CHAPTER 12

INVERSE TRIGONOMETRIC FUNCTION

12.1 INVERSE FUNCTION

If a function is one-to-one and onto from A to B, then function g which associates each element $y \in B$ to one and only one element $x \in A$, such that $y = f(x) \Leftrightarrow x = g(y)$, then g is called the inverse function of f, denoted by $g = f^{-1}$. [Read as f inverse]. Thus, if $f : A \to B$, then $g : B \to A$.

12.1.1 Inverse Trigonometric Functions

The equation $\sin x = y$ and $x = \sin^{-1} y$ are not identical because the former associates many values of x of a single value of y while the latter associates a single x to a particular value of y. To assign a unique angle to a particular value of trigonometric ratio, we introduce a term called **principle range**.

We list below the domain (values of x) and principle ranges (values of y) of all the inverse trigonometric functions and their graph.

Remarks:

1. *$\sin 5\pi/6 = 1/2$ But $5\pi/6 \neq \sin^{-1}(1/2)$ $\therefore$ $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, denotes angles or real number, 'whose sine is x', 'whose cosine is x' and 'whose tangent is x', provided that the answers given are numerically smallest available.*
2. *If there are two angles, one positive and the other negative having same numerical value. Then we shall take the positive value. For example, $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. But we write $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ and $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) \neq -\frac{\pi}{4}$.*
3. *I quadrant is common to all the inverse functions.*
4. *III quadrant is not used in inverse function.*
5. *IV quadrant is used in the clockwise direction i.e., $-\pi/2 \leq y \leq 0$.*

12.2 DOMAIN AND RANGE OF INVERSE FUNCTIONS

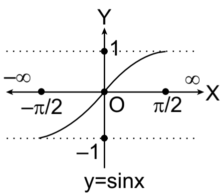
Function	Domain	Range	Principal Value Branch
$y = \sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$	$-\pi/2 \le y \le \pi/2$
$y = \cos^{-1}x$	$[-1\ 1]$	$[0, \pi]$	$0 \le y \le \pi$
$y = \tan^{-1}x$	$\mathbb{R}$	$(-\pi/2, \pi/2)$	$-\pi/2 < y < \pi/2$
$y = \cot^{-1}x$	$\mathbb{R}$	$(0, \pi)$	$0 < y < \pi$
$y = \sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \{\pi/2\}$	$0 \le y \le \pi,\ y \ne \pi/2$
$y = \text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, \pi/2] - \{0\}$	$-\pi/2 \le y \le \pi/2,\ y \ne 0$

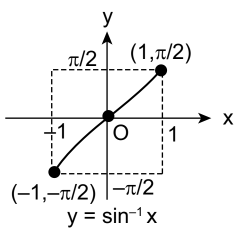
Remark:

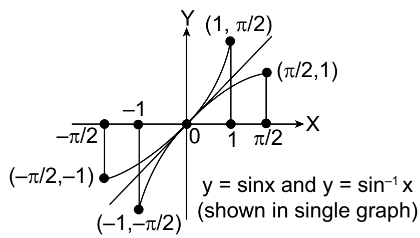
If no branch of an inverse trigonometric function is mentioned, then it means the principal value branch of the function.

12.3 GRAPHS OF INVERSE CIRCULAR FUNCTIONS AND THEIR DOMAIN AND RANGE

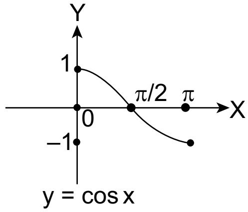
1. Graph of function $y = \sin x$, $y = \sin^{-1}x$:

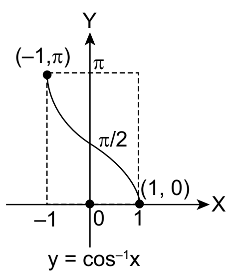


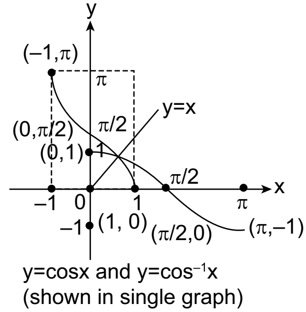




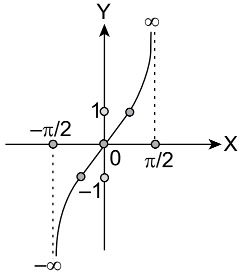
2. Graph of function $y = \cos x$, $y = \cos^{-1}x$:



