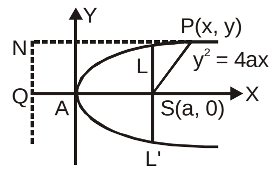
Conic Section

PARABOLA

DEFINITION

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point (called the focus) is equal to its distance from a fixed straight line (called the directrix).



Let S be the focus. QN be the directrix and P be any point on the parabola. Then by definition. PS = PN where PN is the length of the perpendicular from P on the directrix QN.

TERMS RELATED TO PARABOLA

Axis : A straight line passes through the focus and perpendicular to the directrix is called the axis of parabola.

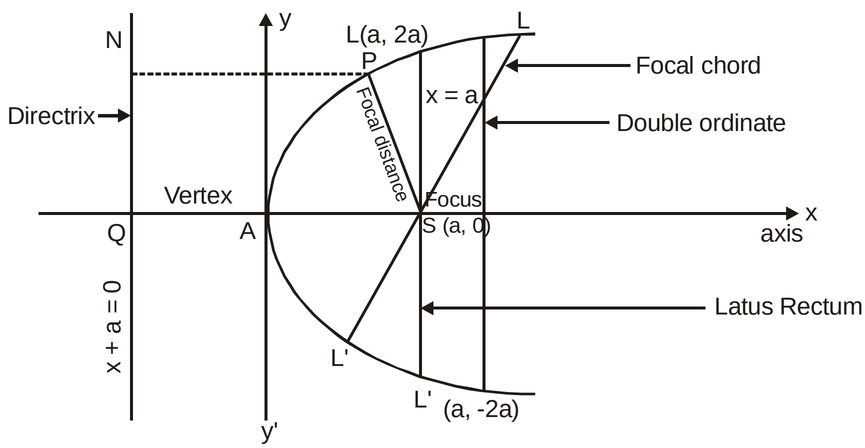
Vertex : The point of intersection of a parabola and its axis is called the vertex of the parabola.

The vertex is the middle point of the focus and the point of intersection of axis and directrix.

Eccentricity : If P be a point on the parabola and PN and PS are the distance from the directrix and focus S respectively then the ratio PS/PN is called the eccentricity of the parabola which is denoted by e.

By the definition for the parabola e = 1.

If $e > 1 \Rightarrow$ hyperbola, $e = 0 \Rightarrow$ circle, $e < 1 \Rightarrow$ ellipse



Latus Rectum

Let the given parabola be $y^2 = 4ax$. In the figure LSL' (a line through focus $\perp$ to axis) is the latus rectum.

Also by definition,

$$LSL' = 2\left(\sqrt{4a.a}\right) = 4a$$

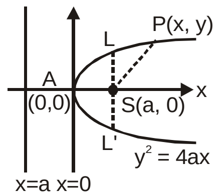
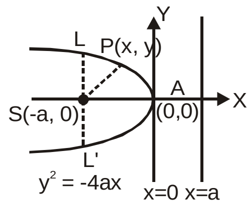
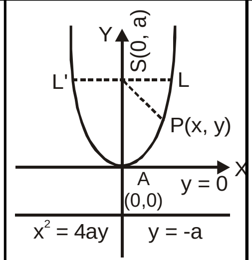
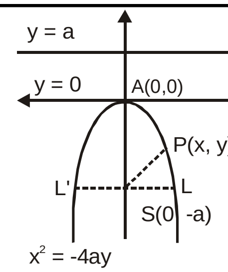
= double ordinate (Any chord of the parabola $y^2 = 4ax$ which is $\perp$ to its axis is called the double ordinate) through the focus S.

Note : Two parabolas are said to be equal when their latus recta are equal.

Focal Chord

Any chord to the parabola which passes through the focus is called a focal chord of the parabola.

FOUR STANDARD FORMS OF THE PARABOLA

Standard Equation	$y^2 = 4ax\ (a>0)$	$y^2 = -4ax (a>0)$	$x^2 = 4ay (a>0)$	$x^2 = -4ay (a > 0)$
Shape of Parabola				
Vertex	A(0, 0)	A(0, 0)	A(0, 0)	A(0, 0)
Focus	S(a, 0)	S(–a, 0)	S(0, a)	S(0, –a)
Equation of directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of latus rectum	4a	4a	4a	4a
Extermities of latus rectum	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
Equation of latus rectum	$x = a$	$x = -a$	$y = a$	$y = -a$
Equation of tangents at vertex	$x = 0$	$x = 0$	$y = 0$	$y = 0$
Focal distance of a point P(x, y)	$x + a$	$x - a$	$y + a$	$y - a$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Eccentricity (e)	1	1	1	1

REDUCTION OF STANDARD EQUATION

If the equation of a parabola contains second degree term either in y or in x(but not in both) then it can be reduced into standard form. For this we change the given equation into the following forms-

$$(y - k)^2 = 4a\,(x - h) \text{ or } (x - p)^2 = 4b\,(y - q)$$

Then we compare from the following table for the results related to parabola.

GENERAL EQUATION OF A PARABOLA

If (h, k) be the locus of a parabola and the euqation of directrix is $ax + by + c = 0$, then its equation is given by

$$\sqrt{(x-h)^2 + (y-k)^2} = \frac{ax+by+c}{\sqrt{a^2+b^2}} \text{ which gives } (bx - ay)^2 + 2gx + 2fy + d = 0$$

where g, f, d are the constant.

Note

· The general equation of second degree $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents a parabola, if

(a) $h^2 = ab$

(b) $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

PARAMETRIC EQUATIONS OF A PARABOLA

Clearly $x = at^2$, $y = 2at$ satisfy the equation $y^2 = 4ax$ for all real values of t. Hence the parametric equation of the parabola $y^2 = 4ax$ are $x = at^2$, $y = 2at$, where t is the parameter.

Also, $(at^2, 2at)$ is a point on the parabola $y^2 = 4ax$ for all real values of t. This point is also described as the point 't' on the parabola.

