2+\sin x_3}{3}$. Hence prove that : if A, B, C are angles of a triangle then maximum value of sinA + sinB + sinC is $\frac{3\sqrt{3}}{2}$.

Solution :



Point A, B, C form a triangle.

y coordinate of centroid G is $\frac{\sin x_1+\sin x_2+\sin x_3}{3}$ and y coordinate of point F is

$\sin\left(\frac{x_1+x_2+x_3}{3}\right)$. Hence $\sin\left(\frac{x_1+x_2+x_3}{3}\right) \geq \frac{\sin x_1+\sin x_2+\sin x_3}{3}$.

If $\quad A + B + C = \pi$, then

$$\sin\left(\frac{A+B+C}{3}\right) \geq \frac{\sin A+\sin B+\sin C}{3}$$

$\Rightarrow \quad \sin\frac{\pi}{3} \geq \frac{\sin A+\sin B+\sin C}{3} \qquad \Rightarrow \quad \frac{3\sqrt{3}}{2} \geq \sin A+\sin B+\sin C$

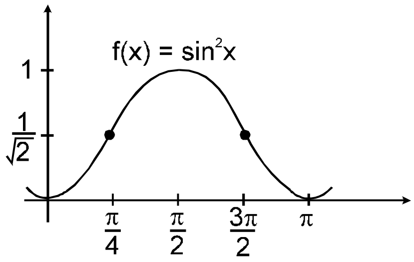
$\Rightarrow \quad$ maximum value of (sinA + sinB + sinC) = $\frac{3\sqrt{3}}{2}$

Example # 54 : Find the points of inflection of the function $f(x) = \sin^2 x \quad x \in [0, 2\pi]$

Solution : $f(x) = \sin^2 x$
$f'(x) = \sin 2x$
$f''(x) = 2\cos 2x$
$f''(0) = 0$

$\Rightarrow \qquad x = \frac{\pi}{4}, \frac{3\pi}{4}$

both these points are inflection points as sign of $f''(x)$ change on either sides of these points.



Example # 55 : Find the inflection point of $f(x) = 3x^4 - 4x^3$. Also draw the graph of f(x) giving due importance to concavity and point of inflection.

Solution : $f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

$f'(x) = 12x^2 (x - 1)$

$f''(x) = 12(3x^2 - 2x)$

$f''(x) = 12x(3x - 2)$

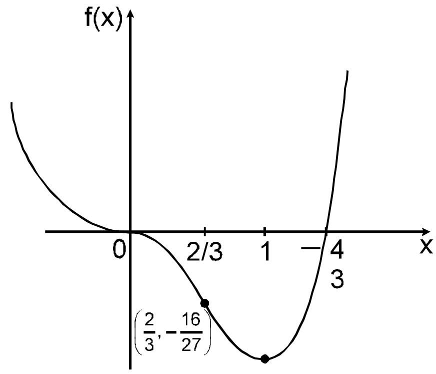
$f''(x) = 0 \qquad \Rightarrow \qquad x = 0, \frac{2}{3}.$

Again examining sign of $f''(x)$

$\underset{0}{+\ \ |}\ \ -\ \ \underset{2/3}{|\ \ +}$

thus $x = 0, \frac{2}{3}$ are the inflection points

Hence the graph of f(x) is



<u>Problems for Self Practice</u>-14 :

(1) Identify which is greater $\frac{1+e^2}{e}$ or $\frac{1+\pi^2}{\pi}$

(2) If $0 < x_1 < x_2 < x_3 < \pi$, then prove that $\sin\left(\frac{2x_1 + x_2 + x_3}{4}\right) > \frac{2\sin x_1 + \sin x_2 + \sin x_3}{4}$

(3) If f(x) is monotonically decreasing function and $f''(x) > 0$. Assuming $f^{-1}(x)$ exists prove that

$$\frac{f^{-1}(x_1) + f^{-1}(x_2)}{2} > f^{-1}\left(\frac{x_1 + x_2}{2}\right).$$

Answer : (1) $\frac{1+e^2}{e}$

14 ROLLE'S THEOREM :

If a function f defined on [a, b] is

(i) continuous on [a, b]

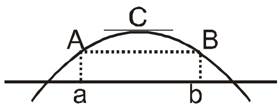
(ii) derivable on (a, b) and

(iii) $f(a) = f(b)$,

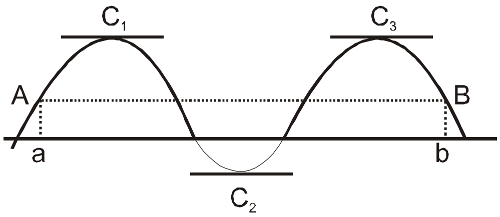
then there exists at least one real number c between a and b ($a < c < b$) such that $f'(c) = 0$

14.1 Geometrical Explanation of Rolle's Theorem :

Let the curve y = f(x), which is continuous on [a, b] and derivable on (a, b), be drawn (as shown in figure).



A(a, f(a)), B(b, f(b)), f(a) = f(b), C(c, f(c)), $f'(c) = 0$.



C_1 (c_1, $f(c_1)$), $f'(c_1) = 0$

C_2 (c_2, $f(c_2)$), $f'(c_2) = 0$

C_3 (c_3, $f(c_3)$), $f'(c_3) = 0$

The theorem simply states that between two points with equal ordinates on the graph of f(x), there exists at least one point where the tangent is parallel to x-axis.

14.2 Algebraic Interpretation of Rolle's Theorem :

Between two zeros a and b of f(x) (i.e. between two roots a and b of f(x) = 0) there exists at least one zero of $f'(x)$

SOLVED EXAMPLE

Example # 56 : Verify Rolle's theorem for $f(x) = (x - a)^n (x - b)^m$, where m, n are positive real numbers, for $x \in [a, b]$.

Solution : Being a polynomial function f(x) is continuous as well as differentiable. Also f(a) = f(b)

$\Rightarrow$ $f'(x) = 0$ for some $x \in (a , b)$

$n(x - a)^{n-1} (x - b)^m + m(x - a)^n (x - b)^{m-1} = 0$

$\Rightarrow$ $(x - a)^{n-1} (x - b)^{m-1}$ $[(m + n) x - (nb + ma)] = 0$

$\Rightarrow$ $x = \dfrac{nb + ma}{m + n}$, which lies in the interval (a, b), as $m, n \in R^+$.

Example # 57 : If 2a + 3b + 6c = 0 then prove that the equation $ax^2 + bx + c = 0$ has at least one real root between 0 and 1.

Solution : Let $f(x) = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx$

f(0) = 0 and $f(1) = \dfrac{a}{3} + \dfrac{b}{2} + c = 2a + 3b + 6c = 0$

If f(0) = f(1) then $f'(x) = 0$ for some value of $x \in (0, 1)$

$\Rightarrow$ $ax^2 + bx + c = 0$ for at least one $x \in (0, 1)$

<u>Problems for Self Practice</u>-15 :

(1) If f(x) satisfies condition in Rolle's theorem then show that between two consecutive zeros of $f'(x)$ there lies at most one zero of f(x).

(2) Show that for any real numbers λ, the polynomial $P(x) = x^7 + x^3 + \lambda$, has exactly one real root.

15. LAGRANGE'S MEAN VALUE THEOREM (LMVT) :

If a function f defined on [a, b] is

(i) continuous on [a, b] and

(ii) derivable on (a, b)

then there exists at least one real numbers between a and b ($a < c < b$) such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

Proof : Let us consider a function $g(x) = f(x) + \lambda x$, $x \in [a, b]$

where λ is a constant to b determined such that $g(a) = g(b)$.

$\therefore \quad \lambda = -\frac{f(b)-f(a)}{b-a}$

Now the function g(x), being the sum of two continuous and derivable functions it self

(i) continuous on [a, b]

(ii) derivable on (a, b) and

(iii) $g(a) = g(b)$.

Therefore, by Rolle's theorem there exists a real number $c \in (a, b)$ such that $g'(c) = 0$

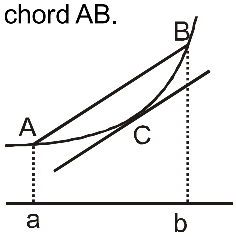
But $g'(x) = f'(x) + \lambda$

$\therefore \quad 0 = g'(c) = f'(c) + \lambda$

$f'(c) = -\lambda = \frac{f(b)-f(a)}{b-a}$

15.1 Geometrical Interpretation of LMVT :

The theorem simply states that between two points A and B of the graph of f(x) there exists at least one point where tangent is parallel to chord AB.



C(c, f(c)), $f'(c)$ = slope of AB.

Alternative Statement : If in the statement of LMVT, b is replaced by a + h, then number c between a and b may be written as $a + \theta h$, where $0 < \theta < 1$. Thus

$\frac{f(a+h)-f(a)}{h} = f'(a + \theta h)$ or $f(a + h) = f(a) + hf'(a + \theta h)$, $0 < \theta < 1$

SOLVED EXAMPLE

Example # 58 : Verify LMVT for $f(x) = -x^2 + 4x - 5$ and $x \in [-1, 1]$

Solution : $f(1) = -2$; $f(-1) = -10$

$\Rightarrow \quad f'(c) = \frac{f(1)-f(-1)}{1-(-1)}$

$\Rightarrow \quad -2c + 4 = 4 \quad \Rightarrow \quad c = 0$

Example # 59 : Using Lagrange's mean value theorem, prove that if b > a > 0,

then $\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}$

Solution : Let $f(x) = \tan^{-1} x$; $x \in [a, b]$ applying LMVT

$f'(c) = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$ for $a < c < b$ and $f'(x) = \frac{1}{1+x^2}$,

Now $f'(x)$ is a monotonically decreasing function

Hence if $a < c < b \Rightarrow f'(b) < f'(c) < f'(a)$

$\Rightarrow \frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$ Hence proved

Example # 60 : Let $f : R \to R$ be a twice differentiable function such that $f(2) = 8$, $f(4) > 64$, $f(7) = 343$ then show that there exists a $c \in (2, 7)$ such that $f''(c) < 6c$.

Solution: Consider $g(x) = f(x) - x^3$

By LMVT

$\frac{g(4)-g(2)}{4-2} = g'(c_1)$, $2 < c_1 < 4$ and $\frac{g(7)-g(4)}{7-4} = g'(c_2)$, $4 < c_2 < 7$

$g'(c_1) > 0$, $g'(c_2) < 0$

By LMVT

$\frac{g'(c_2)-g'(c_1)}{c_2 - c_1} = g''(c)$, $c_1 < c < c_2$ $\Rightarrow g''(c) < 0$ $\Rightarrow f''(c) - 6c < 0$

for same $c \in (c_1, c_2) \subset (2, 7)$

<u>Problems for Self Practice</u>-16 :

(1) If a function f(x) satisfies the conditions of LMVT and $f'(x) = 0$ for all $x \in (a, b)$, then f(x) is constant on [a, b].

(2) Using LMVT, prove that if two functions have equal derivatives at all points of (a, b), then they differ by a constant

(3) If a function f is
(i) continuous on [a, b],
(ii) derivable on (a, b) and (iii) $f'(x) > 0$, $x \in (a, b)$, then show that f(x) is strictly increasing on [a, b].

Exercise # 1

PART-I : SUBJECTIVE QUESTIONS

Section (A) : Equation of Tangent /Normal and Common Tangents /Normals

A-1. Find the equation of tangent and normal to curve

(i) $y = 3x^2 + 4x + 5$ at (0, 5) on it.

(ii) $x^2 + 3xy + y^2 = 5$ at point (1, 1) on it.

(iii) $x = \frac{2at^2}{1+t^2}, y = \frac{2at^3}{1+t^2}$ at the point for which $t = \frac{1}{2}$

(iv) $y = \begin{cases} x^2 \sin 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$ at (0, 0)

A-2. The tangent to $y = ax^2 + bx + \frac{7}{2}$ at (1, 2) is parallel to the normal at the point (–2, 2) on the curve $y = x^2 + 6x + 10$. Find the value of a and b.

A-3. Find all the tangents to the curve $y = \cos(x + y)$, $-2\pi \le x \le 2\pi$, that are parallel to the line $x + 2y = 0$.

A-4. Find the equation of normal to the curve $x^3 + y^3 = 8xy$ at point where it is meet by the curve $y^2 = 4x$, other than origin.

A-5. If the tangent to the curve $xy + ax + by = 0$ at (1, 1) is inclined at an angle $\tan^{-1} 2$ with positive x-axis in anticlockwise, then find a and b ?

A-6. The normal to the curve $5x^5 - 10x^3 + x + 2y + 6 = 0$ at the point P(0, –3) is tangent to the curve at the point(s). Find those point(s)?

A-7. Prove that the length of segment of all tangents to curve $x^{2/3} + y^{2/3} = a^{2/3}$ intercepted between coordinate axes is same.

A-8. If the tangent at (1, 1) on $y^2 = x(2 - x)^2$ meets the curve again at P, then find coordinates of P

A-9. If $ax + by = 1$ is a normal to the parabola $y^2 = 4Px$, then prove that $Pa^3 + 2aPb^2 = b^2$.

A-10. (i) Find equations of tangents drawn to the curve $y^2 - 2x^2 - 4y + 8 = 0$ from the point (1, 2).

(ii) Find equations of tangents to curve $y = x^4$ drawn from point (2, 0).

(iii) Find the equation of tangents to the parabola $y^2 = 9x$, which pass through the point (4, 10).

(iv) Find all the lines that pass through the point (1, 1) and are tangent to the curve represented parametrically as $x = 2t - t^2$ and $y = t + t^2$.

A-11. Find equation of all possible normals to the parabola $x^2 = 4y$ drawn from point (1, 2).

A-12. The number of tangent drawn to the curve $y - 2 = x^5$ which are drawn from point (2, 2) is/are

A-13. The equation of tangent drawn to the curve $x\left(y - \frac{1}{2}\right) = 4$ from point $\left(0, \frac{3}{2}\right)$ can be

A-14. Find length of subnormal to $x = \sqrt{2}\cos t$, $y = -3\sin t$ at $t = \frac{-\pi}{4}$.

A-15. Show that subnormal at any point on the curve $x^2y^2 = a^2(x^2 - a^2)$ varies inversely as the cube of its abscissa.

Section (B) : Angle between curves, Orthogonal curves, Shortest distance between two curves

B-1. Find angle of intersection of the curves $y = 2\sin^2 x$ and $y = \cos 2x$.

B-2. Find the angle of intersection of the curves $y^2 = 4ax$ and $x^2 = 32ay$.

B-3. If the two curves $C_1 : x = y^2$ and $C_2 : xy = k$ cut at right angles find the value of k.

