LIMIT

1. DEFINITION :

Let f(x) be defined on an open interval about 'a' except possibly at 'a' itself. If f(x) gets arbitrarily close to L (a finite number) for all x sufficiently close to 'a' we say that f(x) approaches the limit L as x approaches 'a' and we write $\lim_{x \to a} f(x) = L$ and say "the limit of f(x), as x approaches a, equals L".

2. LEFT HAND LIMIT & RIGHT HAND LIMIT OF A FUNCTION :

Left hand limit (LHL) = $\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a - h), h > 0$.

Right hand limit (RHL) = $\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a + h), h > 0$.

Limit of a function f(x) is said to exist as, $x \to a$ w' en $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ = Finite quantity.

Important note :

In $\lim_{x \to a} f(x)$, $x \to a$ necessarily implies $x \neq a$. That is while evaluating limit at x = a, we are not concerned with the value of the function at x = a. In fact the function may or may not be defined at x = a.

Also it is necessary to note that if f(x) is defined only on one side of **'x = a'**, one sided limits are good enough to establish the existence of limits, & if f(x) is defined on either side of **'a'** both sided limits are to be considered.

3. FUNDAMENTAL THEOREMS ON LIMITS :

Let $\lim_{x\to a} f(x) = l$ & $\lim_{x\to a} g(x) = m$. If l & m exists finitely then :

(a) Sum rule : $\lim_{x\to a}\left[f(x)+g(x)\right] = l + m$

(b) Difference rule : $\lim_{x\to a}\left[f(x)-g(x)\right] = l - m$

(c) Product rule : $\lim_{x\to a} f(x).g(x) = l.m$

(d) Quotient rule : $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{l}{m}$, provided $m \neq 0$

(e) Constant multiple rule : $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$; where k is constant.

(f) Power rule : If m and n are integers, then $\lim_{x\to a}\left[f(x)\right]^{m/n} = l^{m/n}$ provided $l^{m/n}$ is a real number.

(g) $\lim_{x\to a} f\left[g(x)\right] = f\left(\lim_{x\to a} g(x)\right) = f(m)$; provided f(x) is continuous at $x = m$.

For example : $\lim_{x\to a} \ell n(f(x)) = \ell n[\lim_{x\to a} f(x)]$; provided ℓnx is defined at $x = \lim_{t\to a} f(t)$.

4. INDETERMINATE FORMS :

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0\times\infty, 1^{\infty}, 0^0, \infty^0$.

Note :

We cannot plot ∞ on the paper. Infinity (∞) is a symbol & not a number. It does not obey the laws of elementary algebra,

5. GENERAL METHODS TO BE USED TO EVALUATE LIMITS:

(a) Factorization :

Important factors :

(i) $x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + ... + a^{n-1}),\ n \in N$

(ii) $x^n + a^n = (x + a)(x^{n-1} - ax^{n-2} + ... + a^{n-1})$, n is an odd natural number.

Note : $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = na^{n-1}$

(b) Rationalization or double rationalization :

In this method we rationalise the factor containing the square root and simplify.

(c) Limit when $x \to \infty$:

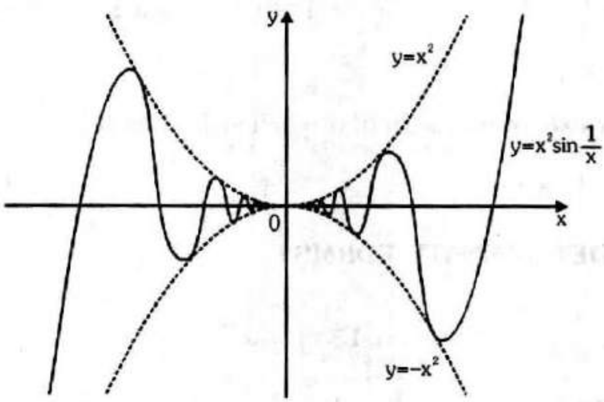
(i) Divide by greatest power of x in numerator and denominator.

(ii) Put $x = 1/y$ and apply $y \to 0$

(d) Squeeze play theorem (Sandwich theorem) :

If $f(x) \le g(x) \le h(x);\ \forall x\ \&\ \lim\limits_{x \to a} f(x) = \ell = \lim\limits_{x \to a} h(x)$ then

$\lim\limits_{x \to a} g(x) = \ell$,



for example : $\lim\limits_{x \to 0} x^2 \sin\dfrac{1}{x} = 0$, as illustrated by the graph given.

6. LIMIT OF TRIGONOMETRIC FUNCTIONS :

$\lim_{x\to 0} \frac{\sin x}{x} = 1 = \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\tan^{-1} x}{x} = \lim_{x\to 0} \frac{\sin^{-1} x}{x}$ [where x is measured in radians]

(a) If $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} \frac{\sin f(x)}{f(x)} = 1$.

(b) Using substitution $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a-h)$ or $\lim_{h\to 0} f(a+h)$ i.e.

by substituting x by a − h or a + h

7. LIMIT OF EXPONENTIAL FUNCTIONS :

(a) $\lim_{x\to 0} \frac{a^x - 1}{x} = \ell n a (a > 0)$ In particular $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$.

In general if $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} \frac{a^{f(x)} - 1}{f(x)} = \ell n a, a > 0$

(b) $\lim_{x\to 0} \frac{\ell n(1+x)}{x} = 1$

(c) $\lim_{x\to 0} (1+x)^{1/x} = e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$

(**Note :** The base and exponent depends on the same variable.)

In general, if $\lim_{x\to a} f(x) = 0$, then $\lim_{x\to a} (1 + f(x))^{1/f(x)} = e$

(d) If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} \phi(x) = \infty$,

then $\lim_{x\to a} [f(x)]^{\phi(x)} = e^k$ where $k = \lim_{x\to a} \phi(x) [f(x) - 1]$

(e) If $\lim_{x\to a} f(x) = A > 0$ & $\lim_{x\to a} \phi(x) = B$ (a finite quantity),

then $\lim_{x\to a} [f(x)]^{\phi(x)} = e^{B \ell n A} = A^B$

8. LIMIT USING SERIES EXPANSION :

Expansion of function like binomial expansion, exponential & logarithmic expansion, expansion of sinx, cosx, tanx should be remembered by heart which are given below :

(a) $a^x = 1 + \frac{x \ell na}{1!} + \frac{x^2 \ell n^2 a}{2!} + \frac{x^3 \ell n^3 a}{3!} + ... a > 0$

(b) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$

(c) $\ell n(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\text{for} -1 < x \le 1$

(d) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$

(e) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$

(f) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(g) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

(h) $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + ...$

(i) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... n \in Q$