CHAPTER

01

Trigonometric Functions and Identities

Session 1

Measurement of Angles

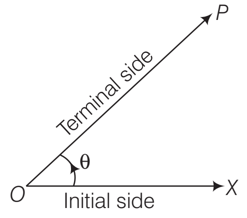
The word 'Trigonometry' is derived from two Greek words.

(i) trigonon (ii) metron

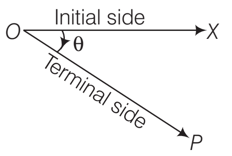
The word trigonon means a triangle and the word metron mean a measure. Hence, trigonometry means measuring the sides and angle of triangle. The subject was originally develop to solve geometric problems involving triangle.

Angle

In trigonometry, as in case of geometry. Angle is measure of rotation from the direction of one ray about its initial point. The original ray called the initial side and the final position of the ray after rotation is called the terminal side of the angle. The point of rotation is called the vertex. If the direction of rotation is anti-clockwise, the angle is said to be positive and if the direction of rotation is clockwise, then the angle is negative



(i) Positive angle



(ii) Negative angle

Measurement of Angles

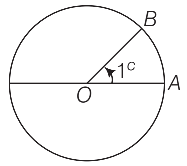
There are three systems used for the measurement of angles.

1. Sexagesimal system or English system (degree)
2. Circular measurement (radian)
3. Centesimal system or French system (grade)

We shall describe the units of measurement of angle which are most commonly used, i.e sexagesimal system (degree measure) and circular measurement (radian measure)

1. **Sexagesimal or Degree measure** If a rotation from the initial side to the terminal side is (1/360)th of a revolution, the angle is said to have a measure of one degree, written as 1°. A degree is divided into 60 minutes, and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute, written as $1'$; one sixtieth of minute is called a second, written as $1''$. Thus, $1^\circ = 60'$ and $1' = 60''$.

2. **Circular measurement or Radian measure** The angle subtended at the centre of a circle by an arc whose length is equal to the radius of the circle is called a radian and denoted by 1^c.



3. **Centesimal or French system** In this system of measurement a right angle is divided into 100 equal parts called **Grades**. Each grade is then divided into 100 equal parts called **minutes** and each minute is further divided into 100 equal parts called **Seconds**.

Thus, right angle $= 100^g$

$$1^\circ = 100'$$
$$1' = 100''$$

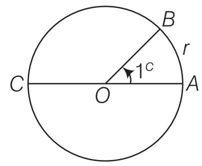
Note

Angle of 90° is called a right angle 1′ of centesimal system ≠ 1′ of sexagesimal system 1″ of centesimal system ≠ 1″ of sexagesimal system.

This system of measurement of angles is not commonly used and so here we will not study this system of measurement of angles.

Radian is a Constant Angle

Let ABC be a circle whose centre is O and radius is r. Let the length of arc AB of the circle by equal to r. Then by the definition of radian.



$$\angle AOB = 1 \text{ radian}$$

Produce AO and let it cut the circle at C. Then AC is a diameter of the circle and arc ABC is equal to half the circumference of the circle.

Also $\angle AOC = 2$ right angle $= 180^\circ$

By geometry, we know that angles subtended at the centre of a circle are proportional to the lengths of the arcs which subtend them

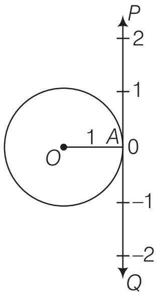
$$\therefore \qquad \frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } ABC} \quad \text{or} \quad \frac{1^c}{180^\circ} = \frac{r}{\frac{2\pi r}{2}}$$

[∵ circumference of the circle $= 2\pi r$]

$$\therefore \qquad 1 \text{ radian} = \frac{180^\circ}{\pi} = \frac{2 \text{ right angle}}{\pi} = \text{constant}$$

[since a right angle and π are constants]

Relation between Radians and Real Numbers



Consider a unit circle with center O. Let A be any point on the circle. Consider OA as the initial side of an angle. Then the length of an arc of the circle gives the radian measure of the angle which the arc subtends at the center of the circle. Consider line PAQ which is tangent to the circle at A. Let point A represents the real number zero, AP represents a positive real number, and AQ represents a negative real number. If we rope line AP in the counter-clockwise direction along the circle, and AQ in the clockwise direction, then every real number corresponds to a radian measure and conversely. Thus, radian measures and real numbers can be considered as one and the same.