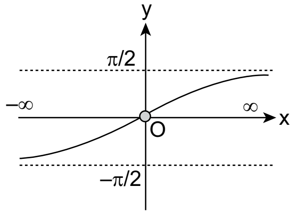




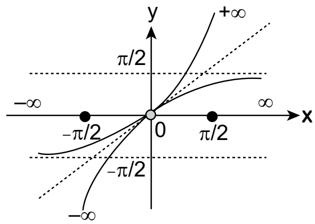
3. Graph of function y = tan x, y = $\tan^{-1}$x:



y = tanx

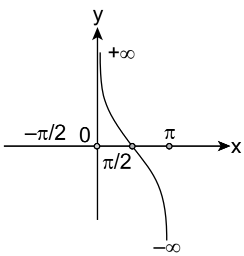


y = $\tan^{-1}$x

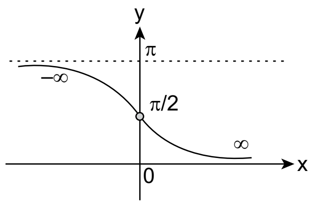


y = tanx and y = $\tan^{-1}$x

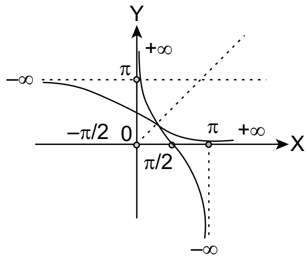
4. Graph of function y = cot x, y = $\cot^{-1}$x:



y = cot x

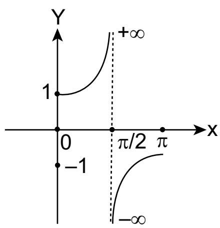


y = $\cot^{-1}$ x

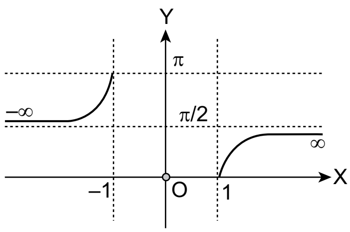


y = cotx and y = $\cot^{-1}$x

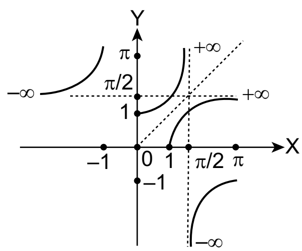
4. Graph of function y = sec x, y = $\sec^{-1}$x:



y = sec x

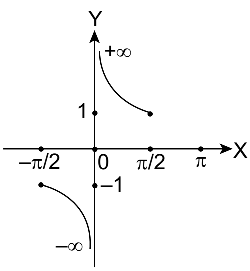


y = $\sec^{-1}$x

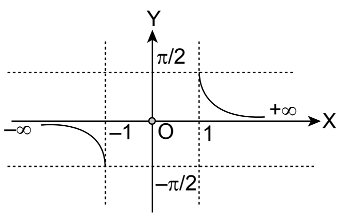


y = secx and y =$\sec^{-1}$x

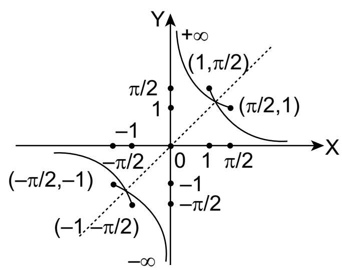
5. Graph of function y = cosec x, y = cosec^{-1}x:



y = cosec x



y = cosec^{-1}x



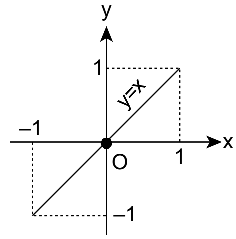
y = cosecx and y = cosec^{-1}x

12.4 COMPOSITIONS OF TRIGONOMETRIC FUNCTIONS AND THEIR INVERSE FUNCTIONS

12.4.1 Trigonometric Functions of Their Corresponding Circular Functions

(i) $\sin(\sin^{-1} x) = x$, for all $x \in [-1, 1]$

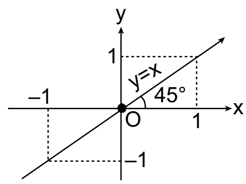
(ii) $\cos(\cos^{-1} x) = x$, for all $x \in [-1, 1]$