B-4. Find the condition that curves $ax^2 + by^2 = 1$ and $a' x^2 + b' y^2 = 1$ may cut each other orthogonally

B-5. Find the co-ordinates of the point on the curve $x^2 = 4y$, which is at least distance from the line $y = x - 4$.

B-6. Find the point on hyperbola $3x^2 - 4y^2 = 72$ which is nearest to the straight line $3x + 2y + 1 = 0$

B-7. Find the shortest distance between the curves $f(x) = -6x^6 - 3x^4 - 4x^2 - 6$ and $g(x) = e^x + e^{-x} + 2$

Section (C) : Rate of change and approximation

C-1. The length x of rectangle is decreasing at a rate of 3 cm/min and width y is increasing at a rate of 2 cm/min. When x = 10 cm and y = 6 cm, find the rate of change of (i) the perimeter, (ii) the area of rectangle.

C-2. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of the second square with respect to the first square.

C-3. A man 1.5 m tall walks away from a lamp post 4.5 m high at a rate of 4 km/hr.
(i) How fast is his shadow lengthening?
(ii) How fast is the farther end of shadow moving on the pavement?

C-4. Water is being poured on to a cylindrical vessel at the rate of 1 m^3/min. If the vessel has a circular base of radius 3 m, find the rate at which the level of water is rising in the vessel.

C-5. Water is dripping out from a conical funnel of semi vertical angle $\pi/4$, at the uniform rate of 2 cm^3/sec through a tiny hole at the vertex at the bottom. When the slant height of the water is 4 cm, find the rate of decrease of the slant height of the water.

C-6. If in a triangle ABC, the side 'c' and the angle 'C' remain constant, while the remaining elements are changed slightly, show that $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$.

C-7. (i) Use differentials to a approximate the values of ; (a) $\sqrt{36.6}$ and (b) $\sqrt[3]{26}$.

C-8. Find the approximate change in volume V of a cube of side 5m caused by increasing its side length by 2%.

Section (D) : Monotonicity on an interval, Monotonicity about a point, local maxima/minima

D-1. Find the intervals of monotonicity for the following functions.

(a) $f(x) = 2.\ e^{x^2 - 4x}$ (b) $f(x) = e^x/x$ (c) $f(x) = x^2 e^{-x}$ (d) $f(x) = 2x^2 - \ln|x|$

(e) $\frac{x^4}{4} + \frac{x^3}{3} - 3x^2 + 5$ (f) $\log_3^2 x + \log_3 x$

D-2. Find the intervals of monotonocity of the functions in $[0, 2\pi]$

(a) $f(x) = \sin x - \cos x$ in $x \in [0, 2\pi]$

(b) $g(x) = 2\sin x + \cos 2x$ in $(0 \le x \le 2\pi)$.

(c) $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$

D-3. Show that $f(x) = \dfrac{x}{\sqrt{1+x}} - \ell n(1 + x)$ is an increasing function for $x > -1$.

D-4. If g(x) is monotonically increasing and f(x) is monotonically decreasing for $x \in R$ and if (gof) (x) is defined for $x \in R$, then prove that (gof)(x) will be monotonically decreasing function. Hence prove that $(g\circ f)(x + 1) \le (g\circ f)(x - 1)$.

D-5. Check monotonicity of f(x) at indicated point :

(i) $f(x) = x^3 - 3x + 1$ at $x = -1, 2$

(ii) $f(x) = |x - 1| + 2|x - 3| - |x + 2|$ at $x = -2, 0, 3, 5$

(iii) $f(x) = x^{1/3}$ at $x = 0$

(iv) $f(x) = x^2 + \dfrac{1}{x^2}$ at $x = 1, 2$

(v) $f(x) = \begin{cases} x^3 + 2x^2 + 5x, & x < 0 \\ 3\sin x, & x \ge 0 \end{cases}$ at $x = 0$

D-6. Let $f(x) = \begin{cases} x^2; & x \ge 0 \\ ax; & x < 0 \end{cases}$. Find real values of 'a' such that f(x) is strictly monotonically increasing at $x = 0$.

D-7. Prove the inequality, $\dfrac{\tan x_2}{\tan x_1} > \dfrac{x_2}{x_1}$ for $0 < x_1 < x_2 < \dfrac{\pi}{2}$.

D-8. For $x \in \left(0, \dfrac{\pi}{2}\right)$ identify which is greater (2sinx + tanx) or (3x). Hence find $\lim\limits_{x\to 0^+} \left[\dfrac{3x}{2\sin x + \tan x}\right]$ where [.] denotes the GIF.

D-9. Let f and g be differentiable on R and suppose $f(0) = g(0)$ and $f'(x) \le g'(x)$ for all $x \ge 0$. Then show that $f(x) \le g(x)$ for all $x \ge 0$.

D-10. Find the points of local maxima/minima of following functions

(i) $f(x) = 2x^3 - 21x^2 + 36x - 20$

(ii) $f(x) = -(x - 1)^3 (x + 1)^2$

(iii) $f(x) = x\,\ell nx$

(iv) $f(x) = (2^x - 1)(2^x - 2)^2$

(v) $f(x) = x^2e^{-x}$

(vi) $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3,\ x \in [0, \pi]$

(vii) $f(x) = 2x + 3x^{2/3}$

(viii) $f(x) = \left|\dfrac{x^2 - 2}{x^2 - 1}\right|$

D-11. Draw graph of $f(x) = x|x - 2|$ and, hence find points of local maxima/minima.

D-12. Let $f(x) = x^2$; $x \in [-1, 2)$. Then show that f(x) has exactly one point of local maxima but global maximum is not defined.

D-13. Let $f(x) = \begin{cases} 3 - x & 0 \le x < 1 \\ x^2 + \ell nb & x \ge 1 \end{cases}$. Find the set of values of b such that f(x) has a local minima at x = 1.

Section (E) : Global maxima, Global minima, Application of Maxima and Minima

E-1. Find the absolute maximum/minimum value of following functions

(i) $f(x) = x^3$; $x \in [-2, 2]$

(ii) $f(x) = \sin x + \cos x$; $x \in [0, \pi]$

(iii) $f(x) = 4x - \frac{x^2}{2}$; $x \in \left[-2, \frac{9}{2}\right]$

(iv) $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$; $x \in [0, 3]$

(v) $f(x) = \sin x + \frac{1}{2}\cos 2x$; $x \in \left[0, \frac{\pi}{2}\right]$

(vi) $y = x + \sin 2x,\ 0 \le x \le 2\pi$

(vii) $y = 2\cos 2x - \cos 4x,\ 0 \le x \le \pi$

E-2. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible when revolved around one of its side .

E-3. John has 'x' children by his first wife and Anglina has 'x + 1' children by her first husband. They both marry and have their own children. The whole family has 24 children. It is given that the children of the same parents don't fight. Then find then maximum number of fights that can take place in the family.

E-4. If the sum of the lengths of the hypotenuse and another side of a right angled triangle is given, show that the area of the triangle is a maximum when the angle between these sides is $\pi/3$.

E-5. Find the volume of the largest cylinder that can be inscribed in a sphere of radius ' r ' cm.

E-6. Show that the semi vertical angle of a right circular cone of maximum volume, of a given slant height is $\tan^{-1}\sqrt{2}$.

E-7. A running track of 440 m. is to be laid out enclosing a football field, the shape of which is a rectangle with semi circle at each end . If the area of the rectangular portion is to be maximum, find the length of its sides.

E-8. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the curve $y = 12 - x^2$.

E-9. Find the minimum and maximum values of y in $4x^2 + 12xy + 10y^2 - 4y + 3 = 0$.

Section (F) : Rolle's Theorem, LMVT

F-1. Verify Rolle's theorem for the function, $f(x) = \log_e\left(\dfrac{x^2+ab}{x(a+b)}\right) + p$, for [a, b] where $0 < a < b$.

F-2. Using Rolle's theorem prove that the equation $3x^2 + px - 1 = 0$ has at least one real root in the interval $(-1, 1)$.

F-3. Using Rolle's theorem show that the derivative of the function $f(x) = \begin{cases} x\sin\dfrac{\pi}{x} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$ vanishes at an infinite set of points of the interval (0, 1).

F-4. Let f(x) be differentiable function and g(x) be twice differentiable function. Zeros of f(x), $g'(x)$ be a, b respectively ($a < b$). Show that there exists at least one root of equation $f'(x)\,g'(x) + f(x)\,g''(x) = 0$ on (a, b).

F-5. f(x) and g(x) are differentiable functions for $0 \le x \le 2$ such that $f(0) = 5$, $g(0) = 0$, $f(2) = 8$, $g(2) = 1$. Show that there exists a number c satisfying $0 < c < 2$ and $f'(c) = 3g'(c)$.

F-6. Assume that f is continuous on [a, b], $a > 0$ and differentiable on an open interval (a, b). Show that if $\dfrac{f(a)}{a} = \dfrac{f(b)}{b}$, then there exist $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$.

F-7. If $f(x) = \begin{vmatrix} \sin^3 x & \sin^3 a & \sin^3 b \\ xe^x & ae^a & be^b \\ \dfrac{x}{1+x^2} & \dfrac{a}{1+a^2} & \dfrac{b}{1+b^2} \end{vmatrix}$

where $0 < a < b < 2\pi$, then show that the equation $f'(x) = 0$ has atleast one root in the interval (a, b)

F-8. A function y = f(x) is defined on [0, 6] as $f(x) = \begin{cases} -8x & ; & 0 \le x \le 1 \\ (x-3)^3 & ; & 1 < x < 4 \\ 2 & ; & 4 \le x \le 6 \end{cases}$

Show that for the function y = f(x), all the three conditions of Rolle's theorem are violated on [0, 6] but still f'(x) vanishes at a point in (0, 6)

F-9. If $f(x) = \tan x$, $x \in \left[0, \dfrac{\pi}{5}\right]$ then show that $\dfrac{\pi}{5} < f\left(\dfrac{\pi}{5}\right) < \dfrac{2\pi}{5}$

F-10. Use LMVT to establish following inequilities

(i) $x \ge \sin x \ \forall\ x \in [0, \infty)$

(ii) $\dfrac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \dfrac{b-a}{1+a^2} \ \forall\ 0 < a < b$

(iii) $\dfrac{b-a}{b} < \ell n\left(\dfrac{b}{a}\right) < \dfrac{b-a}{a} \ \forall\ 0 < a < b$

PART-II : OBJECTIVE QUESTIONS

Section (A) : Equation of Tangent/Normal and Common Tangents/Normals

A-1. The slope of the curve $y = \sin x + \cos^2 x$ is zero at the point, where-

(A) $x = \frac{\pi}{4}$ (B) $x = \frac{\pi}{2}$ (C) $x = \pi$ (D) No where

A-2. The slope of the normal to the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at point $\theta = \pi/2$ is-

(A) 0 (B) 1 (C) –1 (D) $1/\sqrt{2}$

A-3. The equation of tangent at the point (at^2, at^3) on the curve $ay^2 = x^3$ is-

(A) $3tx - 2y = at^3$ (B) $tx - 3y = at^3$ (C) $3tx + 2y = at^3$ (D) None of these

A-4. The equation of normal to the curve $y = x^3 - 2x^2 + 4$ at the point $x = 2$ is-

(A) $x + 4y = 0$ (B) $4x - y = 0$ (C) $x + 4y = 18$ (D) $4x - y = 18$

A-5. The equation of normal to the curve $y = (1 + x)^y + \sin^{-1}(\sin^2 x)$ at $x = 0$ is

(A) $x + y = 1$ (B) $x + 2y = 1$ (C) $x + 3y = 1$ (D) $2x + y = 1$

A-6. The line $x/a + y/b = 1$ touches the curve $y = be^{-x/a}$ at the point-

(A) (0, a) (B) (0, 0) (C) (0, b) (D) (b, 0)

A-7. The curve $y - e^{xy} + x = 0$ has a vertical tangent at

(A) (1, 1) (B) (0, 1) (C) (1, 0) (D) no point

A-8. The abscissa of point on curve $ay^2 = x^3$, normal at which cuts off equal intercepts from the coordinate axes is

(A) $\frac{2a}{9}$ (B) $\frac{4a}{9}$ (C) $-\frac{4a}{9}$ (D) $-\frac{2a}{9}$

A-9. If the tangent at a point P, with parameter t, on the curve $x = 4t^2 + 3$, $y = 8t^3 - 1$, $t \in R$, meets the curve again at a point Q, then the coordinates of Q are :

(A) $(t^2 + 3, t^3 - 1)$ (B) $(t^2 + 3, -t^3 - 1)$

(C) $(16t^2 + 3, -64t^3 - 1)$ (D) $(4t^2 + 3, -8t^3 - 1)$

A-10. The lines $y = -\frac{3}{2}x$ and $y = -\frac{2}{5}x$ intersect the curve $3x^2 + 4xy + 5y^2 - 4 = 0$ at the points P and Q respectively. The tangents drawn to the curve at P and Q:

(A) intersect each other at angle of 45º (B) are parallel to each other

(C) are perpendicular to each other (D) none of these

A-11. The number of tangents drawn to the curve $xy = 4$ from point (0, 1) is

(A) 0 (B) 1 (C) 2 (D) Infinite

A-12. The equation of tangents drawn to curve $y = (x + 1)^3$ from origin can be :

(A) $4y + 27x = 0$ (B) $4y - 27x = 0$ (C) $y = x$ (D) $x = 0$

A-13. The equation of normal drawn to curve $x^2 = 4y$ from (1, 2) can be

(A) $x - 2y + 3 = 0$ (B) $x + y = 3$ (C) $2x - y = 0$ (D) $y - x = 1$

A-14. The equation of normals to $y^2 = 4x$ from point (6, 0) can not be

(A) $y = 0$ (B) $y + 2x - 12 = 0$ (C) $y - 2x + 12 = 0$ (D) $x + 2y - 6 = 0$

A-15. The subtangent, ordinate and subnormal to the parabola $y^2 = 4ax$ at a point (different from the origin) are in

(A) AP (B) GP (C) HP (D) none of these

A-16. The length of subtangent to the curve $x^2 + xy + y^2 = 7$ at the point (1, –3) is-

(A) 3 (B) 5 (C) 15 (D) 3/5

Section (B) : Angle between curves, Orthogonal curves, Shortest distance between two curves

B-1. The lines tangent to the curve $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to-

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

B-2. The angle of intersection between the curves $y^2 = 8x$ and $x^2 = 4y - 12$ at (2, 4) is-

(A) 90° (B) 60° (C) 45° (D) 0°

B-3. If curve $y = 1 - ax^2$ and $y = x^2$ intersect orthogonally then the value of a is

(A) 1/2 (B) 1/3 (C) 2 (D) 3

B-4 The value of a^2 if the curves $\frac{x^2}{a^2} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ cut orthogonally is

(A) 3/4 (B) 1 (C) 4/3 (D) 4

B-5. The coordinates of the point of the parabola $y^2 = 8x$, which is at minimum distance from the circle $x^2 + (y + 6)^2 = 1$ are

(A) (2 , – 4) (B) (18 , –12) (C) (2 , 4) (D) none of these

B-6. If (a, b) be the point on the curve $y = |x^2 - 4x + 3|$ which is nearest to the circle $x^2 + y^2 - 4x - 4y + 7 = 0$, then (a + b) is equal to -

(A) $\frac{7}{4}$ (B) 0 (C) 3 (D) 2

Section (C) : Rate of change and approximation

C-1. Water is poured into an inverted conical vessel of which the radius of the base is 2 m and height 4 m, at the rate of 77 litre/minute. The rate at which the water level is rising at the instant when the depth is 70 cm is (use $\pi = 22/7$)

(A) 10 cm/min (B) 20 cm/min (C) 40 cm/min (D) 30 cm/min

C-2 On the curve $x^3 = 12y$. The interval in which abscissa changes at a faster rate then its ordinate

(A) (–3, 0) (B) $(-\infty, -2) \cup (2, \alpha)$ (C) (–2, 2) (D) (–3, 3)

C-3. A kite is 300 m high and there are 500 m of cord out. If the wind moves the kite horizontally at the rate of 5 km/hr. directly away from the person who is flying it, find the rate at which the cord is being paid?