EQUATION OF A CHORD

(i) The equation of chord joining the points (x_1, y_1) and (x_2, y_2) on the parabola $y^2 = 4ax$ is

$$y(y_1 + y_2) = 4ax + y_1y_2$$

(ii) The equation of chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is-

$$(y - 2at_1) = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$

$$\Rightarrow \quad y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$

$$y(t_1 + t_2) = 2(x + at_1t_2)$$

(iii) Length of the chord $y = mx + c$ to the parabola $y^2 = 4ax$ is given by $\frac{4}{m^2}\sqrt{1+m^2}\sqrt{a(a-mc)}$.

Condition for the Chord to be a Focal Chord

The chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ passes through focus provided $t_1t_2 = -1$.

Length of Focal Chord

The length of a focal chord joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is $(t_2 - t_1)^2$.

Note :

- The length of the focal chord through the point 't' on the parabola $y^2 = 4ax$ is $a(t + 1/t)^2$
- The length of the chord joining two points 't_1' and 't_2' on the parabola $y^2 = 4ax$ is

$$a(t_1 - t_2)\sqrt{(t_1 + t_2)^2 + 4}$$

CONDITION FOR TANGENCY AND POINT OF CONTACT

The line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$ and the coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

Note

- The line $y = mx + c$ touches parabola $x^2 = 4ay$ if $c = -am^2$
- The line $x\cos\alpha + y\sin\alpha = p$ touches the parabola $y^2 = 4ax$ if $a\sin^2\alpha + p\cos\alpha = 0$.

EQUATION OF TANGENT IN DIFFERENT FORMS

(i) Point Form

The equation of the tangent to the parabola $y^2 = 4ax$ at the point (x_1, y_1) is

$$yy_1 = 2a(x + x_1)$$

Note :

(ii) Parametric Form

The equation of the tangent to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$ty = x + at^2.$$

(iii) Slope Form

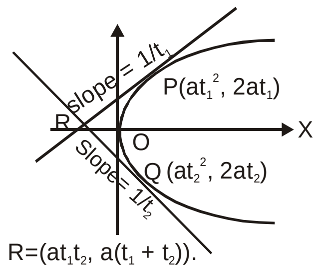
The equation of tangent to the parabola $y^2 = 4ax$ in terms of slope 'm' is

$$y = mx + \frac{a}{m}.$$

The coordinate of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

POINT OF INTERSECTION OF TANGENTS

The point of intersection of tangents drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is



Note :

- Angle between tangents at two points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y_2 = 4ax$ is $\theta = \tan^{-1}\left|\frac{t_2 - t_1}{1 + t_1t_2}\right|$
- The G.M. of the x-coordinates of P and Q (i.e., $\sqrt{at_1^2 \times at_2^2} = at_1t_2$) is the x-coordinate of the point of intersection of tangents at P and Q on the parabola.
- The A.M. of the y-coodinates of P and Q $\left(\text{i.e. } \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)\right)$ is the y-coordinate of the point of intersection of tangents at P and Q on the parabola.
- The orthocentre of the triangle formed by three tangents to the parabola lies on the directrix.

EQUATIONS OF NORMAL IN DIFFERENT FORMS

(i) Point form

The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

(ii) Parametric form

The equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$$y + tx = 2at + at^3.$$

(iii) Slope form

The eqaution of normal to the parabola $y^2 = 4ax$ in terms of slope 'm' is

$$y = mx - 2am - am^3$$

Note : The coordinates of the point of contact are (am^2 – 2am).

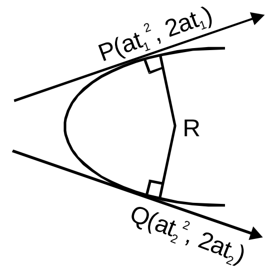
Condition for Normality

The line y = mx + c is normal to the parabola

$$y^2 = 4ax \quad \text{if} \quad c = -2am - am^3 \quad \text{and} \quad x^2 = 4ay \quad \text{if} \quad c = 2a + \frac{a}{m^2}$$

Point of Intersection of Normals

The point of intersection of normals drawn at two different points of contact $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ is

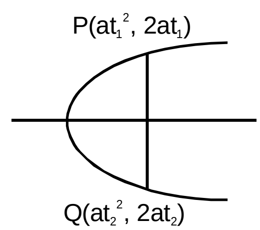


$$R \equiv [2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$$

Note :

- If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ again at $Q(at_2^2, 2at_2)$, then

$$t_2 = -t_1 - \frac{2}{t_1}$$



It is clear that PQ is normal to the parabola at P and not at Q.

- If the normals at the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ meet on the parabola $y^2 = 4ax$, then $t_1t_2 = 2$
- The normal at the extremities of the latus rectum of a parabola intersect at right angle on the axis of the parabola.

Co-normal Points

Any three points on a parabola normals at which pass through a common point are called co-normal points

Note :

This implies that if three normals are drawn through a point (x_1, y_1) then their slopes are the roots of the cubic: $y_1 = mx_1 - 2am - am^3$ which gives three values of m. Let these values are m_1, m_2, m_3 then from the eq^n.

$\Rightarrow \quad am^3 + (2a - x_1)m + y_1 = 0$

- The sum of the slopes of the normals at co-normal points is zero, i.e., $m_1 + m_2 + m_3 = 0$.

  and $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - x_1}{a}$ and $m_1m_2m_3 = -\frac{y_1}{a}$

- The sum of the ordinates of the co-normal points is zero (i.e., $-2am_1 - 2am_2 - 2am_3 = -2a(m_1 + m_2 + m_3) = 0$.
- The centroid of the triangle formed by the co-normal points lies on the axis of the parabola

- The vertices of the triangle formed by the co-normal points are $(am_1^2 - 2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$. Thus, y-coordinate of the centroid becomes

$$\frac{-2a(m_1+m_2+m_3)}{3} = \frac{-2a}{3}\times 0 = 0.$$

i.e., centroid of triangle $\left(\frac{am_1^2+am_2^2+am_3^2}{3}, \frac{2am_1+2am_2+2am_3}{3}\right) = \left(\frac{am_1^2+am_2^2+am_3^2}{3}, 0\right)$

Hence, the centroid lies on the x-axis i.e., axis of the parabola.]