y=sin(sin–1x)=cos(cos–1x)=x

(iii) $\tan(\tan^{-1} x) = x$, for all $x \in \mathbb{R}$

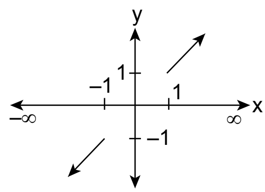
(iv) $\cot(\cot^{-1} x) = x$, for all $x \in \mathbb{R}$



y=tan(tan–1x)=cot(cot–1x)=x

(v) $\text{cosec}(\text{cosec}^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\sec(\sec^{-1} x) = x$, for all $x \in (-\infty, -1] \cup [1, \infty)$



y=sec(sec–1x)=cosec(cosec–1x)=x

12.5 INVERSE CIRCULAR FUNCTIONS OF THEIR CORRESPONDING TRIGONOMETRIC FUNCTIONS ON PRINCIPAL DOMAIN

(i) $\sin^{-1}(\sin x) = x$; for all $x \in [-\pi/2, \pi/2]$

(ii) $\cos^{-1}(\cos x) = x$; for all $x \in [0, \pi]$

(iii) $\tan^{-1}(\tan x) = x$; for all $x \in (-\pi/2, \pi/2)$

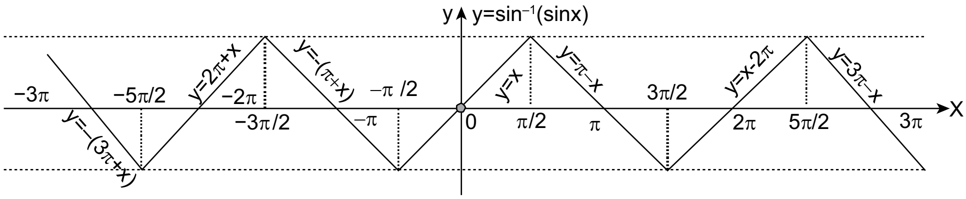
(iv) $\cot^{-1}(\cot x) = x$; for all $x \in (0, \pi)$

(v) $\sec^{-1}(\sec x) = x$; for all $x \in [0, \pi], \sim \{\pi/2\}$

(vi) $\text{cosec}^{-1}(\text{cosec } x) = x$; for all $x \in [-\pi/2, \pi/2] \sim \{0\}$

12.6 INVERSE CIRCULAR FUNCTIONS OF THEIR CORRESPONDING TRIGONOMETRIC FUNCTIONS ON DOMAIN

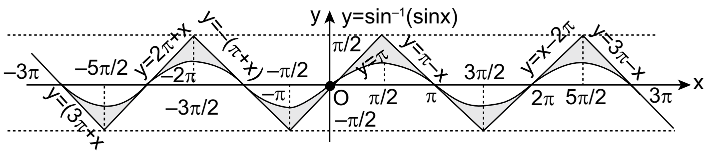
1. $\sin^{-1}(\sin x) = \begin{cases} -\pi - x, & \text{if } x \in [-3\pi/2, -\pi/2] \\ x, & \text{if } x \in [-\pi/2, \pi/2] \\ \pi - x, & \text{if } x \in [\pi/2, 3\pi/2] \\ -2\pi + x, & \text{if } x \in [3\pi/2, 5\pi/2] \end{cases}$ and so on as shown below:



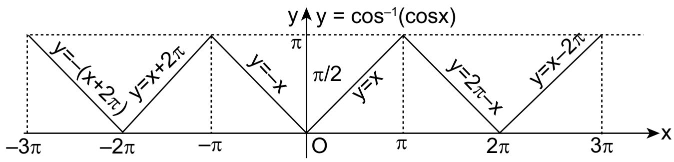
Domain: $\mathbb{R}$ **Range:** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **Period:** 2π

Remark:

$y = \sin^{-1}(\sin x)$ can be formed by tangents of $y = \sin x$ at $x = n\pi$ as shown below:

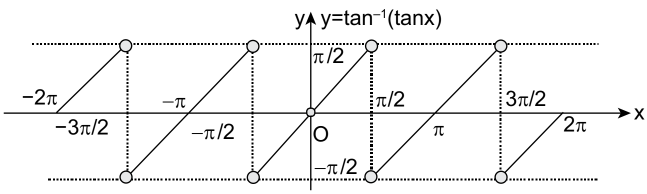


2. $\cos^{-1}(\cos x) = \begin{cases} -x, & \text{if } x \in [-\pi, 0] \\ x, & \text{if } x \in [0, \pi] \\ 2\pi - x, & \text{if } x \in [\pi, 2\pi] \\ -2\pi + x, & \text{if } x \in [2\pi, 3\pi] \end{cases}$ and so on as shown:



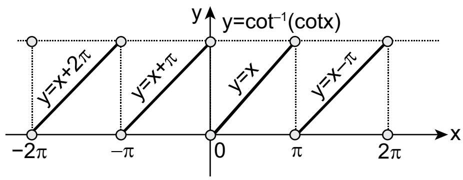
Domain: $\mathbb{R}$ **Range:** $[0, \pi]$ **Period:** 2π

3. $\tan^{-1}(\tan x) = \begin{cases} \pi + x, & \text{if } x \in (-3\pi/2, -\pi/2) \\ x, & \text{if } x \in (-\pi/2, \pi/2) \\ x - \pi, & \text{if } x \in (\pi/2, 3\pi/2) \\ x - 2\pi, & \text{if } x \in (3\pi/2, 5\pi/2) \end{cases}$ and so on, as shown:



Domain: $\mathbb{R} \sim \left\{(2n+1)\frac{\pi}{2}\right\}$ **Range:** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **Period:** π

4. $y=\cot^{-1}(\cot x)=\begin{cases} x+2\pi & \text{for } x\in(-2\pi,-\pi) \\ x+\pi & \text{for } x\in(-\pi,0) \\ x & \text{for } x\in(0,\pi) \\ x-\pi & \text{for } x\in(\pi,2\pi) \\ x-2\pi & \text{for } x\in(2\pi,3\pi) \end{cases}$

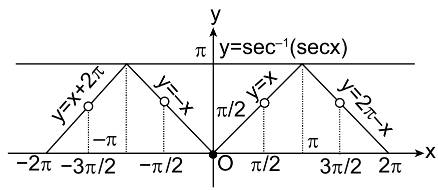


The graph of $\cot^{-1}$ (cotx) is as shown:

Domain: $x \in R - \{n\pi, n \in Z\}$ **Range:** $y \in (0, \pi)$

Period: periodic with period π and and $\cot^{-1}(\cot x) = x\ \forall\ x \in (0,\pi)$

5. $y=\sec^{-1}(\sec x). = \begin{cases} -x & \text{for } x\in[-\pi,0] \\ x & \text{for } x\in[0,\pi]\sim\left\{\frac{\pi}{2}\right\} \\ 2\pi-x & \text{for } x\in[\pi,2\pi]\sim\left\{\frac{3\pi}{2}\right\} \end{cases}$

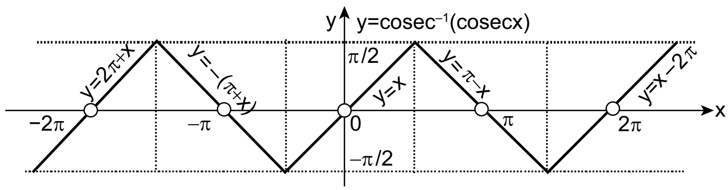


The graph of $y = \sec^{-1}$ (secx) is as shown:

Domain: $x \in \mathbb{R} - \left\{(2n+1)\frac{\pi}{2}, n\in\mathbb{Z}\right\}$ **Range:** $y \in [0, \pi/2) \cup (\pi/2, \pi]$

Period: Periodic with period 2π and $\sec^{-1}(\sec x) = x\ \forall\ x \in [0, \pi/2) \cup (\pi/2, \pi]$

6. $y = \text{cosec}^{-1}(\text{cosec } x) = \begin{cases} -(\pi+x) & \text{for } x\in\left[\frac{-3\pi}{2},\frac{-\pi}{2}\right]\sim\{-\pi\} \\ x & \text{for } x\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]\sim\{0\} \\ \pi-x & \text{for } x\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\sim\{\pi\} \end{cases}$