(A) 4 (B) 8 (C) 3 (D) cannot be determined

C-4. A Spherical balloon is being inflated at the rate of 35cc/min. The rate of increase in the surface area (in cm2/min.) of the balloon when its diameter is 14 cm, is :

(A) $\sqrt{10}$ (B) $10\sqrt{10}$ (C) 100 (D) 10

C-5. The approximate value of tan 46° is (take π = 22/7) :

(A) 3 (B) 1.035 (C) 1.033 (D) 1.135

Section (D) : Monotonicity on an interval, Monotonicity about a point, local maxima/minima

D-1. When $0 \le x \le 1$, $f(x) = |x| + |x - 1|$ is-

(A) strictly increasing (B) strictly decreasing (C) constant (D) None of these

D-2. The function $f(x) = \tan^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ is -

(A) strictly increasing in its domain

(B) strictly decreasing in its domain

(C) strictly decreasing in $(-\infty, 0)$ and strictly increasing in $(0, \infty)$

(D) strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$

D-3. If function $f(x) = 2x^2 + 3x - m\log x$ is monotonic decreasing in the interval (0, 1), then the least value of the parameter m is-

(A) 7 (B) $\dfrac{15}{2}$ (C) $\dfrac{31}{4}$ (D) 8

D-4. The complete set of values of 'a' for which the function $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ decreases for all real values of x is.

(A) $(-\infty, -3]$ (B) $(-\infty, 0]$ (C) $[-3, 0]$ (D) $[-3, \infty)$

D-5. **STATEMENT-1 :** e^{π} is bigger than π^e.

STATEMENT-2 : $f(x) = x^{1/x}$ is a increasing function when $x \in [e, \infty)$

(A) Statement-1 is True, Statement-2 is True

(B) Statement-1 is True, Statement-2 is False

(C) Statement-1 is False, Statement-2 is True

(D) Statement-1 is False, Statement-2 is False

D-6. The function $\dfrac{|x-1|}{x^2}$ is monotonically decreasing at the point

(A) x = 3 (B) x = 1 (C) x = 2 (D) none of these

D-7. If $f(x) = 1 + 2x^2 + 4x^4 + 6x^6 + ... + 100x^{100}$ is a polynomial in a real variable x, then f(x) has:

(A) neither a maximum nor a minimum

(B) only one maximum

(C) only one minimum

(D) one maximum and one minimum

D-8. If $f(x) = a\,\ell n|x| + bx^2 + x$ has its extremum values at x = – 1 and x = 2, then

(A) a = 2, b = – 1 (B) a = 2, b = – 1/2

(C) a = – 2, b = 1/2 (D) none of these

D-9. If $f(x) = \sin^3 x + \lambda \sin^2 x$; $-\pi/2 < x < \pi/2$, then the interval in which λ should lie in order that f(x) has exactly one minima and one maxima

(A) $(-3/2, 3/2) - \{0\}$ (B) $(-2/3, 2/3) - \{0\}$ (C) R (D) $\left[-\frac{3}{2}, 0\right)$

D-10. If $f(x) = x^3 + ax^2 + bx + c$ is minimum at x = 3 and maximum at x = –1, then-

(A) a = –3, b = –9, c = 0 (B) a = 3, b = 9, c = 0

(C) a = –3, b = –9, $c \in R$ (D) none of these

D-11. If $(x - a)^{2m} (x - b)^{2n+1}$, where m and n are positive integers and a > b, is the derivative of a function f, then-

(A) x = a gives neither a maximum, nor a minimum

(B) x = a gives a maximum

(C) x = b gives neither a maximum nor a minimum

(D) None of these

D-12. Let $f(x) = ax^2 - b|x|$, where a and b are constants. Then at x = 0, f(x) has

(A) a maxima whenever a > 0, b > 0

(B) a maxima whenever a > 0, b < 0

(C) minima whenever a > 0, b > 0

(D) neither a maxima nor minima whenever a > 0, b < 0

Section (E) : Global maxima, Global minima, Application of Maxima and Minima

E-1. The greatest value of $x^3 - 18x^2 + 96x$ in the interval (0, 9) is-

(A) 128 (B) 60 (C) 160 (D) 120

E-2. Difference between the greatest and the least values of the function $f(x) = x(\ln x - 2)$ on $[1, e^2]$ is

(A) 2 (B) e (C) e^2 (D) 1

E-3. The greatest, the least values of the function, $f(x) = 2 - \sqrt{1+2x+x^2}$, $x \in [-2, 1]$ are respectively

(A) 2, 1 (B) 2, – 1 (C) 2, 0 (D) –2, 3

E-4. The sum of lengths of the hypotenuse and another side of a right angled triangle is given. The area of the triangle will be maximum if the angle between them is :

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $5\pi/12$

E-5. Two sides of a triangle are to have lengths 'a' cm & 'b' cm. If the triangle is to have the maximum area, then the length of the median from the vertex containing the sides 'a' and 'b' is

(A) $\frac{1}{2}\sqrt{a^2+b^2}$ (B) $\frac{2a+b}{3}$ (C) $\sqrt{\frac{a^2+b^2}{2}}$ (D) $\frac{a+2b}{3}$

E-6. P is a point on positive x-axis, Q is a point on the positive y-axis and 'O' is the origin. If the line passing through P and Q is tangent to the curve $y = 3 - x^2$ then the minimum area of the triangle OPQ, is

(A) 2 (B) 4 (C) 8 (D) 9

E-7. The least area of a circle circumscribing any right triangle of area S is :

(A) πS (B) $2\pi S$ (C) $\sqrt{2}\pi S$ (D) $4\pi S$

E-8. The radius of a right circular cylinder of greatest curved surface which can be inscribed in a given right circular cone is

(A) one third that of the cone (B) $1/\sqrt{2}$ times that of the cone

(C) 2/3 that of the cone (D) 1/2 that of the cone

E-9. The dimensions of the rectangle of maximum area that can be inscribed in the ellipse $(x/4)^2 + (y/3)^2 = 1$ are

(A) $\sqrt{8},\sqrt{2}$ (B) 4, 3 (C) $2\sqrt{8}, 3\sqrt{2}$ (D) $\sqrt{2}$, $\sqrt{6}$

E-10. The largest area of a rectangle which has one side on the x–axis and the two vertices on the curve y = e^{-x^2} is

(A) $\sqrt{2}\ e^{-1/2}$ (B) $2\ e^{-1/2}$ (C) $e^{-1/2}$ (D) none of these

E-11. **Statement 1 :** ABC is given triangle having respective sides a,b,c. D,E,F are points of the sides BC,CA,AB respectively so that AFDE is a parallelogram. The maximum area of the parallelogram is $\frac{1}{4}$ bcsinA.

Statement 2 : Maximum value of $2kx - x^2$ is at x = k.

(A) Statement-1 is True, Statement-2 is True

(B) Statement-1 is True, Statement-2 is False

(C) Statement-1 is False, Statement-2 is True.

(D) Statement-1 is False, Statement-2 is False

E-12. The bottom of the legs of a three legged table are the vertices of an isosceles triangle with sides 5, 5 and 6. The legs are to be braced at the bottom by three wires in the shape of a Y. The minimum length of the wire needed for this purpose, is

(A) $4 + 3\sqrt{3}$ (B) 10 (C) $3 + 4\sqrt{3}$ (D) $1 + 6\sqrt{2}$

Section (F) : Rolle's Theorem, LMVT

F-1. The function $f(x) = x^3 - 6x^2 + ax + b$ satisfy the conditions of Rolle's theorem on [1, 3]. Which of these are correct ?

(A) a =11, $b \in R$ (B) a = 11, b = – 6

(C) a = –11, b = 6 (D) a = – 11, $b \in R$

F-2. The function $f(x) = x(x + 3)e^{-x/2}$ satisfies all the conditions of Rolle's theorem on [–3, 0]. The value of c which verifies Rolle's theorem, is

(A) 0 (B) – 1 (C) – 2 (D) 3

F-3. Consider the function for $x \in [-2, 3]$

$$f(x) = \begin{cases} -6 \ ; & x = 1 \\ \dfrac{x^3 - 2x^2 - 5x + 6}{x-1} \ ; & x \neq 1 \end{cases}$$. The value of c obtained by applying Rolle's theorem for which f′(c) = 0 is

(A) 0 (B) 1 (C) 1/2 (D) 'c' does not exist

F-4. A value of c for which the conclusion of Lagrange's Mean Value Theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is-

(A) $2\log_3 e$ (B) $\frac{1}{2}\log_e 3$ (C) $\log_3 e$ (D) $\log_e 3$

F-5. Given: $f(x) = 4 - \left(\frac{1}{2} - x\right)^{2/3}$ $g(x) = \begin{cases} \frac{\tan[x]}{x} & , x \neq 0 \\ 1 & , x = 0 \end{cases}$

$h(x) = \{x\}$ $k(x) = 5^{\log_2(x+3)}$

then in [0, 1], Lagrange's Mean Value Theorem is NOT applicable to

(A) f, g, h (B) h, k (C) f, g (D) g, h, k

where [x] and {x} denotes the greatest integer and fractional part function.

F-6. If the function $f(x) = 2x^2 + 3x + 5$ satisfies LMVT at x = 2 on the closed interval [1, a], then the value of 'a' is equal to

(A) 3 (B) 4 (C) 6 (D) 1

F-7. If 2a + 3b + 6c = 0, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval-

(A) (0, 1) (B) (1, 2) (C) (2, 3) (D) none

F-8. If f(x) satisfies the requirements of Lagrange's mean value theorem on [0, 2] and if f(0) = 0 and $f'(x) \leq \frac{1}{2}$ $\forall$ $x \in [0, 2]$, then

(A) $|f(x)| \leq 2$ (B) $f(x) \leq 1$

(C) f(x) = 2x (D) f(x) = 3 for at least one x in [0, 2]

PART-III : MATCH THE COLUMN

1.

	Column-I		Column-II
(A)	If curves $y^2 = 4ax$ and $y = e^{-\frac{x}{2a}}$ are orthogonal then 'a' can take value	(p)	36
(B)	If θ is angle between the curves $y = [\,|\sin x| + |\cos x|\,]$, ([·] denote GIF) and $x^2 + y^2 = 5$ then $\text{cosec}^2\theta$ is	(q)	1/2
(C)	Maximum value of $\left(\sqrt{-3+4x-x^2}+4\right)^2 + (x-5)^2$ (where $1 \leq x \leq 3$) is	(r)	5/4
(D)	If normal at point (6, 2) to ellipse passes through its nearest focus (5, 2) having centre at (4, 2) then its eccentricity is	(s)	0

2. **Column-I** **Column-II**

	Column-I		Column-II
(A)	The number of point (s) of maxima of $f(x) = x^2 + \frac{1}{x^2}$ is	(p)	0
(B)	$(\sin^{-1} x)^3 + (\cos^{-1} x)^3$ is maximum at x =	(q)	2
(C)	If [a, b], (b < 1) is largest interval in which $f(x) = 3x^4 + 8x^3 - 6x^2 - 24x + 19$ is strictly increasing then $\frac{a}{b}$ is	(r)	$\frac{8}{3}$
(D)	If a + b = 8, a, b > 0 then minimum value of $\frac{a^3+b^3}{48}$ is	(s)	–1

3. **Column-I** **Column-II**

	Column-I		Column-II
(A)	A rectangle is inscribed in an equilateral triangle of side 4cm. Square of maximum area of such a rectangle is	(p)	65
(B)	The volume of a rectangular closed box is 72 and the base sides are in the ratio 1 : 2. The least total surface area is	(q)	45
(C)	If x and y are two positive numbers such that x + y = 60 and x^3y is maximum then value of x is	(r)	12
(D)	The sides of a rectangle of greatest perimeter which is inscribed in a semicircle of radius $\sqrt{5}$ are a and b. Then $a^3 + b^3$ =	(s)	108

4. **Column-I** **Column-II**

	Column-I		Column-II
(A)	$f(x) = \frac{\sin x}{e^x}, x \in [0,\pi]$	(p)	Conditions in Rolle's theorem are satisfied.
(B)	$f(x) = \text{sgn}((e^x - 1)\,\ell nx), x \in \left[\frac{1}{2},\frac{3}{2}\right]$	(q)	Conditions in LMVT are satisfied.
(C)	$f(x) = (x-1)^{2/5}, x \in [0,3]$	(r)	At least one condition in Rolle's theorem is not satisfied.
(D)	$f(x) = \begin{cases} x\left(\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}\right), & x \in [-1,1]-\{0\} \\ 0, & x=0 \end{cases}$	(s)	At least one condition in LMVT is not satisfied.

Exercise # 2

PART-I : OBJECTIVE

1. Equation of normal drawn to the graph of the function defined as $f(x) = \frac{\sin x^2}{x}$, $x \neq 0$ and $f(0) = 0$ at the origin is
(A) $x + y = 0$ (B) $x - y = 0$ (C) $y = 0$ (D) $x = 0$

2. If tangents are drawn from the origin to the curve $y = \sin x$, then their points of contact lie on the curve
(A) $x - y = xy$ (B) $x + y = xy$ (C) $x^2 - y^2 = x^2y^2$ (D) $x^2 + y^2 = x^2y^2$

3. The x-intercept of the tangent at any arbitrary point of the curve $\frac{a}{x^2} + \frac{b}{y^2} = 1$ is proportional to:
(A) square of the abscissa of the point of tangency
(B) square root of the abscissa of the point of tangency
(C) cube of the abscissa of the point of tangency
(D) cube root of the abscissa of the point of tangency.