- If three normals drawn to any parabola $y^2 = 4ax$ from a given point (h, k) be real, then h > 2a.

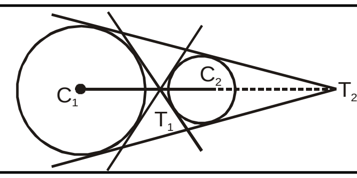
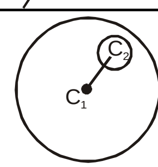
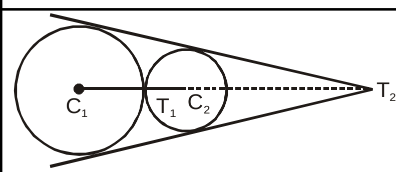
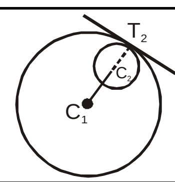
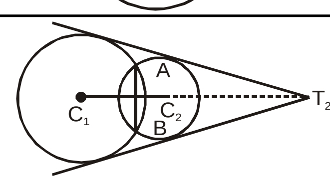
POSITION OF A POINT & LINE W.R.T. A PARABOLA

- The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, = \text{ or } < 0$, respectively.
- The line $y = mx + c$ will intersect a parabola $y^2 = 4ax$ in two real and different, coincident or imaginary point, according as $a - mc >, = < 0$.

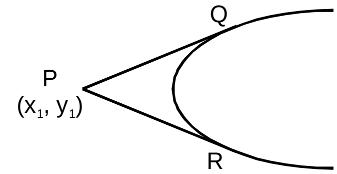
Number of tangents drawn from a point to a parabola

Two tangents can be drawn from a point to a parabola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the parabola.

EQUATION OF THE PAIR OF TANGENTS

	CONDITION	POSITION	DIAGRAM	NO. OF COMMON TANGENTS
(i)	$C_1C_2 > r_1 + r_2$	do not intersect or one outside the other		4
(ii)	$C_1C_2 < \lvert r_1 - r_2 \rvert$	one inside the other		0
(iii)	$C_1C_2 = r_1 + r_2$	external touch		3
(iv)	$C_1C_2 = \lvert r_1 - r_2 \rvert$	internal touch		1
(v)	$\lvert r_1 - r_2 \rvert < C_1C_2 < r_1 + r_2$	intersection at two real points		2

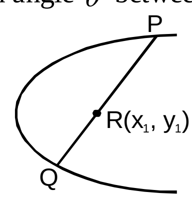
The equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $SS_1 = T^2$.



where $S \equiv y^2 - 4ax$, $S_1 \equiv y_1{}^2 - 4ax_1$ and $T \equiv yy_1 - 2a(x + x_1)$

LOCUS OF POINT OF INTERSECTION

The locus of point of intersection of tangent to the parabola $y^2 = 4ax$ which are having an angle θ between them given by $y^2 - 4ax = (a + x)^2 \tan^2 \theta$

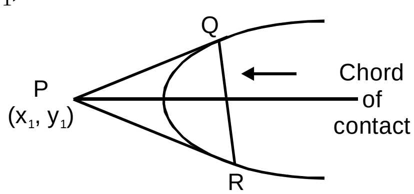


Note :

- If $\theta = 0°$ or p then locus is $(y^2 - 4ax) = 0$ which is the given parabola.
- If $\theta = 90°$, then locus is $x + a = 0$ which is the directrix of the parabola.

CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is $T = 0$ where $T \equiv yy_1 - 2a(x + x_1)$.



Note :

- The chord of contact joining the point of contact of two perpendicular tangents always passes through focus.
- Lengths of the chord of contact is $\frac{1}{a}\sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$
- Area of triangle formed by tangents drawn from (x_1, y_1) and their chord of contact is $\frac{1}{2a}(y_1^2 - 4ax_1)^{3/2}$.

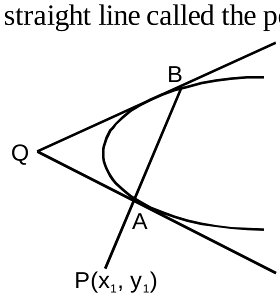
CHORD WITH A GIVEN MID POINT

The equation of the chord of the parabola $y^2 = 4ax$ with $P(x_1, y_1)$ as its middle point is given by

$$T = S_1 \quad \text{where } T \equiv yy_1 - 2a(x + x_1) \text{ and } S_1 \equiv y_1{}^2 - 4ax.$$

POLE AND POLAR

Let P be a given point. Let a line through P intersect the parabola at two points A and B. Let the tangents at A anf B intersect at Q. The locus of point Q is a straight line called the polar of point P with respect to the parabola and the point P is called the pole of the polar.



Equation of Polar of a Point

The polar of a point $P(x_1, y_1)$ with respect to the parabola $y^2 = 4ax$ is $T = 0$ where $T \equiv yy_1 - 2a(x + x_1)$.

Coordinate of pole

The pole of the line $lx + my + n = 0$ with respect to the parabola $y^2 = 4ax$ is $\left(\frac{n}{l}, -\frac{2am}{l}\right)$

Conjugate points and conjugate lines

(i) If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$ then the polar of Q will pass through P and such points are said to be conjugate points.

So two points (x_1, y_1) and (x_2, y_2) are conjugate points with respect to parabola $y^2 = 4ax$ if $yy_1 = 2a(x_1+x_2)$

(ii) If the pole of a line $ax + by + c = 0$ lies on the another line $a_1x + b_1y + c_1 = 0$ then the pole of the second line will lie on the first and such line are said to be conjugate lines.