Relation between Degree and Radian

It follows that the magnitude in radian of one complete revolution (360 degree) is the length of the entire circumference divided by the radius, or $\frac{2\pi r}{r}$ or 2π.

Therefore, 2π radian $= 360^\circ$

or π radian $= 180^\circ$

or $1 \text{ radian} = \frac{180^\circ}{\pi} = 57^\circ 16'$ (approximately)

Again, $180^\circ = \pi$ radian

$\therefore 1^\circ = \frac{\pi}{180}$ radian $= 0.01746$ radian (approximately)

Thus radian measure of an angle $= \frac{\pi}{180} \times$ degree measure of the angle and degree measure of an angle $= \frac{180}{\pi} \times$ radian measure of the angle.

Thus if the measure of an angle in degrees, and radians be D and C respectively, then

$$\frac{D}{180} = \frac{C}{\pi}$$

The Relation between Degree Measures and Radian Measures of Some Common Angles

Degree	30°	45°	60°	90°	180°	270°	360°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Note

(i) Radian is the unit to measure angle and **it does not means that π stands for 180°**, π is a real number. Where as π^c **stands for 180°**.

Remember the relation π **radians = 180 degrees = 200 grade**.

(ii) The number of radians in an angle subtended by an arc of a circle at the centre is equal to $\frac{\text{arc}}{\text{radius}}$.

$$\Rightarrow \qquad \theta = \frac{s}{r}$$

Example 1. Convert 40°21′ into radian measure.

Sol. We know that $180^\circ = \pi$ radian.

Hence $40^\circ 21' = 40\frac{1}{3}$ degree

$= \frac{\pi}{180} \times \frac{121}{3}$ radian $= \frac{121\pi}{540}$ radian.

Therefore $40^\circ 21' = \frac{121\pi}{540}$ radian.

Example 2. Express the following angle in degrees.

(i) $\left(\frac{5\pi}{12}\right)^c$ (ii) $-\left(\frac{7\pi}{12}\right)^c$

(iii) $\frac{1^c}{3}$ (iv) $-\frac{2\pi^c}{9}$

Sol. (i) $\left(\frac{5\pi}{12}\right)^c = \left(\frac{5\pi}{12} \times \frac{180}{\pi}\right)^\circ = (5 \times 15)^\circ = 75^\circ$

(ii) $-\left(\frac{7\pi}{12}\right)^c = -\left(\frac{7\pi}{12} \times \frac{180}{\pi}\right)^\circ$

$= -(7 \times 15)^\circ = -105^\circ$

(iii) $\left(\frac{1}{3}\right)^c = -\left(\frac{1}{3} \times \frac{180}{\pi}\right)^\circ = -\left(\frac{60}{\pi}\right)^\circ = 19^\circ\, 5'\, 27''$

(iv) $-\frac{2\pi^c}{9} = -\left(\frac{2\pi}{9} \times \frac{180}{\pi}\right)^\circ = -(2 \times 20)^\circ = -40^\circ$

Example 3. Express the following angle in degrees, minutes and seconds form

$(321.9)^\circ$

Sol. $(321.9)^\circ = 321^\circ + 0.9^\circ$

$= 321^\circ + (0.9^\circ \times 60)'$

$= 321^\circ + 54' = 321^\circ 54'$

Example 4. In ΔABC, $m\angle A = \frac{2\pi^c}{3}$ and $m\angle B = 45°$. Find $m\angle C$ in both the systems.

Sol. $m\angle A = \frac{2\pi^c}{3} = \left(\frac{2\pi}{3} \times \frac{180}{\pi}\right)^\circ = 120°$

$m\angle B = 45°$

$= \left(45 \times \frac{\pi}{180}\right)^c = \frac{\pi^c}{4}$

In ΔABC, $m\angle A + m\angle B + m\angle C = 180°$

$\because$ The sum of angles of a triangle is 180°

$\Rightarrow \quad 120° + 45° + m\angle C = 180°$

$\Rightarrow \quad 165° + m\angle C = 180°$

$\Rightarrow \quad m\angle C = 180° - 165°$

$\Rightarrow \quad m\angle C = 15°$

$\Rightarrow \quad m\angle C = \left(15 \times \frac{\pi}{180}\right)^\circ$

$\therefore \quad m\angle C = \frac{\pi^c}{12}$

Example 5. The sum of two angles is $5\pi^c$ and their difference is 60°. Find the angles in degrees.