4. A curve is represented parametrically by the equations $x = t + e^{at}$ and $y = -t + e^{at}$ when $t \in R$ and $a > 0$. If the curve touches the axis of x at the point A, then the coordinates of the point A are
(A) (1, 0) (B) (1/e, 0) (C) (e, 0) (D) (2e, 0)

5. A curve is represented by the equations, $x = \sec^2 t$ and $y = \cot t$ where t is a parameter. If the tangent at the point P on the curve where $t = \pi/4$ meets the curve again at the point Q then $|PQ|$ is equal to:
(A) $\frac{5\sqrt{3}}{2}$ (B) $\frac{5\sqrt{5}}{2}$ (C) $\frac{2\sqrt{5}}{3}$ (D) $\frac{3\sqrt{5}}{2}$

6. The tangent to curve $y = x - x^3$ at point P meets the curve again at Q. The locus of one of the point of trisection of PQ is
(A) $y = x + 5x^3$ (B) $y = -x - 5x^3$ (C) $y = x - 5x^3$ (D) $y = x - x^3$

7. Number of tangents drawn from the point (–1/2, 0) to the curve $y = e^{\{x\}}$. (Here { } denotes fractional part function).
(A) 2 (B) 1 (C) 3 (D) 4

8. Let $f(x) = \begin{cases} -x^2 & , x < 0 \\ x^2 + 8 & , x \geq 0 \end{cases}$ Equation of tangent line touching both branches of y = f(x) is
(A) $y = 4x + 1$ (B) $y = 4x + 4$ (C) $y = x + 4$ (D) $y = x + 1$

9. Let f(x) and g(x) be two functions which cut each other orthogonally at their common point of intersection (x_1). Both f(x) and g(x) are equal to 0 at $x = x_1$. Also $|f'(x_1)| = |g'(x_1)|$, then find $\lim_{x \to x_1} [f(x) . g(x)]$, where [.] denotes greatest integer functions.
(A) –2 (B) –1 (C) 0 (D) Does not exits

10. If curves $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $xy = c^2$ intersect orthogonally, then

(A) $a + b = 0$ (B) $a^2 = b^2$ (C) $a + b = c$ (D) none of these

11. The point(s) on the parabola $y^2 = 4x$ which are closest to the circle, $x^2 + y^2 - 24y + 128 = 0$ is/are:

(A) (0, 0) (B) $\left(2, 2\sqrt{2}\right)$ (C) (4, 4) (D) none of these

12. Let $f(x) = x^3 + ax^2 + bx + 5\sin^2 x$ be an increasing function in the set of real numbers R. Then a & b satisfy the condition :

(A) $a^2 - 3b - 15 > 0$ (B) $a^2 - 3b + 15 \leq 0$ (C) $a^2 + 3b - 15 < 0$ (D) $a > 0$ & $b > 0$

13. If $f(x) = \frac{x^2}{2-2\cos x}$; $g(x) = \frac{x^2}{6x-6\sin x}$ where $0 < x < 1$, then

(A) both 'f' and 'g' are increasing functions (B) 'f' is decreasing & 'g' is increasing function

(C) 'f' is increasing & 'g' is decreasing function (D) both 'f' & 'g' are decreasing function

14. If $f : [1, 10] \rightarrow [1, 10]$ is a non-decreasing function and $g : [1, 10] \rightarrow [1, 10]$ is a non-increasing function. Let $h(x) = f(g(x))$ with $h(1) = 1$, then $h(2)$

(A) lies in (1, 2) (B) is more than 2 (C) is equal to 1 (D) is not defined

15. Let $f(x) = 1 - x - x^3$. Find set of all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$

(A) $(-2, 0)$ (B) $(2, \infty)$ (C) $(-2, 0) \cup (2, \infty)$ (D) $(0, 2)$

16. If $f(x) = |ax - b| + c|x|$ is stricly increasing at atleast one point of non differentiability of the function where $a > 0$, $b > 0$, $c > 0$ then

(A) $c > a$ (B) $a > c$ (C) $b > a + c$ (D) $a = b$

17 The set of values of p for which all the points of extremum of the function $f(x) = x^3 - 3px^2 + 3(p^2 - 1)x + 1$ lie in the interval $(-2, 4)$, is:

(A) $(-3, 5)$ (B) $(-3, 3)$ (C) $(-1, 3)$ (D) $(-1, 4)$

18. Let $f(x) = \begin{cases} x^3-x^2+10x-5 & ,x\leq 1 \\ -2x+\log_2\left(b^2-2\right) & ,x>1 \end{cases}$ the set of values of b for which f(x) has greatest value at x = 1 is given by:

(A) $1 \leq b \leq 2$ (B) $b = \{1, 2\}$

(C) $b \in (-\infty, -1)$ (D) $\left[-\sqrt{130}, -\sqrt{2}\right) \cup \left(\sqrt{2}, \sqrt{130}\right]$

19. Four points A, B, C, D lie in that order on the parabola $y = ax^2 + bx + c$. The coordinates of A, B & D are known as A(– 2, 3); B(– 1, 1) and D(2, 7). The coordinates of C for which the area of the quadrilateral ABCD is greatest, is

(A) (1/2, 7/4) (B) (1/2, – 7/4) (C) (–1/2, 7/4) (D) (–1/2, –7/4)

20. In a regular triangular prism the distance from the centre of one base to one of the vertices of the other base is ℓ. The altitude of the prism for which the volume is greatest, is :

(A) $\frac{\ell}{2}$ (B) $\frac{\ell}{\sqrt{3}}$ (C) $\frac{\ell}{3}$ (D) $\frac{\ell}{4}$

21. The maximum area of the rectangle whose sides pass through the angular points of a given rectangle of sides a and b is

(A) 2 (ab) (B) $\frac{1}{2}(a+b)^2$ (C) $\frac{1}{2}(a^2+b^2)$ (D) $\frac{a^3}{b}$

22. There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per additional tree drops by 10 apples. Number of trees that should be added to the existing orchard for maximising the output of the trees, is

(A) 5 (B) 10 (C) 15 (D) 20

23. Square roots of 2 consecutive natural number greater than N^2 is differ by

(A) $> \frac{1}{2N}$ (B) $\geq \frac{1}{2N}$ (C) $< \frac{1}{2N}$ (D) $> \frac{1}{N}$

24. Let h be a twice differentiable positive function on an open interval J. Let

$$g(x) = \ell n\,(h(x))\ \forall\ x \in J$$

Suppose $(h'(x))^2 > h''(x)\,h(x)$ for each $x \in J$. Then

(A) g is increasing on J (B) g is decreasing on J
(C) g is concave upward on J (D) g is concave downward on J

25. **STATEMENT-1 :** If f(x) is increasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.
STATEMENT-2 : If f(x) is decreasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

(A) Statement-1 is True, Statement-2 is True (B) Statement-1 is True, Statement-2 is False
(C) Statement-1 is False, Statement-2 is True (D) Statement-1 is False, Statement-2 is False

26. If $f(x) = (x-4)(x-5)(x-6)(x-7)$ then,

(A) $f'(x) = 0$ has four roots.
(B) three roots of $f'(x) = 0$ lie in $(4, 5) \cup (5, 6) \cup (6, 7)$.
(C) the equation $f'(x) = 0$ has only one real root.
(D) three roots of $f'(x) = 0$ lie in $(3, 4) \cup (4, 5) \cup (5, 6)$.

27. If Rolle's theorem is applicable to the function $f(x) = \frac{\ell nx}{x}$, $(x > 0)$ over the interval [a, b] where $a \in I$, $b \in I$, then the value of $a^2 + b^2$ can be

(A) 20 (B) 25 (C) 45 (D) 10

28. If f(x) is a twice differentiable fucntion such that $f(x) = x\ \forall\ x \in \{1, 2, 3, 4, 5\}$, then minimum number of real roots of $f''(x) = 0$ is

(A) 2 (B) 3 (C) 4 (D) 5

PART-II : NUMERICAL QUESTIONS

1. The number of values of c such that the straight line $3x + 4y = c$ touches the curve $\frac{x^4}{2} = x + y$ is

2. There is a point (p,q) on the graph of $f(x) = x^2$ and a point (r,s) on the graph of $g(x) = -8/x$ where $p > 0$ and $r > 0$. If the line through (p,q) and (r,s) is also tangent to both the curves at these points respectively then find the value of (p + q).

3. Tangent at a point P_1 [other than (0 , 0)] on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3 & so on. Show that the abscissae of $P_1, P_2, P_3, \ldots\ldots\ldots P_n$, form a GP. Also find the ratio

$$4\left(\frac{\text{area of } \Delta(P_1P_2P_3)}{\text{area of } \Delta(P_2P_3P_4)}\right).$$

4. If at any point on a curve the length of subtangent and subnormal are equal, then the tangent is equal to λ |ordinate| then the value of λ is

5. The curves $x^3 + p\,xy^2 = -2$ and $3x^2y - y^3 = 2$ are orthogonal then |p| is

6. Minimum distance between the curves $f(x) = e^x$ & $g(x) = \ell n\, x$ is $\frac{\lambda}{\sqrt{2}}$ then the value of λ is

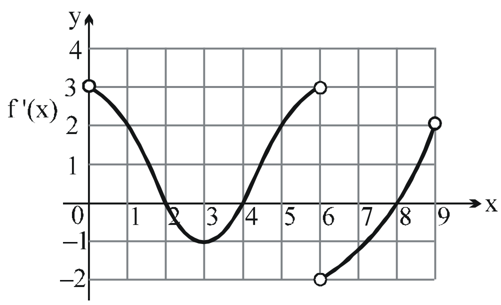
7. A light shines from the top of a pole 50 ft. high. A ball is dropped from the same height from a point 30 ft. away from the light. If the shadow of the ball moving at the rate of 100λ ft/sec along the ground 1/2 sec. later [Assume the ball falls a distance $s = 16\,t^2$ ft. in 't' sec.], then $|\lambda|$ is :

8. A variable ΔABC in the xy plane has its orthocentre at vertex 'B' , a fixed vertex 'A' at the origin and the third vertex 'C' restricted to lie on the parabola $y = 1 + \frac{7x^2}{36}$. The point B starts at the point (0, 1) at time t = 0 and moves upward along the y axis at a constant velocity of 2 cm/sec . If the area of the triangle increasing at the rate of 'p' cm²/sec when $t = \frac{7}{2}$ sec, then p is.

9. If $f(x) = 2e^x - ae^{-x} + (2a + 1)x - 3$ monotonically increases for $\forall x \in R$, then the minimum value of 'a' is

10. If the set of all values of the parameter 'a' for which the function $f(x) = \sin 2x - 8(a + 1)\sin x + (4a^2 + 8a - 14)x$ increases for all $x \in R$ and has no critical points for all $x \in R$, is $(-\infty,\ -m - \sqrt{n}\,) \cup (\sqrt{n}\,, \infty)$ then $(m^2 + n^2)$ is (where m, n are prime numbers) :

11. The graph of the derivative $f'(x)$ of a continuous function $f(x)$ in (0,9). If



(i) f is strictly increasing in the interval (a,b]; [c,d];[e,f) and strictly decreasing in [p, q];[r, s].

(ii) f has a local minima at $x = x_1$ and $x = x_2$.

(iii) $f''(x) > 0$ in (l, m) ; (n, t)

(iv) f has inflection point at x = k

(v) number of critical points of y = f (x) is 'w'.

Find the value of (a + b + c + d + e) + (p + q + r + s) + (l + m + n) + ($x_1 + x_2$) + (k + w).

12. Let P(x) be a polynomial of degree 5 having extremum at x = –1, 1 and $\lim_{x\to 0}\left(\frac{P(x)}{x^3} - 2\right) = 4$.

If M and m are the maximum and minimum value of the function y = P'(x) on the set $A = \{x | x^2 + 6 \le 5x\}$ then find $\frac{m}{M}$.

13. The exhaustive set of values of 'a' for which the function $f(x) = \frac{a}{3}x^3 + (a + 2)x^2 + (a - 1)x + 2$ possess a negative point of minimum is (q, ∞). The value of q is :

14. Least value of the function, $f(x) = 2^{x^2} - 1 + \frac{2}{2^{x^2}+1}$ is:

15. The three sides of a trapezium are equal each being 6 cms long. Let Δ cm² be the maximum area of the trapezium. The value of $\sqrt{3}\,\Delta$ is :

16. A sheet of poster has its area 18 m². The margin at the top & bottom are 75 cms. and at the sides 50 cms. Let ℓ, n are the dimensions of the poster in meters when the area of the printed space is maximum. The value of $\ell^2 + n^2$ is :

17. If the complete set of value(s) of 'a' for which the function $f(x) = \frac{ax^3}{3} + (a + 2)x^2 + (a - 1)x + 2$ possess a negative point of inflection is $(-\infty, \alpha) \cup (\beta, \infty)$, then $|\alpha| + |\beta|$ is :

18. If $p \in (0, 1/e)$ then the number of the distinct roots of the equation $|\ell n\, x| - px = 0$ is:

19. For $-1 \le p \le 1$, the equation $4x^3 - 3x - p = 0$ has 'n' distinct real roots in the interval $\left[\frac{1}{2}, 1\right]$ and one of its root is $\cos(k\cos^{-1}p)$, then the value of $n + \frac{1}{k}$ is :

20. Real root of the equation

$(x - 1)^{2013} + (x - 2)^{2013} + (x - 3)^{2013} + \ldots\ldots\ldots\ldots + (x - 2013)^{2013} = 0$ is a four digit number. Then the sum of the digits is :

21. Let $f(x) = \text{Max.}\{x^2, (1 - x)^2, 2x(1 - x)\}$ where $x \in [0, 1]$ If Rolle's theorem is applicable for f(x) on largest possible interval [a, b] then the value of 2(a + b + c) when $c \in (a, b)$ such that $f'(c) = 0$, is

22. For what value of a + m + b does the function $f(x) = \begin{bmatrix} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \le x \le 2 \end{bmatrix}$

satisfy the hypothesis of the mean value theorem for the interval [0, 2].

23. Let $f(x) = x^3 - ax + b$. Roll'es theorem can be applied on f(x) in [–2, 1]. Further the value of 'c' in (–2, 1) where $f'(c) = 0$ divides the interval (–2, 1) in the ratio 1 : 2, then find the value of $a^2 + b^2$.

