1 Sequence and Series

ARITHMATIC PROGRESSION (A.P.)

KEY FACTS

1. A **sequence** is a set of numbers specified in a definite order by some assigned rule or law.

Ex. 2, 7, 12, 17.... (Each succeeding term is obtained by adding 5 to the preceding term)

1, 2, 4, 8, 16.... (Each succeeding term is obtained by multiplying the preceding term by 2)

A **finite sequence** is that which ends or has a last term.

Ex. 5, 9, 13, 17, 21.

An **infinite sequence** is one which has no last term.

Ex. 3, 6, 12, 24, 48....

In general, $\boldsymbol{a_n}$ or $\boldsymbol{T_n}$ denotes the ***n*th term** of a sequence.

2. An expression consisting of the term of a sequence, alternating with the symbol '+' is called a **series.**

Ex. The sequence $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$, ... expressed as a series is $\frac{2}{3}+\frac{4}{5}+\frac{6}{7}+....$

3. **Arithmetic Progression (A.P.):** A sequence is called an arithmetic progression if its terms continually increase or decrease by the same number. The fixed number by which they increase or decrease is called the **common difference.** Three quantities a, b, c will be in A.P. if $b - a = c - b$, *i.e.*, $\boldsymbol{2b = a + c}$.

(*a*) ***n*th term of an A.P.:** The *n*th term of an A.P. $a, a + d, a + 2d, a + 3d, ...$ is

$$\boldsymbol{T_n = a + (n - 1)d}$$

where $\boldsymbol{T_n}$ denotes *n*th term, $\boldsymbol{a}$ the first term, $\boldsymbol{n}$ the number of terms and $\boldsymbol{d}$ the common difference.

Also, common difference $\boldsymbol{d = T_n - T_{n-1}}$.

Ex. The 9th term of the A.P.: 2, 5, 8.... is

$$T_9 = 2 + (9 - 1) \times 3 = 2 + 24 = \mathbf{26}.$$

Note: Here $a = 2, d = 3$.

(*b*) **Sum of *n* terms of an A.P.**

Let the A.P. be $a, a + d, a + 2d, ...$ Let l be the last term and S the required sum. Then,

$$\boldsymbol{S = \frac{n}{2}(a + l)} = \frac{\text{Number of terms}}{2}(\text{First term + Last term})$$

$$= \frac{n}{2}(a + a + (n - 1)\, d) = \frac{n}{2}(2a + (n - 1)d),$$

where n is the number of terms, a is first term and d is common difference.

Also, *n*th term = Sum of *n* terms – Sum of $(n - 1)$ terms

i.e., $\boldsymbol{T_n = S_n - S_{n-1}}$.

Ex. The sum of 20 terms of the the A.P. 1, 3, 5, 7, 9.... is

$$S_{20} = \frac{20}{2}(2\times1+(20-1)\times2) \qquad (\because a = 1, d = 2, n = 20)$$
$$= 10 \times 40 = \mathbf{400.}$$

(*c*) **Arithmetic Mean:** The Arithmetic Mean between two numbers is the number which when placed between them forms an arithmetic progression with them. Thus if ***x*** is the arithmetic mean of two given numbers ***a*** and ***b***, then *a, x, b* form an A.P.

$$\therefore \quad x - a = b - x \Rightarrow \boldsymbol{x = \frac{a+b}{2}}$$

Ex. (*a*) The arithmetic mean between – 4 and 6 is $\frac{6+(-4)}{2} = \mathbf{1.}$

(*b*) **Find 4 arithmetic means between 3 and 23.**

Let A_1, A_2, A_3, A_4, be the four arithmetic means between 3 and 23.

Then, 3, A_1, A_2, A_3, A_4, 23 form an A.P.

Here, First term = $a = 3$

Sixth term = $T_6 = 23$

Number of terms = $n = 6$

Common difference = d = ?

$T_6 = a + (n - 1) \times d$

$\Rightarrow 23 = 3 + (6 - 1) \times d$

$\Rightarrow 23 = 3 + 5d \Rightarrow 5d = 20 \Rightarrow d = \mathbf{4.}$

$\therefore A_1 = 3 + 4 = 7, A_2 = 7 + 4 = 11, A_3 = 11 + 4 = 15, A_4 = 15 + 4 = 19.$

(*d*) **Some useful facts about an A.P.**

I. ***If each term of a given A.P. is increased or decreased or multiplied or divided by the same number, the resulting progression is also an A.P.***

II. ***If a, b, c are in A.P., then*** $\boldsymbol{\frac{a-b}{b-c} = 1}$, ***i.e.,*** $\frac{\textbf{First term – Second term}}{\textbf{Second term – Third term}} = \mathbf{1.}$

(*e*) If we have to find an odd number of terms in A.P. whose sum is given, it is convenient to take *a* as the middle term and *d* as the common difference. Thus, three terms may by taken as ***a – d, a, a + d*** and five terms as ***a* – 2*d*, *a* – *d*, *a*, *a* + *d*, *a* + 2*d*. [Solved Ex. 13, 14].** If we have to find even number of terms, we take *a* – *d*, *a* + *d* as the middle terms and 2*d* as the common difference. Then, four terms are taken as ***a* –3*d*, *a* – *d*, *a* + *d*, *a* + 3*d*.**

SOLVED EXAMPLES

Ex. 1. The 8th term of a series in A.P. is 23 and the 102th term is 305. Find the series.

Sol. Let *a* be the first term and *d* be the common difference.

Then, $T_8 = a + (8 - 1)\,d \Rightarrow 23 = a + 7d$ ...(*i*)

$T_{102} = a + (102 - 1)\,d \Rightarrow 305 = a + 101d$ ...(*ii*)

$\therefore$ Eqn (*ii*) – Eqn (*i*)

$\Rightarrow 94d = 282 \Rightarrow d = 3$

Now substituting $d = 3$ in (*i*), we get $23 = a + 21 \Rightarrow a = 2$.

$\therefore a = 2, d = 3 \Rightarrow$ Series is 2 + 5 + 8 + 11 + ...

Ex. 2. If *a*, *b* and *c* be respectively the *p*th, *q*th and *r*th terms of an A.P., prove that $\boldsymbol{a(q-r) + b(r-p) + c(p-q) = 0}$.

Sol. Let *A* be the first term and *D* the common difference of the given A.P.

Then, $T_p = A + (p - 1)\,D = a$ ...(*i*)

$T_q = A + (q - 1)\,D = b$ ...(*ii*)

$T_r = A + (r - 1)\,D = c$ ...(*iii*)

$\therefore$ $(i) \times (q-r) + (ii) \times (r-p) + (iii) \times (p-q)$

$\Rightarrow A\,[(q-r)+(r-p)+(p-q)] + D\,[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]$

$= a(q-r) + b(r-p) + c(p-q)$

$\Rightarrow A\,[0] + D\,[pq - q - pr + r + qr - r - pq + p + pr - p - qr + q] = a(q-r) + b(r-p) + c(p-q)$

$\Rightarrow a(q-r) + b(r-p) + c(p-q) = A \times 0 + D \times 0 = \mathbf{0}.$

Ex. 3. Which term of the progression 19, $18\frac{1}{5}, 17\frac{2}{5}$, ... is the first negative term?

Sol. Here $a = 19$, $d = 18\frac{1}{5} - 19 = -\frac{4}{5}$

Let the nth term be the first negative term. Then,

$$T_n < 0 \Rightarrow a + (n-1)\,d < 0$$

$$\Rightarrow 19 + (n-1)\,(-4/5) < 0$$

$$\Rightarrow 19 - \frac{4}{5}n + \frac{4}{5} < 0$$

$$\Rightarrow \frac{99}{5} - \frac{4}{5}n < 0 \quad \Rightarrow n > \frac{99}{5} \times \frac{5}{4} \Rightarrow n > \frac{99}{4} = 24\frac{3}{4}$$

$$\Rightarrow n = \mathbf{25.}$$

Ex. 4. Find the sum of the series 101 + 99 + 97 + ... + 47.

Sol. In this case, we have to first find the number of terms.

Here $a = 101$, $l = T_n = 47$, $d = 99 - 101 = -2$

$\therefore$ $47 = 101 + (n-1) \times (-2)$, where n = number of terms

$\Rightarrow$ $47 = 101 - 2n + 2$

$\Rightarrow$ $2n = 103 - 47 = 56 \Rightarrow n = 28$

$\therefore$ $S_n = \frac{n}{2}(a + l)$

$\Rightarrow$ $S_{28} = \frac{28}{2}(101 + 47) = 14 \times 148 = \mathbf{2072.}$

Ex. 5. The sums of n terms of two arithmetic series are in the ratio $2n + 1 : 2n - 1$. Find the ratio of their 10th terms.

Sol. Let the two arithmetic series be $a, a + d, a + 2d$ ..... and $A, A + D, A + 2D$, ...

Given that, $\frac{n/2[2a + (n-1)\,d]}{n/2[2A + (n-1)\,D]} = \frac{2n+1}{2n-1}$

$$\Rightarrow \frac{2a + (n-1)\,d}{2A + (n-1)\,D} = \frac{2n+1}{2n-1} \qquad ...(i)$$

Ratio of the 10th terms of these series = $\frac{t_{10}}{T_{10}} = \frac{a + 9d}{A + 9D} = \frac{2a + 18d}{2A + 18D}$

$\therefore$ Putting $n = 19$ in (i), we have $\frac{t_{10}}{T_{10}} = \frac{2a + 18d}{2A + 18D} = \frac{2 \times 19 + 1}{2 \times 19 - 1} = \mathbf{\frac{39}{37}}.$

Ex. 6. The sum of the first n terms of the arithmetical progression 3, $5\frac{1}{2}$, 8, ... is equal to the 2nth term of the A.P. $16\frac{1}{2}, 28\frac{1}{2}, 40\frac{1}{2}$, ... Calculate the value of n.

Sol. For the A.P. : 3, $5\frac{1}{2}$, 8, ... $a = 3, d = 2\frac{1}{2}$, number of terms = n

$$\Rightarrow \quad S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}\left(6 + (n-1)\times\frac{5}{2}\right) \quad ...(i)$$

For the A.P.: $16\frac{1}{2}, 28\frac{1}{2}, 40\frac{1}{2}$ ... $a = 16\frac{1}{2}, d = 12.$

$$\therefore \quad T_{2n} = a + (2n-1)d = 16\frac{1}{2} + (2n-1)\times 12 \quad ...(ii)$$

Given, $S_n = T_{2n} \Rightarrow \frac{n}{2}\left(6 + (n-1)\frac{5}{2}\right) = \frac{33}{2} + (2n-1)\times 12$ (From (*i*) and (*ii*))

$\Rightarrow \frac{6n}{2} + \frac{5n^2}{4} - \frac{5n}{4} = \frac{33}{2} + 24n - 12$ $\Rightarrow \frac{7n}{4} + \frac{5n^2}{4} = \frac{9}{2} + 24n$

$\Rightarrow \frac{5n^2}{4} - \frac{89n}{4} - \frac{9}{2} = 0$ $\Rightarrow 5n^2 - 89n - 18 = 0$

$\Rightarrow 5n^2 - 90n + n - 18 = 0$ $\Rightarrow 5n(n-18) + 1(n-18) = 0$

$\Rightarrow (n-18)(5n+1) = 0$ $\Rightarrow n = 18$ or $-\frac{1}{5}$

$\Rightarrow$ Neglecting – *ve* value, we have n = **18.**

Ex. 7. Let a_1, a_2, a_3, ... be the terms of an A.P. If $\frac{a_1 + a_2 + a_3 + ... + a_p}{a_1 + a_2 + a_3 + ... + a_q} = \frac{p^2}{q^2}$ $(p \neq q)$, then find $\frac{a_6}{a_{21}}$.

(*AIEEE 2006*)

Sol. Let d be the common difference for the A.P.; a_1, a_2, a_3, ...

Then, $\frac{S_p}{S_q} = \frac{p/2\,[2a_1 + (p-1)d]}{q/2\,[2a_1 + (q-1)d]} = \frac{p^2}{q^2}$

$\Rightarrow \frac{2a_1 + (p-1)d}{2a_1 + (q-1)d} = \frac{p}{q}$ $\Rightarrow q[2a_1 + (p-1)d] = p[2a_1 + (q-1)d]$

$\Rightarrow 2a_1q + pqd - qd = 2a_1p + pqd - pd$ $\Rightarrow pd - qd = 2a_1p - 2a_1q$

$\Rightarrow d(p-q) = 2a_1(p-q)$ $\Rightarrow d = 2a_1$

Now $\frac{\text{Term 6}}{\text{Term 21}} = \frac{a_6}{a_{21}} = \frac{a_1 + (6-1)d}{a_1 + (21-1)d} = \frac{a_1 + 5\times 2a_1}{a_1 + 20\times 2a_1} = \frac{11a_1}{41a_1} = \mathbf{\frac{11}{41}}.$

Ex. 8. If the number of terms of an A.P. is $(2n + 1)$, then what is the ratio of the sum of the odd terms to the sum of even terms? **(*NDA/NA 2008*)**

Sol. Let the A.P. be $a, a + d, a + 2d, ..., a + (2n-1)d, a + 2nd$

Then, the progression of odd terms is $a, a + 2d, a + 4d, ..., a + 2nd.$

This progression has $(n + 1)$ terms.

$$\text{Its sum} = \frac{n+1}{2}[a + a + 2nd] = \frac{n+1}{2}[2a + 2nd] = (n+1)(a+nd) \quad ...(i)$$

The progression of even terms is $a + d, a + 3d, ... a + (2n-1)d.$

This progression has n terms.

Its sum = $\frac{n}{2}\left[(a+d)+(a+(2n-1)d\right] = \frac{n}{2}[2a+2nd] = n(a+nd)$ ...(*ii*)

$\therefore \quad \frac{\text{Sum of odd terms}}{\text{Sum of even terms}} = \frac{(n+1)(a+nd)}{n(a+nd)} = \boldsymbol{\frac{n+1}{n}}.$

Ex. 9. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then show that ab^2, ca^2, bc^2 are in A.P. (***DCE***)

Sol. Let α, β be the roots of the equation $ax^2 + bx + c = 0$.

Then $\alpha + \beta = -b/a$, $\alpha\beta = c/a$

Given, Sum of roots = Sum of squares of reciprocals of roots

$\Rightarrow \quad \alpha+\beta = \frac{1}{\alpha^2}+\frac{1}{\beta^2} \qquad \Rightarrow \quad \alpha+\beta = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2} \qquad \Rightarrow \quad \alpha+\beta = \frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$

$\Rightarrow \quad -b/a = \frac{(-b/a)^2 - \frac{2c}{a}}{(c/a)^2} \qquad \Rightarrow \quad -\frac{b}{a} = \frac{b^2-2ca}{c^2} \qquad \Rightarrow \quad -bc^2 = ab^2 - 2ca^2$

$\Rightarrow \quad 2ca^2 = ab^2 + bc^2 \qquad \Rightarrow \quad ab^2, ca^2, bc^2$ are in A.P. $\quad (\because a,b,c \text{ in A.P.} \Rightarrow 2b = a + c)$

Ex. 10. Find the value of n, if $\boldsymbol{\frac{a^{n+1}+b^{n+1}}{a^n+b^n}}$ is the arithmetic mean between a and b. (***WBJEE 2009***)

Sol. The arithmetic mean between a and b is $\frac{a+b}{2}$.

Given, $\frac{a+b}{2} = \frac{a^{n+1}+b^{n+1}}{a^n+b^n}$

$\Rightarrow (a^n + b^n)(a + b) = 2(a^{n+1} + b^{n+1}) \qquad \Rightarrow a^{n+1} + a^nb + b^na + b^{n+1} = 2a^{n+1} + 2b^{n+1}$

$\Rightarrow a^{n+1} + b^{n+1} = a^nb + b^na \qquad \Rightarrow (a^{n+1} - ba^n) - (b^na - b^{n+1}) = 0$

$\Rightarrow a^n(a - b) - b^n(a - b) = 0 \qquad \Rightarrow (a^n - b^n)(a - b) = 0$

$\Rightarrow (a^n - b^n) = 0$ or $(a - b) = 0$

But $a \neq b \Rightarrow a - b \neq 0$

$\therefore \quad a^n - b^n = 0 \Rightarrow a^n = b^n \Rightarrow \left(\frac{a}{b}\right)^n = 1 \quad \Rightarrow \quad \left(\frac{a}{b}\right)^n = \left(\frac{a}{b}\right)^0 \quad \Rightarrow \quad \boldsymbol{n = 0.}$

Ex. 11. If $\log_{10}2$, $\log_{10}(2^x - 1)$ and $\log_{10}(2^x + 3)$ be three consecutive terms of an A.P., then find the value of x. (***AMU 2012***)

Sol. Given, $\log_{10}2$, $\log_{10}(2^x - 1)$, $\log_{10}(2^x + 3)$ are in A.P. Then,

$\log_{10}(2^x - 1) - \log_{10}2 = \log_{10}(2^x + 3) - \log_{10}(2^x - 1)$

$\Rightarrow \quad \log_{10}\left(\frac{2^x-1}{2}\right) = \log_{10}\left(\frac{2^x+3}{2^x-1}\right)$

$\Rightarrow \quad \frac{2^x-1}{2} = \frac{2^x+3}{2^x-1} \qquad \Rightarrow \quad (2^x - 1)^2 = 2(2^x + 3)$

$\Rightarrow \quad 2^{2x} - 2.2^x + 1 = 2.2^x + 6 \qquad \Rightarrow \quad 2^{2x} - 4.2^x - 5 = 0$

$\Rightarrow 2^{2x} - 5.2^x + 2^x - 5 = 0 \qquad \Rightarrow \quad 2^x(2^x - 5) + 1(2^x - 5) = 0$

$\Rightarrow (2^x - 5)(2^x + 1) = 0 \qquad \Rightarrow \quad 2^x = 5 \qquad (\because 2^x \neq -1)$

$\Rightarrow x = \boldsymbol{\log_2 5.}$

Ex. 12. If $a_1, a_2, a_3, \ldots a_n$ be an A.P. of non-zero terms, then find the sum: $\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots+\frac{1}{a_{n-1}\,a_n}$. (*AMU 2009*)

Sol. Let $a_2 - a_1 = a_3 - a_2 = \ldots\ldots = a_n - a_{n-1} = d$ (common difference)

Then, $\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots\ldots+\frac{1}{a_{n-1}a_n} = \frac{1}{d}\left[\frac{d}{a_1a_2}+\frac{d}{a_2a_3}+\ldots\ldots+\frac{d}{a_{n-1}a_n}\right]$

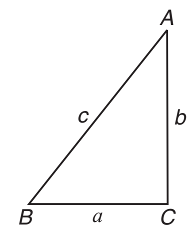
$= \frac{1}{d}\left[\frac{a_2-a_1}{a_1a_2}+\frac{a_3-a_2}{a_2a_3}+\ldots\ldots+\frac{a_n-a_{n-1}}{a_{n-1}a_n}\right] = \frac{1}{d}\left[\frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_2}-\frac{1}{a_3}+\ldots\ldots+\frac{1}{a_{n-1}}-\frac{1}{a_n}\right]$

$= \frac{1}{d}\left[\frac{1}{a_1}-\frac{1}{a_n}\right] = \frac{1}{d}\left[\frac{a_n-a_1}{a_1a_n}\right] = \frac{1}{d}\left[\frac{(a_1+(n-1)\,d)-a_1}{a_1a_n}\right] = \boldsymbol{\frac{n-1}{a_1a_n}}.$

Ex. 13. If the sides of a right angled triangle form an A.P., then find the sines of the acute angles. (*VITEE 2008*)

Sol. Let the ΔABC be right angled at C.

Then, $AB = c$, $BC = a$, $AC = b$



Given, the sides of the right angled Δ are in A.P. ⇒ a, b, c are in A.P.

Now, let $a = x - d$, $b = x$, $c = x + d$

(***d*** being a +ve quantity as ***c*** being the hypotenuse is the greatest side)

∴ $c^2 = a^2 + b^2$ (Pythagoras' Theorem)

⇒ $(x + d)^2 = (x - d)^2 + x^2$

⇒ $x^2 + 2xd + d^2 = x^2 - 2xd + d^2 + x^2$

⇒ $4xd = x^2$ ⇒ $d = \frac{x}{4}$.

∴ $a = x - d = x - \frac{x}{4} = \frac{3x}{4}$, $b = x$, $c = x + d = x + x/4 = \frac{5x}{4}$

C being the right angle, A and B are the acute angles.

∴ $\sin A = \frac{a}{c} = \frac{3x/4}{5x/4} = \frac{3}{5}$ and $\sin B = \frac{b}{c} = \frac{x/5}{x/4} = \frac{x}{\frac{5x}{4}} = \frac{4}{5}$

∴ The sines of the acute angles are $\boldsymbol{\frac{3}{5}, \frac{4}{5}}$.

Ex. 14. a_1, a_2, a_3, a_4, a_5 are the first five terms of an A.P. such that $a_1 + a_3 + a_5 = -12$ and $a_1.a_2.a_3 = 8$. Find the first term and common difference.