So two lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ are conjugate lines with respect to parabola $y_2 = 4ax$ if $(l_1n_2 + l_2n_1) = 2\,am_1m_2$

Note

· The polar of focus is directrix and pole of directrix is focus.

· The polars of all points on directrix always pass through a fixed point and this fixed point is focus.

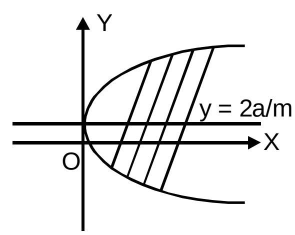
· The pole of a focal chord lies on directrix and locus of poles of focal chord is a directrix.

· The chord of contact and polar of any point on the directrix always passes through focus.

DIAMETER OF A PARABOLA

Diameter of a parabola is the locus of middle points of a series of its parallel chords.

The equation of the diameter bisecting chords of slope m of the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.



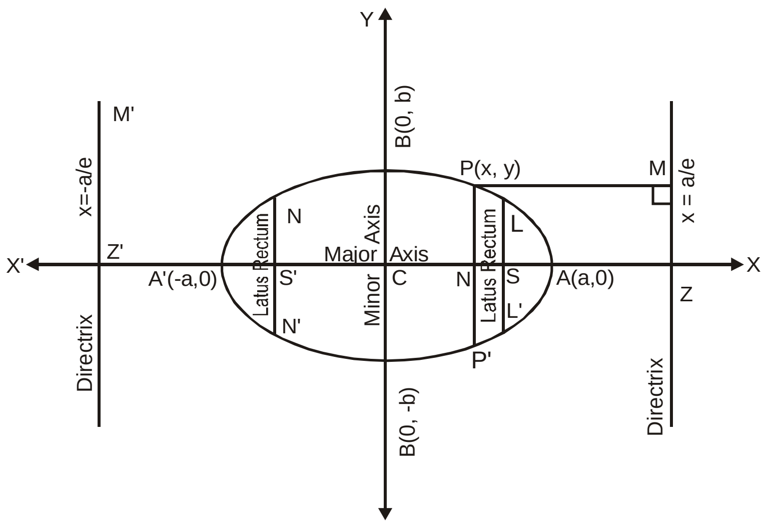
ELLIPSE

EQUATION OF AN ELLIPSE IN STANDARD FORM

The Standard form of the equation of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b),$$

where a and b are constants.



TERMS RELATED TO AN ELLIPSE

A sketch of the locus of a moving point satisfying the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$, has been shown in the figure given above.

Symmetry

(a) On replacing y by –y, the above equation remains unchanged. So, the curve is symmetrical about x-axis.

(b) On replacing x by –x, the above equation remains unchanged. So, the curve is symmetrical about y-axis

Foci

If S and S' are the two foci of the ellipse and their coordinates are (ae, 0) and (–ae, 0) respectively, then distance between foci is given by

$$SS' = 2ae.$$

Directrices

If ZM and Z' M' are the two directrices of the ellipse and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively, then the distance between directrices is given by

$$ZZ' = \frac{2a}{e}.$$

Axes

The lines AA' and BB' are called the major axis and minor axis respectively of the ellipse.

The length of major axis = AA' = 2a

The length of minor axis = BB' = 2b

Centre

The point of intersection C of the axes of the ellipse is called the centre of the ellipse. All chords, passing through C are bisected at C.

Vertices

The end points A and A' of the major axis are known as the vertices of the ellipse

$$A \equiv (a, 0) \text{ and } A' \equiv (-a, 0)$$

Focal chord

A chord of the ellipse passing through its focus is called a focal chord.

Ordinate and Double Ordinate

Let P be a point on the ellipse. From P, draw PN $\perp$ AA' (major axis of the ellipse) and produce PN to meet the ellipse at P'. Then PN is called an ordinate and PNP' is called the double ordinate of the point P.

Latus Rectum

If LL' and NN' are the latus rectum of the ellipse, then these lines are $\perp$ to the major axis AA' passing through the foci S and S' respectively.

$$L \equiv \left(ae, \frac{b^2}{a}\right), \qquad L' \equiv \left(ae, \frac{-b^2}{a}\right)$$

$$N \equiv \left(-ae, \frac{b^2}{a}\right), \qquad N' \equiv \left(-ae, \frac{-b^2}{a}\right)$$

Length of latus rectum = LL' = $\frac{2b^2}{a} = NN'$.

By definition SP = ePM = $e\left(\frac{a}{e} - x\right)$ = a – ex and $S'P = e\left(\frac{a}{e} + x\right) = a + ex.$

Thus implies that distances of any point P(x, y) lying on the ellipse from foci are : (a – ex) and (a + ex). In other words SP + S'P = 2a

i.e., sum of distances of any point P(x, y) lying on the ellipse from foci is constant.

Eccentricity

Since, SP = ePM, therefore

$$SP^2 = e^2 \, PM^2$$

or

$$(x - ae)^2 + (y - 0)^2 = e^2 \left(\frac{a}{e} - x\right)^2$$

$$(x - ae)^2 + y^2 = (a - ex)^2$$

$$x^2 + a^2e^2 - 2aex + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2 (1 - e^2) + y^2 = a^2 (1 - e^2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

On comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$b^2 = a^2 (1 - e^2) \quad \text{or} \quad e = \sqrt{1 - \frac{b^2}{a^2}}$$

Auxillary Circle

The circle drawn on major axis AA' as diameter is known as the Auciliary circle.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then the equation of its auxiliary circle is

$$x^2 + y^2 = a^2.$$

Let Q be a point on auxiliary circle so that QM, perpendicular to major axis meets the ellipse at P. The points P and Q are called as corresponding point on the ellipse and auxiliary circle respectively.

The angle θ is known as eccentric angle of the point P on the ellipse.

It may be noted that the CQ and not CP is inclined at θ with x-axis.