24. f(x) is a twice differentiable function such that line joining p(a, f(a)) and Q(b, f(b)) meets y = f(x) at R, S, T, U, V also. (Note R, S, T, U, V do not coincide with P or Q). Find minimum number of roots of $f''(x) = 0$, given $b > a$.

PART - III : ONE OR MORE THAN ONE CORRECT

1. If tangent to curve $2y^3 = ax^2 + x^3$ at point (a, a) cuts off intercepts α, β on co-ordinate axes, where $\alpha^2 + \beta^2 = 61$, then the value of 'a' is equal to

(A) 20 (B) 25 (C) 30 (D) – 30

2. The equation of normal to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ (n is even natural number) at the point with abscissa equal to 'a' can be:

(A) $ax + by = a^2 - b^2$ (B) $ax + by = a^2 + b^2$ (C) $ax - by = a^2 - b^2$ (D) $bx - ay = a^2 - b^2$

3. The equation of line which is tangent at a point on curve $4x^3 = 27y^2$ and normal at other point are :

(A) $y = \sqrt{2}x - 2\sqrt{2}$ (B) $y = -\sqrt{2}x - 2\sqrt{2}$ (C) $y = \sqrt{2}x + 2\sqrt{2}$ (D) $y = -\sqrt{2}x + 2\sqrt{2}$

4. If the curves $y = 2(x - a)^2$ and $y = e^{2x}$ touches each other, then 'a' is less than-

(A) –1 (B) 0 (C) 1 (D) 2

5. For the curve C : $y = e^{2x}\cos x$, which of the following statement(s) is/are true ?

(A) equation of the tangent where C crosses y-axis is $y = 3x + 1$

(B) equation of the tangent where C crosses y-axis is $y = 2x + 1$

(C) number of points in $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ where tangent on the curve C is parallel to x-axis is 4.

(D) number of points in $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ where tangent on the curve C is parallel to x-axis is 2.

6. For the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$, at point (2, – 1)

(A) length of subtangent is 7/6. (B) slope of tangent = 6/7

(C) length of tangent = $\sqrt{(85)}/6$ (D) none of these

7. Given that g(x) is a non constant linear function defined on R-

(A) $y = g(x)$ and $y = g^{-1}(x)$ are orthogonal (B) $y = g(x)$ and $y = g^{-1}(-x)$ are orthogonal

(C) $y = g(-x)$ and $y = g^{-1}(x)$ are orthogonal (D) $y = g(-x)$ and $y = g^{-1}(-x)$ are orthogonal

8. Let A(p,q) and B(h,k) are points on the curve $4x^2 + 9y^2 = 1$, which are nearest and farthest from the line $72 + 8x = 9y$ respectively, then -

(A) $p + q = -\frac{1}{5}$ (B) $p + q = \frac{1}{5}$ (C) $h + k = -\frac{1}{5}$ (D) $h + k = \frac{1}{5}$

9. Which of the following statements is/are correct ?

(A) x + sinx is increasing function

(B) sec x is neither increasing nor decreasing function

(C) x + sinx is decreasing function

(D) sec x is an increasing function

10. Let $f(x) = x^{m/n}$ for $x \in R$ where m and n are integers, m even and n odd and $0 < m < n$. Then

(A) f(x) decreases on $(-\infty, 0]$ (B) f(x) increases on $[0, \infty)$

(C) f(x) increases on $(-\infty, 0]$ (D) f(x) decreases on $[0, \infty)$

11. Let f and g be two differentiable functions defined on an interval I such that $f(x) \geq 0$ and $g(x) \leq 0$ for all $x \in I$ and f is strictly decreasing on I while g is strictly increasing on I then

(A) the product function fg is strictly increasing on I

(B) the product function fg is strictly decreasing on I

(C) fog(x) is monotonically increasing on I

(D) fog (x) is monotonically decreasing on I

12. Let $g(x) = 2f(x/2) + f(1 - x)$ and $f''(x) < 0$ in $0 \leq x \leq 1$ then g(x)

(A) decreases in $\left[0, \frac{2}{3}\right]$

(B) decreases $\left[\frac{2}{3}, 1\right]$

(C) increases in $\left[0, \frac{2}{3}\right]$

(D) increases in $\left[\frac{2}{3}, 1\right]$

13. Let $\phi(x) = (f(x))^3 - 3(f(x))^2 + 4f(x) + 5x + 3 \sin x + 4 \cos x \ \forall x \in R$, where f(x) is a differentiable function $\forall x \in R$, then

(A) ϕ is increasing whenever f is increasing

(B) ϕ is increasing whenever f is decreasing

(C) ϕ is decreasing whenever f is decreasing

(D) ϕ is decreasing if $f'(x) = -11$

14. Let $f(x) = (x^2 - 1)^n (x^2 + x + 1)$. f(x) has local extremum at x = 1 if

(A) n = 2 (B) n = 3 (C) n = 4 (D) n = 6

15. If $f(x) = \frac{x}{1 + x\tan x}$, $x \in \left(0, \frac{\pi}{2}\right)$, then

(A) f(x) has exactly one point of minima

(B) f(x) has exactly one point of maxima

(C) f(x) is increasing in $\left(0, \frac{\pi}{2}\right)$

(D) maxima occurs at x_0 where $x_0 = \cos x_0$

16. If $f(x) = \begin{cases} -\sqrt{1-x^2} & , \ 0 \leq x \leq 1 \\ -x & , \ x > 1 \end{cases}$, then

(A) Maximum of f(x) exist at x = 1

(B) Maximum of f (x) doesn't exists

(C) Minimum of $f^{-1}(x)$ exist at x = – 1

(D) Minimum of $f^{-1}(x)$ exist at x = 1

17. Let $f(x) = \frac{40}{3x^4 + 8x^3 - 18x^2 + 60}$. Which of the following statement(s) about f(x) is (are) correct ?

(A) f(x) has local minima at x = 0.

(B) f(x) has local maxima at x = 0.

(C) Absolute maximum value of f(x) is not defined.

(D) f(x) is local maxima at x = – 3, x = 1.

18. Let $f(x, y) = \sqrt{x^2 + y^2} + \sqrt{x^2 + y^2 - 2x + 1} + \sqrt{x^2 + y^2 - 2y + 1} + \sqrt{x^2 + y^2 - 6x - 8y + 25} \quad \forall x, y \in R$, then-

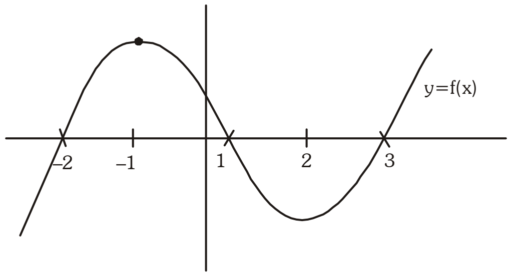
(A) Minimum value of $f(x,y) = 5 + \sqrt{2}$

(B) Minimum value of $f(x,y) = 5 - \sqrt{2}$

(C) Minimum value occurs of $f(x,y)$ for $x = \frac{3}{7}$

(D) Minimum value occurs of $f(x,y)$ for $y = \frac{4}{7}$

19. Graph of y = f(x) is given, then



(A) $y = H(x) = \max\{f(t) : t \le x\}\ \forall\ x \in R$ has no point of extrema

(B) y = |f(x)| has 5 point of extrema

(C) y = f(|x|) has 5 points of extrema

(D) y = f(|x|) has 3 points of extrema

20. For the function $f(x) = x^4 (12\,\ell n\, x - 7)$

(A) the point (1, –7) is the point of inflection

(B) $x = e^{1/3}$ is the point of minima

(C) the graph is concave downwards in (0, 1)

(D) the graph is concave upwards in (1, ∞)

21. The curve $y = \frac{x+1}{x^2+1}$ has

(A) x = 1, as point of inflection

(B) $x = -2 + \sqrt{3}$, as point of inflection

(C) x = –1, as point of minimum

(D) $x = -2 - \sqrt{3}$, as point of inflection

22. Let $f(x) = ax^3 + bx^2 + cx + 1$ have extrema at $x = \alpha, \beta$ such that $\alpha\beta < 0$ and $f(\alpha) \cdot f(\beta) < 0$, then which of the following can be true for the equation $f(x) = 0$?

(A) three equal roots

(B) three distinct real roots

(C) one positive root if $f(\alpha) < 0$ and $f(\beta) > 0$

(D) one negative root if $f(\alpha) > 0$ and $f(\beta) < 0$.

23. For the function $f(x) = x \cot^{-1}x,\ x \ge 0$

(A) there is atleast one $x \in (0, 1)$ for which $\cot^{-1}x = \frac{x}{1+x^2}$

(B) for atleast one x in the interval (0, ∞), $f\left(x + \frac{2}{\pi}\right) - f(x) < 1$

(C) number of solution of the equation f(x) = sec x is 1

(D) f'(x) is strictly decreasing in the interval (0, ∞)

24. Which of the following statements are true :

(A) $|\tan^{-1} x - \tan^{-1} y| \le |x - y|$, where x, y are real numbers.

(B) The function $x^{100} + \sin x - 1$ is strictly increasing in [0, 1]

(C) If a, b, c are is A.P, then at least one root of the equation $3ax^2 - 4bx + c = 0$ is positive

(D) The number of solution(s) of equation $3 \tan x + x^3 = 2$ in $(0, \pi/4)$ is 2

25. Let f(x) be a differentiable function and $f(\alpha) = f(\beta) = 0$ $(\alpha < \beta)$, then in the interval (α, β)

(A) $f(x) + f'(x) = 0$ has at least one root

(B) $f(x) - f'(x) = 0$ has at least one real root

(C) $f(x) . f'(x) = 0$ has at least one real root

(D) none of these

26. For all x in [1, 2]

Let f"(x) of a non-constant function f(x) exist and satisfy $|f''(x)| \le 2$. If f(1) = f(2), then

(A) There exist some $a \in (1, 2)$ such that $f'(a) = 0$

(B) f(x) is strictly increasing in (1, 2)

(C) There exist atleast one $c \in (1, 2)$ such that $f'(c) > 0$

(D) $|f'(x)| < 2 \ \forall \ x \in [1, 2]$

27. Which of the following is/are correct ?

(A) Between any two roots of $e^x \cos x = 1$, there exists atleast one root of $\tan x = 1$.

(B) Between any two roots of $e^x \sin x = 1$, there exists atleast one root of $\tan x = -1$.

(C) Between any two roots of $e^x \cos x = 1$, there exists atleast one root of $e^x \sin x = 1$.

(D) Between any two roots of $e^x \sin x = 1$, there exists atleast one root of $e^x \cos x = -1$.

28. For the equation $\frac{e^{-x}}{x+1} = a$; which of the following statement(s) is/are correct ?

(A) If $a \in (0,\infty)$, then equation has 2 real and distinct roots

(B) If $a \in (-\infty,-e^2)$, then equation has 2 real & distinct roots.

(C) If $a \in (0,\infty)$, then equation has 1 real root

(D) If $a \in (-e,0)$, then equation has no real root.

29. Let $f(x) = \sin^2 x$ & $0 < A < B$, where $A, B \in \left(0, \frac{\pi}{4}\right)$, then

(A) $(B - A) \sin 2A < \sin^2 B - \sin^2 A < (B - A)\sin 2B$

(B) $(B - A)\sin 2A > \sin^2 B - \sin^2 A > (B - A)\sin 2B$

(C) $\cos 2B < \cos 2A$

(D) $B\cos 2A > A\cos 2B$

30. Let $f : \left[0, \frac{\pi}{2}\right] \to [0,1]$ be a differentiable function such that f(0) = 0, $f\left(\frac{\pi}{2}\right) = 1$, then

(A) $f'(\alpha) = \sqrt{1-f^2(\alpha)}$ for all $\alpha \in \left(0, \frac{\pi}{2}\right)$.

(B) $f'(\alpha) = \frac{2}{\pi}$ for all $\alpha \in \left(0, \frac{\pi}{2}\right)$.

(C) $f(\alpha) f'(\alpha) = \frac{1}{\pi}$ for atleast one $\alpha \in \left(0, \frac{\pi}{2}\right)$

(D) $f'(\alpha) = \frac{8\alpha}{\pi^2}$ for atleast one $\alpha \in \left(0, \frac{\pi}{2}\right)$.

31. Let f(x) be a non-negative function. Further x = 0 and x = 2 are two consecutive roots of f(x) = 0. If f(x) is twice differentiable $\forall\ x \in [0,2]$, then

(A) f'(x) = 0 has a root in (0, 2)

(B) f''(x) < 0 for atleast one x $\in$ (0, 2)

(C) f(x) has atleast one point of maxima in (0, 2)

(D) If $|f''(x)| \le 2,\ \forall\ x \in [0,2]$, then $|f'(x)| \le 4,\ \forall\ x \in [0,2]$

PART - IV : COMPREHENSION

Comprehension # 1

Let P(x) be a polynomial of degree 4 and it vanishes at x = 0. Given P(–1) = 55 and P(x) has relative maximum/relative minimum at x = 1, 2, 3.