Sol. Let $a_1 = a_3 - 2d$, $a_2 = a_3 - d$, $a_3 = a_3$, $a_4 = a_3 + d$, $a_5 = a_3 + 2d$

Then $a_1 + a_3 + a_5 = -12$, (given) ⇒ $a_3 - 2d + a_3 + a_3 + 2d = -12$ ⇒ $3a_3 = -12$ ⇒ $a_3 = -4$

Also, $a_1 \,.\, a_2 \,.\, a_3 = 8$ (given)

⇒ $a_1 \,.\, a_2 = -2$ ($\because a_3 = -4$) ⇒ $(a_3 - 2d)\,(a_3 - d) = -2$

⇒ $(-4 - 2d)\,(-4 - d) = -2$ ⇒ $(2 + d)\,(4 + d) = 1$

⇒ $d^2 + 6d + 9 = 0$ ⇒ $(d + 3)^2 = 0$ ⇒ $d = \mathbf{-3}$

∴ $a_1 = a_3 - 2d = -4 - 2(-3) = \mathbf{2}$.

Ex. 15. If a, b, c are in A.P. show that $\frac{a\,(b+c)}{bc}, \frac{b\,(c+a)}{ca}, \frac{c\,(a+b)}{ab}$ are in A.P.

Sol. *a, b, c* are in A.P.

⇒ $\frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc}$ are in A.P. (Dividing each term by *abc*)

$\Rightarrow \ \frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}$ are in A.P.

$\Rightarrow \ \frac{ab+bc+ca}{bc}, \frac{ab+bc+ca}{ac}, \frac{ab+bc+ca}{ab}$ are in A.P. (Multiplying each term by $ab + bc + ca$)

$\Rightarrow \ \frac{ab+ca}{bc}+1, \frac{ab+bc}{ac}+1, \frac{bc+ca}{ab}+1$ are in A.P.

$\Rightarrow \ \frac{ab+ca}{bc}, \frac{ab+bc}{ac}, \frac{bc+ca}{ab}$ are in A.P. (Subtracting 1 from each term)

$\Rightarrow \ \frac{a(b+c)}{bc}, \frac{b(a+c)}{ac}, \frac{c(b+a)}{ab}$ are in A.P.

Ex. 16. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P., prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A.P.

Sol. $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A.P.

$\Rightarrow \ \frac{b+c-a}{a}+2, \frac{c+a-b}{b}+2, \frac{a+b-c}{c}+2$ are in A.P. (Adding 2 to each term of A.P.)

$\Rightarrow \ \frac{b+c+a}{a}, \frac{c+a+b}{b}, \frac{a+b+c}{c}$ are in A.P.

$\Rightarrow \ \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P. (Dividing each term by $a + b + c$)

Ex. 17. If *a, b, c* are in A.P., show that $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P. (*NDA/NA 2010*)

Sol. Given a, b, c are in A.P. $\Rightarrow 2b = a + c$

Now $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P., if

$$\frac{1}{\sqrt{c}+\sqrt{a}} - \frac{1}{\sqrt{b}+\sqrt{c}} = \frac{1}{\sqrt{a}+\sqrt{b}} - \frac{1}{\sqrt{c}+\sqrt{a}}$$

$\Rightarrow \ \frac{2}{\sqrt{c}+\sqrt{a}} = \frac{1}{\sqrt{b}+\sqrt{c}} + \frac{1}{\sqrt{a}+\sqrt{b}} \quad \Rightarrow \quad \frac{2}{\sqrt{c}+\sqrt{a}} = \frac{\sqrt{a}+\sqrt{b}+\sqrt{b}+\sqrt{c}}{(\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b})}$

$\Rightarrow 2\,(\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}) = (\sqrt{c}+\sqrt{a})(\sqrt{a}+2\sqrt{b}+\sqrt{c})$

$\Rightarrow 2\,(\sqrt{ab}+b+\sqrt{ac}+\sqrt{bc}) = (\sqrt{ac}+2\sqrt{bc}+c+a+2\sqrt{ba}+\sqrt{ac})$

$\Rightarrow 2\,\sqrt{ab}+2b+2\sqrt{ac}+2\sqrt{bc} = 2\sqrt{ac}+2\sqrt{bc}+c+a+2\sqrt{ba}$

$\Rightarrow 2b = a + c$, which is true as a, b, c are in A.P.

$\Rightarrow \ \frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$ are in A.P.

Ex. 18. What is the sum of all two-digit numbers which leave remainder 5 when they are divided by 7?

Sol. The two digit natural numbers which leave a remainder 5, when divided by 7 are 12, 19, 26 ...., 89, 96.

$\therefore$ 12, 19, 26, ..., 89, 96 is an A.P. whose first term $a = 12$ and common difference $d = 7$.

Let the last or nth term be T_n.

Then, $T_n = a + (n - 1)\, d$, where n is the number of terms in A.P.

$\Rightarrow$ 96 = 12 + (n – 1) 7

$\Rightarrow$ 84 = (n – 1) 7 $\Rightarrow n - 1 = 12 \Rightarrow n = 13$

$\therefore$ Required Sum = $\frac{n}{2}(a+l) = \frac{13}{2}(12+96) = \frac{13}{2}\times 108 = 13\times 54 =$ **702.**

Ex. 19. What is the sum of the series $1^2 - 2^2 + 3^2 - 4^2 + ... + 99^2 - 100^2$. **(*Orissa JEE 2006*)**

Sol. $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + ... + 99^2 - 100^2$

$= (1-2)(1+2) + (3-4)(3+4) + (5-6)(5+6) + ... + (99-100)(99+100)$

$= -(1+2) - (3+4) - (5+6) ..... - (99+100)$

$= -(1+2+3+4+5+6+ ..... + 99+100) = -\left(\frac{100}{2}(1+100)\right)$ $\left(\because S_n = \frac{n}{2}\text{ (first term + last term)}\right)$

$= -50\times 101 =$ **– 5050.**

Ex. 20. If the first, second and last terms of an arithmetic series are *a, b* and *c* respectively, then what is the number of terms? **(*MPPET 2009*)**

Sol. $\left.\begin{array}{l}\text{First term} = a\\ \text{Second term} = b\end{array}\right\} \Rightarrow$ common difference $(d) = b - a$

Last term = c

Let the number of terms be n. Then,

$c = a + (n-1)\times d \Rightarrow c = a + (n-1)\times(b-a) \Rightarrow c - a = (n-1)(b-a)$

$\Rightarrow (n-1) = \frac{c-a}{b-a} \Rightarrow n = \frac{c-a}{b-a} + 1 = \frac{c-a+b-a}{b-a} = \mathbf{\frac{c+b-2a}{b-a}}$.

Ex. 21. After inserting *x* A.M's. between 2 and 38, the sum of the resulting progression is 200. What is the value of *x*? **(*AMU 2001*)**

Sol. After inserting x A.M's between 2 and 38, we get an A.P. of $(x + 2)$ terms with first term as 2 and last term as 38.

Now, sum of n terms of an A.P. = $\frac{n}{2}(a+l)$, where a = first term, l = last term

$\therefore$ Here, $200 = \frac{(x+2)}{2}(2+38) \Rightarrow 200 = 20(x+2)$

$\Rightarrow 20x + 40 = 200 \Rightarrow 20x = 160 \Rightarrow x =$ **8.**

PRACTICE SHEET

1. Find the 26th term of the A.P: 10, 6, 2, –2, – 6, –10, ... ?

(*a*) –86 (*b*) 96 (*c*) –90 (*d*) –106

2. The 2nd, 31st and last term of an A.P. are $7\frac{3}{4}, \frac{1}{2}$ and $-6\frac{1}{2}$ respectively. The number of terms of the A.P. is

(*a*) 48 (*b*) 60 (*c*) 52 (*d*) 59

3. If p times the pth term of an A.P. is q times the qth term, then what is $(p + q)$th term equal to?

(*a*) $p + q$ (*b*) pq (*c*) 1 (*d*) 0

(*NDA/NA 2010*)

4. The 59th term of an A.P. is 449 and the 449th term is 59. Which term is equal to 0?

(*a*) 501st term (*b*) 502nd term
(*c*) 508th term (*d*) 509th term

(*NDA/NA 2010*)

5. Let T_r be the rth term of an A.P. whose first term is a and common difference is d. If for some positive integers m and n, $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then $(a - d)$ equals

(*a*) $1/mn$ (*b*) 1 (*c*) 0 (*d*) $\frac{1}{m}+\frac{1}{n}$

(*AIEEE 2004, UPSEE 2007*)

6. If a, b, c be in Arithmetic Progression, then the value of $(a + 2b - c)(2b + c - a)(a + 2b + c)$ is

(*a*) $3abc$ (*b*) $4abc$ (*c*) $8abc$ (*d*) $16abc$

(*WBJEE 2008*)

7. Find the sum of 24 terms of the series $2\frac{1}{2}, 3\frac{1}{3}, 4\frac{1}{6}, 5,$ ......?

(*a*) 200 (*b*) 185 (*c*) 290 (*d*) 250

8. If the sum of the 12th and 22nd terms of an A.P. is 100, then the sum of the first 33 terms of the A.P. is

(*a*) 1650 (*b*) 2340 (*c*) 3300 (*d*) 3400

(*Kerala PET 2008*)

9. If $S_1 = a_2 + a_4 + a_6 + ... +$ upto 100 terms and $S_2 = a_1 + a_3 + a_5 + ... +$ upto 100 terms of a certain A.P., then its common difference is

(*a*) $S_1 - S_2$ (*b*) $S_2 - S_1$

(*c*) $\frac{S_1 - S_2}{2}$ (*d*) None of these

(*AMU 2010*)

10. The sum of n terms of an A.P. is $2n + 3n^2$. Which term of this A.P. is equal to 299?

(*a*) 11th (*b*) 50th (*c*) 35th (*d*) 29th

11. If S_n denotes the sum of first n terms of an A.P. $a_1 + a_2 + a_3 + ...$, such that $\frac{S_m}{S_n} = \frac{m^2}{n^2}$, then $\frac{a_m}{a_n} = ?$

(*a*) $m - 1 : n - 1$ (*b*) $m - n : m + n$

(*c*) $2m - 1 : 2n - 1$ (*d*) $m + 1 : n + 1$

(*J&K CET 2013*)

12. If the sum of the first ten terms of an A.P. is 4 times the sum of the first five terms, then the ratio of the first term to the common difference is:

(*a*) 1 : 2 (*b*) 2 : 1 (*c*) 1 : 4 (*d*) 4 : 1

(*NDA/NA 2003*)

13. What is the sum of numbers lying between 107 and 253, which are divisible by 5?

(*a*) 5250 (*b*) 5210 (*c*) 5220 (*d*) 5000

14. The ratio between the sum of n terms of two A.P.'s is $(7n + 1) : (4n + 27)$. The ratio of their 11th terms is

(*a*) 125 : 106 (*b*) 148 : 111 (*c*) 131 : 89 (*d*) 127 : 108

15. Let S_n denote the sum of first n terms of an A.P. If $S_{2n} = 3S_n$, then the ratio $\frac{S_{3n}}{S_n}$ is equal to

(*a*) 4 (*b*) 6 (*c*) 8 (*d*) 10

(*MAT 2002, Rajasthan PET 2006*)

16. If the sum of $2n$ terms of the A.P. 2, 5, 8, 11,... is equal to the sum of n terms of A.P. 57, 59, 61, 63, ..., then n is equal to

(*a*) 10 (*b*) 11 (*c*) 12 (*d*) 13

(*IIT 2001*)

17. If S_1, S_2, S_3 denote respectively the sum of first n_1, n_2 and n_3 terms of an A.P., then

$\frac{S_1}{n_1}(n_2 - n_3) + \frac{S_2}{n_2}(n_3 - n_1) + \frac{S_3}{n_3}(n_1 - n_2)$ is equal to

(*a*) 0 (*b*) $n_1 + n_2 + n_3$

(*c*) $n_1 n_2 n_3$ (*d*) $S_1 S_2 S_3$ **(*DCE 2007*)**

18. If $S_n = nP + \frac{n}{2}(n - 1)Q$, where S_n denotes the sum of first n terms of an A.P., then the common difference of the A.P. is

(*a*) Q (*b*) $P + Q$ (*c*) $P + 2Q$ (*d*) $2P + 3Q$

(*DCE 2001*)

19. n arithmetic means are inserted between 3 and 17. If the ratio of the last and the first arithmetic mean is 3 : 1, then n is equal to

(*a*) 5 (*b*) 6 (*c*) 7 (*d*) 9

(*DEC 2008*)

20. If $a_1, a_2, a_3, ..., a_n$ are in A.P. and $a_1 = 0$, then the value of $\left(\frac{a_3}{a_2} + \frac{a_4}{a_3} + ... + \frac{a_n}{a_{n-1}}\right) - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_{n-2}}\right)$ is equal to

(*a*) $n + \frac{1}{n}$ (*b*) $n + \frac{1}{n-1}$

(*c*) $(n-1) + \frac{1}{(n-1)}$ (*d*) $(n-2) + \frac{1}{(n-2)}$

(*Kerala PET 2011*)

21. Let a_1, a_2, a_3, a_4 be in A.P. If $a_1 + a_4 = 10$ and $a_2 a_3 = 24$, then the least term of them is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

(*Kerala PET 2013*)

22. An A.P. has a property that the sum of first ten terms is half the sum of next ten terms. If the second term is 13, then the common difference is

(*a*) 3 (*b*) 2 (*c*) 5 (*d*) 4

(*Kerala PET 2013*)

23. The sides of a triangle are in A.P. and its area is $\frac{3}{5}$th the area of an equilateral triangle of same perimeter. The sides of the triangle are in the ratio.

(*a*) $1 : 2 : \sqrt{7}$ (*b*) 2 : 3 : 5 (*c*) 1 : 6 : 7 (*d*) 3 : 5 : 7

24. If $\log_3 2$, $\log_3 (2^x - 5)$ and $\log_3 (2^x - 7/2)$ are in A.P., the value of x is

(*a*) 0 (*b*) $\frac{1}{3}$ (*c*) 2 (*d*) 3

(*IIT 1990*)

25. If $a_1, a_2, a_3, ..., a_n$ are in A.P with common difference $d \neq 0$, then the value of $\sin d$ (cosec a_1 cosec a_2 + cosec a_2 cosec a_3 + ... + cosec a_{n-1} cosec a_n) will be

(*a*) $\sec a_1 - \sec a_n$ (*b*) $\tan a_1 - \tan a_n$

(*c*) $\text{cosec } a_1 - \text{cosec } a_n$ (*d*) $\cot a_1 - \cot a_n$

(*Rajasthan PET 2000*)

ANSWERS

1. (*c*)	**2.** (*d*)	**3.** (*d*)	**4.** (*c*)	**5.** (*c*)	**6.** (*d*)	**7.** (*c*)	**8.** (*a*)	**9.** (*d*)	**10.** (*b*)
11. (*c*)	**12.** (*a*)	**13.** (*c*)	**14.** (*b*)	**15.** (*b*)	**16.** (*b*)	**17.** (*a*)	**18.** (*a*)	**19.** (*b*)	**20.** (*d*)
21. (*b*)	**22.** (*b*)	**23.** (*d*)	**24.** (*d*)	**25.** (*d*)					

HINTS AND SOLUTIONS

1. $T_n = a + (n-1)d \Rightarrow T_{26} = a + 25d$

Here $a = 10$, $d = -4$

$\therefore T_{26} = 10 + 25 \times (-4) = 10 - 100 = \mathbf{-90.}$

2. Let the first term, common difference and number of terms of the A.P. be a, d and n respectively. Then,

$$a + d = 7\frac{3}{4} \quad ...(i)$$

$$a + 30d = \frac{1}{2} \quad ...(ii)$$

and $a + (n-1)d = -6\frac{1}{2}$ ...(iii)

Eqn (ii) – Eqn (i)

$\Rightarrow 29d = \frac{1}{2} - \frac{31}{4} = -\frac{29}{4}$

$\Rightarrow d = -\frac{1}{4}$

Putting $d = -\frac{1}{4}$ in (i), we get

$a - \frac{1}{4} = 7\frac{3}{4} \Rightarrow a = 7\frac{3}{4} + \frac{1}{4}$

$\Rightarrow a = 8$

$\therefore$ Putting the values of a and d in (iii), we have

$8 + (n-1)\left(-\frac{1}{4}\right) = -\frac{13}{2}$

$\Rightarrow 8 + \frac{1}{4} - \frac{1}{4}n = -\frac{13}{2}$

$\Rightarrow -\frac{1}{4}n = -\frac{13}{2} - \frac{33}{4}$

$\Rightarrow -\frac{1}{4}n = -\frac{59}{4}$

$\Rightarrow \mathbf{n = 59.}$

3. Given, $p(a + (p-1)d) = q(a + (q-1)d)$, where a and d are the first term and common difference of the A.P.

$\Rightarrow (p - q)a = (q^2 - q - p^2 + p)d$

$\Rightarrow -(q - p)a = (q - p)((q + p) - 1)d$

$\Rightarrow -a = ((q + p) - 1)d$

$\Rightarrow a + ((p + q) - 1)d = 0$

$\Rightarrow \mathbf{t_{p+q} = 0.}$

4. Let a and d be the first term and common difference of the given A.P.

Then, $a + 58d = 449$ ...(i)

$a + 448d = 59$ ...(ii)

Solving eqns (i) and (ii) simultaneously, we get

$a = 507, d = -1$

Now assume that the nth term is zero.

$\therefore 0 = a + (n-1)d$

$\Rightarrow 0 = 507 + (n-1)(-1)$

$\Rightarrow 507 = n - 1$

$\Rightarrow \mathbf{n = 508.}$

5. Given, $T_m = \frac{1}{n} \Rightarrow a + (m-1)d = \frac{1}{n}$ ...(i)

$T_n = \frac{1}{m} \Rightarrow a + (n-1)d = \frac{1}{m}$ ...(ii)

Eq. (ii) – Eq. (i) $\Rightarrow (n - m)d = \frac{1}{m} - \frac{1}{n}$

$\Rightarrow (n - m)d = \frac{n-m}{mn} \Rightarrow d = \frac{1}{mn}$

Putting $d = \frac{1}{mn}$ in (i), we get $a = \frac{1}{mn}$

$\therefore a - d = \frac{1}{mn} - \frac{1}{mn} = \mathbf{0.}$

6. a, b, c are in A.P. $\Rightarrow 2b = a + c$

$\therefore (a + 2b - c)(2b + c - a)(a + 2b + c)$

$= (a + a + c - c)(a + c + c - a)(2b + 2b)$

$= 2a \cdot 2c \cdot 4b = \mathbf{16abc}.$

7. Here $a = \frac{5}{2}$, $d = \frac{10}{3} - \frac{5}{2} = \frac{20-15}{6} = \frac{5}{6}$

$\therefore S_{24} = \frac{24}{2}\left[2 \times \frac{5}{2} + (24-1)\frac{5}{6}\right]$

$\left[\because S_n = \frac{n}{2}[2a + (n-1)d]\right]$

$= 12\left[5 + \frac{23 \times 5}{6}\right] = 2[30 + 115] = 2 \times 145 = \mathbf{290.}$

8. Let the first term of the A.P. be a and let the common difference be d.

Then, $t_{12} + t_{22} = 100 \Rightarrow (a + 11d) + (a + 21d) = 100$

$\Rightarrow 2a + 32d = 100$ ...(i)

Now sum of first 33 terms of the A.P

$= \frac{33}{2}(2a + 32d) \left(\because S_n = \frac{n}{2}(2a + (n-1)d\right)$

$= \frac{33}{2} \times 100$ (From (i))

$\mathbf{= 1650.}$

9. Given that,

$S_1 = a_2 + a_4 + a_6 + ... +$ upto 100 terms ...(i)

$S_2 = a_1 + a_3 + a_5 + ... +$ upto 100 terms ...(ii)

Let d be the common difference of the given A.P. Then,

$d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 =$ ...(iii)

Subtracting eqn (ii) from eqn (i), we have

$S_1 - S_2 = (a_1 - a_2) + (a_4 - a_3) + (a_6 - a_5) + ... +$ upto 100 terms

$= d + d + d + ... +$ upto 100 terms

$= 100\, d$

$\Rightarrow d = \mathbf{\frac{S_1 - S_2}{100}}.$

10. $S_n = 3n^2 + 2n$

$\therefore \quad S_{n-1} = 3(n-1)^2 + 2(n-1)$

$= 3n^2 - 6n + 3 + 2n - 2 = 3n^2 - 4n + 1$

$\therefore \quad T_n = S_n - S_{n-1}$

$= (3n^2 + 2n) - (3n^2 - 4n + 1) = 6n - 1.$

Given, $T_n = 299$

$\Rightarrow \quad 6n - 1 = 299$

$\Rightarrow \quad 6n = 300 \quad \Rightarrow \quad \boldsymbol{n = 50.}$

11. $$\frac{S_m}{S_n} = \frac{m/2\,[2a_1 + (m-1)d]}{n/2\,[2a_1 + (n-1)d]} = \frac{m^2}{n^2}$$

($d \to$ common difference of A.P.)

$\Rightarrow n[2a_1 + (m-1)d] = m[2a_1 + (n-1)d]$

$\Rightarrow \quad 2a_1(n-m) = d(n-m) \quad \Rightarrow \quad d = 2a_1$

$$\therefore \quad \frac{a_m}{a_n} = \frac{m\text{th term}}{n\text{th term}} = \frac{a_1 + (m-1)d}{a_1 + (n-1)d}$$

$$= \frac{a_1 + (m-1)\cdot 2a_1}{a_1 + (n-1)\cdot 2a_1} = \frac{-a_1 + 2a_1 m}{-a_1 + 2a_1 n}$$

$$= \frac{a_1(2m-1)}{a_1(2n-1)} = \boldsymbol{\frac{2m-1}{2n-1}}.$$

12. Let a and d be the first term and common difference respectively of the A.P.

$$S_{10} = \frac{10}{2}[2a + 9d]$$

$$S_5 = \frac{5}{2}[2a + 4d]$$

Given, $S_{10} = 4S_5$

$$\Rightarrow \quad 5(2a + 9d) = 4 \times \frac{5}{2}[2a + 4d]$$

$\Rightarrow \quad 10a + 45d = 20a + 40d$

$\Rightarrow \quad 5d = 10a$

$$\Rightarrow \quad \frac{a}{d} = \frac{5}{10} = \frac{1}{2} \quad \Rightarrow \quad \boldsymbol{a : d = 1 : 2.}$$

13. The numbers between 107 and 253 divisible by 5 are

110, 115, 120, ... , 245, 250.