GENERAL EQUATION OF THE ELLIPSE

The general equation of an ellipse whose focus is (h, k) and the directrix is the line ax + by + c = 0 and the eccentricity will be e. Then let $P(x_1, y_1)$ be any point on the ellipse which moves such that

$$SP = CPM$$

$$\Rightarrow \qquad (x_1 - h)^2 + (y_1 - k)^2 = \frac{e^2(ax_1+by_1+c)^2}{a^2+b^2}$$

Hence the locus of (x_1, y_1) will be given by

$$(a^2 + b^2)\,[(x - h)^2 + (y - k)^2] = e^2\,(ax + by + c)^2$$

Which is the equation of second degree from which we can say that any equation of second degree represent equation of an ellipse.

Note : Condition for second degree in X and Y to represent an ellipse is thay if $h^2 = ab < 0$ & $\Delta = abc + 2\,fgh - af^2 - by^2 - ch^2 \neq 0$.

PARAMETRIC EQUATION OF THE ELLIPSE

The coordinates x = a cos θ and y = b sin θ satisfy the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

for all real values of θ. Thus, x = a cos θ, y = b sin θ are the parametric equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where the parameter $0 \le\ < 2\pi$.

Hence the coordinates of any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may be taken as (a cos θ, b sin θ). This point is also called the point 'θ'.

The angle θ is called the eccentric angle of the point (a cos θ, b sin θ) on the ellipse.

Equation of Chord

The equation of the chord joining the points P $\equiv$ (acos θ_1, bsin θ_1) and Q $\equiv$ (acos θ_2, bsin θ_2) is

$$\frac{x}{a}\cos\left(\frac{\theta_1+\theta_2}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1-\theta_2}{2}\right).$$

POSITION OF A POINT WITH RESPECT TO AN ELLIPSE

The point $P(x_1, y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ according as

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1>0,=0 \qquad \text{or} < 0.$$

CONDITION OF TANGENCY AND POINT OF CONTACT

The condition for the line y = mx + c to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is that $c^2 = a^2m^2 + b^2$ and the coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

Note

- x cos a + y sin a = p is a tangent if $p^2 = a^2 \cos^2 a + b^2 \sin^2 a$.
- lx + my + n = 0 is a tangent if $n^2 = a^2l^2 + b^2m^2$.

EQUATION OF TANGENT IN DIFFERENT FORMS

(i) Point Form

The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

at the point (x_1, y_1) is $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$

Note :

The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y^2 by yy_1.x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$, and xx by $\frac{xy_1+x_1y}{2}$. This method is used only when the equation of ellipse is a polynomial of second degree in x and y.

(ii) Parametric Form

The equation of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (acos q, bsin q) is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

(iii) Slope Form

The equation of tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of slope 'm' is

$$y = mx \pm \sqrt{a^2m^2+b^2}$$

The coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2+b^2}}, \mp\frac{b^2}{\sqrt{a^2m^2+b^2}}\right)$$

Note :

- Number of Tangent Drawn From a Point

  Two tangents can be drawn from a point to an ellipse. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the ellipse.

- Director Circle

  It is the locus of points from which perpendicular tangents are drawn to the ellipse. The equation of Director Circle of the ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } x^2 + y^2 = a^2 + b^2.$$

The product of perpendicualr from the foci on any tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is equal to b^2.

EQUATION OF NORMAL IN DIFFERENT FORMS

(i) Point Form

The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is

$$\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2.$$

(ii) Parametric Form

The equation of the normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (acos q, bsin q) is

$$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2.$$

or $$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2.$$

(iii) Slope Form

The equation of normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in terms of slope 'm' is

$$y = mx \pm \frac{m(a^2-b^2)}{\sqrt{a^2+b^2m^2}}$$

Notes :

- The coordinates of the points of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2+b^2m^2}}, \pm\frac{mb^2}{\sqrt{a^2+b^2m^2}}\right)$$

- Condition for normality The line = mx + c is normal to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ if } c^2=\frac{m^2(a^2-b^2)^2}{(a^2+b^2m^2)}$$

- Number of Normals

In general, four normals can be drawn to an ellipse from a point in its plane i.e., there are four points on the ellipse, the normals at which it will pass through a given point. These four points are called the co-normal points.

- if $\alpha,\beta,\gamma,\delta$ are the eccentric angles of the four points on the ellipse such that the normals at these points are concurrent, then $(\alpha+\beta+\gamma+\delta)$ is an odd multiple of π.

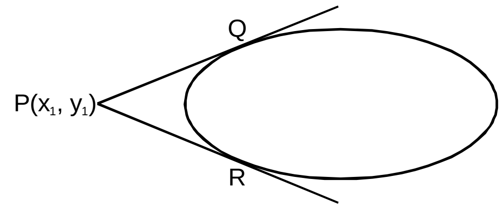
- If α,β,γ are the eccentric angles of three points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the normals at which are concurrent, then

$$\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)=0.$$

EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point

$P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is



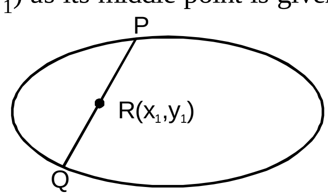
$$SS_1 = T^2$$

where $S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1,\ S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$

and $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$

CHORD WITH A GIVEN MID POINT

The equation of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $P(x_1, y_1)$ as its middle point is given by



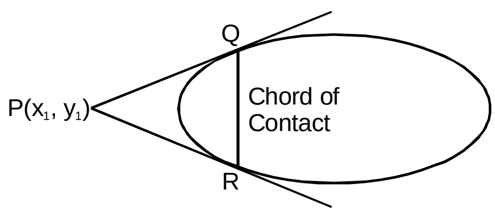
$$T = S_1$$

where $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$ and $S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$.