Domain: $x \in \mathbb{R} \sim \{n\pi : n \in \mathbb{Z}\}$ **Range:** $y \in [-\pi/2, \pi/2] \sim \{0\}$

Period: Periodic with period 2π and $\text{cosec}^{-1}(\text{cosec } x) = x$ for $x \in [-\pi/2, \pi/2] \sim \{0\}$

12.7 INVERSE TRIGONOMETRIC FUNCTIONS OF NEGATIVE INPUTS

(i) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, for all $x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in R$

(iv) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, for all $x \in R$

12.8 INVERSE TRIGONOMETRIC FUNCTIONS OF RECIPROCAL INPUTS

(i) $\sin^{-1}(1/x) = \text{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(ii) $\cos^{-1}(1/x) = \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(iii) $\tan^{-1}(1/x) = \begin{cases} \cot^{-1} x, & \text{for } x > 0 \\ -\pi + \cot^{-1} x, & \text{for } x < 0 \end{cases}$

12.9 INTER CONVERSION OF INVERSE TRIGONOMETRIC FUNCTIONS

(a) $$\sin^{-1}x = \begin{Bmatrix} \cos^{-1}\sqrt{1-x^2} & \text{if } 0 \le x \le 1; \\ -\cos^{-1}\sqrt{1-x^2} & \text{if } -1 \le x \le 0 \end{Bmatrix} = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \text{ if } \forall\, x \in (-1, 1)$$

$$= \begin{Bmatrix} \cot^{-1}\dfrac{\sqrt{1-x^2}}{x} & \text{if } 0 < x \le 1; \\ -\pi + \cot^{-1}\dfrac{\sqrt{1-x^2}}{x} & \text{if } -1 \le x < 0 \end{Bmatrix} = \begin{Bmatrix} \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & \text{if } x \in [0, 1) \\ -\sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & \text{if } x \in (-1, 0] \end{Bmatrix}$$

$$= \text{cosec}^{-1}\left(\frac{1}{x}\right) \text{ if } x \in [-1, 1] \sim \{0\}$$

(b) $$\cos^{-1}x = \begin{Bmatrix} \sin^{-1}\sqrt{1-x^2} & \text{for } x \in [0, 1] \\ \pi - \sin^{-1}\sqrt{1-x^2} & \text{for } x \in [-1, 0] \end{Bmatrix} = \begin{Bmatrix} \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right) & \text{for } x \in (0, 1] \\ \pi + \tan^{-1}\dfrac{\sqrt{1-x^2}}{x} & \text{for } x \in [-1, 0) \end{Bmatrix}$$

$$= \left\{ \cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \text{for } x \in (-1, 1) \right\} = \sec^{-1}\left(\frac{1}{x}\right) \text{for } x \in [-1, 1] \sim \{0\}$$

$$= \begin{Bmatrix} \text{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & \text{for } x \in [0, 1) \\ \pi - \text{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) & \text{for } x \in (-1, 0] \end{Bmatrix}$$

(c) $$\tan^{-1}x = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) \text{for } x \in \mathbb{R} = \begin{cases} \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) & \text{for } x \in [0, 1] \\ -\cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) & \text{for } x \in [-1, 0] \end{cases}$$

$$= \begin{cases} \cot^{-1}\left(\dfrac{1}{x}\right) & \text{for } x > 0 \\ -\pi + \cot^{-1}\left(\dfrac{1}{x}\right) & \text{for } x < 0 \end{cases} = \begin{cases} \sec^{-1}\left(\sqrt{1+x^2}\right) & \text{for } x > 0 \\ -\sec^{-1}\left(\sqrt{1+x^2}\right) & \text{for } x < 0 \end{cases} = \left\{ \text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right) \text{for } x \in \mathbb{R} \sim \{0\} \right.$$