This is an A.P with first term (a) = 110 and common difference (d) = 5. Let the last term be the nth term.

$\therefore \quad T_n = a + (n-1)d$

$\Rightarrow 110 + (n-1) \times 5 = 250$

$\Rightarrow \quad 5n = 250 - 105 = 145$

$\Rightarrow \quad n = 29.$

$$\therefore \quad \text{Required sum} = \frac{n}{2}(a + T_n) = \frac{29}{2}(110 + 250)$$

$$= \frac{29}{2} \times 360 = \mathbf{5220.}$$

14. Let the two A.P.'s be $a, a + d, a + 2d$, ... and $A, A + D, A + 2D$, ...

$$\text{Given,} \quad \frac{n/2\,[2a + (n-1)\,d]}{n/2\,[2A + (n-1)\,D]} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \quad \frac{2a + (n-1)\,d}{2A + (n-1)\,D} = \frac{7n+1}{4n+27} \quad ...(i)$$

Now, we have to find the ratio $\dfrac{t_{11}}{T_{11}} = \dfrac{a + 10d}{A + 10D} = \dfrac{2a + 20d}{2A + 20D}$

Putting $n = 21$ in (i), we get

$$\frac{2a + 20\,d}{2A + 20\,D} = \frac{7 \times 21 + 1}{4 \times 21 + 27} = \frac{147 + 1}{84 + 27} = \frac{148}{111}$$

$\therefore \quad t_{11} : T_{11} = \mathbf{148 : 111.}$

15. Let a be the first term and d the common difference of the given A.P.

$$\text{Then,} \quad S_n = \frac{n}{2}[2a + (n-1)\,d]$$

$$S_{2n} = \frac{2n}{2}[2a + (2n-1)\,d]$$

$$S_{3n} = \frac{3n}{2}[2a + (3n-1)\,d]$$

Given, $S_{2n} = 3S_n$

$$\Rightarrow \quad \frac{2n}{2}[2a + (2n-1)\,d] = \frac{3n}{2}[2a + (n-1)\,d]$$

$\Rightarrow \quad 4a + 4nd - 2d = 6a + 3nd - 3d$

$\Rightarrow \quad d + nd = 2a$

$$\Rightarrow \quad a = \frac{d(n+1)}{2} \quad ...(i)$$

$$\text{Now,} \quad \frac{S_{3n}}{S_n} = \frac{\frac{3n}{2}[2a + (3n-1)\,d]}{\frac{n}{2}[2a + (n-1)\,d]}$$

$$= \frac{3[(n+1)d + (3n-1)d]}{[(n+1)d + (n-1)\,d]} \quad \text{(From } (i))$$

$$= \frac{3nd + 3d + 9nd - 3d}{nd + d + nd - d}$$

$$= \frac{12nd}{2nd} = \mathbf{6.}$$

16. For the Ist A.P., 2, 5, 8, 11, ... ,

First term (a_1) = 2, common difference (d_1) = 3

$\therefore$ The sum of this A.P. to $2n$ terms

$$= \frac{2n}{2}[2a_1 + (2n-1)d_1]$$

$= n[4 + (2n-1)3]$

$= 4n + 6n^2 - 3n = 6n^2 + n = n(6n + 1)$

For the second A.P., 57, 59, 61, 63, ... ,

First term (b_1) = 57, common difference (d_2) = 2

$\therefore$ The sum of this A.P. to n term is

$$\text{Sum}_n = \frac{n}{2}[2b_1 + (n-1)\,d_2]$$

$$= \frac{n}{2}[114 + (n-1)\,2]$$

$= 57n + n^2 - n = n^2 + 56n = n(n + 56)$

Given, $S_{2n} = \text{Sum}_n$

$\Rightarrow \quad n(6n + 1) = n(n + 56)$

$\Rightarrow \quad 6n + 1 = n + 56 \Rightarrow 5n = 55 \Rightarrow \mathbf{n = 11.}$

17. Let the first term of the given A.P. be a and the common difference be d.

Then, $S_1 = \frac{n_1}{2}\,[2a + (n_1 - 1)\,d]$

$S_2 = \frac{n_2}{2}\,[2a + (n_2 - 1)\,d]$

$S_3 = \frac{n_3}{2}\,[2a + (n_3 - 1)\,d]$

$\therefore \quad \frac{S_1}{n_1}(n_2 - n_3) + \frac{S_2}{n_2}(n_3 - n_1) + \frac{S_3}{n_3}(n_1 - n_2)$

$= \frac{\frac{n_1}{2}}{n_1}\,[2a + (n_1 - 1)\,d]\,(n_2 - n_3) + \frac{\frac{n_2}{2}}{n_2}\,[2a + (n_2 - 1)\,d]$

$(n_3 - n_1) + \frac{\frac{n_3}{2}}{n_3}\,[2a + (n_3 - d)]\,(n_1 - n_2)$

$= \frac{1}{2}\,[2a\,(n_2 - n_3) + n_1\,(n_2 - n_3) - d\,(n_2 - n_3)]$

$+ \frac{1}{2}\,[2a\,(n_3 - n_1) + n_2\,(n_3 - n_1) - d\,(n_3 - n_1)]$

$+ \frac{1}{2}\,[2a(n_1 - n_2) + n_3\,(n_1 - n_2) - d\,(n_1 - n_2)]$

$= \frac{1}{2}\,[2an_2 - 2an_3 + 2an_3 - 2an_1 + 2an_1 - 2an_2) + (n_1n_2$

$- n_1\,n_3 + n_2n_3 - n_1n_2 + n_3n_1 - n_3n_2)$

$- d\,(n_2 - n_3 + n_3 - n_1 + n_1 - n_2)]$

$= \mathbf{0.}$

18. nth term of an A.P. is given by

T_n = Sum of n terms – Sum of $(n - 1)$ terms

$= S_n - S_{n-1}$

$= nP + \frac{n}{2}\,(n - 1)Q - \left[(n-1)P + \frac{(n-1)}{2}\,(n - 2)\,Q\right]$

$= nP - nP + P + \frac{Q}{2}\,[n^2 - n - n^2 + 3n - 2]$

$= P + \frac{Q}{2}\,(2n - 2)$

$= P + (n - 1)\,Q.$

$\therefore$ Common difference $(d) = T_n - T_{n-1}$

$= P + (n - 1)Q - (P + (n - 2)Q)$

$= \boldsymbol{Q.}$

19. Let $a_1, a_2, a_3, \ldots, a_n$ be the n arithmetic means between 3 and 17. Then,

$3, a_1, a_2, a_3, \ldots, a_n, 17$ form an A.P.

Let d be the common difference of this A.P. Then,

$a_1 = 3 + d \quad \text{and} \quad a_n = 17 - d$

Given, $\frac{a_n}{a_1} = \frac{3}{1} \Rightarrow \frac{17 - d}{3 + d} = \frac{3}{1}$

$\Rightarrow \quad 17 - d = 9 + 3d$

$\Rightarrow \quad 4d = 8 \Rightarrow d = 2.$

Also 17 is the $(n + 2)$th term of the given A.P.

$\therefore \quad 17 = 3 + (n + 2 - 1)2$

$\Rightarrow \quad 17 = 3 + (n + 1)2$

$\Rightarrow \quad 14 = (n + 1)2 \Rightarrow n + 1 = 7 \Rightarrow n = \mathbf{6.}$

20. Given, $a_1, a_2, a_3, \ldots, a_n$ are in A.P and $a_1 = 0$.

Let d be the common difference of the the A.P. So,

$a_2 = a_1 + d = d$

$\therefore \quad a_3 = a_1 + 2d = 2d,\ a_4 = 3d,\ a_5 = 4d, \ldots, a_n = (n - 1)d$

$\therefore \quad \left(\frac{a_3}{a_2} + \frac{a_4}{a_3} + \ldots + \frac{a_n}{a_{n-1}}\right) - a_2\left(\frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_{n-2}}\right)$

$= \left[\frac{2d}{d} + \frac{3d}{2d} + \ldots + \frac{(n-1)\,d}{(n-2)\,d}\right]$

$- d\left[\frac{1}{d} + \frac{1}{2d} + \ldots + \frac{1}{(n-3)d}\right]$

$= \left[2 + \frac{3}{2} + \ldots + \frac{n-1}{n-2}\right] - \left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-3}\right]$

$= \left\{(1+1) + \left(1 + \frac{1}{2}\right) + \ldots \left(1 + \frac{1}{n-2}\right)\right\}$

$- \left\{1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-3}\right\}$

$= 1 + 1 + 1 \ldots$ to $(n - 2)$ term

$- \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{(n-3)} + \frac{1}{(n-2)}\right)$

$- \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-3}\right)$

$= \mathbf{(n - 2) + \frac{1}{(n - 2)}}.$

21. Let $a_1 = a - 3d,\ a_2 = a - d,\ a_3 = a + d,\ a_4 = a + 3d$

Given, $a_1 + a_4 = 10 \Rightarrow a - 3d + a + 3d = 10$

$\Rightarrow \quad 2a = 10 \Rightarrow a = 5$

Also, $a_2\,a_3 = 24 \Rightarrow (a - d)\,(a + d) = 24$

$\Rightarrow \quad a^2 - d^2 = 24$

$\Rightarrow \quad 25 - d^2 = 24 \Rightarrow d^2 = 1 \Rightarrow \mathbf{d = 1}$

$\therefore$ The least term is $a - 3d = 5 - 3 = \mathbf{2.}$

22. Sum of first 10 terms = $\frac{1}{2}$ (Sum of next 10 terms)

$\Rightarrow \quad S_{10} = \frac{1}{2}(S_{20} - S_{10}) \quad \Rightarrow \quad 3S_{10} = S_{20}$

$\Rightarrow \quad 3 \times \frac{10}{2}[2a + 9d] = \frac{20}{2}[2a + 19d]$

$$\left[\because S_n = \frac{n}{2}[2a + (n-1)d]\right]$$

$\Rightarrow$ $3(2a + 9d) = 2(2a + 19d)$

$\Rightarrow$ $6a + 27d = 4a + 38d$

$\Rightarrow$ $2a = 11d$

$\Rightarrow$ $2(13 + d) = 11d$ ($\because$ Second term = 13 and $13 - a = d$)

$\Rightarrow$ $26 - 2d = 11d$

$\Rightarrow$ $13d = 26 \quad \Rightarrow \quad d =$ **2.**

23. Let the sides of the triangle be $a - d, a, a + d$.

Perimeter of the triangle = $a - d + a + a + d = 3a$

$\therefore$ Each side of the equilateral triangle = $\frac{3a}{3} = a$

$\therefore$ Area of equilateral triangle = $\frac{\sqrt{3}}{4}a^2$.

Area of the given $\Delta = \sqrt{s(s-(a-d))(s-a)(s-(a+d))}$

where $s = \frac{3a}{2}$

$$= \sqrt{\frac{3a}{2}\left(\frac{3a}{2} - a + d\right)\left(\frac{3a}{2} - a\right)\left(\frac{3a}{2} - a - d\right)}$$

$$= \sqrt{\frac{3a}{2}\left(\frac{1}{2}a + d\right)\left(\frac{1}{2}a\right)\left(\frac{1}{2}a - d\right)}$$

$$= \sqrt{\frac{3}{4}a^2\left(\frac{1}{4}a^2 - d^2\right)}$$

Given, $\sqrt{\frac{3}{4}a^2\left(\frac{1}{4}a^2 - d^2\right)} = \frac{3}{5} \times \frac{\sqrt{3}}{4}a^2$

$\Rightarrow \quad \frac{3}{16}a^4 - \frac{3}{4}a^2d^2 = \left(\frac{3\sqrt{3}}{20}a^2\right)^2$

$\Rightarrow \quad \frac{3}{16}a^4 - \frac{3}{4}a^2d^2 = \frac{27}{400}a^4$

$\Rightarrow \quad \frac{3}{16}a^4 - \frac{27}{400}a^4 = \frac{3}{4}a^2d^2$

$\Rightarrow \quad \frac{1}{16}a^2 - \frac{9}{400}a^2 = \frac{1}{4}d^2$

$\Rightarrow \quad \frac{25a^2 - 9a^2}{400} = \frac{1}{4}d^2$

$\Rightarrow \quad \frac{16a^2}{400} = \frac{1}{4}d^2$

$\Rightarrow \quad \frac{a^2}{d^2} = \frac{400}{64} \quad \Rightarrow \quad \frac{a}{d} = \frac{20}{8} = k$ (say)

$\therefore \quad (a - d) : a : (a + d) = (20k - 8k) : 20k : (20k + 8k)$

$= 12k : 20k : 28k =$ **3 : 5 : 7.**

24. $\log_3 2$, $\log_3(2^x - 5)$ and $\log_3\left(2^x - \frac{7}{2}\right)$ are in A.P.

$\Rightarrow \quad 2\log_3(2^x - 5) = \log_3 2 + \log_3(2^x - 7/2)$

($\because$ a, b, c in A.P $\Rightarrow$ $2b = a + c$)

$\Rightarrow \quad \log_3(2^x - 5)^2 = \log_3[2(2^x - 7/2)]$

($\because$ $a \log b = \log b^a$ and $\log a + \log b = \log ab$)

$\Rightarrow \quad (2^x - 5)^2 = 2^{x+1} - 7$

Let $2^x = y$. Then,

$(y - 5)^2 = 2y - 7$

$\Rightarrow \quad y^2 - 10y + 25 = 2y - 7$

$\Rightarrow \quad y^2 - 12y + 32 = 0$

$\Rightarrow \quad (y - 8)(y - 4) = 0$

$\Rightarrow \quad y = 8$ or $4 \quad \Rightarrow \quad 2^x = 8$ or $2^x = 4$

$2^x = 8 \Rightarrow$ $\boldsymbol{x = 3}$

($\because$ $2^x = 4$ shall make the term $\log_3(2^x - 5)$ negative which is not possible)

25. $d = a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots\ldots = a_n - a_{n-1}$.

Now, $\sin d(\text{cosec } a_1 \text{ cosec } a_2 + \text{cosec } a_2 \text{ cosec } a_3 + \ldots\ldots + \text{cosec } a_{n-1} \text{ cosec } a_n)$

$$= \frac{\sin d}{\sin a_1 \sin a_2} + \frac{\sin d}{\sin a_2 \sin a_3} + \ldots\ldots + \frac{\sin d}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin(a_2 - a_1)}{\sin a_1 \sin a_2} + \frac{\sin(a_3 - a_2)}{\sin a_2 \sin a_3} + \ldots\ldots + \frac{\sin(a_n - a_{n-1})}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\sin a_2 \cos a_1 - \cos a_2 \sin a_1}{\sin a_1 \sin a_2} + \frac{\sin a_3 \cos a_2 - \cos a_3 \sin a_2}{\sin a_2 \sin a_3} + \ldots\ldots + \frac{\sin a_n \cos a_{n-1} - \cos a_n \sin a_{n-1}}{\sin a_{n-1} \sin a_n}$$

$$= \frac{\cos a_1}{\sin a_1} - \frac{\cos a_2}{\sin a_2} + \frac{\cos a_2}{\sin a_2} - \frac{\cos a_3}{\sin a_3} + \ldots\ldots + \frac{\cos a_{n-1}}{\sin a_{n-1}} - \frac{\cos a_n}{\sin a_n}$$

= $\mathbf{\cot a_1 - \cot a_n}$.

GEOMETRIC PROGRESSION (G.P.)

KEY FACTS

1. A **Geometric Progression (G.P)** is one in which the ratio of any term to its predecessor is always the same number. This ratio is called the **common ratio.** Thus, a sequence, $a_1, a_2, \ldots, a_n$ is said to be in G.P.,

if $$\frac{a_2}{a_1} = \frac{a_3}{a_2} = \frac{a_4}{a_3} = \ldots = \frac{a_n}{a_{n-1}} = r$$

where $a_1, a_2, a_3 \ldots$ are all non-zero terms of the G.P. and r is the **common ratio**.

Examples of G.P.

	Progression	Common ratio
(*i*)	1, 3, 9, 27, 81, ...	$r = 3$
(*ii*)	16/27, – 8/9, 4/3, – 2	$r = -3/2$
(*iii*)	$x, x^2, x^3, x^4, \ldots$	$r = x$

If a denotes the first term of a G.P. and r, the common ratio, then a standard G.P. is $a, ar, ar^2, \ldots$.

2. **The *n*th term of a G.P.**

For a standard G.P with **first term = *a*, common ratio = *r*** and **number of terms = *n*,** we know that:

First term	Second term	Third term	Fourth term	
$t_1 = a$	$t_2 = ar$	$t_3 = ar^2$	$t_4 = ar^3$	and so on.

So, **nth term = $t_n = ar^{n-1}$.**

Ex. The sixth of the G.P. – 3, 6, – 12, 24, ... is:

Here $a = -3,\ r = \frac{6}{-3} = -2$

$\therefore$ $t_6 = ar^{6-1} = -3 \times (-2)^5$

$= -3 \times -32 = \mathbf{96}.$

3. **Geometric Mean:** When three real numbers form a geometric progression, then the middle one is called the geometric mean of the other two quantities.

Thus, if G is the geometric mean of two non-zero numbers a and b, then a, G, b forms a geometric progression, so that $\frac{G}{a} = \frac{b}{G}$

$\therefore$ $$G^2 = ab \Rightarrow G = \pm\sqrt{ab}$$

Maths Alert. $G = \sqrt{ab}$ if a and b are positive numbers.

$G = -\sqrt{ab}$ if a and b are negative numbers.

Also, if there are n positive integers $a_1, a_2, \ldots, a_n$, then their geometric mean is defined to be equal to

$$(a_1 . a_2 . a_3 . \ldots . a_n)^{1/n}.$$

Ex. (*i*) G.M. of 4 and 36 = $\sqrt{4 \times 36} = \sqrt{144} = \mathbf{12}.$

(*ii*) G.M. of $\frac{1}{3}$, 1, 3, 9, 27, 81 and 243 = $\left(\frac{1}{3} \times 1 \times 3 \times 9 \times 27 \times 81 \times 243\right)^{1/7}$

$= (3^2 \times 3^3 \times 3^4 \times 3^5)^{1/7} = (3^{14})^{1/7} = 3^2 = \mathbf{9}.$

4. **Sum of *n* terms of a Geometric Progression:**

Consider the geometric progression $a, ar, ar^2, \ldots, ar^{n-1}$ of n terms.

Then, **the sum of *n* terms** $S_n = \frac{a(1 - r^n)}{1 - r} (r \neq 1)$ or $\frac{a(r^n - 1)}{(r - 1)} (r \neq 1)$

where, a = first term, r = common ratio, n = number of terms

Also, $\boldsymbol{S_n = \dfrac{a - lr}{1 - r}}$ or $\boldsymbol{\dfrac{lr - a}{r - 1}}$, where l is the last term.

Ex. The sum of the G.P. $4 + 2 + 1 + \dfrac{1}{2} + \dfrac{1}{4} +$ .... to 10 term is:

Here $\quad a = 4, r = \dfrac{1}{2}, \; n = 10$

$$\therefore \quad S_{10} = \frac{4\left[1-\left(\frac{1}{2}\right)^{10}\right]}{1-\frac{1}{2}} = \frac{4\left(1-\frac{1}{1024}\right)}{\frac{1}{2}} = \frac{8\times 1023}{1024} = \mathbf{8} \text{ (approx.)}$$

5. Sum of an infinite G.P.

$\boldsymbol{S_\infty = \dfrac{a}{1 - r}}$, where a = first term
r = common ratio.

Note: Sum of an infinite series exists only when r is numerically less than 1, *i.e.*, $|r| < 1$.

Ex. Sum of the G.P. 16, – 8, 4, ... to infinity is:

Here $\quad a = 16, r = -\dfrac{8}{16} = -\dfrac{1}{2}$

$$\therefore \quad S_\infty = \frac{a}{1-r} = \frac{16}{1-\left(\frac{1}{-2}\right)} = \frac{16}{1+\frac{1}{2}} = \frac{16}{\frac{3}{2}} = \mathbf{\frac{32}{3}}.$$

6. Properties of a G.P.

(*a*) If each term of a given geometric progression is multiplied or divided by the same number, then the resulting progression is also a G.P.

If a_1, a_2, a_3, a_4, ... be a G.P. with common ratio r, then

a_1c, a_2c, a_3c, a_4c, ... and $\dfrac{a_1}{c}, \dfrac{a_2}{c}, \dfrac{a_3}{c}, \dfrac{a_4}{c}$, ... are also G.P.s with common ratio r. c is the constant number.

(*b*) The reciprocals of the terms of a G.P. also form a G.P.

If a_1, a_2, a_3, a_4, ... be a G.P. with common ratio r, then $\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \dfrac{1}{a_4}$, ... is also a G.P. with common ratio $\dfrac{1}{r}$.

(*c*) If each term of a G.P. be raised to the same power, then the resulting progression is also a G.P.