CHORD OF CONTACT

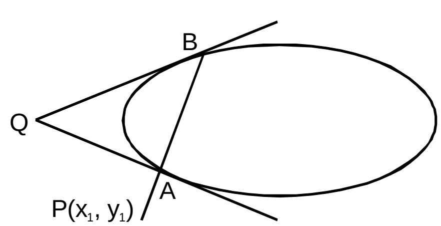
The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is T = 0, where

$$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1.$$



POLE AND POLAR

Let P be a given point. Let a line through P intersect the ellipse at two points A and B. Let the tangents at A and B intersect at Q. The locus of point Q is a straight line called the polar of point P w.r.t. the ellipse and the point P is called the pole of the polar.



Equation of polar of a Point

The polar of a point $P(x_1, y_1)$ w.r.t. the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is T = 0, where $T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$.

Notes :

- Polar of the focus is the directrix.

- Any tangent is the polar of its points of contact.

- Pole of a given line lx + my + n = 0 w.r.t. the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } \left(\frac{-a^2 l}{n}.\frac{-b^2 m}{n}\right)$$

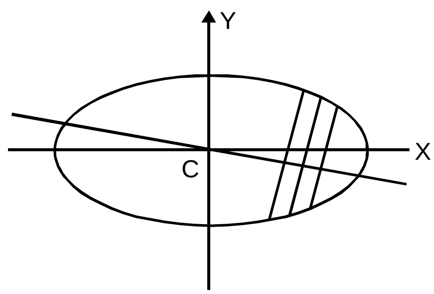
- If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$ then the polar of Q will pass through P and such points are said to be conjugate points.

- If the pole of a line lx + my + n = 0 lies on the another line l'x + m'y + n' = 0, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

- The point of intersection of any two lines is the pole of the line joining the pole of the two lines.

DIAMETER OF AN ELLIPSE

The locus of the middle points of a system of a parallel chords of an ellipse is called a diameter of the ellipse.



The equation of the diameter bisecting chords of slope m of the ellipse

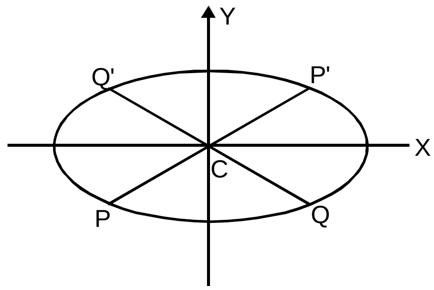
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is $$y=\frac{b^2}{a^2 m}x.$$

Note : Diameter of an ellipse always passes through its centre. Thus a diameter of an ellipse is its chord passing through its centre.

CONJUGATE DIAMETERS

Two diameters of an ellipse are said to be conjugate diameters if each bisects the chord parallel to the other.



Note :

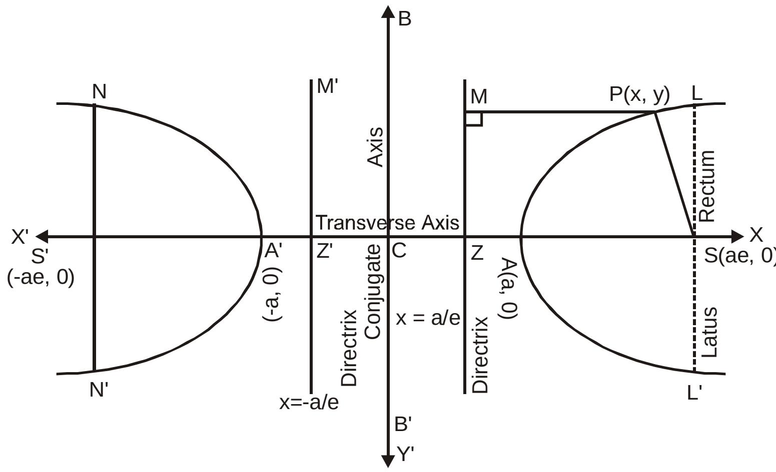
- Major and minor axes of an ellipse is also a pair of conjugate diameters.
- If m_1 and m_2 be the slopes of the conjugate diameters of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $m_1m_2 = \frac{-b^2}{a^2}$.
- The eccentric angles of the ends of a pair of conjugate diameters differ by a right angle.
  i.e., if PCP' and QCQ' is a pair of conjugate diameters and if eccentric angle of P is θ, then eccentric angles of Q, P', Q' (proceeding in anticlockwise direction) will be $\theta + \frac{\pi}{2}$, $\theta + \pi$ and $\theta + \frac{3\pi}{2}$ respectively. Ans.(4)

HYPERBOLA

EQUATION OF HYPERBOLA IN STANDARD FORM

The general form of standard hyperbola is

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where a and b are constants.



TERMS RELATED TO A HYPERBOLA

A sketch of the locus of a moving point satisfying the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, has been shown in the figure given above.

Symmetry Since only even powers of x and y occur in the above equation, so the curve is symmetrical about both the axes.

Foci If S and S' are the two foci of the hyperbola and their coordinatesd are (ae, 0) and (–ae, 0) respectively, then distance between foci is given by SS' = 2ae.

Directries ZM and Z' M' are the two directrices of the hyperbola and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$ respectively, then the distance directrices is given by zz' = $\frac{2a}{e}$.

Axes The lines AA' and BB' are called the transverse axis and conjugate axis respectively of the hyperbola.

The length of transverse axis = AA' = 2a

The length of conjugate axis = BB' = 2b

Centre The point of intersection C of the axes of hyperbola is called the centre of the hyperbola. All chords, passing through C, are bisected at C.

Vertices The points A $\equiv$ (a, 0) and A' $\equiv$ (–a, 0) where the curve meets the line joining the foci S and S', are called the vertices of the hyperbola.

Focal Chord A chord of the hyperbola passing through its focus is called a focal chord.

Focal Distances of a Point The difference of the focal distances of any point on the hyperbola is constant and equal to the length of the transverse axis of the hyperbola. If P is any point on the hyperbola, then

S'P – SP = 2a = Transverse axis.