1. Area of the triangle formed by extremum points of P(x), is

(A) 1 (B) 2 (C) 3 (D) 4

2. If the number of negative integers in the range of P(x) is λ, then λ equals–

(A) 6 (B) 7 (C) 8 (D) 9

3. If P(x) + μ = 0 has four distinct roots. then μ lies in interval

(A) (1, 2) (B) (8, 9) (C) (3, 4) (D) (–5, 7)

Comprehension # 2

Consider a function f defined by f(x) = $\sin^{-1} \sin\left(\frac{x + \sin x}{2}\right)$, $\forall$ x $\in$ [0, π], which satisfies

f(x) + f(2π – x) = π, $\forall$ x $\in$ [π, 2π] and f(x) = f(4π – x) for all x $\in$ [2π, 4π], then

4. If α is the length of the largest interval on which f(x) is increasing, then α =

(A) $\frac{\pi}{2}$ (B) π (C) 2π (D) 4π

5. If f(x) is symmetric about x = β, then β =

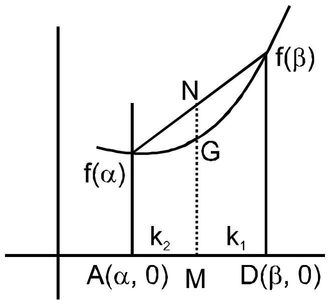
(A) $\frac{\alpha}{2}$ (B) α (C) $\frac{\alpha}{4}$ (D) 2α

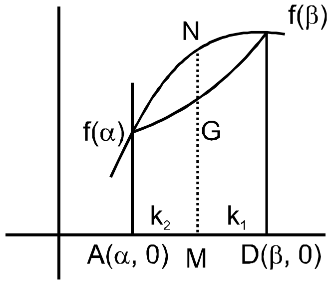
6. Maximum value of f(x) on [0, 4π] is :

(A) $\frac{\beta}{2}$ (B) β (C) $\frac{\beta}{4}$ (D) 2β

Comprehension # 3

For a double differentiable function f(x) if $f''(x) \geq 0$ then f(x) is concave upward and if $f''(x) \leq 0$ then f(x) is concave downward





Here $M\left(\frac{k_1\alpha + k_2\beta}{k_1 + k_2}, 0\right)$

If f(x) is a concave upward in [a, b] and $\alpha, \beta \in [a, b]$ then $\frac{k_1 f(\alpha) + k_2 f(\beta)}{k_1 + k_2} \geq f\left(\frac{k_1\alpha + k_2\beta}{k_1 + k_2}\right)$, where $k_1, k_2 \in R^+$

If f(x) is a concave downward in [a, b] and $\alpha, \beta \in [a, b]$ then $\frac{k_1 f(\alpha) + k_2 f(\beta)}{k_1 + k_2} \leq f\left(\frac{k_1\alpha + k_2\beta}{k_1 + k_2}\right)$, where $k_1, k_2 \in R^+$

then answer the following

7. Which of the following is true

(A) $\frac{\sin\alpha + \sin\beta}{2} > \sin\left(\frac{\alpha+\beta}{2}\right)$; $\alpha, \beta\ (0, \pi)$

(B) $\frac{\sin\alpha + \sin\beta}{2} < \sin\left(\frac{\alpha+\beta}{2}\right)$; $\alpha, \beta \in (\pi, 2\pi)$

(C) $\frac{\sin\alpha + \sin\beta}{2} < \sin\left(\frac{\alpha+\beta}{2}\right)$; $\alpha, \beta \in (0, \pi)$

(D) None of these

8. Which of the following is true

(A) $\frac{2^{\alpha} + 2^{\beta+1}}{3} \leq 2^{\frac{\alpha+2\beta}{3}}$

(B) $\frac{2\ell n\,\alpha + \ell n\beta}{3} \geq \ell n\left(\frac{2\alpha+\beta}{3}\right)$

(C) $\frac{\tan^{-1}\alpha + \tan^{-1}\beta}{2} \leq \tan^{-1}\left(\frac{\alpha+\beta}{2}\right)$ a, b $\in R^-$

(D) $\frac{e^{\alpha} + 2e^{\beta}}{3} \geq e^{\frac{\alpha+2\beta}{3}}$

9. Let α , β and γ are three distinct real numbers and $f''(x) < 0$. Also f(x) is increasing function and let

$A = \frac{f^{-1}(\alpha) + f^{-1}(\beta) + f^{-1}(\gamma)}{3}$ and $B = f^{-1}\left(\frac{\alpha+\beta+\gamma}{3}\right)$, then order relation between A and B is ?

(A) A > B (B) A < B (C) A = B (D) none of these

Exercise # 3

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. Let f be a function defined on **R** (the set of all real numbers) such that

$f'(x) = 2010\,(x - 2009)\,(x - 2010)^2\,(x - 2011)^3\,(x - 2012)^4$, for all $x \in$ **R**.

If g is a function defined on **R** with values in the interval $(0, \infty)$ such that $f(x) = \ell n\,(g(x))$, for all $x \in$ **R**, then the number of points in **R** at which g has a local maximum is **[IIT-JEE 2010, Paper-2, (3, 0)/ 79]**

2. Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote, respectively, the absolute maximum of f, g and h on [0, 1], then

[IIT-JEE 2010, Paper-1, (3, –1)/ 84]

(A) $a = b$ and $c \neq b$ (B) $a = c$ and $a \neq b$

(C) $a \neq b$ and $c \neq b$ (D) $a = b = c$

3. Match the statements given in **Column-I** with the intervals/union of intervals given in **Column-II**

[IIT-JEE 2011, Paper-2, (8, 0), 80]

	Column-I		**Column-II**
(A)	The set $\left\{\mathrm{Re}\left(\dfrac{2iz}{1-z^2}\right) : z \text{ is a complex number}, \lvert z\rvert = 1, z \neq \pm 1\right\}$ is	(p)	$(-\infty, -1) \cup (1, \infty)$
(B)	The domain of the function $f(x) = \sin^{-1}\left(\dfrac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q)	$(-\infty, 0) \cup (0, \infty)$
(C)	If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \leq \theta < \dfrac{\pi}{2}\right\}$ is	(r)	$[2, \infty)$
(D)	If $f(x) = x^{3/2}(3x - 10)$, $x \geq 0$, then f(x) is increasing in	(s)	$(-\infty, -1] \cup [1, \infty)$
		(t)	$(-\infty, 0] \cup [2, \infty)$

4. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is **[IIT-JEE 2011, Paper-2, (4, 0), 80]**

5. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If $p(1) = 6$ $p(3) = 2$, then $p'(0)$ is **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

6. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **[IIT-JEE 2012, Paper-1, (4, 0), 70]**

7. The number of points in $(-\infty, \infty)$, for which $x^2 - x \sin x - \cos x = 0$, is

[JEE (Advanced) 2013, Paper-1, (2, 0)/60]

(A) 6 (B) 4 (C) 2 (D) 0

8. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. The lengths of the sides of the rectangular sheet are **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

(A) 24 (B) 32 (C) 45 (D) 60

9. A vertical line passing through the point (h, 0) intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max\limits_{1/2 \le h \le 1} \Delta(h)$ and $\Delta_2 = \min\limits_{1/2 \le h \le 1} \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ **[JEE (Advanced) 2013, Paper-1, (4, – 1)/60]**

10. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at x =

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A) -2 (B) $\frac{-2}{3}$ (C) 2 (D) $\frac{2}{3}$

Paragraph for Question Nos. 11 and 12

Let $f : [0, 1] \to R$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \ge e^x$, $x \in [0, 1]$.

11. Which of the following is true for $0 < x < 1$? **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]**

(A) $0 < f(x) < \infty$

(B) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(C) $-\frac{1}{4} < f(x) < 1$

(D) $-\infty < f(x) < 0$

12. If the function $e^{-x} f(x)$ assumes its minimum in the interval [0, 1] at $x = \frac{1}{4}$, which of the following is true ?

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

(A) $f'(x) < f(x)$, $\frac{1}{4} < x < \frac{3}{4}$

(B) $f'(x) > f(x)$, $0 < x < \frac{1}{4}$

(C) $f'(x) < f(x)$, $0 < x < \frac{1}{4}$

(D) $f'(x) < f(x)$, $\frac{3}{4} < x < 1$

13. A line L : $y = mx + 3$ meets y - axis at E(0, 3) and the arc of the parabola $y^2 = 16x$, $0 \le y \le 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the y-axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum.

Match List I with List II and select the correct answer using the code given below the lists :

[JEE (Advanced) 2013, Paper-2, (3, –1)/60]

	List - I		**List - II**
P.	m =	1.	$\frac{1}{2}$
Q.	Maximum area of ΔEFG is	2.	4
R.	y_0 =	3.	2
S.	y_1 =	4.	1

Codes :

	P	Q	R	S
(A)	4	1	2	3
(B)	3	4	1	2
(C)	1	3	2	4
(D)	1	3	4	2

14. Let $a \in R$ and let $f : R \to R$ be given by $f(x) = x^5 - 5x + a$. Then

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) f(x) has three real roots if $a > 4$
(B) f(x) has only one real root if $a > 4$
(C) f(x) has three real roots if $a < -4$
(D) f(x) has three real roots if $-4 < a < 4$

15. A cylindrical container is to be made from certain solid material with the following constraints: It has fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container. If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is **[JEE (Advanced) 2015, P-1 (4, 0) /88]**

16. Let $F(x) = \int_{x}^{x^2+\frac{\pi}{6}} 2\cos^2 t \, dt$ for all $x \in R$ and $f:\left[0,\frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0,\frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then f(0) is

[JEE (Advanced) 2015, P-1 (4, 0) /88]

17. Let $f, g : [-1, 2] \to R$ be continuous function which are twice differentiable on the interval (–1, 2). Let the values of f and g at the points –1, 0 and 2 be as given in the following table :

	x = –1	x = 0	x = 2
f(x)	3	6	0
g(x)	0	1	– 1

In each of the intervals (–1, 0) and (0, 2) the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is (are) **[JEE (Advanced) 2015, P-2 (4, –2)/ 80]**

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$
(B) $f'(x) - 3g'(x) = 0$ has exactly one solution in (–1, 0)
(C) $f'(x) - 3g'(x) = 0$ has exactly one solution in (0, 2)
(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in (–1, 0) and exactly two solutions in (0, 2)

18. Let $f : \mathbb{R} \to (0, \infty)$ and $g : \mathbb{R} \to \mathbb{R}$ be twice differentiable function such that f'' and g'' are continuous functions on $\mathbb{R}$. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim\limits_{x \to 2} \dfrac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

[JEE(Advanced)-2016, P-2 (4, –2)/62]

(A) f has a local minimum at x = 2 (B) f has a local maximum at x = 2

(C) $f''(2) > f(2)$ (D) $f(x) - f''(x) = 0$ for at least one $x \in \mathbb{R}$

Answer Q.19, Q.20 and Q.21 by appropriately matching the information given in the three columns of the following table.

Let $f(x) = x + \log_e x - x \log_e x$, $x \in (0,\infty)$.

* Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.
* Column 2 contains information about the limiting behavior of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.
* Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

	Column 1		**Column 2**		**Column 3**
(I)	$f(x) = 0$ for some $x \in (1,e^2)$	(i)	$\lim_{x\to\infty} f(x) = 0$	(P)	f is increasing in (0,1)
(II)	$f'(x) = 0$ for some $x \in (1,e)$	(ii)	$\lim_{x\to\infty} f(x) = -\infty$	(Q)	f is decreasing in (e,e^2)
(III)	$f'(x) = 0$ for some $x \in (0,1)$	(iii)	$\lim_{x\to\infty} f'(x) = -\infty$	(R)	f' is increasing in (0,1)
(IV)	$f''(x) = 0$ for some $x \in (1,e)$	(iv)	$\lim_{x\to\infty} f''(x) = 0$	(S)	f' is decreasing in (e,e^2)

19. Which of the following options is the only **CORRECT** combination ?

[JEE(Advanced)-2017, P-1 (3, –1)/61]

(A) (IV) (i) (S) (B) (I) (ii) (R) (C) (III) (iv) (P) (D) (II) (iii) (S)

20. Which of the following options is the only **CORRECT** combination ?

[JEE(Advanced)-2017, P-1 (3, –1)/61]

(A) (III) (iii) (R) (B) (I) (i) (P) (C) (IV) (iv) (S) (D) (II) (ii) (Q)

21. Which of the following options is the only **INCORRECT** combination ?

[JEE(Advanced)-2017, P-1 (3, –1)/61]

(A) (II) (iii) (P) (B) (II) (iv) (Q) (C) (I) (iii) (P) (D) (III) (i) (R)

22. If f(x) = $\begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then **[JEE(Advanced)-2017, P-2 (4, –2)/61]**

(A) f'(x) = 0 at exactly three points in $(-\pi, \pi)$ (B) f(x) attains its maximum at x = 0

(C) f(x) attains its minimum at x = 0 (D) f'(x) = 0 at more than three points in $(-\pi, \pi)$

23. If $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in \mathbb{R}$, and $f\left(\frac{1}{2}\right) = \frac{1}{2},\ f(1) = 1$, then

[JEE(Advanced)-2017, P-2 (4, –2)/61]

(A) $0 < f'(1) \leq \frac{1}{2}$ (B) $f'(1) \leq 0$ (C) $f'(1) > 1$ (D) $\frac{1}{2} < f'(1) \leq 1$

24. For every twice differentiable function $f : \mathbb{R} \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, P-1 (4, –2)/60]**

(A) There exist r, s $\in \mathbb{R}$, where r < s, such that f is one-one on the open interval (r, s).

(B) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \le 1$

(C) $\lim\limits_{x\to\infty} f(x) = 1$

(D) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

25. Let $f : \mathbb{R} \to \mathbb{R}$ be given by **[JEE(Advanced)-2019, P-1 (4, –1)/63]**

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \ge 3 \end{cases}$$

Then which of the following options is/are correct ?

(1) f' has a local maximum at x = 1 (2) f is onto

(3) f is increasing on $(-\infty, 0)$ (4) f' is NOT differentiable at x = 1

26. Let $f(x) = \frac{\sin \pi x}{x^2}, x > 0$ **[JEE(Advanced)-2019, P-2 (4, –1)/63]**

Let $x_1 < x_2 < x_3 < ... < x_n < ...$ be all the points of local maximum of f and $y_1 < y_2 < y_3 < ... < y_n < ...$ be all the points of local minimum of f.

Then which of the following options is/are correct ?

(1) $|x_n - y_n| > 1$ for every n (2) $x_1 < y_1$

(3) $x_n \in \left(2n, 2n + \frac{1}{2}\right)$ for every n (4) $x_{n+1} - x_n > 2$ for every n

PART - II : JEE(MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. Let f : **R** $\to$ **R** be defined by **[AIEEE 2010(8, –2), 144]**

$$f(x) = \begin{cases} k - 2x, & \text{if } x \le -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$$

If f has a local minimum at x = – 1, then a possible value of k is

(1) 0 (2) $-\frac{1}{2}$ (3) –1 (4) 1

2. Let f : **R** $\to$ **R** be a continuous function defined by $f(x) = \dfrac{1}{e^x + 2e^{-x}}$ **[AIEEE 2010(8, –2), 144]**

Statement -1 : $f(c) = \dfrac{1}{3}$, for some $c \in R$.

Statement -2 : $0 < f(x) \le \dfrac{1}{2\sqrt{2}}$, for all $x \in R$.

(1) Statement -1 is true, Statement-2 is true ; Statement -2 is not a correct explanation for Statement -1.
(2) Statement-1 is true, Statement-2 is false.
(3) Statement -1 is false, Statement -2 is true.
(4) Statement -1 is true, Statement -2 is true; Statement-2 is a correct explanation for Statement-1.

3. Let f be a function defined by - **[AIEEE 2011 (4, –1), 120]**

$$f(x) = \begin{cases} \dfrac{\tan x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Statement - 1 : x = 0 is point of minima of f
Statement - 2 : $f'(0) = 0$.
(1) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for statement-1.
(2) Statement-1 is true, statement-2 is true; statement-2 is not a correct explanation for statement-1
(3) Statement-1 is true, statement-2 is false.
(4) Statement-1 is false, statement-2 is true.

4. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is : **[AIEEE 2012(4, –1), 120]**

(1) $\dfrac{9}{7}$ (2) $\dfrac{7}{9}$ (3) $\dfrac{2}{9}$ (4) $\dfrac{9}{2}$

5. Let $a, b \in R$ be such that the function f given by $f(x) = \ell n\,|x| + bx^2 + ax$, $x \neq 0$ has extreme values at x = – 1 and x = 2.
Statement-1 : f has local maximum at x = – 1 and at x = 2. **[AIEEE 2012 (4, –1), 120]**

Statement-2 : $a = \dfrac{1}{2}$ and $b = \dfrac{-1}{4}$.

(1) Statement-1 is false, Statement-2 is true.
(2) Statement-1 is true, statement-2 is true; statement-2 is a correct explanation for Statement-1.
(3) Statement-1 is true, statement-2 is true; statement-2 is **not** a correct explanation for Statement-1.
(4) Statement-1 is true, statement-2 is false.

6. The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1]
[AIEEE - 2013, (4, –1/4),120]

(1) lies between 1 and 2 (2) lies between 2 and 3
(3) lies between –1 and 0 (4) does not exist.

7. If f and g are differentiable functions in [0, 1] satisfying f(0) = 2 = g(1), g(0) = 0 and f(1) = 6, then for some $c \in]0, 1[$: **[JEE(Main) 2014, (4, – ¼), 120]**

(1) f'(c) = g'(c) (2) f'(c) = 2g'(c) (3) 2f'(c) = g'(c) (4) 2f'(c) = 3g'(c)

8. If x = –1 and x = 2 are extreme points of $f(x) = \alpha\log|x| + \beta x^2 + x$ then : **[JEE(Main) 2014, (4, –1), 120]**

(1) $\alpha = 2, \beta = -\frac{1}{2}$ (2) $\alpha = 2, \beta = \frac{1}{2}$ (3) $\alpha = -6, \beta = \frac{1}{2}$ (4) $\alpha = -6, \beta = -\frac{1}{2}$

9. Let f(x) be a polynomial of degree four having extreme values at x = 1 and x =2. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right]=3$, then f(2) is equal to : **[JEE(Main) 2015, (4, – ¼), 120]**

(1) – 8 (2) – 4 (3) 0 (4) 4

10. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then : **[JEE(Main) 2016, (4, – 1),120]**

(1) 2x = r (2) $2x = (\pi + 4)r$ (3) $(4 - \pi)x = \pi r$ (4) x = 2r

11. Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $x \in \left(0, \frac{\pi}{2}\right)$. A normal to y = f(x) at $x = \frac{\pi}{6}$ also passes through the point : **[JEE(Main) 2016, (4, – 1),120]**

(1) $\left(\frac{\pi}{4}, 0\right)$ (2) (0, 0) (3) $\left(0, \frac{2\pi}{3}\right)$ (4) $\left(\frac{\pi}{6}, 0\right)$

12. The normal to the curve y(x – 2)(x – 3) = x + 6 at the point where the curve intersects the y-axis passes through the point : **[JEE(Main) 2017, (4, – 1),120]**

(1) $\left(\frac{1}{2}, \frac{1}{3}\right)$ (2) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (3) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (4) $\left(\frac{1}{2}, -\frac{1}{3}\right)$

13. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower -bed, is :- **[JEE(Main) 2017, (4, – 1),120]**

(1) 30 (2) 12.5 (3) 10 (4) 25

14. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is : **[JEE(Main) 2018, (4, – 1),120]**

(1) $\frac{7}{2}$ (2) 4 (3) $\frac{9}{2}$ (4) 6

15. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of h(x) is : **[JEE(Main) 2018, (4, – 1),120]**

(1) –3 (2) $-2\sqrt{2}$ (3) $2\sqrt{2}$ (4) 3

16. The maximum volume (in cu.m) of the right circular cone having slant height 3m is

[JEE(Main) 2019, Oniline (09-01-20) P-1 (4, – 1),100]

(1) 6π (2) $3\sqrt{3}\pi$ (3) $\frac{4}{3}\pi$ (4) $2\sqrt{3}\pi$

17. Let A(4, –4) and B (9, 6) be points on the parabola $y^2 = 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of ΔACB is maximum. Then, the area (in sq. units) of ΔACB, is :

[JEE(Main) 2019, Oniline (09-01-20) P-1 (4, – 1),100]

(1) $31\frac{3}{4}$ (2) 32 (3) $30\frac{1}{2}$ (4) $31\frac{1}{4}$

18. Let the function, $f : [-7, 0] \to R$ be continuous on $[-7, 0]$ and differentiable on $(-7, 0)$. If $f(-7) = -3$ and $f'(x) \le 2$, for all $x \in (-7, 0)$, then for all such functions f, $f(-1) + f(0)$ lies in the interval :

[JEE(Main) 2020, Oniline (07-01-20) P-1 (4, – 1),100]

(1) $[-6, 20]$ (2) $(-\infty, 20]$ (3) $(-\infty, 11]$ (4) $[-3, 11]$

19. The value of c in the Lagrange's mean value theorem for the function $f(x) = x^3 - 4x^2 + 8x + 11$, when $x \in [0, 1]$ is :

[JEE(Main) 2020, Oniline (07-01-20) P-2 (4, – 1),100]

(1) $\frac{2}{3}$ (2) $\frac{\sqrt{7}-2}{3}$ (3) $\frac{4-\sqrt{5}}{3}$ (4) $\frac{4-\sqrt{7}}{3}$

20. Let f(x) be a polynomial of degree 5 such that x = ±1 are its critical points. If $\lim_{x\to 0}\left(2+\frac{f(x)}{x^3}\right)=4$, then which one of the following is not true?

[JEE(Main) 2020, Oniline (07-01-20) P-2 (4, – 1),100]

(1) f is an odd function

(2) x = 1 is a point of minima and x = –1 is a point of maxima of f.

(3) x = 1 is a point of maxima and x = –1 is a point of minimum of f.

(4) $f(1) - 4f(-1) = 4$

Answers

Exercise # 1

PART - I

SECTION-(A)

A-1. (i) tangent : y = 4x + 5, normal: x + 4y – 20 = 0
(ii) tangent : y + x = 2, normal: y = x
(iii) tangent : 16x + 13y = 9a,
normal: 13x – 16y = 2a
(iv) tangent : y = 0, normal: x = 0

A-2. a = 1, b = $\frac{-5}{2}$

A-3. x + 2y = π/2 & x + 2y = – 3π/2

A-4. y = x **A-5.** a = 1, b = –2

A-6. (1, –1), (–1, –5) **A-8.** (9/4, 3/8)

A-10. (i) 2x + y = 4, y = 2x
(ii) y = 0, $y - \frac{4096}{81} = \frac{2048}{81}(x - 8/3)$
(iii) 4y = 9x + 4, 4y = x + 36
(iv) x = 1, 5x – 4y = 1

A-11. x + y = 3 **A-12.** 2

A-13. 16 (y – 1) = –(x – 8)

A-14. (9/2)

SECTION-(B)

B-1. $\frac{\pi}{3}$

B-2. $\frac{\pi}{2}$ at (0, 0), $\tan^{-1}\left(\frac{3}{5}\right)$ at (16a,8a)

B-3. $\pm\frac{1}{2\sqrt{2}}$ **B-4.** $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$

B-5. (2, 1) **B-6.** (– 6, 3)

B-7. 10

SECTION-(C)

C-1. (i) – 2 cm/min, (ii) 2 cm^2/min

C-2. $2x^2 - 3x + 1$ **C-3.** (i) 2 km/hr, (ii) 6 km/h

C-4. 1/9π m/min **C-5.** $\frac{\sqrt{2}}{4\pi}$ cm/s

C-7. (i) (a) 6.05, (b) $\frac{80}{27}$

C-8. 7.5 m^3

SECTION-(D)

D-1. (a) Strictly increasing in [2 , ∞) & Strictly decreasing in (– ∞, 2]
(b) Strictly increasing in [1 , ∞) & Strictly decreasing in (– ∞ , 0) ; (0 , 1]
(c) Strictly increasing in [0, 2] & Strictly decreasing in (– ∞, 0] ; [2 , ∞)
(d) Strictly increasing in $\left[-\frac{1}{2},0\right);\left[\frac{1}{2},\infty\right)$ & Strictly decreasing in $\left(-\infty,-\frac{1}{2}\right];\left(0,\frac{1}{2}\right]$
(e) Strictly increasing in [– 3 , 0]; [2 , ∞) & Strictly decreasing in (– ∞, –3]; [0 , 2]
(f) Strictly increasing in $\left[\frac{1}{\sqrt{3}},\infty\right)$ & Strictly decreasing in $\left(0,\frac{1}{\sqrt{3}}\right]$

D-2. (a) Strictly increasing in [0, 3π/4]; [7π/4 , 2 π] & Strictly decreasing in [3π/4 , 7 π/4]
(b) Strictly increasing in [0 , π/6]; [π/2 , 5π/6]; [3π/2 , 2 π] & Strictly decreasing in [π/6 , π/2]; [5π/6, 3 π/2]
(c) Strictly increasing in [0, π/2] ∪ [3π/2, 2π] and Strictly decreasing in [π/2, 3π/2]

D-5. (i) Neither increasing nor decreasing, increasing
(ii) at x = – 2 decreasing
at x = 0 decreasing
at x = 3 neither increasing nor decreasing
at x = 5 increasing
(iii) Strictly increasing at x = 0
(iv) Strictly increasing at x = 2, neither I nor D at x = 1
(v) Strictly increasing at x = 0

D-6. $a \in R^+$

D-8. 2sinx + tanx, 0

D-10. (i) local max at x = 1, local min at x = 6

(ii) local max. at $x = -\frac{1}{5}$, local min. at x = – 1

(iii) local mini at $x = \frac{1}{e}$, No local maxima

(iv) local maxima at $x = \log_2 \frac{4}{3}$ and local minima at x = 1

(v) local min at 0, local max at 2

(vi) local max at x = 0, $\frac{2\pi}{3}$, local min at $x = \frac{\pi}{2}$, π

(vii) local maxima at –1 and local minma at 0

(viii) local minima at $x = \pm\sqrt{2}$, 0

D-11. local max at x = 1, local min at x = 2.

D-13. $b \in (0, e]$

SECTION-(E)

E-1. (i) max = 8, min. = – 8

(ii) max = $\sqrt{2}$, min = – 1

(iii) max. = 8, min. = – 10

(iv) max. = 25, min = – 39

(v) max. = 3/4; min. = 1/2

(vi) max = 2π, min. = 0

(vii) max = 3/2, min = – 3

E-2. 12cm, 6 cm **E-3.** F = 191

E-5. $\frac{4\pi r^3}{3\sqrt{3}}$ **E-7.** 110 m , $\frac{220}{\pi}$ m

E-8. 32 sq. units **E-9.** 1, 3 (respective)

PART-II :

SECTION-(A)

A-1. (B) **A-2.** (C)
A-3. (A) **A-4.** (C)
A-5. (A) **A-6.** (C)
A-7. (C) **A-8.** (B)
A-9. (B) **A-10.** (C)
A-11. (B) **A-12.** (B)
A-13. (B) **A-14.** (D)
A-15. (B) **A-16.** (C)

SECTION-(B)

B-1. (D) **B-2.** (D)
B-3. (B) **B-4** (C)
B-5. (A) **B-6.** (C)

SECTION-(C)

C-1. (B) **C-2** (C)
C-3. (A) **C-4.** (D)
C-5. (B)

SECTION-(D)

D-1. (C) **D-2.** (D)
D-3. (A) **D-4.** (A)
D-5. (B) **D-6.** (A)
D-7. (C) **D-8.** (B)
D-9. (A) **D-10.** (C)
D-11. (A) **D-12.** (A)

SECTION-(E)

E-1. (C) **E-2.** (B)
E-3. (C) **E-4.** (C)
E-5. (A) **E-6.** (B)
E-7. (A) **E-8.** (D)
E-9. (C) **E-10.** (A)
E-11. (A) **E-12.** (A)

SECTION-(F)

F-1. (A) **F-2.** (C)
F-3. (C) **F-4.** (A)
F-5. (A) **F-6.** (A)
F-7. (A) **F-8.** (B)

PART-III

1. (A) $\rightarrow$ (p, q, r); (B) $\rightarrow$ (r) ; (C) $\rightarrow$ (p) ; (D) $\rightarrow$ (q)

2. (A) $\rightarrow$ (p), (B) $\rightarrow$ (s), (C) $\rightarrow$ (q), (D) $\rightarrow$ (r)

3. (A) $\rightarrow$ (r), (B) $\rightarrow$ (s), (C) $\rightarrow$ (q), (D) $\rightarrow$ (p)

4. (A) $\rightarrow$ (p,q),(B) $\rightarrow$ (r,s), (C) $\rightarrow$ (r,s), (D) $\rightarrow$ (r,s)

Exercise # 2

PART - I

1.	(A)	**2.**	(C)
3.	(C)	**4.**	(D)
5.	(D)	**6.**	(C)
7.	(B)	**8.**	(B)
9.	(B)	**10.**	(B)
11.	(C)	**12.**	(B)
13.	(C)	**14.**	(C)
15.	(C)	**16.**	(A)
17	(C)	**18.**	(D)
19.	(A)	**20.**	(B)
21.	(B)	**22.**	(C)
23.	(C)	**24.**	(D)
25.	(C)	**26.**	(B)
27.	(A)	**28.**	(B)

PART-II

1.	1	**2.**	20
3.	0.25	**4.**	1.41
5.	3	**6.**	2
7.	15	**8.**	9.42 or 9.43
9.	0	**10.**	29
11.	74	**12.**	6
13.	1	**14.**	1
15.	81	**16.**	39
17.	2	**18.**	3
19.	4	**20.**	8
21.	3	**22.**	8
23.	13	**24.**	5

PART - III

1.	(C,D)	**2.**	(A,C)
3.	(A,D)	**4.**	(B,C,D)
5.	(B,D)	**6.**	(A,B,C)
7.	(B,C)	**8.**	(A,D)
9.	(A,C)	**10.**	(A,B)
11.	(A,D)	**12.**	(B,D)
13.	(A,D)	**14.**	(A,C,D)
15.	(B,D)	**16.**	(A,C)
17.	(A,C,D)	**18.**	(A,C,D)
19.	(A,B,D)	**20.**	(A,B,C,D)
21.	(A,B,D)	**22.**	(B,C,D)
23.	(B,D)	**24.**	(A,B,C)
25.	(A,B,C)	**26.**	(A,C,D)
27.	(A,B,C,D)	**28.**	(B,C,D)
29.	(A,C,D)	**30.**	(C,D)
31.	(A,B,C,D)		

PART - IV

1.	(A)	**2.**	(D)
3.	(B)	**4.**	(C)
5.	(B)	**6.**	(A)
7.	(C)	**8.**	(D)
9.	(A)		

Exercise # 3

PART - I

1.	1	**2.**	(D)
3.	(A) $\rightarrow$ (s), (B) $\rightarrow$ (t), (C) $\rightarrow$ (r), (D) $\rightarrow$ (r)		
4.	2	**5.**	(9)
6.	(5)	**7.**	(C)
8.	(A,C)	**9.**	9
10.	(A, B)	**11.**	(D)
12.	(C)	**13.**	(A)
14.	(B,D)	**15.**	4
16.	3	**17.**	(B,C)
18.	(A,D)	**19.**	(D)
20.	(D)	**21.**	(D)
22.	(B,D)	**23.**	(C)
24.	(A,B,D)	**25.**	(1,2,4)
26.	(1,3,4)		

PART - II

1.	(3)	**2.**	(4)
3.	(2)	**4.**	(3)
5.	(2)	**6.**	(4)
7.	(2)	**8.**	(1)
9.	(3)	**10.**	(4)
11.	(3)	**12.**	(3)
13.	(4)	**14.**	(3)
15.	(3)	**16.**	(4)
17.	(4)	**18.**	(2)
19.	(4)	**20.**	(2)

Reliable Ranker Problems

1. Tangent at any point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ meet the coordinate axes at P & Q. Find locus of mid point of PQ.