If a_1, a_2, a_3, a_4, ... be a G.P. with common ratio r, then $a_1^k, a_2^k, a_3^k, a_4^k$, ... is also a G.P. with common ratio r.

(*d*) If a_1, a_2, a_3, a_4, ... and b_1, b_2, b_3, b_4, ... be two G.P.s with common ratio r_1 and r_2 respectively, then the progressions a_1b_1, a_2b_2, a_3b_3, ... is a G.P. with common ratio r_1r_2, and $\dfrac{a_1}{b_1}, \dfrac{a_2}{b_2}, \dfrac{a_3}{b_3}, \dfrac{a_4}{b_4}$, ... is a G.P. with common ratio $\dfrac{r_1}{r_2}$.

(*e*) If a_1, a_2, a_3, ... be a G.P. with common ratio r, such that each term of the progression is a positive number, then $\log a_1, \log a_2, \log a_3$, ... is an A.P. with common difference $\log r$.

Conversely, if $\log a_1, \log a_2, \log a_3$, ... are terms of an A.P., then a_1, a_2, a_3, ... are terms of a G.P.

(*f*) The Arithmetic mean between two positive numbers is always greater than equal to their Geometric Mean.

A.M. ≥ G.M.

(*g*) If A and G be the arithmetic and geometric mean respectively between two numbers, then the numbers are $A \pm \sqrt{A^2 - G^2}$.

7. **Useful note:** When product is known, suppose the numbers in G.P. as under:

(*i*) If the odd number of terms are to be considered, suppose them as follows by taking a as the mid-term and common ratio r.

$$\frac{a}{r},\ a,\ ar; \qquad \frac{a}{r^2},\ \frac{a}{r},\ a,\ ar,\ ar^2$$

(*ii*) If even number of terms are to be considered, suppose them as under by taking $\frac{a}{r}$, ar as the two mid-terms and r^2 as the common ratio.

$$\frac{a}{r^3},\ \frac{a}{r},\ ar,\ ar^3 \quad \text{[4 terms]}$$

$$\frac{a}{r^5},\ \frac{a}{r^3},\ \frac{a}{r},\ ar,\ ar^3,\ ar^5 \quad \text{[6 terms]}$$

[See Solved Ex. (7)]

SOLVED EXAMPLES

Ex. 1. Write the G.P. whose 4th term is 54 and 7th term is 1458.

Sol. $t_4 = ar^3 = 54$ ...(*i*)

$t_7 = ar^6 = 1458$ ...(*ii*)

∴ Dividing (*ii*) by (*i*), we get

$$\frac{t_7}{t_4} = r^3 = \frac{1458}{54} = 27$$

⇒ $r = 3$

∴ From (*i*), $a.(3)^3 = 54$

⇒ $27a = 54$

⇒ $a = 2$

∴ The G.P. is 2, 2 × 3, 2 × 3^2, ... , *i.e.* 2, 6, 18, 54, ...

Ex. 2. If the 10th term of a G.P. is 9 and the 4th term is 4, then what is its 7th term?

Sol. Let the first term and common ratio of the given G.P. be a and r respectively.

$T_{10} = ar^9 = 9$ ...(*i*)

$T_4 = ar^3 = 4$ ...(*ii*)

$$\frac{T_{10}}{T_4} = \frac{ar^9}{ar^3} = \frac{9}{4}$$

$$r^6 = \frac{9}{4} \Rightarrow (r^3)^2 = \frac{9}{4} \Rightarrow r^3 = \frac{3}{2} \quad ...(iii)$$

From (*i*), $ar^9 = 9 \Rightarrow a(r^3)^3 = 9$

∴ Substituting the value of r^3 from (*iii*), we get

$$a \cdot \left(\frac{3}{2}\right)^3 = 9 \Rightarrow a = \frac{9 \times 8}{27} = \frac{8}{3}.$$

∴ $T_7 = ar^6 = \frac{8}{3} \times \frac{9}{4} = \mathbf{6.}$

Ex. 3. What is the sum of 10 terms of the series $\sqrt{2}+\sqrt{6}+\sqrt{18}+$ .....? **(*DCE*)**

Sol. $$S_n = \frac{a\,(r^n-1)}{(r-1)}$$

Here $$a = \sqrt{2},\; r = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3},\; n = 10$$

$\therefore$ $$S_{10} = \frac{\sqrt{2}\left((\sqrt{3})^{10}-1\right)}{\sqrt{3}-1} = \frac{\sqrt{2}\,(3^5-1)}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{\sqrt{2}\,(243-1)\,(\sqrt{3}+1)}{(3-1)}$$

$$= \frac{\sqrt{2}\,(242)\,(\sqrt{3}+1)}{2} = \mathbf{121\,(\sqrt{6}+\sqrt{2}).}$$

Ex. 4. What is the sum to infinity of the series $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+$ ...... ? **(*NDA/NA 2012*)**

Sol. Given series is an infinite G.P. with $a = 1$, $r = -\frac{1}{2}\Big/1 = -\frac{1}{2}$.

$\therefore$ $$S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{1}{\frac{3}{2}} = \mathbf{\frac{2}{3}}.$$

Ex. 5. If *a*, *b*, *c* are the three consecutive terms of an A.P. and *x*, *y*, *z* are three consecutive terms of a G.P., then prove that $x^{b-c}.\ y^{c-a}.\ z^{a-b} = 1$.

Sol. a, b, c are in A.P. $\Rightarrow$ $2b = a + c$ ...(*i*)

x, y, z are in G.P. $\Rightarrow$ $y = xr,\ z = xr^2$, where r is the common ratio

$\therefore$ $x^{b-c}.\, y^{c-a}.\, z^{a-b} = x^{b-c}.\,(xr)^{c-a}.\,(xr^2)^{a-b}$

$= x^{b-c}\,.\,x^{c-a}\,.\,r^{c-a}.\ x^{a-b}\,.\,r^{2a-2b} = x^{b-c+c-a+a-b}\,.\,r^{c-a+2a-2b}$

$= x^0\,.\,r^{c+a-2b} = x^0.r^{2b-2b} = x^0.r^0 = 1$ (From (*i*))

Ex. 6. Let two numbers have arithmetic mean 9 and geometric mean 4. Then find the equation which has these two numbers as its roots. **(*AIEEE 2004*)**

Sol. Let the two numbers be a and b.

Then $\frac{a+b}{2} = 9 \;\Rightarrow\; a + b = 18$

$\sqrt{ab} = 4 \;\Rightarrow\; ab = 16$

$\therefore$ Required eqn is x^2 – Sum of roots x + Product of roots = 0

$\Rightarrow x^2 - 18x + 16 = 0.$

Ex. 7. Insert 3 geometric means between 16 and 256.

Sol. Let G_1, G_2, G_3 be the required means.

Then 16, G_1, G_2, G_3, 256 form a G.P.

Let r be the common ratio.

$\Rightarrow$ $256 = \text{5th term} = ar^4 = 16 \times r^4$

$\Rightarrow$ $16r^4 = 256 \;\Rightarrow\; r^4 = 16 \;\Rightarrow\; r = 2$

$\therefore$ $G_1 = ar = 16 \times 2 = 32$

$G_2 = ar^2 = 16 \times 4 = 64$

$G_3 = ar^3 = 16 \times 8 = 128$

Hence 32, 64 and 128 are the required G.M's between 16 and 256.

Ex. 8. If one AM 'A' and two GM p and q are inserted between two given numbers, then find the value of $\frac{p^2}{q} + \frac{q^2}{p}$ in terms of A? ***(VITEEE 2011)***

Sol. Let a and b be the two given numbers.

Then, A.M. $= \frac{a+b}{2} \Rightarrow A = \frac{a+b}{2} \Rightarrow 2A = a + b$ ...(*i*)

As, p and q are G.M's between a and b, a, p, q, b forms a G.P.

$\therefore \quad \frac{p}{a} = \frac{q}{p} = \frac{b}{q} \Rightarrow \frac{p^2}{q} = a$ and $\frac{q^2}{p} = b$

Now, $\frac{p^2}{q} + \frac{q^2}{p} = a + b = \mathbf{2A}$.

Ex. 9. What is the sum of the series $\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} + ...$ to n terms? ***(KCET 2001)***

Sol. The series $\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} + ...$ to n terms can be written as

$$\left(1-\frac{1}{3}\right)+\left(1-\frac{1}{9}\right)+\left(1-\frac{1}{27}\right)+\left(1-\frac{1}{81}\right)+ \text{ ..... to } n \text{ terms}$$

$$= (1+1+1+1+ \text{.... to } n \text{ terms}) - \left(\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+ \text{..... to } n \text{ terms}\right)$$

$$= n - \frac{\frac{1}{3}\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}} \qquad \left(S_n = \frac{a(1-r^n)}{1-r}, \text{ here } a = \frac{1}{3}, r = \frac{1}{3}\right)$$

$$= n - \frac{\frac{1}{3}\left(1-\frac{1}{3^n}\right)}{2/3} = \mathbf{n - \frac{1}{2}(1 - 3^{-n})}.$$

Ex. 10. If $1 + \sin x + \sin^2 x + \sin^3 x + ... \infty = 4 + 2\sqrt{3}$, $0 < x < \pi$, then find the value of x. ***(Kerala PET 2004)***

Sol. Given, $1 + \sin x + \sin^2 x + \sin^3 x + ... \infty = 4 + 2\sqrt{3}$

$\left[\because S_\infty = \frac{a}{1-r}.\right.$ Here $a = 1, r = \sin x$. Also $|\sin x| < 1$ as $|\sin x| = 1$ will make the the sum of infinte G.P. infinite$\left.\right)$

$$\Rightarrow \quad \frac{1}{1-\sin x} = 4 + 2\sqrt{3}$$

$$\Rightarrow \quad 1 - \sin x = \frac{1}{4+2\sqrt{3}}$$

$$\Rightarrow \quad \sin x = 1 - \frac{1}{4+2\sqrt{3}} = \frac{4+2\sqrt{3}-1}{4+2\sqrt{3}} = \frac{3+2\sqrt{3}}{4+2\sqrt{3}} \times \frac{4-2\sqrt{3}}{4-2\sqrt{3}}$$

$$= \frac{12+8\sqrt{3}-6\sqrt{3}-12}{16-12} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad x = \mathbf{\frac{\pi}{3}} \text{ or } \mathbf{\frac{2\pi}{3}} \text{ as } 0 < x < \pi.$$

Ex. 11. If a, b, c are in G.P. and $4a, 5b, 4c$ are in A.P. such that $a + b + c = 70$, then what is the value of the smallest of the numbers a, b and c?

Sol. a, b, c are in G.P. $\Rightarrow$ $b^2 = ac$ ...(*i*)

$4a, 5b, 4c$ are in A.P. $\Rightarrow$ $2 \times 5b = 4a + 4c$ $\Rightarrow$ $10b = 4a + 4c \Rightarrow 5b = 2a + 2c$ ...(*ii*)

Also, given $a + b + c = 70$ ...(*iii*)

$\Rightarrow$ $2a + 2b + 2c = 140$ $\Rightarrow$ $5b + 2b = 140$ (From (*ii*))

$\Rightarrow$ $7b = 140$ $\Rightarrow$ $b = 20.$

Now, from (*i*), $400 = ac$. ($\because$ $b = 20$)

Also, from (*ii*), $a + 20 + c = 70$ $\Rightarrow$ $a + c = 50$

$$\therefore \quad (a - c)^2 = (a + c)^2 - 4ac$$
$$= 2500 - 1600 = 900$$
$$\Rightarrow \quad a - c = \pm 30$$

$$\therefore \left.\begin{array}{l} a + c = 50 \\ a - c = \pm 30 \end{array}\right\} \Rightarrow a = 40, c = 10 \quad \text{or} \quad a = 10, c = 40.$$

$\therefore$ The least value out of a, b and c is **10.**

Ex. 12. Find the sum of n terms of the series $1 + (1 + x) + (1 + x + x^2) + ...?$ ***(AMU 2003)***

Sol. $1 + (1 + x) + (1 + x + x^2) + ...\ n$ terms

$$\Rightarrow \quad \text{Required sum} = \frac{1}{(1-x)}[(1 - x) + (1 - x)(1 + x) + (1 - x)(1 + x + x^2) + ...\ n \text{ terms}]$$

$$= \frac{1}{(1-x)}[(1 - x) + (1 - x^2) + (1 - x^3) + ...\ n \text{ terms}]$$

$$= \frac{1}{(1-x)}[(1 + 1 + 1 + ...\ n \text{ terms}) - (x + x^2 + x^3 + ...\ n \text{ terms}]$$

$$= \frac{1}{(1-x)}\left[n - \frac{x(1-x^n)}{(1-x)}\right] \qquad \left(\because\ S_n = \frac{a(1-r^n)}{(1-r)}, \text{Here } a = x, r = x\right)$$

$$= \boldsymbol{\frac{n(1-x) - x(1-x^n)}{(1-x)^2}}.$$

Ex. 13. Five numbers are in A.P. with common difference $\neq 0$. If 1st, 3rd and 4th terms are in G.P., then which term is always zero? ***(WBJEE 2013)***

Sol. Let the five numbers in A.P. be $a - 2d, a - d, a, a + d, a + 2d$.

Since 1st, 3rd and 4th terms are in G.P.

$$(t_3)^2 = t_1 \cdot t_4$$
$$\Rightarrow \quad a^2 = (a - 2d)(a + d)$$
$$\Rightarrow \quad a^2 = a^2 - ad - 2d^2$$
$$\Rightarrow \quad -ad = 2d^2$$
$$\Rightarrow \quad a = -2d \qquad (\because\ d \neq 0)$$
$$\Rightarrow \quad a + 2d = 0 \quad \Rightarrow \quad t_5 = 0$$

$\therefore$ 5th term is always zero.

Ex. 14. Find the sum to n terms of the series, $7 + 77 + 777 + ...$.

Sol.
$$S_n = 7 + 77 + 777 + ...\text{ to } n \text{ terms}$$
$$= 7\,[1 + 11 + 111 + ...\text{ to } n \text{ terms}]$$
$$= \frac{7}{9}[9 + 99 + 999 + ...\text{ to } n \text{ terms}]$$

$= \frac{7}{9}$ [(10 – 1) + (100 – 1) + (1000 – 1) + ... to n terms]

$= \frac{7}{9}$ [{10 + 10^2 + 10^3 + ... to n terms} – {1 + 1 + 1 + ... to n terms}]

$$= \frac{7}{9}\left[\frac{10\,(10^n - 1)}{10 - 1} - n\right] = \mathbf{\frac{7\,(10^{n+1} - 10)}{81} - \frac{7}{9}n.}$$

Ex. 15. What is the sum of n terms of the series 0.2 + 0.22 + 0.222 + ...? ***(WBJEE 2009)***

Sol. 0.2 + 0.22 + 0.222 + ... to n terms

= 2[0.1 + 0.11 + 0.111 + ... to n terms]

$= \frac{2}{9}$ [0.9 + 0.99 + 0.999 + ... to n terms]

$= \frac{2}{9}$ [(1 – 0.1) + (1 – 0.01) + (1 + 0.001) + ... to n terms]

$= \frac{2}{9}$ [(1 + 1 + 1 + ... to n terms) – (0.1 + 0.01 + 0.001 + ... to n terms)]

$$= \frac{2}{9}\left[n - \frac{0.1\,(1 - (0.1)^n}{(1 - 0.1)}\right] = \frac{2}{9}\left[n - \frac{\frac{1}{10}\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}}\right] = \mathbf{\frac{2}{9}\left[n - \frac{1}{9}\left(1 - \frac{1}{10^n}\right)\right].}$$

Ex. 16. Find the value of $0.2\overline{34}$ regarding it as a geometric series.

Sol. $0.2\overline{34}$ = 0.234 34 34 ....

= 0.2 + 0.034 + 0.00034 + 0.0000034 + ... + ∞

$$= \frac{2}{10} + \frac{34}{1000} + \frac{34}{100000} + \frac{34}{10000000} + \;...... + \infty$$

$$= \frac{2}{10} + \frac{34}{10^3}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + ...\infty\right]$$

$$= \frac{2}{10} + \frac{34}{10^3} \times \left(\frac{1}{1 - \frac{1}{10^2}}\right) \qquad \left(\because S_\infty = \frac{a}{1 - r}. \text{ Here } a = 1, r = \frac{1}{10^2}\right)$$

$$= \frac{2}{10} + \frac{34}{1000} \times \frac{100}{99} = \frac{198 + 34}{990} = \frac{232}{990} = \mathbf{\frac{116}{495}.}$$

Ex. 17. Six positive numbers are in G.P. such that their product is 1000. If the fourth term is 1, then find the last term. ***(WBJEE 2013)***

Sol. Let the six numbers in G.P. be $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$.

Then, Product = 1000

$$\Rightarrow \quad \frac{a}{r^5} \times \frac{a}{r^3} \times \frac{a}{r} \times ar \times ar^3 \times ar^5 = 1000$$

$$\Rightarrow \quad a^6 = 1000 \quad \Rightarrow \quad a = \sqrt{10}$$

Given, Fourth term = $t_4 = ar = 1$

$$\Rightarrow \quad \sqrt{10} \times r = 1 \quad \Rightarrow \quad r = \frac{1}{\sqrt{10}}$$

$$\therefore \quad \text{Last term} = ar^5 = \sqrt{10} \times \left(\frac{1}{\sqrt{10}}\right)^5 = \mathbf{\frac{1}{100}.}$$

Ex. 18. If a, b, c, d are in G.P., prove that $(a^2 + b^2 + c^2), (ab + bc + cd), (b^2 + c^2 + d^2)$ are in G.P.

Sol. a, b, c, d are in G.P $\Rightarrow$ $b = ar, c = ar^2, d = ar^3$, where **$r$ = common ratio**

$$\therefore \quad (ab + bc + cd)^2 = (a.ar + ar \cdot ar^2 + ar^2 \cdot ar^3)^2 = [a^2 r(1 + r^2 + r^4)]^2 \qquad \text{...}(i)$$

$$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (a^2 + a^2r^2 + a^2r^4)(a^2r^2 + a^2r^4 + a^2r^6)$$
$$= a^2(1 + r^2 + r^4) \cdot a^2r^2(1 + r^2 + r^4) = a^4r^2(1 + r^2 + r^4)^2$$
$$= [a^2r(1 + r^2 + r^4)]^2 = (ab + bc + cd)^2 \qquad \text{(From } (i))$$

$\therefore$ $(a^2 + b^2 + c^2), (ab + bc + cd), (b^2 + c^2 + d^2)$ are in G.P.

Ex. 19. If x, y, z are all positive and are the pth, qth and rth terms of a geometric progression respectively, then find the value of the determinant $\begin{vmatrix} \log x & p & 1 \\ \log y & q & 1 \\ \log z & r & 1 \end{vmatrix}$. ***(VITEEE 2009)***

Sol. Let a be the first term and R the common ratio of the G.P.

Then,
$$T_p = aR^{p-1} = x$$
$$T_q = aR^{q-1} = y$$
$$T_r = aR^{r-1} = z$$

$$\Rightarrow \quad \log x = \log(aR^{p-1}) = \log a + (p-1)\log R$$
$$\log y = \log(aR^{q-1}) = \log a + (q-1)\log R$$
$$\log z = \log(aR^{r-1}) = \log a + (r-1)\log R$$

$$\therefore \quad \begin{vmatrix} \log x & p & 1 \\ \log y & q & 1 \\ \log z & r & 1 \end{vmatrix} = \begin{vmatrix} \log a + (p-1)\log R & p & 1 \\ \log a + (q-1)\log R & q & 1 \\ \log a + (r-1)\log R & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log a & p & 1 \\ \log a & q & 1 \\ \log a & r & 1 \end{vmatrix} + \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix}$$

$$= \log a \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} + \log R \begin{vmatrix} (p-1) & p & 1 \\ (q-1) & q & 1 \\ (r-1) & r & 1 \end{vmatrix}$$

$$= \log a \times 0 + \log R \begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} - \log R \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix}$$

$$= 0 + \log R \times 0 - \log R \times 0 = 0$$

($\because$ The value of a determinants with two identical rows or columns is zero).

Ex. 20. If $A = 1 + r^a + r^{2a} + r^{3a} + \ldots\ldots \infty$, $B = 1 + r^b + r^{2b} + r^{3b} + \ldots\ldots \infty$, then show that $\dfrac{a}{b} = \log_{\left(\frac{B-1}{B}\right)}\left(\dfrac{A-1}{A}\right)$.