Latus Rectum If LL' and NN' are the latus rectum of the hyperbola then these lines are perpendicular to the transverse axis AA', passing through the foci S and S' respectively.

$$L \equiv \left(ae, \frac{b^2}{a}\right), \qquad L' \equiv \left(ae, \frac{-b^2}{a}\right),$$

$$N \equiv \left(-ae, \frac{b^2}{a}\right), \qquad N' \equiv \left(-ae, \frac{-b^2}{a}\right).$$

Length of latus rectum = LL' = $\frac{2b^2}{a}$ = NN'.

Eccentricity of the Hyperbola We know that

$$SP = e\,PM \quad \text{or} \quad SP^2 = e^2\,PM^2$$

or $$(x - ae)^2 + (y - 0)^2 = e^2\ N' \equiv \left(x - \frac{a}{e}\right)^2$$

$$(x - ae)^2 + y^2 = (ex - a)^2$$

$$x^2 + a^2e^2 - 2aex + y^2 = e^2x^2 - 2aex + a^2$$

$$x^2 (e^2 - 1) - y^2 = a^2 (e^2 - 1)$$

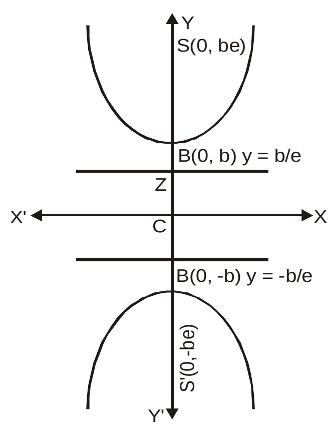
$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1.$$

On comparing with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we get $b^2 = a^2 (e^2 - 1)$ or $e = \sqrt{1 + \frac{b^2}{a^2}}$

PARAMETRIC EQUATIONS OF THE HYPERBOLA

Since coordinates x = a secθ and y = b tanθ satisfy the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

for all real values of q therefore, x = a sec θ, y = b tan θ are the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where the parameter $0 \le \theta < 2\pi$.



Hence, the coordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ may be taken as (a sec θ, b tan θ). This point is also called the point 'θ'.

The angle θ is called the eccentric angle of the point (a secθ, b tanθ) on the hyperbola.

Equation of Chord The equation of the chord joining the points

$$P \equiv (a \sec \theta_1, b \tan \theta_1) \text{ and } Q \equiv (a \sec \theta_2, b \tan \theta_2) \text{ is}$$

$$\frac{x}{a}\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\frac{y}{b}\sin\left(\frac{\theta_1+\theta_2}{2}\right)=\cos\left(\frac{\theta_1+\theta_2}{2}\right) \quad \text{or} \quad \begin{vmatrix} x & y & 1 \\ a\sec\theta_1 & b\tan\theta_1 & 1 \\ a\sec\theta_2 & b\tan\theta_2 & 1 \end{vmatrix}=0$$

CONJUGATE HYPERBOLA

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola is called the conjugate hyperbola of the given hyperbola.

The conjugate hyperbola of the hyperbola.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ is } \qquad -\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \left(\text{i.e., } \frac{x^2}{a^2}-\frac{y^2}{b^2}=-1\right)$$

PROPERTIES OF HYPERBOLA AND ITS CONJUGATE

	Hyperbola	Conjugate Hyperbola
Standard equation	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\frac{-x^2}{a^2}+\frac{y^2}{b^2}=1$ or $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$
Centre	(0, 0)	(0, 0)
Equation of transverse axis	$y = 0$	$x = 0$
Equation of conjugate axis	$x = 0$	$y = 0$
Length of transverse axis	2a	2b
Length of Conjugate axis	2b	2a
Foci	$(\pm ae, 0)$	$(0, \pm be)$
Equation of directrices	$x = \pm a/e$	$y = \pm b/e$
Vertices	$(\pm a, 0)$	$(0, \pm b)$
Eccentricity	$e=\sqrt{\frac{a^2+b^2}{a^2}}$	$e=\sqrt{\frac{a^2+b^2}{b^2}}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Parameter Coordinates	$(a \sec\theta, b\tan\theta)$	$(b\sec\theta, a\tan\theta)$
Focal radii	$SP = ex_1 - a$ and $S'P = ex_1 + a$	$SP = ey_1 - b$ and $S'P = ey_1 + b$
Difference of focal radii (S'P – SP)	2a	2b
Tangent at the vertices	$x = \pm a$	$y = \pm b$

POSITION OF A POINT WITH RESPECT TO A HYPERBOLA

The point $P(x_1, y_1)$ lies outside, on or inside the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ according as $\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1>0.$ $= 0$ or < 0.

CONDITION FOR TRANGENCY AND POINTS OF CONTACT

The condition for the line y = mx + c to be a tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is that $c^2 = a^2m^2 - b^2$ and the coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$$

EQUATION OF TANGENT IN DIFFERENT FORMS

Point Form The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1$.

Note : The equation of tangent at (x_1, y_1) can also be obtained by replacing x^2 by xx_1, y_2 by yy_1, x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$ and xy by $\frac{xy_1+x_1y}{2}$. This method is used only when the equation of hyperbola is a polynomial of second degree in x and y.

Parametric Form The eq^n of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (a secθ, b tanθ) is $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$

Slope Form The equation of tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of slope 'm' is

$$y = mx \pm \sqrt{a^2m^2-b^2}$$

The coordinates of the points of contact are

$$\left(\pm\frac{a^2m}{\sqrt{a^2m^2-b^2}}, \pm\frac{b^2}{\sqrt{a^2m^2-b^2}}\right)$$

Notes :

· Number of Tangents From a Point Two tangents can be drawn from a point to a hyperbola. The two tangents are real and distinct or coincident or imaginary according as the given point lies outside, on or inside the hyperbola.