Sol. Let the angles be x and y in degrees.

Then, $x + y = 5\pi^c \Rightarrow x + y = \left(5\pi \times \frac{180}{\pi}\right)^\circ$

$\therefore \quad x + y = 900° \quad$ ...(i)

$x - y = 60° \quad$ ...(ii)

On adding Eqs. (i) and (ii), we get

$2x = 960°$

$\therefore \quad x = 480°$

On putting $x = 480°$ in Eq. (i), we get

$480° + y = 900°$

$\therefore \quad y = 420°$

$\therefore$ Hence, the angles are 480° and 420°.

Example 6. One angle of a quadrilateral has measure $\frac{2\pi^c}{5}$ and the measures of other three angles are in the ratio 2 : 3 : 4. Find their measures in radians and in degrees.

Sol. One angle $= \frac{2\pi^c}{5} = \left(\frac{2\pi}{5} \times \frac{180}{\pi}\right)^\circ = 72°$

Since, measures of other three angles are in the ratio 2 : 3 : 4. Let the angle be $2k$, $3k$ and $4k$ measured in degree.

$\therefore$ Sum of all angles of quadrilateral = 360°

$\Rightarrow \quad 72° + 2k + 3k + 4k = 360°$

$\Rightarrow \quad 9k = 288° \Rightarrow k = 32°$

$\therefore$ The other three angles are

$2k = 2 \times 32 = 64°$

$3k = 3 \times 32 = 96°$

$4k = 4 \times 32 = 128°$

$\therefore$ The other three angles measured in degree are 64°, 96° and 128°.

The angles in radians are

$$64° = \left(64 \times \frac{\pi}{180}\right)^c = \frac{16\pi^c}{45}$$

$$96° = \left(96 \times \frac{\pi}{180°}\right)^c = \frac{8\pi^c}{15}$$

$$128° = \left(128 \times \frac{\pi}{180}\right)^c = \frac{32\pi^c}{45}$$

$\therefore$ The other three angles measured in radian are

$$\frac{16\pi^c}{45}, \frac{8\pi^c}{15} \text{ and } \frac{32\pi^c}{45}.$$

Example 7. Express the following angles in radians.

(i) 120° (ii) $-600°$

(iii) $-144°$

Sol. (i) $120° = \left(120 \times \frac{\pi}{180}\right)^c = \frac{2\pi^c}{3}$

(ii) $-600° = -\left(600 \times \frac{\pi}{180}\right)^c = -\frac{10\pi}{3}$

(iii) $-144° = \left(-144 \times \frac{\pi}{180}\right) = -\frac{4\pi^c}{5}$

Example 8. If the three angles of a quadrilateral are 60°, 60^g and $\frac{5\pi}{6}$. Then, find the fourth angle.

Sol. First angle = 60°

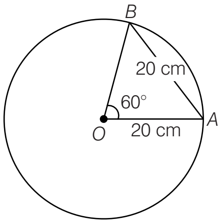
Second angle $= 60^g = 60 \times \frac{90}{100}$ degrees $= 54°$

Third angle $= \frac{5\pi}{6}$ radian $= \frac{5 \times 180}{6} = 150°$

$\therefore$ Fourth angle $= 360° - (60° + 54° + 150°) = 96°$

Example 9. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of minor arc corresponding to the chord.

Sol. Let arc $AB = S$. It is given that $OA = 20$ cm and chord $AB = 20$ cm. Therefore, ΔOAB is an equilateral triangle.



Hence, $\angle AOB = 60°$

$= \left(60 \times \frac{\pi}{180}\right)^c = \left(\frac{\pi}{3}\right)^c$

Now, $\theta = \frac{\text{arc}}{\text{radius}}$

$\Rightarrow \quad \frac{\pi}{3} = \frac{S}{20} \Rightarrow S = \frac{20\pi}{3}$ cm

Example 10. In the circle of 5 cm. radius, what is the length of the arc which subtends and angle of 33°15′ at the centre.