Sol.
$$A = 1 + r^a + r^{2a} + r^{3a} + \ldots\ldots \infty$$

$$\Rightarrow \quad A = \frac{1}{1 - r^a} \qquad \left(\because S_\infty = \frac{a}{1-r}. \text{ Here } a = 1, \text{ common ratio} = r^a\right)$$

$$\Rightarrow \quad 1 - r^a = \frac{1}{A} \quad \Rightarrow \quad r^a = 1 - \frac{1}{A} = \frac{A-1}{A} \quad \Rightarrow \quad a \log r = \log\left(\frac{A-1}{A}\right)$$

$$\Rightarrow \quad a = \frac{\log\left(\dfrac{A-1}{A}\right)}{\log r} \qquad \text{...}(i)$$

Now, $B = 1 + r^b + r^{2b} + r^{3b} + ... \infty$

$\Rightarrow \quad B = \dfrac{1}{1-r^b} \Rightarrow 1 - r^b = \dfrac{1}{B} \Rightarrow r^b = 1 - \dfrac{1}{B} = \dfrac{B-1}{B} \Rightarrow \log r^b = \log\left(\dfrac{B-1}{B}\right)$

$\Rightarrow \quad b \log r = \log\left(\dfrac{B-1}{B}\right) \Rightarrow b = \dfrac{\log\left(\dfrac{B-1}{B}\right)}{\log r}$ ...(*ii*)

$\therefore \quad \dfrac{a}{b} = \dfrac{\log\left(\dfrac{A-1}{A}\right)}{\log r} \times \dfrac{\log r}{\log\left(\dfrac{B-1}{B}\right)}$ (From (*i*) and (*ii*))

$\Rightarrow \quad \dfrac{a}{b} = \dfrac{\log\left(\dfrac{A-1}{A}\right)}{\log\left(\dfrac{B-1}{B}\right)} = \mathbf{\log_{\left(\frac{B-1}{B}\right)}\left(\dfrac{A-1}{A}\right)}.$ $\left[\because \dfrac{\log a}{\log b} = \log_b a\right].$

Ex. 21. $S_1, S_2, S_3, ..., S_n$ **are the sums of** n **infinite geometric progressions. The first term** S **of these progressions are 1,** $2^2 - 1, 2^3 - 1, 2^4 - 1, ..., 2^n - 1$ **and the common ratios are** $\dfrac{1}{2}, \dfrac{1}{2^2}, \dfrac{1}{2^3}, ..., \dfrac{1}{2^n}$**. Calculate the sum** $S_1 + S_2 + S_3 + ... + S_n$**.**

Sol. Since $S_\infty = \dfrac{a}{1-r}$,

$$S_1 = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$$

$$S_2 = \frac{2^2-1}{1-\frac{1}{2^2}} = \frac{2^2-1}{\frac{2^2-1}{2^2}} = 2^2$$

$$S_3 = \frac{2^3-1}{1-\frac{1}{2^3}} = \frac{2^3-1}{\frac{2^3-1}{2^3}} = 2^3$$

....

....

$$S_n = \frac{2^n-1}{1-\frac{1}{2^n}} = \frac{2^n-1}{\frac{2^n-1}{2^n}} = 2^n$$

$\therefore \quad S_1 + S_2 + S_3 + ... + S_n = 2 + 2^2 + 2^3 + ... + 2^n$

$= \dfrac{2(2^n-1)}{2-1} = \mathbf{2(2^n - 1)}$ $\left(\because S_n = \dfrac{a(r^n-1)}{r-1}, \text{ Here } a = 2, r = 2\right).$

PRACTICE SHEET

Level–1

1. Which term of the G.P. 3, $3\sqrt{3}$, 9, ... is 2187 ?

(*a*) 13 (*b*) 14 (*c*) 15 (*d*) 16

(*Kerala 2004*)

2. If $n!$, $3 \times (n!)$ and $(n + 1)!$ are in G.P, then the value of n will be

(*a*) 3 (*b*) 4 (*c*) 8 (*d*) 10

(*NDA/NA 2011*)

3. If 1, x, y, z, 16 are in G.P., then what is the value of $x + y + z$?

(*a*) 8 (*b*) 12 (*c*) 14 (*d*) 16

(*NDA/NA 2008*)

4. The geometric mean of 1, 2, 2^2, ... , 2^n is

(*a*) $2^{n/2}$ (*b*) $n^{(n+1)/2}$ (*c*) $2^{(n-1)/2}$ (*d*) $2^{n(n+1)/2}$

(*MPPET 2009*)

5. If the first term of a G.P. is 729 and its 7th term is 64, then the sum of the first seven terms is

(*a*) 2187 (*b*) 2059 (*c*) 1458 (*d*) 2123

(*Kerala 2013*)

6. If the third term of a G.P. is 3, then the product of its first 5 terms is:

(*a*) 15 (*b*) 81

(*c*) 243 (*d*) Cannot be determined

(*J & K CET 2009*)

7. In a G.P, $t_2 + t_5 = 216$ and $t_4 : t_6 = 1 : 4$ and all the terms are integers, then its first term is

(*a*) 16 (*b*) 14 (*c*) 12 (*d*) None of these

(*AMU 2010*)

8. The sum of the first *n* terms of the series

$\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+......$ is

(*a*) $2^n - n - 1$ (*b*) $1 - 2^{-n}$

(*c*) $n + 2^{-n} - 1$ (*d*) $2^n - 1$ **(*AMU 2003*)**

9. In a geometric progression consisting of positive terms, each term equals the sum of next two terms. Then, the common ratio of the progression equals

(*a*) $\frac{\sqrt{5}}{2}$ (*b*) $\sqrt{5}$ (*c*) $\frac{\sqrt{5}-1}{2}$ (*d*) $\frac{\sqrt{5}+1}{2}$

(*AIEEE 2007*)

10. If in an infinite G.P, the first term equal to twice the sum of all the successive terms, then the common of this G.P. is

(*a*) $\frac{1}{3}$ (*b*) $\frac{1}{4}$ (*c*) $\frac{2}{5}$ (*d*) $\frac{2}{3}$

(*Rajasthan 2002*)

Level–2

11. Consider an infinite geometric series with first term *a* and common ratio *r*. If its sum is 4 and the second term is $\frac{3}{4}$, then

(*a*) $a = 2, r = \frac{1}{2}$ (*b*) $a = 2, r = \frac{3}{8}$

(*c*) $a = 1, r = \frac{3}{4}$ (*d*) None of these

(*AMU 2013*)

12. If x, $2x + 2$, $3x + 3$, are the first three terms of a G.P, then the fourth term is

(*a*) $\frac{-27}{2}$ (*b*) $\frac{27}{2}$ (*c*) $\frac{-33}{2}$ (*d*) $\frac{33}{2}$

(*NDA/NA 2009, Rajasthan PET 2005*)

13. Six positive numbers are in G.P, such that their product is 1000. If the fourth term is 1, then the last term is

(*a*) 1000 (*b*) 100 (*c*) $\frac{1}{100}$ (*d*) $\frac{1}{1000}$

(*WBJEE 2013*)

14. If *a* is the A.M. of *b* and *c* and the two geometric means are G_1 and G_2, then $G_1^3 + G_2^3$ is equal to

(*a*) $\frac{abc}{2}$ (*b*) abc (*c*) $2abc$ (*d*) $\frac{3}{2}abc$

(*BCECE 2009, IIT 1997*)

15. If *a*, *b*, *c* are unequal numbers such that *a*, *b*, *c* are in A.P. and $b - a$, $c - b$, a are in G.P, then $a : b : c$ is

(*a*) 1 : 2 : 3 (*b*) 1 : 2 : 4 (*c*) 1 : 3 : 4 (*d*) 2 : 3 : 4

(*AMU 2005*)

16. What is the sum of the 100 terms of the series

9 + 99 + 999 + ...?

(*a*) $\frac{10}{9}(10^{100} - 1) - 100$ (*b*) $\frac{10}{9}(10^{99} - 1) - 100$

(*c*) $100\,(100^{10} - 1)$ (*d*) $\frac{9}{100}(10^{100} - 1)$

(*NDA/NA 2008*)

17. The sum of the first 20 terms of the sequence

0.7, 0.77, 0.777, ... is

(*a*) $\frac{7}{81}(179 - 10^{-20})$ (*b*) $\frac{7}{9}(99 - 10^{-20})$

(*c*) $\frac{7}{81}(179 + 10^{-20})$ (*d*) $\frac{7}{9}(99 + 10^{-20})$

(*IIT JEE 2013*)

18. If $\frac{2}{3} = \left(x - \frac{1}{y}\right) + \left(x^2 - \frac{1}{y^2}\right) + ...$ to ∞ and $xy = 2$, then the value of *x* and *y* under the condition $x < 1$ are

(*a*) $x = \frac{1}{3}, y = 6$ (*b*) $x = \frac{1}{2}, y = 4$

(*c*) $x = \frac{1}{4}, y = 8$ (*d*) $x = \frac{1}{6}, y = 12$

19. The first two terms of a geometric progression add upto 12. The sum of the third and fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then the first term is

(*a*) – 4 (*b*) – 12 (*c*) 12 (*d*) 4

(*AIEEE 2008*)

20. If 64, 27 and 36 are the *P*th, *Q*th and *R*th terms of a G.P, then $P + 2Q$ is equal to

(*a*) R (*b*) $2R$ (*c*) $3R$ (*d*) $4R$

(*WBJEE 2012*)

Level–3

21. In a sequence of 21 terms, the first 11 terms are in A.P. with common difference 2 and the last 11 terms are in G.P with common ratio 2. If the middle term of A.P. be equal to the middle term of G.P., then the middle term of the entire sequence is

(*a*) $\frac{-10}{31}$ (*b*) $\frac{10}{31}$ (*c*) $\frac{32}{31}$ (*d*) $\frac{-31}{32}$

(*AMU 2009*)

22. If $x = 1 + a + a^2 + ... \infty$ and $y = 1 + b + b^2 + ... \infty$, where a and b are proper fractions, then $1 + ab + a^2b^2 + ... \infty$ equals

(*a*) $\frac{x+y}{x-y}$ (*b*) $\frac{x^2+y^2}{x-y}$ (*c*) $\frac{x^2-y^2}{x+y-1}$ (*d*) $\frac{xy}{x+y-1}$

(*Punjab CET 2007*)

23. The first term of an infinite G.P. is x and its sum is 5. Then,

(*a*) $-10 < x < 0$ (*b*) $0 < x < 10$

(*c*) $0 \le x \le 10$ (*d*) $x > 10$

(*IIT 2004*)

24. If $x > 0$ and $\log_3 x + \log_3 \sqrt{x} + \log_3(\sqrt[4]{x}) + \log_3(\sqrt[8]{x}) + \log_3 \sqrt[16]{x} + ... = 4$, then x equals

(*a*) 1 (*b*) 9 (*c*) 27 (*d*) 81

(*VITEEE 2007*)

25. If a, b, c, d are in G.P, then $(a + b + c + d)^2$ is equal to

(*a*) $(a + b)^2 + (c + d)^2 + 2(b + c)^2$

(*b*) $(a + b)^2 + (c + d)^2 + 2(a + c)^2$

(*c*) $(a + b)^2 + (c + d)^2 + 2(b + d)^2$

(*d*) $(a + b)^2 + (c + d)^2 + (b + c)^2$ **(*Kerala PET 2012*)**

ANSWERS

1. (*a*)	**2.** (*c*)	**3.** (*c*)	**4.** (*a*)	**5.** (*b*)	**6.** (*c*)	**7.** (*c*)	**8.** (*c*)	**9.** (*c*)	**10.** (*a*)
11. (*c*)	**12.** (*a*)	**13.** (*c*)	**14.** (*c*)	**15.** (*a*)	**16.** (*a*)	**17.** (*c*)	**18.** (*b*)	**19.** (*b*)	**20.** (*c*)
21. (*a*)	**22.** (*d*)	**23.** (*b*)	**24.** (*b*)	**25.** (*a*)					

HINTS AND SOLUTIONS

1. Here first term $a = 3$, common ratio $r = \frac{3\sqrt{3}}{3} = \sqrt{3}$

Let the nth term be 2187. Then,

$$T_n = ar^{n-1} = 2187$$

$\Rightarrow$ $3 \times (\sqrt{3})^{n-1} = 2187$

$\Rightarrow$ $(\sqrt{3})^{n-1} = 729$

$\Rightarrow$ $(\sqrt{3})^{n-1} = 3^6 = (\sqrt{3})^{12}$

$\Rightarrow$ $n - 1 = 12 \Rightarrow n = \mathbf{13.}$

2. Given, $n!$, $3 \times n!$ and $(n + 1)!$ are in G.P

$\Rightarrow$ $(3 \times n!)^2 = n! \times (n + 1)!$

($\because$ a, b, c in G.P. $\Rightarrow b^2 = ac$)

$\Rightarrow$ $9 \times (n!)^2 = n! \times (n + 1)!$

$\Rightarrow$ $9 \times n! = (n + 1)!$

$\Rightarrow$ $9n! = (n + 1) . n!$ ($\because$ $n! = 1.2.3. ...... n$)

$\Rightarrow$ $n + 1 = 9 \Rightarrow n = \mathbf{8.}$

3. 1, x, y, z, 16 are in G.P.

$\therefore$ $a = 1, n = 5, T_5 = ar^4 = 16$

$\Rightarrow$ $r^4 = 16 \Rightarrow r = 2$

$\therefore$ $x = ar = 2, y = ar^2 = 4, z = ar^3 = 8$

$\therefore$ $x + y + z = 2 + 4 + 8 = \mathbf{14.}$

4. In the G.P, 1, 2, 2^2, ..., 2^n, there are $(n + 1)$ terms.

$\therefore$ Geometric mean of this G.P. $= (1 \times 2 \times 2^2 \times ..... \times 2^n)^{\frac{1}{n+1}}$

$$= (2^{1+2+3+.....+n})^{\frac{1}{n+1}}$$

$$= \left(2^{\frac{n}{2}(n+1)}\right)^{\frac{1}{n+1}} = \mathbf{2^{n/2}.}$$

5. Let the first term and common ratio of the G.P. be a and r respectively.

Then, $T_1 = a = 729$, $T_7 = ar^6 = 64$

$$\therefore \frac{ar^6}{a} = \frac{64}{729} \Rightarrow r^6 = \frac{64}{729} = \frac{2^6}{3^6} \Rightarrow r = \frac{2}{3}.$$

$$\therefore S_7 = \frac{a(1-r^n)}{1-r} = \frac{729\left(1-\left(\frac{2}{3}\right)^7\right)}{1-2/3}$$

$$= \frac{729\left(1-\frac{128}{2187}\right)}{\frac{1}{3}} = 2187\left(\frac{2187-128}{2187}\right) = \mathbf{2059.}$$

6. Let a and r be the first term and common ratio respectively of the given G.P. Then,

$$T_3 = ar^2 = 3$$

Required product $= a . ar . ar^2 . ar^3 . ar^4$

$= a^5 r^{15} = (ar^3)^5 = 3^5 = \mathbf{243.}$

7. Let a and r be the first term and common ratio respectively of the given G.P. Then, $t_2 = ar$, $t_4 = ar^3$, $t_5 = ar^4$, $t_6 = ar^5$

Given, $t_2 + t_5 = ar + ar^4 = 216$ ...(*i*)

and $\frac{t_4}{t_6} = \frac{ar^3}{ar^5} = \frac{1}{4} \Rightarrow \frac{1}{r^2} = \frac{1}{4} \Rightarrow \frac{1}{r} = \frac{1}{2}$ ...(*ii*)

Now putting $r = 2$, in (*i*) we get

$2a + 16a = 216 \Rightarrow 18a = 216 \Rightarrow a = \mathbf{12.}$

8. Let $S_n = \frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + ...$ to n terms

$= \left(1-\frac{1}{2}\right) + \left(1-\frac{1}{4}\right) + \left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right) + ...$ to n terms

$= (1 + 1 + 1 + 1 + ...$ to n terms$)$

$- \left\{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} .....\text{ to } n \text{ terms}\right\}$

$$= n - \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = n - (1 - 2^{-n}) = \mathbf{n + 2^{-n} = 1.}$$

9. Let the G.P. be $a, ar, ar^2, ...$

As all the terms of the given G.P. are positive, $a > 0, r > 0$.

Given, $a = ar + ar^2$

$\Rightarrow \; ar^2 + ar - a = 0$

$\Rightarrow \; r^2 + r - 1 = 0.$

$\therefore \; r = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

$\Rightarrow \; r = \mathbf{\frac{\sqrt{5}-1}{2}}.$ $\left(\because \frac{-1-\sqrt{5}}{2} \text{ is a negative quantity and } r > 0\right)$

10. Let the given G.P. be $a, ar, ar^2, ar^3, ... , \infty$.

As the given infinite G.P. has a finite sum, $|r| < 1$

Also, given $a = 2(ar + ar^2 + ar^3 + ... + \infty)$

$\Rightarrow \; a = 2\left(\frac{ar}{1-r}\right) \quad \left(\because S_\infty = \frac{a}{1-r}. \text{ Here } a = ar, r = r\right)$

$\Rightarrow a - ar = 2ar$

$\Rightarrow \; 1 - r = 2r \; \Rightarrow \; 3r = 1 \; \Rightarrow \; r = \mathbf{\frac{1}{3}}.$

11. Sum of an infinite G.P $= \frac{a}{1-r}$, where first term $= a$, common ratio $= r$ and $|r| < 1$

Given, $\frac{a}{1-r} = 4$ and $ar = \frac{3}{4}$

$\Rightarrow \; a = 4 - 4r$ and $a = \frac{3}{4r}$

$\Rightarrow \; \frac{3}{4r} = 4 - 4r \; \Rightarrow \; 3 = 16r - 16r^2$

$\Rightarrow \; 16r^2 - 16r + 3 = 0$

$\Rightarrow (4r - 3)(4r - 1) = 0$

$\Rightarrow \; 4r = 3$ or $4r = 1 \; \Rightarrow \; r = \frac{3}{4}$ or $\frac{1}{4}$

Now when $r = \frac{3}{4}, \; a = \frac{3}{4 \times \frac{3}{4}} = 1$

$r = \frac{1}{4}, \; a = \frac{3}{4 \times \frac{1}{4}} = 3$

$\therefore \; (a, r) = \mathbf{\left(1, \frac{3}{4}\right) \text{ or } \left(3, \frac{1}{4}\right)}.$

12. $x, 2x + 2, 3x + 3$, are in G.P.

$\Rightarrow \; (2x + 2)^2 = x(3x + 3)$

$\Rightarrow \; 4x^2 + 8x + 4 = 3x^2 + 3x$

$\Rightarrow \; x^2 + 5x + 4 = 0$

$\Rightarrow \; (x + 4)(x + 1) = 0$

$\Rightarrow \; x = -1$ or -4

Let the fourth term of the given G.P. be a. Then,

$r = \frac{a}{3x+3} = \frac{2x+2}{x}$

$\Rightarrow \; a = \frac{(2x+2)(3x+3)}{x}$

When $x = -4$, Fourth term $a = \frac{(-8+2)(-24+3)}{-4}$

$= \frac{-6 \times -21}{-4} = -\frac{27}{2}$

When $x = -1, \; a = \frac{(-2+2)(-3+3)}{-1} = \mathbf{0.}$

$\therefore$ Fourth term $= \mathbf{-\frac{27}{2}}.$

13. Let the six numbers in G.P. be $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$

Given, $\frac{a}{r^5} \times \frac{a}{r^3} \times \frac{a}{r} \times ar \times ar^3 \times ar^5 = 1000$

$\Rightarrow \; a^6 = 1000 \; \Rightarrow \; a = \sqrt{10}$

Given, $T_4 = ar = 1 \; \Rightarrow \; \sqrt{10}r = 1 \; \Rightarrow \; r = \frac{1}{\sqrt{10}}$

$\therefore$ Last term of G.P. $= ar^5 = \sqrt{10} \times \frac{1}{(\sqrt{10})^5} = \mathbf{\frac{1}{100}}.$

14. a is the A.M of b and $c \Rightarrow 2a = b + c$

Given, G_1 and G_2 are G.Ms between b and c

$\Rightarrow \; b, G_1, G_2, c$ are in G.P.

Let r be the common ratio of the G.P.