· Director Circle It is the locus of points from which $\perp$ tangents are drawn to the hyperbola. The equation of director circle of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ is } x^2 + y^2 = a^2 - b^2.$$

EQUATION OF NORMAL IN DIFFERENT FORMS

Point Form The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (x_1, y_1) is

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$$

Parametric Form The equation of the normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point (a secθ, b tanθ) is $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

Slope Form The equation of normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in terms of slope 'm' is

$$y = mx \pm \frac{m(a^2+b^2)}{\sqrt{a^2-b^2m^2}}$$

Notes :

The coordinates of the points of contact are

$$\left(\pm\frac{a^2}{\sqrt{a^2-b^2m^2}}, \mp\frac{mb^2}{\sqrt{a^2-b^2m^2}}\right)$$

· Number of Normals

In general, four normals can be drawn to a hyperbola from a point in its plane i.e., there are four points on the hyperbola, the normals at which will pass through a given point. These four points are called the co-normal points.

· Tangent drawn at any point bisects the angle between the lines joining the point to the foci, whereas normal bisects the supplementary angle between the lines.

EQUATION OF THE PAIR OF TANGENTS

The equation of the pair of tangents drawn from a point

$P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$SS_1 = T^2$

where $S \equiv \frac{x^2}{a^2}-\frac{y^2}{b^2}-1$, $S_1 \equiv \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

and $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

CHORD WITH A GIVEN MID POINT

The equation of chord of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with $P(x_1, y_1)$ as its middle point is given by $T = S_1$ where $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$ and $S_1 \equiv \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1$

CHORD OF CONTACT

The equation of chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $T = 0$, where $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$.

POLE AND POLAR

The polar of a point $P(x_1, y_1)$ w.r.t. the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $T = 0$, where $T \equiv \frac{xx_1}{a^2}-\frac{yy_1}{b^2}-1$

Notes :

· Pole of a given line $lx + my + n = 0$ w.r.t. the hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $\left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right)$

- Polar of the focus is the directrix.
- Any tangent is the polar of its point of contact.
- If the polar of $P(x_1, y_1)$ passes through $Q(x_2, y_2)$ then the polar of Q will pass through P and such points are said to the conjugate points.
- If the pole of a line $lx + my + n = 0$ lies on the another line $l'x + m'y + n' = 0$, then the pole of the second line will lie on the first and such lines are said to be conjugate lines.

EQUATION OF A DIAMETER OF A HYPERBOLA

The equation of the diameter bisecting chords of slope m of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $y=\frac{b^2}{a^2m}$.

CONJUGATE DIAMETERS

Two diameters of a hyperbola are said to be conjugate diameters if each bisects the chord parallel to the other. If m_1 and m_2 be the slopes of the conjugate diameters of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then $m_1m_2 = \frac{b^2}{a^2}$

ASYMPTOTES OF HYPERBOLA

The lines $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ i.e., $y = \pm\frac{bx}{a}$ are called the asymptotes of the hyperbola.

The curve comes close to these lines as $x \to \infty$ or $x \to -\infty$ but never meets them. In other words, asymtote to a curve touches the curve at infinity.

Note :

- The angle between the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is $2\tan^{-1}\left(\frac{b}{a}\right)$.
- Asymptotes are the diagonals of the rectangle passing through A, B, A', B' with sides parallel to axes.
- A hyperbola and its conjugate hyperbola have the same asymptotes.
- The asymptotes pass through the centre of the hyperbola.
- The bisector of the angle between the asymptotes are the coordinates axes.
- The product of the perpendicular from any point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ to its asymptotes is a constant equal to $\frac{a^2b^2}{a^2+b^2}$.
- Any line drawn parallel to the asymptote of the hyperbola would meet the curve only at one point.
- A hyperbola and its conjugate hyperbola have the same asymptotes.

RECTANGULAR HYPERBOLA

If asymptotes of the standard hyperbola are perpendicular to each other, then it is known as Rectangular Hyperbola. Then

$$2\tan^{-1}\frac{b}{a} = \frac{\pi}{2} \Rightarrow b = a \text{ or } x^2 - y^2 = a^2$$

is general form of the equation of the rectangular hyperbola.

If we take the coordinate axes along the asymptotes of a rectangle hyperbola, then equation of rectangular hyperbola becomes : $xy = c^2$, where c is any constat.

In parametric form, the equation of rectangular hyperbola

$x = ct$, $y = c/t$, where t is the parameter.

The point $(ct, c/t)$ on the hyperbola $xy = c^2$ is generally referred as the point 't'.

Properties of Rectangular Hyperbola, $x^2 - y^2 = t^2$

· The equations of asymptotes of the rectangular hyperbola are $y = \pm x$.

· The transverse and conjugate axes of a rectangular hyperbola are equal in length.

· Eccentricity, $e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{2}$.

Properties of Rectangular hyperbola $xy = c^2$

· Equation of the chord joining 't_1' and 't_2' is

$$x + yt_1t_2 - c(t_1 + t_2) = 0$$

· Equation of tangnet at (x_1, y_1) is

$$xy_1 + x_1y = 2c^2 \text{ or } \frac{x}{x_1} + \frac{y}{y_1} = 2$$

· Equation of tangent at 't' is : $\frac{x}{t} + yt = 2c$.

· Point of intersection of tangents at 't_1' and 't_2' is $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1t_2}\right)$

· Equation of normal at (x_1, y_1) is $xx_1 - yy_1 = x_1^2 - y_1^2$.

· Equation of normal at 't' is: $xt^3 - yt - ct^4 + c = 0$

· The equation of the chord of the hyperbola $xy = c^2$ whose middle point is (x_1, y_1) is $T = S_1$ i.e., $xy_1 + x_1y = 2x_1y_1$.

· The slope of the tangent at the point $(ct, c/t)$ is $-1/t^2$, which is always negative. Hence tangents drawn at any point to $xy = c^2$ would always make an obtuse angle with the x-axis.

· The slope of the normal at the point $(ct, c/t)$ is t^2 which is always positive. Hence normal drawn to $xy = c^2$ at any point would always make an acute angle with the x-axis.