Mind Map-4

DETERMINANTS

1. Determinant

Every square matrix associates to an expression or a number which is known as its determinant. If $A = [a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by det (A) or |A| or Δ.

2. Determinant of a Square Matrix of Order Three

Consider $A = [a_{ij}]_{3\times3}$

Then, $|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

Expansion along first Row (R_1)

$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$

$= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$

3. Properties of Determinants

(i) The value of a determinant remains unchanged if its rows and columns are interchanged.

(ii) If any two rows (or columns) of a determinant are interchanged, then sign of determinant changes.

(iii) If any two rows (or columns) of a determinant are identical, then the value of determinant is zero.

(iv) If each element of a row (or a column) of a determinant is multiplied by a constant k, then its value gets multiplied by k.

(v) If some or all the elements of a row or column of a determinant are expressed as a sum of two (or more) terms, then the determinant can be expressed as a sum of two (or more) determinants.

(vi) If the equimultiples of corresponding elements of other row (or column) are added to each element of any row or column of a determinant, then the value of the determinant remains the same.

(vii) $|A^T| = |A|$, where A^T = transpose of A.

(viii) If $A = [a_{ij}]_{3\times3}$, then $|kA| = k^3|A|$.

(ix) The determinant of the product of matrices is equal to product of their respective determinants, i.e., |AB| = |A| |B|, where A & B are square matrices of same order

4. Area of a Triangle

Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

5. Minor and Cofactor of an Element of a Determinant

Minor: The determinant that is left by cancelling the row and column intersecting at a particular element of a determinant is called the minor of that element of the determinant. Minor of an element a_{ij} of a determinant is denoted by M_{ij}.

Cofactor: The cofactor of an element a_{ij} of a determinant is denoted by A_{ij} (or C_{ij}) and is equal to $(-1)^{i+j} M_{ij}$.

6. Adjoint of a Matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then adj $A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$,

where A_{ij} is the cofactor of a_{ij}

If A be any given square matrix of order n, then
A(adj A) = (adj A) A = |A| I
where I is the identity matrix of order n.

7. Singular and Non-SIngular Matrices

A square matrix A is said to be singular if |A| = 0, otherwise it is called non-singular matrix.
If A & B are non-singular matrices of same order, then AB & BA are also non-singular matrices of same order.

8. Inverse of a Matrix

If A and B are two matrices such that
AB = I = BA
then B is called the inverse of A and it is denoted by A^{-1}.

Also, $A^{-1} = \frac{\text{adj}A}{|A|}$, if $|A| \neq 0$

Properties of Inverse Matrix

Let A and B are two invertible matrices of the same order, then

(i) $(AB)^{-1} = B^{-1}A^{-1}$

(ii) $(A^T)^{-1} = (A^{-1})^T$

(iii) $\text{adj}(A^{-1}) = (\text{adj } A)^{-1}$

9. Applications of Determinants and Matrices

Solution of System of Linear Equations using Inverse of a Matrix

Consider the system of equations

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then, we can write, AX = B i.e.,

- Unique solution of the equation AX = B is given by $X = A^{-1}B$, when $|A| \neq 0$
- A system of equations is said to be consistent or inconsistent according as its solution exists or not.
- For a square matrix A in the matrix equation AX = B
  - (i) If $|A| \neq 0$, there exists a unique solution and the system of equations is consistent.
  - (ii) If |A| = 0, and (adj A) B ≠ 0, then there exists no solution and the system of equations is inconsistent
  - (iii) If |A| = 0 and (adj A) B = 0, then the system may or may not be consistent according as the system has either infinitely many solutions or no solution.