$\Rightarrow \; G_1 = br, G_2 = br^2, c = br^3$

Now $c = br^3 \; \Rightarrow \; r^3 = \frac{c}{b} \; \Rightarrow \; r = \left(\frac{c}{b}\right)^{\frac{1}{3}}$

$\therefore \; G_1 = b \times \frac{c^{1/3}}{b^{1/3}} = b^{2/3} c^{1/3} = (b^2c)^{1/3}$

$G_2 = b \times \left(\frac{c}{b}\right)^{\frac{2}{3}} = \frac{b \times c^{2/3}}{b^{2/3}} = b^{1/3} c^{2/3} = (bc^2)^{1/3}$

Now $G_1^{\,3} + G_2^{\,3} = \left((b^2c)^{1/3}\right)^3 + \left((bc^2)^{1/3}\right)^3$

$= b^2c + bc^2 = bc(b + c) = bc \cdot 2a = \mathbf{2abc.}$

15. a, b, c are in A.P. $\Rightarrow \; b - a = c - b$ ...(*i*)

$(b - a), (c - b), a$ are in G.P $\Rightarrow \; (c - b)^2 = (b - a)a$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*)

$\Rightarrow \; (b - a)^2 = (b - a)a$

$\Rightarrow \; (b - a)[(b - a) - a] = 0$

$\Rightarrow \; b - a = 0$ or $b - 2a = 0$

$\Rightarrow \; b = 2a$

($\because$ a and b are distinct and $b - a \neq 0$)

a, b, c are in A.P. $\Rightarrow \; 2b = a + c$

$\Rightarrow \; 4a = a + c \; \Rightarrow \; c = 3a.$

$\therefore \; a : b : c = a : 2a : 3a = \mathbf{1 : 2 : 3.}$

16. Let $S_{100} = 9 + 99 + 999 + \ldots\ldots$ upto 100 terms

$= (10 - 1) + (100 - 1) + (1000 - 1) + \ldots\ldots +$ upto 100 terms

$= (10 + 10^2 + 10^3 + \ldots.$ upto 100 terms$) - (1 + 1 + 1 + \ldots\ldots$ upto 100 terms$)$

$$= \frac{10\,(10^{100} - 1)}{10 - 1} - 100$$

$$\left(\because\ S_n = \frac{a\,(r^n - 1)}{r - 1} \text{ when } r > 1\right)$$

$$= \mathbf{\frac{10}{9}\,(10^{100} - 1) - 100}.$$

17. Let $S_{20} = 0.7 + 0.77 + 0.777 + \ldots\ldots$ upto 20 terms

$= 7(0.1 + 0.11 + 0.111 + \ldots\ldots$ upto 20 terms$)$

$= \frac{7}{9}(0.9 + 0.99 + 0.999 + \ldots\ldots$ upto 20 terms$)$

$= \frac{7}{9}[(1 - 0.1) + (1 - 0.01) + (1 - 0.001) +$ upto 20 terms$)$

$= \frac{7}{9}[(1 + 1 + 1 + \ldots\ldots$ upto 20 terms$) - (0.1 + 0.01 + 0.001 + \ldots\ldots$ upto 20 terms$)]$

$$= \frac{7}{9}\left[20 - \frac{0.1\,\{1 - (0.1)^2\}}{(1 - 0.1)}\right]$$

$$\left(\because\ S_n = \frac{a\,(1 - r^n)}{1 - r}, \text{ when } r < 1\right)$$

$$= \frac{7}{9}\left[20 - \frac{1}{9}\left(1 - \left(\frac{1}{10}\right)^{20}\right)\right] = \frac{7}{9}\left[20 - \frac{1}{9} + \frac{10^{-20}}{9}\right]$$

$$= \frac{7}{9}\left[\frac{179 + 10^{-20}}{9}\right] = \mathbf{\frac{7}{81}\,(179 + 10^{-20})}.$$

18. Given $\left(x - \frac{1}{y}\right) + \left(x^2 - \frac{1}{y^2}\right) + \ldots\ldots \text{ to } \infty = \frac{2}{3}$

$\Rightarrow\ (x + x^2 + x^3 + \ldots. \text{ to } \infty) - \left(\frac{1}{y} + \frac{1}{y^2} + \ldots\ldots \text{ to } \infty\right) = \frac{2}{3}$

$\Rightarrow\ \frac{x}{1 - x} - \frac{\frac{1}{y}}{1 - \frac{1}{y}} = \frac{2}{3} \quad \Rightarrow \quad \frac{x}{1 - x} - \frac{1}{y - 1} = \frac{2}{3}$

$\Rightarrow\ \frac{\frac{2}{y}}{1 - \frac{2}{y}} - \frac{1}{y - 1} = \frac{2}{3} \qquad (\because\ xy = 2 \ \Rightarrow\ x = 2/y)$

$\Rightarrow\ \frac{2}{y - 2} - \frac{1}{y - 1} = \frac{2}{3} \quad \Rightarrow \quad \frac{2y - 2 - y + 2}{y^2 - 3y + 2} = \frac{2}{3}$

$\Rightarrow\ 3y = 2(y^2 - 3y + 2)$

$\Rightarrow\ 2y^2 - 9y + 4 = 0 \quad \Rightarrow \quad 2y^2 - 8y - y + 4 = 0$

$\Rightarrow\ (2y - 1)(y - 4) = 0.$

$\Rightarrow\ y = \frac{1}{2}$ or $y = 4$

When $y = \frac{1}{2}$, $x = 4$ and $y = 4$, $x = \frac{1}{2}$

$\because\ x < 1 \quad \therefore\ \mathbf{y = 4, x = \frac{1}{2}}$ **is true.**

19. Let a and r be the first term and common ratio respectively of the given G.P.

Then $\left.\begin{aligned} a + ar &= 12 \\ ar^2 + ar^3 &= 48 \end{aligned}\right\}$ ...(*i*) ...(*ii*)

$\Rightarrow\ \frac{ar^2\,(1 + r)}{a\,(1 + r)} = \frac{48}{12}$ (Dividing (*ii*) by (*i*))

$\Rightarrow\ r^2 = 4 \ \Rightarrow\ r \pm 2 \ \Rightarrow\ r = -2$

as the terms of the G.P. are alternately positive and negative.

Now $a\,(1 + r) = 12 \ \Rightarrow\ a\,(1 - 2) = 12 \ \Rightarrow\ \mathbf{a = -12.}$

20. Let a and r be the first term and common ratio respectively of the given G.P.

Pth term of G.P $= ar^{P-1} = 64 = 2^6$ ...(*i*)

Qth term of G.P $= ar^{Q-1} = 27 = 3^3$ ...(*ii*)

Rth term of G.P $= ar^{R-1} = 36 = 2^2 . 3^2$ ...(*iii*)

Now from (*i*), $2 = a^{1/6}\, r^{(P-1)/6}$ ...(*iv*)

From (*ii*), $3 = a^{1/3}\, r^{(Q-1)/3}$ ...(*v*)

From (*iii*), $2.3 = a^{1/2}\, r^{(R-1)/2}$ ...(*vi*)

$\therefore$ From (*iv*), (*v*) and (*vi*) we have

$\Rightarrow\ a^{\frac{1}{6}}\, r^{\frac{P-1}{6}} . a^{\frac{1}{3}} . r^{\frac{Q-1}{3}} = a^{\frac{1}{2}}\, r^{\frac{R-1}{2}}$

$\Rightarrow\ a^{\frac{1}{6}+\frac{1}{3}}\, r^{\frac{P-1}{6}+\frac{Q-1}{3}} = a^{\frac{1}{2}}\, r^{\frac{R-1}{2}}$

$\Rightarrow\ a^{\frac{1}{2}}\, r^{\frac{P-1+2Q-2}{6}} = a^{\frac{1}{2}}\, r^{\frac{R-1}{2}}$

$\Rightarrow\ \frac{P - 1 + 2Q - 2}{6} = \frac{R - 1}{2}$

$\Rightarrow\ P + 2Q - 3 = 3R - 3$

$\Rightarrow\ \mathbf{P + 2Q = 3R.}$

21. Let the first term of the A.P. be a and common difference d.

Given $d = 2 \ \Rightarrow\ T_{11}$ of A.P. $= a + 10d = a + 20.$

Let the first term of the G.P. be b and common ratio r.

Given $r = 2.$

Now, the middle term of A.P = middle term of G.P

$\Rightarrow\ T_6$ of A.P $= a + 5d = T_6$ of G.P $= br^5$

$\Rightarrow\ a + 5d = br^5$

$\Rightarrow\ a + 10 = 32b \qquad (\because\ r = 2)$ ...(*i*)

Also the last term of A.P. is the first term of G.P.

$\therefore\ b = T_{11}$ of A.P $= a + 20$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*)

$a + 10 = 32.(a + 20)$

$\Rightarrow\ 31a = -630 \ \Rightarrow\ a = \frac{-630}{31}$

$\therefore$ Middle term of the entire sequence of 21 terms = 11th term

$$= a + 10d$$

$$= \frac{-630}{31} + 20 = \frac{-630+620}{31} = \mathbf{\frac{-10}{31}}.$$

22. Since a and b are proper fractions, $|a| < 1, |b| < 1$

$$\therefore \quad x = 1 + a + a^2 + \ldots\ldots \infty = \frac{1}{1-a} \quad \left(\because S_\infty = \frac{a}{1-r}\right)$$

and $\quad y = 1 + b + b^2 + \ldots\ldots \infty = \dfrac{1}{1-b}$

Also, $\quad 1 + ab + a^2b^2 + \ldots\ldots \infty = \dfrac{1}{1-ab}$ ...(*i*)

Now $\quad x = \dfrac{1}{1-a} \Rightarrow x - xa = 1$

$\Rightarrow \quad xa = x - 1 \Rightarrow a = \dfrac{x-1}{x}$ ...(*ii*)

$\quad y = \dfrac{1}{1-b} \Rightarrow y - yb = 1$

$\Rightarrow \quad yb = y - 1 \Rightarrow b = \dfrac{y-1}{y}$ ...(*iii*)

$\therefore$ Putting the values of a & b from (*ii*) and (*iii*) in (*i*), we get

$$\text{Reqd. sum} = \frac{1}{1-ab} = \frac{1}{1-\left(\frac{x-1}{x}\right)\left(\frac{y-1}{y}\right)}$$

$$= \frac{xy}{xy-(xy-x-y+1)} = \mathbf{\frac{xy}{x+y-1}}.$$

23. Let the common ratio of the given G.P be r. Then,

$$S_\infty = \frac{x}{1-r} \Rightarrow 5 = \frac{x}{1-r} \Rightarrow 5 - 5r = x$$

$$\Rightarrow \quad r = \frac{5-x}{5} = 1 - \frac{x}{5}$$

$\because$ Sum to infinity of the given series is a finite quantity, $|r| < 1$.

$$\therefore \quad \left|1-\frac{x}{5}\right| < 1 \Rightarrow -1 < \left(1-\frac{x}{5}\right) < 1$$

$$\Rightarrow \quad -1 < (x/5 - 1) < 1$$

(Multiplying the inequality by (– 1))

$$\Rightarrow \quad 0 < \frac{x}{5} < 2 \Rightarrow \mathbf{0 < x < 10.}$$

24. $\log_3 x + \log_3 \sqrt{x} + \log_3 \sqrt[4]{x} + + \log_3 \sqrt[8]{x} + \ldots\ldots \infty = 4$

$\Rightarrow \log_3 x + \log_3 x^{1/2} + \log_3 x^{1/4} + \log_3 (x^{1/8}) + \ldots\ldots \infty = 4$

$\Rightarrow \log_3 (x \,.\, x^{1/2} \,.\, x^{1/4} \,.\, x^{1/8} \ldots\ldots \infty) = 4$

$\Rightarrow \log_3 (x^{1 + 1/2 + 1/4 + 1/8} \ldots\ldots \infty) = 4$

$\Rightarrow \quad \log_3 x^{\frac{1}{1-\frac{1}{2}}} = 4 \qquad (\because S_\infty = a/1-r)$

$\Rightarrow \log_3 x^2 = 4 \Rightarrow x^2 = 3^4 = (3^2)^2 \Rightarrow x = \mathbf{9}.$

25. a, b, c, d are in G.P.

$\Rightarrow b = ar, c = ar^2, d = ar^3$, where r is the common ratio of the G.P.

$\text{LHS} = (a + b + c + d)^2 = (a + ar + ar^2 + ar^3)^2$

$= \{a(1 + r + r^2 + r^3)\}^2 = \{a((1 + r) + r^2(1 + r))\}^2$

$= \{a(1 + r)(1 + r^2)\}^2 = a^2(1 + r)^2(1 + r^2)^2$

$= a^2(1 + r)^2(r^4 + 2r^2 + 1)$

$= a^2(r^4(1 + r)^2 + 2r^2(1 + r)^2 + (1 + r)^2)$

$= a^2r^4(1 + r)^2 + 2a^2r^2(1 + r)^2 + a^2(1 + r)^2$

$= (ar^2 + ar^3)^2 + 2(ar + ar^2)^2 + (a + ar)^2$

$= \mathbf{(c + d)^2 + 2(b + c)^2 + (a + b)^2}.$

HARMONIC PROGRESSION (H.P.)

KEY FACTS

1. A sequence of numbers is said to form a **Harmonic Progression,** ***when the reciprocals of the numbers form an Arithmetic progression.***

   Ex. (*i*) $1 + 2 + 3 + 4 + \ldots\ldots$ is an A.P

   $\Rightarrow 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots$ is an H.P.

   (*ii*) $a + (a + d) + (a + 2d) + (a + 3d) \ldots\ldots$ is an A.P.

   $\Rightarrow \frac{1}{a} + \frac{1}{(a+d)} + \frac{1}{(a+2d)} + \frac{1}{(a+3d)} + \ldots\ldots$ is an H.P.

2. ***n*th term of an H.P.**

   The nth term of an H.P. is the reciprocal of the nth term of the A.P. formed by the reciprocals of the terms of the H.P.

   If the given H.P. is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots\ldots$, then its nth term is $\mathbf{\frac{1}{a+(n-1)d}}$.

Ex. Find the 7th term of the H.P. $\frac{2}{13}, \frac{1}{6}, \frac{2}{11}, ...$.

The reciprocals of the terms of the given H.P, *i.e.*, $\frac{13}{2}, 6, \frac{11}{2}, ...$ form an A.P with first term = $\frac{13}{2}$ and common difference = $6 - \frac{13}{2} = -\frac{1}{2}$.

$\therefore$ 7th term of this A.P. = $a + 6d = \frac{13}{2} + \left(-\frac{1}{2}\right) \times 6 = \frac{7}{2}$.

Hence 7th of the given H.P. = $\frac{1}{7/2} = \mathbf{\frac{2}{7}}$.

3. Harmonic Mean

If *a, H*, *b* are three quantities in H.P., then **H** is said to be the **Harmonic Mean** between *a* and *b*.

a, *H*, *b* are in H.P

$$\Rightarrow \frac{1}{a}, \frac{1}{H}, \frac{1}{b} \text{ are in A.P.}$$

$$\Rightarrow \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H} \Rightarrow \frac{2}{H} = \frac{1}{b} + \frac{1}{a} \Rightarrow \frac{2}{H} = \frac{a+b}{ab} \Rightarrow \boldsymbol{H = \frac{2ab}{a+b}}.$$

Ex. The **Harmonic Mean** between 3 and – 5 is:

$$H = \frac{2 \times 3 \times (-5)}{3 + (-5)} = \frac{-30}{-2} = \mathbf{15.}$$

4. If $a_1, a_2, a_3, a_4, ... a_n$ are *n* non-zero numbers in H.P., then their

$$\text{Harmonic Mean} = \frac{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + ... + \frac{1}{a_n}\right)}{n}$$

5. Relations between the three series, *i.e.*, A.P, G.P and H.P.

(*a*) Three numbers *a*, *b*, *c* will be in A.P, G.P or in H.P. according as $\frac{a-b}{b-c} = \frac{a}{a}, \frac{a-b}{b-c} = \frac{a}{b}, \frac{a-b}{b-c} = \frac{a}{c}$ respectively.

(*i*) $\frac{a-b}{b-c} = \frac{a}{a} \Rightarrow a^2 - ba = ab - ac \Rightarrow 2ab = a^2 + ac \Rightarrow \mathbf{2b = a + c} \Rightarrow$ ***a*, *b*, *c* are in A.P.**

(*ii*) $\frac{a-b}{b-c} = \frac{a}{b} \Rightarrow ab - b^2 = ab - ca \Rightarrow \mathbf{b^2 = ac} \Rightarrow$ ***a*, *b*, *c* are in G.P.**

(*iii*) $\frac{a-b}{b-c} = \frac{a}{c} \Rightarrow ac - bc = ab - ac \Rightarrow 2ac = ab + bc \Rightarrow \boldsymbol{b = \frac{2ac}{a+c}} \Rightarrow$ ***a*, *b*, *c* are H.P.**

(*b*) If *A*, *G*, *H* are respectively the Arithmetic, the Geometric and the Harmonic means between any two unequal positive numbers, then

(*i*) ***A*, *G*, *H* are in G.P** (*ii*) ***A* > *G* > *H*.**

(*i*) Let the two positive numbers be *a* and *b*. Then,

$$A = \frac{a+b}{2}, \; G = \sqrt{ab}, \; H = \frac{2ab}{a+b}$$

Then $$A \times H = \left(\frac{a+b}{2}\right) \times \frac{2ab}{a+b} = ab = G^2$$

$\Rightarrow$ $G = \sqrt{AH} \Rightarrow$ ***A, G, H* are in G.P.**

(*ii*) $A - G = \dfrac{a+b}{2} - \sqrt{ab} = \dfrac{a+b-2\sqrt{ab}}{2} = \dfrac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2} > 0 \Rightarrow A - G > 0 \quad \Rightarrow \quad A > G$

Also $\quad G - H = \sqrt{ab} - \dfrac{2ab}{a+b} = \dfrac{(a+b)\sqrt{ab} - 2ab}{(a+b)} = \dfrac{\sqrt{ab}\,(a+b-2\sqrt{ab})}{a+b} = \dfrac{\sqrt{ab}}{a+b}\left(\sqrt{a}-\sqrt{b}\right)^2 > 0$

$\Rightarrow \quad G - H > 0 \quad \Rightarrow \quad G > H$

$\therefore \quad$ ***A > G > H.***

SOLVED EXAMPLES

Ex. 1. The third and seventh terms of an H.P are $\dfrac{7}{11}$ and $\dfrac{7}{31}$. Find the first term and the eighth term.

Sol. Third and seventh terms of an H.P are $\dfrac{7}{11}$ and $\dfrac{7}{31}$

$\Rightarrow$ Third and seventh terms of an A.P are $\dfrac{11}{7}$ and $\dfrac{31}{7}$

Let the first term of the A.P. be *a* and common difference *d*.

Then, $\quad T_3 = a + 2d = \dfrac{11}{7} \quad$ ...(*i*)

$T_7 = a + 6d = \dfrac{31}{7} \quad$ ...(*ii*)

$(ii) - (i) \quad \Rightarrow \quad 4d = \dfrac{20}{7} \quad \Rightarrow \quad d = \dfrac{5}{7}$

$\therefore$ From (*i*) $\quad a + \dfrac{10}{7} = \dfrac{11}{7} \quad \Rightarrow \quad a = \dfrac{1}{7}$

$$T_8 = a + 7d = \frac{1}{7} + 7\times\frac{5}{7} = \frac{1}{7} + 5 = \frac{36}{7}$$

$\therefore$ The first and eighth terms of the H.P. are respectively the reciprocals of the first and eighth terms of the A.P, *i.e.* **7 and $\dfrac{7}{36}$.**

Ex. 2. Insert three harmonic means between 5 and 6.

Sol. 3 harmonic means between 5 and 6

$\Rightarrow$ 3 arithmetic means between $\dfrac{1}{5}$ and $\dfrac{1}{6}$

Let A_1, A_2, A_3 be the arithmetic means between $\dfrac{1}{5}$ and $\dfrac{1}{6}$.

Then, $\dfrac{1}{5}, A_1, A_2\ A_3, \dfrac{1}{6}$ form an A.P,

where $t_1 = a = \dfrac{1}{5},\ t_5 = a + 4d = \dfrac{1}{6}$

$\therefore \quad \dfrac{1}{5} + 4d = \dfrac{1}{6} \quad \Rightarrow \quad 4d = \dfrac{1}{6} - \dfrac{1}{5} = -\dfrac{1}{30} \quad \Rightarrow \quad d = -\dfrac{1}{120}$

$\therefore \quad A_1 = a + d = \dfrac{1}{5} + \left(-\dfrac{1}{120}\right) = \dfrac{23}{120}$

$$A_2 = a + 2d = \frac{1}{5} + 2\times\left(-\frac{1}{120}\right) = \frac{1}{5} - \frac{1}{60} = \frac{11}{60}$$

$$A_3 = a + 3d = \frac{1}{5} + 3 \times \left(-\frac{1}{120}\right) = \frac{1}{5} - \frac{1}{40} = \frac{7}{40}$$

$\therefore$ Required harmonic means are $\mathbf{\frac{120}{23}, \frac{60}{11}, \frac{40}{7}}$.

Ex. 3. If the *m*th term of an H.P. is *n* and *n*th term is *m*, show that the *r*th term is $\frac{mn}{r}$.

Sol. Let the corresponding A.P. be a, $a + d$, $a + 2d$, ...

Since the *m*th *term* and *nth term* of the H.P. are n and m respectively, then for the A.P.,

$$m\text{th term} = a + (m - 1)\, d = \frac{1}{n} \quad ...(i)$$

$$n\text{th term} = a + (n - 1)\, d = \frac{1}{m} \quad ...(ii)$$

$(ii) - (i) \Rightarrow (n - 1)d - (m - 1)d = \frac{1}{m} - \frac{1}{n}$

$\Rightarrow \quad (n - m)d = \frac{n - m}{mn} \Rightarrow d = \frac{1}{mn}$

Putting in (i), $a + (m - 1) \times \frac{1}{mn} = \frac{1}{n}$

$\Rightarrow \quad a = \frac{1}{n} - \frac{1}{n} + \frac{1}{mn} = \frac{1}{mn}$

$\therefore \quad t_r = a + (r - 1)d = \frac{1}{mn} + (r - 1)\frac{1}{mn} = \frac{r}{mn}$

$\therefore$ rth term of H.P. = $\mathbf{\frac{mn}{r}}$.

Ex. 4. If *H* be the harmonic mean between *x* and *y*, then prove that $\frac{H+x}{H-x} + \frac{H+y}{H-y} = 2$.

Sol. H being the H.M. between x and y

$\Rightarrow \quad H = \frac{2xy}{x + y} \Rightarrow \frac{H}{x} = \frac{2y}{x + y}$ and $\frac{H}{y} = \frac{2x}{x + y}$

$\Rightarrow \quad \frac{H + x}{H - x} = \frac{2y + x + y}{2y - (x + y)}$ and $\frac{H + y}{H - y} = \frac{2x + x + y}{2x - (x + y)}$

(Using Componendo and Dividendo)

$\Rightarrow \quad \frac{H + x}{H - x} = \frac{3y + x}{y - x}$ and $\frac{H + y}{H - y} = \frac{3x + y}{x - y}$

$\therefore \quad \frac{H + x}{H - x} + \frac{H + y}{H - y} = \frac{3y + x}{y - x} + \frac{3x + y}{x - y} = \frac{3y + x - 3x - y}{y - x} = \frac{2\,(y - x)}{y - x} = \mathbf{2.}$

Ex. 5. In an H.P., *p*th term is *qr* and *q*th term is *pr*, show that *r*th term is *pq*.