(d) $$\cot^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) & \text{for } x \ge 0 \\ \pi - \sin^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right) & \text{for } x \le 0 \end{cases} = \left\{ \cos^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) \forall x \in \mathbb{R} \right.$$

$$= \begin{cases} \tan^{-1}\left(\dfrac{1}{x}\right) & \text{for } x > 0 \\ \pi + \tan^{-1}\left(\dfrac{1}{x}\right) & \text{for } x < 0 \end{cases} = \left\{ \sec^{-1}\left(\dfrac{\sqrt{1+x^2}}{x}\right) \forall x \in \mathbb{R} \sim \{0\} \right. = \begin{cases} \mathrm{cosec}^{-1}\left(\sqrt{1+x^2}\right) & \text{for } x > 0 \\ \pi - \mathrm{cosec}^{-1}\left(\sqrt{1+x^2}\right) & \text{for } x < 0 \end{cases}$$

(e) $$\sec^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{\sqrt{x^2-1}}{x}\right) & \text{for } x > 0 \\ \pi + \sin^{-1}\left(\dfrac{\sqrt{x^2-1}}{x}\right) & \text{for } x < 0 \end{cases} = \cos^{-1}\left(\dfrac{1}{x}\right) \forall x \in \mathbb{R} \sim \{0\} = \begin{cases} \tan^{-1}\left(\sqrt{x^2-1}\right) & \text{for } x > 0 \\ \pi - \tan^{-1}\sqrt{x^2-1} & \text{for } x < 0 \end{cases}$$

$$= \begin{cases} \cot^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right) & \text{for } x > 0\, x \ne 1 \\ \pi - \cot^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right) & \text{for } x < 0; x \ne -1 \end{cases} = \begin{cases} \mathrm{cosec}^{-1}\left(\dfrac{x}{\sqrt{x^2-1}}\right) & \text{for } x > 0 \\ \pi + \mathrm{cosec}^{-1}\left(\dfrac{x}{\sqrt{x^2-1}}\right) & \text{for } x < 0 \end{cases}$$

(f) $$\mathrm{cosec}^{-1}x = \sin^{-1}\frac{1}{x} \text{ for } x \in \mathbb{R} \sim \{0\} = \begin{cases} \cos^{-1}\left(\dfrac{\sqrt{x^2-1}}{x}\right) & \text{for } x > 0 \\ -\pi + \cos^{-1}\left(\dfrac{\sqrt{x^2-1}}{x}\right) & \text{for } x < 0 \end{cases}.$$

$$= \begin{cases} \tan^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right) & \text{for } x > 0; \ne 1 \\ -\tan^{-1}\left(\dfrac{1}{\sqrt{x^2-1}}\right) & \text{for } x < 0; \ne -1 \end{cases} = \begin{cases} \cot^{-1}\left(\sqrt{x^2-1}\right) & \text{for } x > 0 \\ -\cot^{-1}\left(\sqrt{x^2-1}\right) & \text{for } x < 0 \end{cases}$$

$$= \begin{cases} \sec^{-1}\left(\dfrac{x}{\sqrt{x^2-1}}\right) & \text{for } x > 0; \ne 1 \\ -\pi + \sec^{-1}\left(\dfrac{x}{\sqrt{x^2-1}}\right) & \text{for } x < 0; \ne -1 \end{cases}$$

12.10 THREE IMPORTANT IDENTITIES OF INVERSE TRIGONOMETRIC FUNCTIONS

(i) $\sin^{-1}x + \cos^{-1}x = \pi/2$, for all $x \in [-1, 1]$

(ii) $\tan^{-1}x + \cot^{-1}x = \pi/2$, for all $x \in R$

(iii) $\sec^{-1}x + \mathrm{cosec}^{-1}x = \pi/2$, for all $x \in (-\infty, -1] \cup [1, \infty)$

12.11 MULTIPLES OF INVERSE TRIGONOMETRIC FUNCTIONS

Property (1):

$$2\sin^{-1}x=\begin{cases}\sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -\frac{1}{\sqrt{2}}\le x\le\frac{1}{\sqrt{2}}\\ \pi-\sin^{-1}(2x\sqrt{1-x^2}), & \text{if } \frac{1}{\sqrt{2}}\le x\le 1\\ -\pi-\sin^{-1}(2x\sqrt{1-x^2}), & \text{if } -1\le x\le-\frac{1}{\sqrt{2}}\end{cases}$$