Sol. Here, $r = 5$ cm; $15' = \frac{15}{60} = \left(\frac{1}{4}\right)^\circ$

$\therefore \quad \theta = 33°15' = 33 + \frac{1}{4} = \frac{133}{4}$ degrees

$= \frac{133}{4} \times \frac{\pi}{180} = \frac{133}{4} \times \frac{22}{7 \times 180} = \frac{1463}{2520}$ radians

Now, $\theta = \frac{l}{r}$

$\therefore \quad l = \theta r = \frac{1463}{2520} \times 5 = 2\frac{65}{72}$ cm (approx.)

Example 11. The minute hand of a watch is 35 cm long. How far does its tip move in 18 minutes? $\left(\text{use } \pi = \frac{22}{7}\right)$

Sol. The minute hand of a watch completes one revolution in 60 minutes. Therefore the angle traced by a minute hand in 60 minutes = 360° = 2π radians.

∴ Angle traced by the minute hand in 18 minutes

$= 2\pi \times \frac{18}{60}$ radians $= \frac{3\pi}{5}$ radians

Let the distance moved by the tip in 18 minutes be l, then

$l = r\theta$

$= 35 \times \frac{3\pi}{5} = 21\pi = 21 \times \frac{22}{7} = 66$ cm

Example 12. The wheel of a railway carriage is 40 cm. in diameter and makes 6 revolutions in a second; how fast is the train going?

Sol. Diameter of the wheel = 40 cm

∴ radius of the wheel = 20 cm

Circumference of the wheel $= 2\pi r = 2\pi \times 20 = 40\pi$ cm

Number of revolutions made in 1 second = 6

∴ Distance covered in 1 second $= 40\pi \times 6 = 240\pi$ cm

∴ Speed of the train = 240π cm/sec.

Example 13. Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find the height of the letters that he can read at a distance of 12 metres.

Sol. Let the height of the letters be h metres.

Now, h many be considered as the arc of a circle of radius 12 m, which subtends an angle of 5′ at its centre.

$\therefore \quad \theta = 5' = \left(\frac{5}{60} \times \frac{\pi}{180}\right)$ radians $= \left(\frac{\pi}{12 \times 180}\right)$ radian

and $\quad r = 12$ m

$\therefore \quad h = r\theta = 12 \times \frac{\pi}{12 \times 180} = \left(\frac{\pi}{180}\right)$ metres ≈ 1.7 cm

Exercise for Session 1

1. The difference between two acute angles of a right angle triangle is $\frac{3\pi}{10}$ rad. Find the angles in degree.
2. Find the length of an arc of a circle of radius 6 cm subtending an angle of 15° at the centre.
3. A horse is tied to post by a rope. If the horse moves along circular path always keeping the tight and describes 88 m, when it has traced out 72° at centre, find the length of rope.
4. Find the angle between the minute hand and hour hand of a clock, when the time is 7 : 30 pm.
5. If OQ makes 4 revolutions in 1s, find the angular velocity in radians per second.
6. If a train is moving on the circular path of 1500 m radius at the rate of 66 km/h, find the angle in radian, if it has in 10 second.
7. Find the distance from the eye at which a coin of 2.2 cm diameter should be held so as to conceal the full moon with angular diameter 30′.
8. The wheel of a railway carriage is 40 cm in diameter and makes 7 revolutions in a second, find the speed of train.
9. Assuming that a person of normal sight can read print at such a distance that the letters subtend an angle of 5′ at his eye, find the height of letters that he can read a distance of 12 m.
10. For each natural number k, let C_K denotes the circle with radius k cm and centre at origin. On the circle C_K, a particle moves k cm in the counter-clockwise direction. After completing its motion on C_K, the particle moves on C_{K+1} in the radial direction. The motion of the particle continues in this manner. The particle starts at (1, 0). If the particle crosses the positive direction of the x-axis for the first time on the circle C_n, then n is equal to

Answers

Exercise for Session 1

1. 72°, 18° **2.** $\frac{\pi}{2}$ cm **3.** 70 m **4.** 45° **5.** 8π

6. $\left(\frac{11}{90}\right)^c$ **7.** 252 cm **8.** 880 cm/s **9.** 1.7 cm **10.** 7