Sol. T_p of H.P $= qr \Rightarrow T_p$ of A.P. $= \frac{1}{qr} \Rightarrow a + (p - 1)d = \frac{1}{qr}$ ...(i)

Where a and d are the first term and common difference respectively of the A.P.

Also T_q of H.P $= pr \Rightarrow T_q$ of A.P $= \frac{1}{pr} \Rightarrow a + (q - 1)d = \frac{1}{pr}$ ...(ii)

Eqn (*ii*) – Eqn (*i*) $\Rightarrow$ $(q-1)d-(p-1)d=\dfrac{1}{pr}-\dfrac{1}{qr}$ $\Rightarrow$ $(q-p)d=\dfrac{q-p}{pqr}$ $\Rightarrow$ $d=\dfrac{1}{pqr}$

$\Rightarrow$ $a+(p-1)\dfrac{1}{pqr}=\dfrac{1}{qr}$ $\Rightarrow$ $a=\dfrac{1}{qr}-(p-1)\dfrac{1}{pqr}=\dfrac{1}{qr}-\dfrac{1}{qr}+\dfrac{1}{pqr}=\dfrac{1}{pqr}.$

$\therefore$ T_r of A.P $=a+(r-1)d=\dfrac{1}{pqr}+(r-1)\dfrac{1}{pqr}=\dfrac{1}{pqr}+\dfrac{1}{pq}-\dfrac{1}{pqr}=\dfrac{1}{pq}$

$\Rightarrow$ T_r of H.P $=\boldsymbol{pq}$**.**

Ex. 6. Let a, b, c be in A.P and $|a|<1$, $|b|<1$ and $|c|<1$. If

$x = 1 + a + a^2 + ...$ to ∞

$y = 1 + b + b^2 + ...$ to ∞

$z = 1 + c + c^2 + ...$ to ∞, then show that x, y, z are in H.P.

Sol. $x = 1 + a + a^2 + ...\text{ to }\infty = \dfrac{1}{1-a}$

$y = 1 + b + b^2 + ...\text{ to }\infty = \dfrac{1}{1-b}$

$z = 1 + c + c^2 + ...\text{ to }\infty = \dfrac{1}{1-c}.$

Now, a, b, c are in A.P.

$\Rightarrow$ $1-a, 1-b, 1-c$ are in A.P.

$\Rightarrow$ $\dfrac{1}{1-a},\dfrac{1}{1-b},\dfrac{1}{1-c}$ are in H.P.

$\Rightarrow$ **x, y, z are in H.P.**

Ex. 7. If $a_1, a_2, a_3, ..., a_n$ are in H.P. then what will $(a_1a_2 + a_2a_3 + ... + a_{n-1}a_n)$ equal to? ***(AMU 2003)***

Sol. $a_1, a_2, a_3 ....., a_n$ are in H.P.

$\Rightarrow$ $\dfrac{1}{a_1},\dfrac{1}{a_2},\dfrac{1}{a_3}.....,\dfrac{1}{a_n}$ are in A.P

If d is the common difference of the A.P., then,

$$\frac{1}{a_2}-\frac{1}{a_1}=d,\frac{1}{a_3}-\frac{1}{a_2}=d,......,\frac{1}{a_n}-\frac{1}{a_{n-1}}=d \Rightarrow \frac{a_1-a_2}{a_1a_2}=d,\frac{a_2-a_3}{a_2a_3}=d,.....,\frac{a_{n-1}-a_n}{a_{n-1}a_n}=d$$

$\Rightarrow$ $a_1-a_2=a_1a_2d,\ a_2-a_3=a_2a_3d,\ .....,\ a_{n-1}-a_n=a_{n-1}a_nd$

$\Rightarrow$ $(a_1-a_2+a_2-a_3+.....+a_{n-1}-a_n)=a_1a_2d+a_2a_3d+.....+a_{n-1}a_nd$

$\Rightarrow$ $(a_1-a_n)=(a_1a_2+a_2a_3+.....+a_{n-1}a_n)d$ ...(*i*)

Also, $\dfrac{1}{a_1},\dfrac{1}{a_2},.....\dfrac{1}{a_n}$ is an A.P with common difference d

$\Rightarrow$ $\dfrac{1}{a_n}=\dfrac{1}{a_1}+(n-1)d$ $\Rightarrow$ $(n-1)d=\dfrac{1}{a_n}-\dfrac{1}{a_1}$ $\Rightarrow$ $(n-1)d=\dfrac{a_1-a_n}{a_1a_n}$

$\Rightarrow$ $a_1-a_n=(n-1)d\,a_1a_n$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*)

$$(n-1)d\,a_1a_n=(a_1a_2+a_2a_3+....+a_{n-1}a_n)\,d$$

$\Rightarrow$ $(a_1a_2+a_2a_3+....+a_{n-1}a_n)=\boldsymbol{(n-1)\,a_1a_n}$**.**

Ex. 8. If $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ are in H.P, show that $b + c, c + a, a + b$ are in H.P.

Sol. We know by the property of H.P, if x, y, z are H.P, $\frac{x-y}{y-z} = \frac{x}{z}$

$\therefore \quad \frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ are in H.P. $\Rightarrow \frac{\frac{1}{a^2} - \frac{1}{b^2}}{\frac{1}{b^2} - \frac{1}{c^2}} = \frac{\frac{1}{a^2}}{\frac{1}{c^2}} \Rightarrow \frac{\frac{b^2 - a^2}{a^2 b^2}}{\frac{c^2 - b^2}{b^2 c^2}} = \frac{c^2}{a^2} \Rightarrow \frac{c^2 (b^2 - a^2)}{a^2 (c^2 - b^2)} = \frac{c^2}{a^2}$

$\Rightarrow \quad b^2 - a^2 = c^2 - b^2 \Rightarrow 2b^2 = c^2 + a^2$ ...(*i*)

Now, for $b + c, c + a, a + b$ to be in H.P

$$\frac{(b+c)-(c+a)}{(c+a)-(a+b)} = \frac{b+c}{a+b} \Rightarrow \frac{b-a}{c-b} = \frac{b+c}{a+b}$$

$\Rightarrow \quad (b - a)(b + a) = (c + b)(c - b)$

$\Rightarrow \quad b^2 - a^2 = c^2 - b^2$ or $2b^2 = c^2 + a^2$, which is true from (*i*)

$\therefore \quad b + c, c + a, a + b$ are in H.P.

Ex. 9. The sum of the reciprocals of three numbers in H.P. is 15 and the product of the numbers is $\frac{1}{105}$. Find the numbers.

Sol. Let the three numbers in H.P. be $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

$\therefore \quad (a - d) + a + (a + d) = 15$

$\Rightarrow 3a = 15 \Rightarrow a = 5$

Also $\quad \frac{1}{(a-d)} \cdot \frac{1}{a} \cdot \frac{1}{(a+d)} = \frac{1}{105} \Rightarrow \frac{1}{a(a^2 - d^2)} = \frac{1}{105} \Rightarrow \frac{1}{5(25 - d^2)} = \frac{1}{105}$

$\Rightarrow \quad 25 - d^2 = 21$

$\Rightarrow \quad d^2 = 4 \Rightarrow d = \pm 2$

Hence the numbers are $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ when $d = 2$

or $\frac{1}{7}, \frac{1}{5}, \frac{1}{3}$ when $d = -2$.

Ex. 10. Insert between $\frac{1}{6}$ and $\frac{1}{16}$ two numbers such that the first three may be in H.P. and the last three in G.P.

Sol. Let a and b be the two numbers between $\frac{1}{6}$ and $\frac{1}{16}$. Then $\frac{1}{6}$, a, b are in H.P.

$$\Rightarrow \quad a = \frac{2 \times b \times \frac{1}{6}}{\frac{1}{6} + b} \qquad \left(\because a, b, c \text{ in H.P} \Rightarrow b = \frac{2ac}{a+c}\right)$$

$\Rightarrow \quad \frac{a}{6} + ab = \frac{b}{3} \Rightarrow a + 6ab = 2b$ ...(*i*)

Also, $\quad a, b, \frac{1}{16}$ are in G.P.

$\Rightarrow$ $b^2 = a\times\frac{1}{16} \Rightarrow b^2 = \frac{a}{16} \Rightarrow a = 16b^2$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*)

$\Rightarrow$ $16b^2 + 6 \times 16b^2 \times b = 2b \Rightarrow 16b^2 + 96b^3 - 2b = 0 \Rightarrow b(16b + 96b^2 - 2) = 0$

$\Rightarrow$ $b(96b^2 + 16b - 2) = 0 \Rightarrow b(48b^2 + 8b - 1) = 0$

$\Rightarrow$ $b\,(12b - 1)\,(4b + 1) = 0$

$\Rightarrow$ $b = \frac{1}{12}$ or $-\frac{1}{4}$ or 0

Since $-\frac{1}{4}$ and 0 do not lie between $\frac{1}{6}$ and $\frac{1}{16}$, therefore, $b = \mathbf{\frac{1}{12}}$

Hence $a = 16b^2 = 16\times\frac{1}{144} = \mathbf{\frac{1}{9}}$.

$\therefore$ The required numbers are $\frac{1}{4}$ and $\frac{1}{12}$.

Ex. 11. If A.M. between two numbers is to their G.M. as 5 : 4 and the difference of their G.M. and H.M. is 16/5, find the numbers.

Sol. Let the two numbers be *a* and *b*. Then,

$$\text{A.M} = \frac{a+b}{2}$$

$$\text{G.M.} = \sqrt{ab}$$

$$\text{H.M.} = \frac{2ab}{a+b}$$

Given, A.M : G.M = 5 : 4

$\Rightarrow$ $\frac{(a+b)/2}{\sqrt{ab}} = \frac{5}{4} \Rightarrow \frac{a+b}{2\sqrt{ab}} = \frac{5}{4}$ ...(*i*)

Also, G.M. – H.M. = 16/5

$\Rightarrow$ $\sqrt{ab} - \frac{2ab}{a+b} = \frac{16}{5}$ ...(*ii*)

From (*i*), $a + b = \frac{5\times 2}{4}\sqrt{ab} \Rightarrow a + b = \frac{5}{2}\sqrt{ab}$. ...(*iii*)

Putting this value of $(a + b)$ in (*ii*), we have

$$\sqrt{ab} - \frac{2ab}{\sqrt{ab}}\times\frac{2}{5} = \frac{16}{5} \Rightarrow \sqrt{ab} - \frac{4}{5}\sqrt{ab} = \frac{16}{5}$$

$\Rightarrow$ $\frac{1}{5}\sqrt{ab} = \frac{16}{5} \Rightarrow \sqrt{ab} = 16 \Rightarrow ab = 256$

$\therefore$ From (*iii*), $a + b = \frac{5}{2}\times 16 = 40$.

$\therefore$ $(a - b)^2 = (a + b)^2 - 4ab = 40^2 - 4 \times 256 = 1600 - 1024 = 576$

$\Rightarrow$ $a - b = \pm 24$

Now solving $a + b = 40$ and $a - b = \pm 24$, we get

$$a = 32, b = 8 \quad \text{or} \quad a = 8, b = 32.$$

$\therefore$ The numbers are **8 and 32.**

Ex. 12. If $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$ and p, q, r are in A.P., show that x, y, z are in H.P.

Sol. Let $$\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz} = k$$

Then, $$\frac{a-x}{px} = k \Rightarrow \frac{a-x}{kx} = p \Rightarrow p = \frac{1}{k}\left(\frac{a}{x}-1\right)$$

Similarly, $$q = \frac{1}{k}\left(\frac{a}{y}-1\right),\ r = \frac{1}{k}\left(\frac{a}{z}-1\right)$$

Now, p, q, r are in A.P.

$\Rightarrow \frac{1}{k}\left(\frac{a}{x}-1\right), \frac{1}{k}\left(\frac{a}{y}-1\right), \frac{1}{k}\left(\frac{a}{z}-1\right)$ are in A.P.

$\Rightarrow \frac{a}{x}-1, \frac{a}{y}-1, \frac{a}{z}-1$ are in A.P. (Multiplying each term by k)

$\Rightarrow \frac{a}{x}, \frac{a}{y}, \frac{a}{z}$ are in A.P. (Adding 1 to each term)

$\Rightarrow \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P. (Dividing each term by a)

$\Rightarrow$ **x, y, z are in H.P.**

Ex. 13. Let a, b be two positive real numbers. If a, A_1, A_2, b are in arithmetic progression, a, G_1, G_2, b are in geometric progression and a, H_1, H_2, b are in harmonic progression, then show that

$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a+b)(a+2b)}{9ab}$$

Sol. a, A_1, A_2, b are in A.P. $\Rightarrow A_1 - a = b - A_2 \Rightarrow A_1 + A_2 = a + b$ ...(*i*)

a, G_1, G_2, b are in G.P. $\Rightarrow \frac{G_1}{a} = \frac{b}{G_2} \Rightarrow G_1 G_2 = ab$ ...(*ii*)

a, H_1, H_2, b are in H.P.

$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P.

$$\Rightarrow \frac{1}{H_1} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H_2}$$

$$\Rightarrow \frac{1}{H_1} + \frac{1}{H_2} = \frac{1}{a} + \frac{1}{b}$$

$$\Rightarrow \frac{H_1 + H_2}{H_1H_2} = \frac{a+b}{ab} = \frac{A_1 + A_2}{G_1G_2}$$ (From (*i*) and (*ii*))

$$\Rightarrow \frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}.$$ Hence proved.

Now $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}$ are in A.P ($\because a, H_1, H_2,$ are in H.P.)

$$\Rightarrow \frac{1}{H_1} - \frac{1}{a} = \frac{1}{H_2} - \frac{1}{H_1}$$

$$\Rightarrow \quad \frac{2}{H_1} - \frac{1}{H_2} = \frac{1}{a} \quad ...(iii)$$

Also, $\frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A.P (∵ H_1, H_2, b are in H.P.)

$$\Rightarrow \quad \frac{1}{H_2} - \frac{1}{H_1} = \frac{1}{b} - \frac{1}{H_2} \quad \Rightarrow \quad \frac{2}{H_2} - \frac{1}{H_1} = \frac{1}{b} \quad ...(iv)$$

Eq. (*iii*) + 2 × Eqn. (*iv*) $\Rightarrow \left(\frac{2}{H_1} - \frac{1}{H_2}\right) + \left(\frac{4}{H_2} - \frac{2}{H_1}\right) = \frac{1}{a} + \frac{2}{b}$

$$\Rightarrow \frac{3}{H_2} = \frac{b+2a}{ab} \quad \Rightarrow \quad H_2 = \frac{3ab}{2a+b}$$

Eq. (*iv*) + 2 × Eq. (*iii*) $\Rightarrow \left(\frac{2}{H_2} - \frac{1}{H_1}\right) + \left(\frac{4}{H_1} - \frac{2}{H_2}\right) = \frac{1}{b} + \frac{2}{a}$

$$\Rightarrow \frac{3}{H_1} = \frac{a+2b}{ab} \quad \Rightarrow \quad H_1 = \frac{3ab}{a+2b}$$

$$\therefore \quad \frac{G_1G_2}{H_1H_2} = \frac{ab}{\frac{3ab}{(a+2b)} \times \frac{3ab}{(2a+b)}} = \frac{\mathbf{(a+2b)(2a+b)}}{\mathbf{9ab}}$$ Hence proved.

Ex. 14. If $x > 1, y > 1, z > 1$ are in G.P., then show that $\frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z}$ are in H.P.

(*IIT 1999, Manipal 2012*)

Sol. x, y, z are in G.P.

$\Rightarrow \; y^2 = xz \;\Rightarrow\; 2\log y = \log x + \log z$

$\Rightarrow$ $\log x, \log y, \log z$ are in A.P.

$\Rightarrow$ $1 + \log x, 1 + \log y, 1 + \log z$ are in A.P. (Adding 1 to each term)

$\Rightarrow$ $\frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z}$ **are in H.P.**

Ex. 15. Find the harmonic mean of the roots of the equation

$$\mathbf{(5+\sqrt{2})\,x^2 - \left(4+\sqrt{5}\right)x + (8+2\sqrt{5}) = 0.}$$

(*IIT 1999, MPPET 2010, EAMCET 2013*)

Sol. Let the roots of the equation be α and β. Then,

$$\text{Sum of roots} = \alpha + \beta = \frac{(4+\sqrt{5})}{5+\sqrt{2}}$$

$$\text{Product of roots} = \alpha\beta = \frac{8+2\sqrt{5}}{5+\sqrt{2}}$$

Now, Harmonic Mean of the roots, α and β = $\frac{2\alpha\beta}{\alpha+\beta} = \frac{2\left(\frac{8+2\sqrt{5}}{5+\sqrt{2}}\right)}{\frac{(4+\sqrt{5})}{5+\sqrt{2}}} = \frac{4\,(4+\sqrt{5})}{(4+\sqrt{5})} = \mathbf{4.}$

PRACTICE SHEET

1. The *n*th term of the H.P. $4 + 4\frac{2}{7} + 4\frac{8}{13} + 5 + ...$ is

(*a*) $\frac{7}{26 - 2n}$ (*b*) $\frac{3}{3n - 2}$

(*c*) $\frac{60}{16 - n}$ (*d*) None of these

2. The harmonic mean of $\frac{a}{1 - ab}$ and $\frac{a}{1 + ab}$ is

(*a*) a (*b*) $\frac{1}{1 - a^2b^2}$

(*c*) $\frac{a}{1 - a^2b^2}$ (*d*) $\frac{a}{\sqrt{1 - a^2b^2}}$

(*AMU 2002*)

3. The 5th and 11th term of an H.P. are $\frac{1}{45}$ and $\frac{1}{69}$ respectively. Then, its 16th term will be

(*a*) $\frac{1}{77}$ (*b*) $\frac{1}{81}$ (*c*) $\frac{1}{85}$ (*d*) $\frac{1}{89}$

(*Rajasthan PET 2003*)

4. If the A.M. and G.M. of two numbers be 27 and 18 respectively, then what is their H.M. equal to?

(*a*) 24 (*b*) 12 (*c*) 16 (*d*) 28

5. If the first two terms of an H.P. are $\frac{2}{5}$ and $\frac{12}{13}$ respectively, then the largest term is

(*a*) 2nd term (*b*) 3rd term (*c*) 4th term (*d*) 6th term

(*AMU 2007*)

6. If a and b are two real numbers such that $0 < a < b$ and the arithmetic mean between a and b is $\frac{4}{3}$ times the harmonic mean between them, then b/a is equal to

(*a*) $\frac{2}{\sqrt{3}}$ (*b*) $\frac{3}{2}$ (*c*) 3 (*d*) $\frac{8}{3}$

7. G.M. and H.M. of two numbers are 10 and 8 respectively. The numbers are

(*a*) 1, 100 (*b*) 2, 50 (*c*) 4, 25 (*d*) 5, 20

(*WBJEE 2010*)

8. Five numbers are in H.P. The middle term is 1 and the ratio of the second and fourth terms is 2 : 1. Then, the sum of the first three terms is

(*a*) 11/2 (*b*) 5 (*c*) 2 (*d*) 14/2

(*WBJEE 2013*)

9. If for two numbers the ratio of their H.M. to G.M. is 20:29, then the numbers are in the ratio

(*a*) 3 : 40 (*b*) 4 : 25 (*c*) 1 : 22 (*d*) 2 : 27

(*Type IIT*)

10. If $2(y - a)$ is the H.M. between $y - x$ and $y - z$, then $(x - a)$, $(y - a)$, $(z - a)$ are in

(*a*) A.P. (*b*) G.P. (*c*) H.P. (*d*) None of these

(*Rajasthan PET 2001*)

11. If a, b, c are in H.P., then $\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)$ is equal to

(*a*) $\frac{4}{b^2} - \frac{3}{ac}$ (*b*) $\frac{3}{b^2} - \frac{4}{ac}$ (*c*) $\frac{4}{ac} - \frac{3}{b^2}$ (*d*) $\frac{3}{b^2} + \frac{4}{ac}$

(*Kerala PET*)

12. If the *l*th, *m*th and *n*th terms of an H.P. are in H.P. then l, m, n, are in

(*a*) H.P. (*b*) A.P. (*c*) G.P. (*d*) None of these

13. If $\log(a + c)$, $\log(c - a)$ and $\log(a - 2b + c)$ are in A.P, then

(*a*) a, b, c are in A.P. (*b*) a^2, b^2, c^2 are in A.P.

(*c*) a, b, c are in G.P. (*d*) a, b, c are in H.P.

(*DCE 2002*)

14. If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to sum of their squares, then $\frac{c}{a}, \frac{b}{a}, \frac{c}{b}$ are in

(*a*) A.P. (*b*) G.P (*c*) H.P (*d*) None of these

(*Punjab CET 2008*)

15. If $a^x = b^y = c^z$ and x, y, z are in H.P., then a, b, c are in

(*a*) G.P. (*b*) A.P. (*c*) H.P. (*d*) None of these

16. If $x^2 + 9y^2 + 25z^2 = xyz\left(\frac{15}{x} + \frac{5}{y} + \frac{3}{z}\right)$, then x, y, z are in

(*a*) G.P. (*b*) H.P. (*c*) A.P. (*d*) None of these

(*DCE 2004*)

17. If a, b, c are in A.P. and a^2, b^2, c^2 are in H.P., then which of the following statement can be true?

(*a*) $a, b, -\frac{c}{2}$ are in G.P. (*b*) $a = b = c$

(*c*) Any of these (*d*) None of these

(*IIT 2003*)

18. The H.M. of two numbers is 4. Their A.M. is A and G.M. is G. If $2A + G^2 = 27$, then A is equal is

(*a*) $\frac{9}{2}$ (*b*) 18 (*c*) $\frac{27}{2}$ (*d*) 27

(*WBJEE 2011*)

19. Let the positive numbers a, b, c, d be in A.P. Then, abc, abd, acd, bcd are

(*a*) NOT in A.P./G.P./H.P (*b*) In A.P.