2. Show that the equation of the tangent to the curve represented parametrically by the equations.

$x = a\left\{\frac{\phi(t)}{f(t)}\right\}$ and $y = a\left\{\frac{\psi(t)}{f(t)}\right\}$ can be expressed in the form $\begin{vmatrix} x & y & a \\ \phi(t) & \psi(t) & f(t) \\ \phi'(t) & \psi'(t) & f'(t) \end{vmatrix} = 0$

where f, g and h are the differentiable functions.

3. Show that the condition, for curves $x^{2/3} + y^{2/3} = c^{2/3}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to touch, is c = a + b.

4. If the relation between subnormal SN and subtangent ST at any point S on the curve $by^2 = (x + a)^3$ is $p(SN) = q\,(ST)^2$, then find value of $\frac{p}{q}$ in terms of b and a.

5. Show that in the curve $y = a.\,ln\,(x^2 - a^2)$, sum of the length of tangent & subtangent varies as the product of the coordinates of the point of contact.

6. Show that the angle between the tangent at any point 'A' of the curve $ln\,(x^2 + y^2) = C\tan^{-1}\frac{y}{x}$ and the line joining A to the origin is independent of the position of A on the curve.

7. A water tank has the shape of a right circular cone with its vertex down. Its altitude is 10 cm and the radius of the base is 15 cm. Water leaks out of the bottom at a constant rate of 1cu. cm/sec. Water is poured into the tank at a constant rate of C cu. cm/sec. Compute C so that the water level will be rising at the rate of 4 cm/sec at the instant when the water is 2 cm deep.

8. Water is flowing out at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius R = 13 m. The volume of water in the hemispherical bowl is given by $V = \frac{\pi}{3}\cdot y^2(3R - y)$ when the water is y meter deep. Find

(a) At what rate is the water level changing when the water is 8 m deep.

(b) At what rate is the radius of the water surface changing when the water is 8 m deep.

9. A particle moving on a curve has the position at time t given by $x = f'(t)\sin t + f''(t)\cos t$, $y = f'(t)\cos t - f''(t)\sin t$, where f is a thrice differentiable function. Then prove that the velocity of the particle at time t is $f'(t) + f'''(t)$.

10. Find the values of 'a' for which the function $f(x) = \sin x - a\sin 2x - \frac{1}{3}\sin 3x + 2ax$ strictly increases throughout the number line.

11. Find the interval of increasing and decreasing for the function $g(x) = 2f\left(\frac{x^2}{2}\right) + f\left(\frac{27}{2} - x^2\right)$, where $f''(x) < 0$ for all $x \in R$.

12. Find the interval to which b may belong so that the function $f(x) = \left(1 - \frac{\sqrt{21-4b-b^2}}{b+1}\right)x^3 + 5x + \sqrt{6}$ is increasing at every point of its domain.

13. Using calculus , prove that $\log_2 3 > \log_3 5 > \log_4 7$.

14. Prove that $e^x + \sqrt{1+e^{2x}} \geq (1 + x) + \sqrt{2+2x+x^2}\ \ \forall x \in R$

15. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0$, $\forall x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then find the intervals of monotonicity of g(x).

16. Find which of the two is larger $\ell n(1 + x)$ or $\frac{\tan^{-1} x}{1+x}$.

17. Prove the following inequalities

(i) $1 + x^2 > (x \sin x + \cos x)$ for $x \in [0, \infty)$.

(ii) $\sin x - \sin 2x \leq 2x$ for all $x \in \left[0, \frac{\pi}{3}\right]$

(iii) $\frac{x^2}{2} + 2x + 3 \geq (3 - x)e^x$ for all $x \geq 0$

(iv) $0 < x \sin x - \frac{\sin^2 x}{2} < \frac{1}{2}(\pi - 1)$ for $0 < x < \frac{\pi}{2}$

18. Find the set of values of the parameter 'a' for which the function ;
$f(x) = 8ax - a \sin 6x - 7x - \sin 5x$ increases & has no critical points for all $x \in R$, is

19. If all the function $f(x) = a^2x^3 - \frac{a}{2}x^2 - 2x - b$ are positive and the minimum is at the point $x_0 = \frac{1}{3}$ then show that when $a = -2 \Rightarrow b < \frac{-11}{27}$ and when $a = 3 \Rightarrow b < -\frac{1}{2}$

20. Find the values of the parameter 'k' for which the equation $x^4 + 4x^3 - 8x^2 + k = 0$ has all roots real.

Comprehension (Q. No. 21 to 23)

A function f(x) having the following properties;

(i) f(x) is continuous except at x = 3

(ii) f(x) is differentiable except at x = – 2 and x = 3

(iii) $f(0) = 0$, $\lim_{x\to 3} f(x) \to -\infty$, $\lim_{x\to -\infty} f(x) = 3$, $\lim_{x\to\infty} f(x) = 0$

(iv) $f'(x) > 0\ \forall x \in (-\infty, -2) \cup (3, \infty)$ and $f'(x) \leq 0\ \forall x \in (-2, 3)$

(v) $f''(x) > 0\ \forall x \in (-\infty, -2) \cup (-2, 0)$ and $f''(x) < 0\ \forall x \in (0, 3) \cup (3, \infty)$

then answer the following questions

21. Find he Maximum possible number of solutions of $f(x) = |x|$

22. Show that graph of function $y = f(-|x|)$ is continuous but not differentiable at two points, if $f'(0) = 0$

23. Show that $f(x) + 3x = 0$ has five solutions if $f'(0) > -3$ and $f(-2) > 6$

24. With the usual meaning for a, b, c and s, if Δ be the area of a triangle, prove that the error in Δ resulting from a small error in the measurement of c, is given by $d\Delta = \frac{\Delta}{4}\left\{\frac{1}{s} + \frac{1}{s-a} + \frac{1}{s-b} - \frac{1}{s-c}\right\}dc$

25. Find the possible values of 'a' such that the inequality $3 - x^2 > |x - a|$ has atleast one negative solution

26. If $(m - 1)\, a_1{}^2 - 2m\, a_2 < 0$, then prove that $x^m + a_1 x^{m-1} + a_2 x^{m-2} + ... + a_{m-x} + a_0 = 0$ has at least one non real root $(a_1, a_2, ..., a_m \in R)$

27. If $f'(x) > 0$, $f''(x) > 0\ \forall\ x \in (0, 1)$ and $f(0) = 0$, $f(1) = 1$, then prove that $f(x)\, f^{-1}(x) < x^2\ \forall\ x \in (0, 1)$

28. Using calculus prove that H.M $\le$ G.M. $\le$ A.M for positive real numbers.

29. Find positive real numbers 'a' and 'b' such that $f(x) = ax - bx^3$ has two point of extrema in $(-1, 1)$ at each of which $|f(x)| = 1 = |f(1)| = |f(-1)|$

30. For any acute angled ΔABC, find the maximum value of $\frac{\sin A}{A} + \frac{\sin B}{B} + \frac{\sin C}{C}$

31. Suppose p,q,r,s are fixed real numbers such that a quadrilateral can be formed with sides p,q,r,s in clockwise order. Prove that the vertices of the quadrilateral of maximum area lie on a circle .

32. For what real values of 'a' and 'b' all the extrema of the function $f(x) = \frac{5a^2}{3}x^3 + 2ax^2 - 9x + b$ are positive and the maximum is at the point $x_0 = \frac{-5}{9}$

33. Find the minimum value of $f(x) = 8^x + 8^{-x} - 4(4^x + 4^{-x})\ \forall\ x \in R$

34. Find the cosine of the angle at the vertex of an isosceles triangle having the greatest area for the given constant length ℓ of the median drawn to its lateral side .

35. A tangent to the curve $y = 1 - x^2$ is drawn so that the abscissa x_0 of the point of tangency belongs to the interval (0, 1]. The tangent at x_0 meets the x–axis and y–axis at A & B respectively. Then find the minimum area of the triangle OAB, where O is the origin

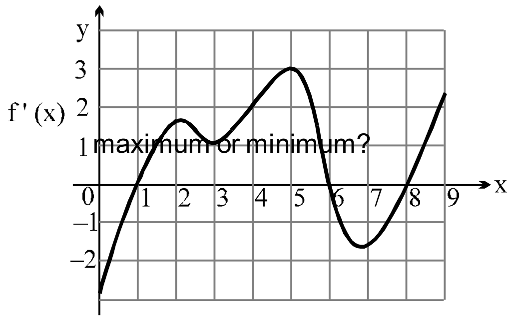
36. A cone is made from a circular sheet of radius $\sqrt{3}$ by cutting out a sector and keeping the cut edges of the remaining piece together. Then find the maximum volume attainable for the cone

37. Suppose velocity of waves of wave length λ in the Atlantic ocean is $k\sqrt{\left\{\left(\frac{\lambda}{a}\right) + \left(\frac{a}{\lambda}\right)\right\}}$, where k and a are constants. Show that minimum velocity attained by the waves is independent of the constant a.

38. Find the minimum distance of origin from the curve $ax^2 + 2bxy + ay^2 = c$ where $a > b > c > 0$

39. A beam of rectangular cross section must be sawn from a round log of diameter d. What should the width x and height y of the cross section be for the beam to offer the greatest resistance (a) to compression; (b) to bending. Assume that the compressive strength of a beam is proportional to the area of the cross section and the bending strength is proportional to the product of the width of section by the square of its height.

40. The graph of the derivative f' of a continuous function f is shown with f (0) = 0



(i) On what intervals f is strictly increasing or strictly decreasing?

(ii) At what values of x does f have a local maximum or minimum?

(iii) On what intervals is $f'' > 0$ or $f'' < 0$

(iv) State the x-coordinate(s) of the point(s) of inflection.

(v) Assuming that f (0) = 0, sketch a graph of f.

41. Find the set of value of m for the cubic $x^3 - \frac{3}{2}x^2 + \frac{5}{2} = \log_{1/4}(m)$ has 3 distinct solutions.

42. A cylinder is obtained by revolving a rectangle about the x – axis, the base of the rectangle lying on the x-axis and the entire rectangle lying in the region between the curve $y = \frac{x}{x^2+1}$ & the x-axis. Find the maximum possible volume of the cylinder.

43. Let f(x) and g(x) be differentiable functions having no common zeros so that $f(x)\,g'(x) \neq f'(x)\,g(x)$. Prove that between any two zeros of f(x), there exist atleast one zero of g(x).

44. If $\phi(x)$ is a differentiable function $\forall\ x \in R$ and $a \in R^+$ such that $\phi(0) = \phi(2a)$, $\phi(a) = \phi(3a)$ and $\phi(0) \neq \phi(a)$ then show that there is at least one root of equation $\phi'(x + a) = \phi'(x)$ in (0, 2a)

45. Prove that, $x^2 - 1 > 2x\,\ln x > 4(x - 1) - 2\,\ln x$ for $x > 1$.

46. Let a > 0 and f be continuous in [–a, a]. Suppose that $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that f(0) = 0.

47. Let f be continuous on [a, b] and differentiable on (a, b). If f (a) = a and f (b) = b, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

1. $4(x^2+y^2)=a^2$ **4.** $\frac{8}{27}|b|$ **6.** $\theta=\tan^{-1}\frac{2}{C}$

7. $1+36\pi$ cu. cm/sec **8.** (a) $-\frac{1}{24\pi}$ m/min., (b) $-\frac{5}{288\pi}$ m/min. **10.** $[1,\infty)$

11. g(x) is increasing if $x\in(-\infty,-3]\cup[0,3]$
g(x) is decreasing if $x\in[-3,0]\cup[3,\infty)$

12. $[-7,-1)\cup[2,3]$ **15.** Increasing when $x\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$, decreasing when $x\in\left(0,\frac{\pi}{4}\right)$.

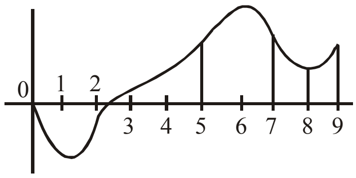
16. $\ell n(1+x)$ **18.** $a\in[6,\infty)$ **20.** $k\in[0,3]$ **21.** 3

25. $a\in\left(-\frac{13}{4},3\right)$ **29.** a = 3 , b = 4 **30.** $\frac{9\sqrt{3}}{2\pi}$

32. If $a=\frac{-9}{5}$, then $b>\frac{36}{5}$; If $a=\frac{81}{25}$ then $b>\frac{400}{243}$ **33.** – 10 **34.** cos A = 0.8

35. $\frac{4\sqrt{3}}{9}$ **36.** $2\pi/3$ **38.** $\sqrt{\frac{c}{a+b}}$ **39.** (a) $x=y=\frac{d}{\sqrt{2}}$, (b) $x=\frac{d}{\sqrt{3}}$, $y=\sqrt{\frac{2}{3}}d$

40. (i) I in [1, 6] ; [8, 9] and D in [0, 1] ; [6, 8];
(ii) L.Min. at x = 1 and x = 8; L.Max. x = 6
(iii) $(0,2)\cup(3,5)\cup(7,9)$ and $(2,3)\cup(5,7)$;
(iv) x = 2, 3, 5, 7

(v) Graph is 

41. $m\in\left(\frac{1}{32},\frac{1}{16}\right)$ **42.** $\pi/4$