Property (2):

$$3\sin^{-1}x=\begin{cases}\sin^{-1}(3x-4x^3), & \text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ \pi-\sin^{-1}(3x-4x^3), & \text{if } \frac{1}{2}\le x\le 1\\ -\pi-\sin^{-1}(3x-4x^3), & \text{if } -1\le x\le-\frac{1}{2}\end{cases}$$

Property (3):

$$2\cos^{-1}x=\begin{cases}\cos^{-1}(2x^2-1); & \text{if } 0\le x\le 1\\ 2\pi-\cos^{-1}(2x^2-1); & \text{if } -1\le x\le 0\end{cases}$$

Property (4):

$$3\cos^{-1}x=\begin{cases}\cos^{-1}(4x^3-3x); & \text{if } \frac{1}{2}\le x\le 1\\ 2\pi-\cos^{-1}(4x^3-3x); & \text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ 2\pi+\cos^{-1}(4x^3-3x); & \text{if } -1\le x\le-\frac{1}{2}\end{cases}$$

Property (5):

$$2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{2x}{1-x^2}\right); & \text{if } -1<x<1\\ \pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right) & \text{if } x>1\\ -\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right); & \text{if } x<-1\\ \frac{\pi}{2} \text{ for } x=1 & \end{cases}$$

Property (6):

$$3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right); & \text{if } -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}\\ \pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right); & \text{if } x>\frac{1}{\sqrt{3}}\\ -\pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right); & \text{if } x<-\frac{1}{\sqrt{3}}\\ \frac{\pi}{2} \text{ for } x=\frac{1}{\sqrt{3}} & \end{cases}$$

Property (7):

$$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right); & \text{if } x < -1 \end{cases}$$

Property (8):

$$2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right); & \text{if } -\infty < x \le 0 \end{cases}$$

12.12 SUM AND DIFFERENCE OF INVERSE TRIGONOMETRIC FUNCTIONS

Property (1):

$$\sin^{-1}x + \sin^{-1}y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}, & \text{if } x^2+y^2 \le 1 \\ & \text{or if } xy < 0 \text{ and } x^2+y^2 > 1; \text{ where } x, y \in [-1,1] \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}, & \text{if } 0 < x, y \le 1 \text{ and } x^2+y^2 > 1 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}, & \text{if } -1 \le x, y < 0 \text{ and } x^2+y^2 > 1 \end{cases}$$

Property (2):

$$\sin^{-1}x - \sin^{-1}y = \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, & \text{if } x^2+y^2 \le 1 \\ & \text{or } xy > 0 \text{ and } x^2+y^2 > 1; \text{ where } x, y \in [-1, 1] \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, & \text{if } 0 < x \le 1, -1 \le y \le 0 \text{ and } x^2+y^2 > 1 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, & \text{if } -1 \le x < 0, 0 < y \le 1 \text{ and } x^2+y^2 > 1 \end{cases}$$

Property (3):

$$\cos^{-1}x + \cos^{-1}y = \begin{cases} \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}, \\ \text{if } -1 \le x, y \le 1 \text{ and } x+y \ge 0 \\ 2\pi - \cos^{-1}\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\}, \\ \text{if } -1 \le x, y \le 1 \text{ and } x+y \le 0 \end{cases}$$

Property (4):

$$\cos^{-1}x - \cos^{-1}y = \begin{cases} \cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}, \\ \text{if } -1 \le x, y \le 1 \text{ and } x \le y \\ -\cos^{-1}\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\}, \\ \text{if } -1 \le x, y \le 1 \text{ and } x \ge y. \end{cases}$$

Property (5):

$$\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{if } x > 0,\ y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), \text{if } x < 0, y < 0 \text{ and } xy > 1 \\ \frac{\pi}{2} \text{ for } x > 0, y > 0 \text{ and } xy = 1 \\ -\frac{\pi}{2} \text{ for } x < 0, y < 0 \text{ and } xy = 1 \end{cases}$$

Property (6):

$$\tan^{-1}x - \tan^{-1}y = \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right), \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), \text{if } x > 0,\ y < 0 \text{ and } xy > -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), \text{if } x < 0,\ y > 0 \text{ and } xy > -1 \\ \frac{\pi}{2} \text{ for } x > 0, y > 0 \text{ and } xy = -1 \\ -\frac{\pi}{2} \text{ for } x < 0, y < 0 \text{ and } xy = -1 \end{cases}$$