(*c*) In G.P. (*d*) In H.P. **(*IIT 2001*)**

20. If $a^x = b^y = c^z = d^u$ and a, b, c, d are in GP, then x, y, z, u are in

(*a*) A.P. (*b*) G.P. (*c*) H.P. (*d*) None of these

(*VITEEE 2010*)

21. If $a, a_1, a_2, \ldots, a_{2n}, b$ are in arithmetic progression and $a, g_1, g_2, \ldots, g_{2n}, b$ are in geometric progression, and h is the harmonic mean of a and b, then $\frac{a_1+a_{2n}}{g_1g_{2n}}+\frac{a_2+a_{2n-1}}{g_2g_{2n-1}}+\ldots..+\frac{a_n+a_{n+1}}{g_n\,g_{n+1}}$ is equal to

(*a*) $2nh$ (*b*) n/h (*c*) nh (*d*) $2n/h$

(*DCE 2009*)

22. Find the value of n for which $\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$ is the harmonic mean between a and b.

(*a*) -2 (*b*) $-\frac{3}{2}$ (*c*) -1 (*d*) $-\frac{1}{2}$

23. If $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$ and $b \neq (a+c)$, then a, b, c are in

(*a*) A.P (*b*) G.P (*c*) H.P (*d*) None of these

24. If a, b, c are in H.P. and half the middle term be subtracted from the three terms, then the resulting series will be in

(*a*) A.P. (*b*) G.P (*c*) H.P (*d*) None of these

25. If p, q, r are in H.P and the $(p+1)$th, $(q+1)$th and $(r+1)$th terms of an A.P. are in G.P., then the ratio of the first term to the common difference of the A.P. is equal to

(*a*) $\frac{-q}{2}$ (*b*) $\frac{-pr}{q}$ (*c*) $\frac{-pr}{q^2}$ (*d*) $\frac{-2q}{pr}$

ANSWERS

1. (*c*)	**2.** (*a*)	**3.** (*d*)	**4.** (*b*)	**5.** (*a*)	**6.** (*c*)	**7.** (*d*)	**8.** (*a*)	**9.** (*b*)	**10.** (*b*)
11. (*c*)	**12.** (*b*)	**13.** (*d*)	**14.** (*c*)	**15.** (*a*)	**16.** (*b*)	**17.** (*c*)	**18.** (*a*)	**19.** (*d*)	**20.** (*c*)
21. (*d*)	**22.** (*c*)	**23.** (*c*)	**24.** (*b*)	**25.** (*a*)					

HINTS AND SOLUTIONS

1. $4, 4\frac{2}{7}, 4\frac{8}{13}, 5, \ldots$ are in H.P.

$\Rightarrow$ $\frac{1}{4}, \frac{7}{30}, \frac{13}{60}, \frac{1}{5}, \ldots$ are in A.P.

Now, here $a=\frac{1}{4}, d=\frac{7}{30}-\frac{1}{4}=-\frac{1}{60}$

$\therefore$ $T_n = a+(n-1)d = \frac{1}{4}+(n-1)\left(-\frac{1}{60}\right)$

$=\frac{1}{4}-\frac{n}{60}+\frac{1}{60}=\frac{15+1}{60}-\frac{n}{60}=\frac{16-n}{60}$

$\therefore$ nth term of the H.P. $=\mathbf{\frac{60}{16-n}}$.

2. $\because$ Harmonic mean of two quantities a and b is $\frac{2ab}{a+b}$

$\therefore$ Reqd. H.M. $=\frac{2\times\left(\frac{a}{1-ab}\right)\times\left(\frac{a}{1+ab}\right)}{\frac{a}{1-ab}+\frac{a}{1+ab}}$

$=\frac{\frac{2a^2}{(1-a^2b^2)}}{\frac{a+a^2b+a-a^2b}{(1-a^2b^2)}}=\frac{2a^2}{a}=\boldsymbol{a}.$

3. Given, 5th term of an H.P. $=\frac{1}{45}$

$\Rightarrow$ 5th term of corresponding A.P. = 45

11th term of an H.P. $=\frac{1}{69}$

$\Rightarrow$ 11th term of corresponding A.P. = 69

Let a and d be the first term and common difference of this A.P.

So $a+4d=45$ ...(*i*)

$a+10d=69$ ...(*ii*)

$(ii)-(i) \Rightarrow 6d=24 \Rightarrow d=4$

From (*i*) $a+16=45 \Rightarrow a=29.$

$\therefore$ $T_{16}=a+15d=29+15\times 4=29+69=89$

$\therefore$ 16th term of corresponding H.P. $=\mathbf{\frac{1}{89}}$.

4. $\because$ $(\text{G.M})^2 = (\text{A.M}).(\text{H.M.})$

$\therefore$ $(18)^2 = 27\times\text{H.M.} \Rightarrow \text{H.M.}=\frac{18\times 18}{27}=\mathbf{12.}$

5. First term of H.P $=\frac{2}{5}$ $\Rightarrow$ First term of A.P $=\frac{5}{2}=\frac{30}{12}$

Second term of H.P $=\frac{12}{13}$ $\Rightarrow$ Second term of A.P $=\frac{13}{12}$

$\therefore$ The first and second terms of the corresponding A.P. are $\frac{30}{12}$ and $\frac{13}{12}$ respectively. The common difference

$d=\frac{13}{12}-\frac{30}{12}=-\frac{17}{12}$

$\therefore$ The corresponding A.P. is $\frac{30}{12}, \frac{13}{12}, -\frac{4}{12}, -\frac{21}{12}, \ldots$

$\therefore$ The corresponding H.P is $\frac{12}{30}, \frac{12}{13}, -\frac{12}{4}, -\frac{12}{21}, \ldots$

All the terms after the second term are negative, so the largest term is out of the first two terms, *i.e.*, between $\frac{12}{30}$ and $\frac{12}{13}$.

$\because \quad \frac{12}{13} > \frac{12}{30}$, so, $\frac{12}{13}$ *i.e.*, second term is the largest term of the H.P.

6. Arithmetic mean between a and b is $A = \frac{a+b}{2}$

Harmonic mean betwen a and b is $H = \frac{2ab}{a+b}$

Given, $A = \frac{4}{3}H \Rightarrow \frac{a+b}{2} = \frac{4}{3}\left(\frac{2ab}{a+b}\right)$

$\Rightarrow 3(a+b)^2 = 16ab \Rightarrow 3b^2 - 10ab + 3a^2 = 0$

$\Rightarrow 3\left(\frac{b}{a}\right)^2 - 10\left(\frac{b}{a}\right) + 3 = 0$ (Dividing throughout by a^2)

$\Rightarrow \left(\frac{3b}{a} - 1)\right)\left(\frac{b}{a} - 3\right) = 0$

$\Rightarrow \frac{b}{a} = \frac{1}{3}$ or $\frac{b}{a} = 3$ $\left[\because 0 < a < b \Rightarrow \frac{b}{a} \neq \frac{1}{3}\right]$

$\Rightarrow \mathbf{\frac{b}{a} = 3.}$

7. Let the two required numbers be a and b.

Then, G.M $= \sqrt{ab} = 10 \Rightarrow ab = 100$ ...(*i*)

H.M. $= \frac{2ab}{a+b} = 8 \Rightarrow a + b = \frac{ab}{4} = \frac{100}{4} = 25$ ...(*ii*)

$\therefore \quad (a-b)^2 = (a+b)^2 - 4ab$

$= 625 - 400 = 225$

$\Rightarrow \quad a - b = \pm 15$ ...(*iii*)

Solving (*i*) and (*ii*), we get $a = 20$, $b = 5$ or $a = 5$, $b = 20$

$\therefore$ The required numbers are **20** and **5.**

8. Let the terms of the corresponding A.P. be

$a - 2d, a - d, a, a + d, a + 2d.$

Given, $a = 1$ and $\frac{a-d}{a+d} = \frac{1}{2}$

$\Rightarrow 2a - 2d = a + d$

$\Rightarrow \quad a = 3d \Rightarrow d = \frac{1}{3}a = \frac{1}{3}.$ $(\because a = 1)$

$\therefore$ First three terms of A.P are $1 - \frac{2}{3}, 1 - \frac{1}{3}, 1$, *i.e.*, $\frac{1}{3}, \frac{2}{3}, 1$

$\Rightarrow$ First three terms of corresponding H.P are $3, \frac{3}{2}, 1$

$\therefore$ Required sum $= 3 + \frac{3}{2} + 1 = 5\frac{1}{2} = \mathbf{\frac{11}{2}.}$

9. Let the two numbers be a and b.

Given, $\frac{\text{H.M}}{\text{G.M}} = \frac{20}{29} \Rightarrow \frac{2ab/a+b}{\sqrt{ab}} = \frac{20}{29}$

$\Rightarrow \quad \frac{2\sqrt{ab}}{a+b} = \frac{20}{29} \Rightarrow 58\sqrt{ab} = 20(a+b)$

$\Rightarrow \quad 20a - 58\sqrt{ab} + 20b = 0$

$\Rightarrow \quad 20\frac{a}{b} - 58\sqrt{\frac{a}{b}} + 20 = 0$ (Dividing all terms by b)

$\Rightarrow \quad 20x^2 - 58x + 20 = 0$ (where $x = \sqrt{\frac{a}{b}}$)

$\Rightarrow \quad 20x^2 - 50x - 8x + 20 = 0$

$\Rightarrow 10x(2x - 5) - 4(2x - 5) = 0$

$\Rightarrow \quad (10x - 4)(2x - 5) = 0$

$\Rightarrow \quad x = \frac{2}{5}$ or $\frac{5}{2}$

$\Rightarrow \quad \sqrt{\frac{a}{b}} = \frac{2}{5}$ or $\frac{5}{2}$

$\Rightarrow \quad \frac{a}{b} = \frac{4}{25}$ or $\frac{25}{4}$

Thus from the given options, the two numbers are in the ratio **4 : 25.**

10. Given, $2(y - a)$ is the H.M. between $(y - x)$ and $(y - z)$

$\Rightarrow \quad (y - x), 2(y - a), (y - z)$ are in H.P.

$\Rightarrow \quad \frac{1}{y-x}, \frac{1}{2(y-a)}, \frac{1}{y-z}$ are in A.P.

$\Rightarrow \quad \frac{1}{2(y-a)} - \frac{1}{(y-x)} = \frac{1}{(y-z)} - \frac{1}{2(y-a)}$

$\Rightarrow \quad \frac{y - x - 2y + 2a}{2(y-a)(y-x)} = \frac{2y - 2a - y + z}{(y-z)\,2(y-a)}$

$\Rightarrow \quad \frac{-x - y + 2a}{(y-x)} = \frac{y + z - 2a}{(y-z)}$

$\Rightarrow \quad \frac{x + y - 2a}{x - y} = \frac{y + z - 2a}{y - z}$

$\Rightarrow \quad \frac{(x-a)+(y-a)}{(x-a)-(y-a)} = \frac{(y-a)+(z-a)}{(y-a)-(z-a)}$

Now applying componendo and dividendo, we have

$\left(\text{By comp. and div., } \frac{x}{y} = \frac{a}{b} \Rightarrow \frac{x+y}{x-y} = \frac{a+b}{a-b}\right)$

$\Rightarrow \quad \frac{2(x-a)}{2(y-a)} = \frac{2(y-a)}{2(z-a)}$

$\Rightarrow \quad (x - a)(z - a) = (y - a)^2$

$\Rightarrow \quad (x - a), (y - a), (z - a)$ are in G.P.

11. If a, b, c are in H.P, then $b = \frac{2ac}{a+c}$

$\therefore \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)$

$= \left(\frac{1}{a} + \frac{a+c}{2ac} - \frac{1}{c}\right)\left(\frac{a+c}{2ac} + \frac{1}{c} - \frac{1}{a}\right)$

$= \left(\frac{2c + a + c - 2a}{2ac}\right)\left(\frac{a + c + 2a - 2c}{2ac}\right)$

$= \left(\frac{3c - a}{2ac}\right)\left(\frac{3a - c}{2ac}\right) = \frac{10\,ac - 3a^2 - 3c^2}{4a^2c^2}$

$$= \frac{16ac-(3a^2+3c^2+6ac)}{4a^2c^2} = \frac{16ac-3(a+c)^2}{4a^2c^2}$$

$$= \frac{4}{ac} - 3\left(\frac{a+c}{2ac}\right)^2 = \mathbf{\frac{4}{ac} - \frac{3}{b^2}}$$

12. Let the lth, mth and nth terms of the corresponding A.P. be

$T_l = a+(l-1)d$, $T_m = a+(m-1)d$, $T_n = a+(n-1)d$

$\Rightarrow 2T_m = T_l + T_n$

$\Rightarrow 2[a+(m-1)d] = a+(l-1)d+a+(n-1)d$

$\Rightarrow (2m-2)d = (l+n-2)d$

$\Rightarrow 2m = l+n$

$\Rightarrow l, m, n$ are in A.P.

13. $\log(a+c)$, $\log(c-a)$ and $\log(a-2b+c)$ are in A.P.

$\Rightarrow 2\log(c-a) = \log(a+c) + \log(a-2b+c)$

$\Rightarrow \log(c-a)^2 = \log[(a+c)(a-2b+c)]$

$\Rightarrow c^2+a^2-2ac = a^2+ca-2ba-2bc+ac+c^2$

$\Rightarrow 2ab+2bc = 4ac \Rightarrow b(a+c) = 2ac$

$\Rightarrow b = \frac{2ac}{a+c} \Rightarrow a, b, c$ are in H.P.

14. Let α, β be the roots of the equation $ax^2+bx+c=0$.

Then, $\alpha+\beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

Also, given $\alpha+\beta = \alpha^2+\beta^2 \Rightarrow \alpha+\beta = (\alpha+\beta)^2 - 2\alpha\beta$

$\Rightarrow -\frac{b}{a} = \left(-\frac{b}{a}\right)^2 - \frac{2c}{a}$

$\Rightarrow -ba = b^2 - 2ac \Rightarrow b^2+ab = 2ac$

$\Rightarrow b(b+a) = 2ac \Rightarrow \frac{b}{c} + \frac{a}{c} = \frac{2a}{b}$

$\Rightarrow$ $\mathbf{\frac{c}{a}, \frac{a}{b}, \frac{b}{c}}$ **are in A.P.**

15. x, y, z are in H.P $\Rightarrow y = \frac{2xz}{x+z}$ ...(i)

Given, $a^x = b^y = c^z = k$ (say)

Then $x\log a = \log k$, $y\log b = \log k$, $z\log c = \log k$

$\Rightarrow x = \frac{\log k}{\log a}$, $y = \frac{\log k}{\log b}$, $z = \frac{\log k}{\log c}$

$\therefore$ Putting these values of x, y, z in (i), we get

$$\frac{\log k}{\log b} = \frac{2\left(\frac{\log k}{\log a}\right)\left(\frac{\log k}{\log c}\right)}{\frac{\log k}{\log a} + \frac{\log k}{\log c}}$$

$\Rightarrow \frac{\log k}{\log b} = \frac{2(\log k)^2}{\log k \log c + \log a \log k}$

$\Rightarrow \frac{(\log k)^2}{\log b} = \frac{2(\log k)^2}{\log c + \log a}$

$\Rightarrow 2\log b = \log c + \log a \Rightarrow \log b^2 = \log(ac)$

$\Rightarrow b^2 = ac \Rightarrow a, b, c$ are in G.P.

16. Given, $x^2+9y^2+25z^2 = xyz\left(\frac{15}{x} + \frac{5}{y} + \frac{3}{z}\right)$

$\Rightarrow x^2+9y^2+25z^2-15yz-5xz-3xy = 0$

$\Rightarrow 2x^2+18y^2+50z^2-30yz-10xz-6xy = 0$

$\Rightarrow (x-3y)^2+(3y-5z)^2+(x-5z)^2 = 0$

$\Rightarrow x = 3y, 3y = 5z, 5z = x \Rightarrow x = 3y = 5z$

$\Rightarrow \frac{x}{1} = \frac{y}{\frac{1}{3}} = \frac{z}{\frac{1}{5}} \Rightarrow \frac{x}{15} = \frac{y}{5} = \frac{z}{3}$

$\Rightarrow x, y, z$ are in H.P. $\left(\because y = \frac{2xz}{x+z}, \; y = \frac{2\times15\times3}{15+3} = \mathbf{5}\right)$

17. a, b, c are in A.P. $\Rightarrow 2b = a+c$ ...(i)

a^2, b^2, c^2 are in H.P. $\Rightarrow b^2 = \frac{2a^2c^2}{a^2+c^2}$ ...(ii)

From eqn (ii)

$b^2(a^2+c^2) = 2a^2c^2 \Rightarrow b^2\{(a+c)^2-2ac\} = 2a^2c^2$

$\Rightarrow b^2\{4b^2-2ac\} = 2a^2c^2$ (From (i) $2b = a+c$)

$\Rightarrow 2b^4 - b^2ac - a^2c^2 = 0$ ...(iii)

$\Rightarrow (b^2-ac)(2b^2+ac) = 0$

$\Rightarrow b^2 - ac = 0$ or $2b^2+ac = 0$

$\Rightarrow \left(\frac{a+c}{2}\right)^2 - ac = 0$ $\quad b^2 = -\frac{ac}{2}$

$\Rightarrow (a-c)^2 = 0$ $\quad a, b, -\frac{c}{2}$ are in G.P.

$\Rightarrow 2b = 2c$ (From (i))

$\Rightarrow$ $\mathbf{b = c.}$

18. Let the two numbers be a and b. Then,

H.M $= \frac{2ab}{a+b}, = 4 \Rightarrow a+b = \frac{ab}{2}$ ...(i)

Also given, $A = \frac{a+b}{2}$, $G = \sqrt{ab}$

$\therefore 2A + G^2 = 27 \Rightarrow 2\times\frac{a+b}{2} + ab = 27$

$\Rightarrow a+b+ab = 27 \Rightarrow (a+b)+2(a+b) = 27$

$\Rightarrow 3(a+b) = 27 \Rightarrow a+b = 9 \Rightarrow \frac{a+b}{2} = \frac{9}{2} = \mathbf{A}.$

19. If a, b, c, d are in A.P.

$\Rightarrow d, c, b, a$ are in A.P.

$\Rightarrow \frac{d}{abcd}, \frac{c}{abcd}, \frac{b}{abcd}, \frac{a}{abcd}$ are in A.P.

(Dividing all terms by $abcd$)

$\Rightarrow \frac{1}{abc}, \frac{1}{abd}, \frac{1}{acd}, \frac{1}{bcd}$ are in A.P.

$\Rightarrow abc, abd, acd, bcd$ are in H.P.

20. Given, $a^x = b^y = c^z = d^u = k$ (say)

Then, $a = k^{1/x}, b = k^{1/y}, c = k^{1/z}, d = k^{1/u}$ ...(i)

Since a, b, c, d are in G.P.

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

$$\Rightarrow \quad \frac{k^{1/y}}{k^{1/x}} = \frac{k^{1/z}}{k^{1/y}} = \frac{k^{1/u}}{k^{1/z}} \quad \text{(Using } (i)$$

$$\Rightarrow \quad k^{1/y - 1/x} = k^{1/z - 1/y} = k^{1/u - 1/z}$$

$$\Rightarrow \quad \frac{1}{y} - \frac{1}{x} = \frac{1}{z} - \frac{1}{y} = \frac{1}{u} - \frac{1}{z}$$

$\Rightarrow \quad \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{u}$ are in A.P $\Rightarrow$ x, y, z, u are in H.P.

21. Given, $a, a_1, a_2, a_3, \ldots\ldots, a_{2n}, b$ are in A.P.

$$\Rightarrow \quad a_1 + a_{2n} = a_2 + a_{2n-1} = \ldots\ldots = a + b$$

$$(\because a_1 - a = b - a_{2n},\ a_2 - a = b - a_{2n-1})$$

Also, $a, g_1, g_2, \ldots\ldots g_{2n}, b$ are in G.P.

$$\Rightarrow \quad g_1 g_{2n} = g_2 g_{2n-1} = \ldots\ldots = ab$$

$$\therefore \quad \frac{a_1 + a_{2n}}{g_1 g_{2n}} + \frac{a_2 + a_{2n-1}}{g_2 g_{2n-1}} + \ldots\ldots + \frac{a_n + a_{n+1}}{g_n g_{n+1}}$$

$$= \frac{a+b}{ab} + \frac{a+b}{ab} + \ldots\ldots + \frac{a+b}{ab} \quad (n \text{ times})$$

$$= \frac{n(a+b)}{ab} = \frac{\mathbf{2n}}{\mathbf{h}}, \text{ where,}$$

[Given, $h = \frac{2ab}{a+b}$ is the harmonic mean between a and b]

22. If $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ is the harmonic mean between a and b, then

$$\frac{a^{n+1} + b^{n+1}}{a^n + b^n} = \frac{2ab}{a+b}$$

$$\Rightarrow \quad a^{n+1} \cdot a + a^{n+1} b + b^{n+1} a + b^{n+1} \cdot b = 2ab \cdot a^n + 2ab\, b^n$$

$$\Rightarrow \quad a^{n+2} + a^{n+1} b + b^{n+1} a + b^{n+2} = 2a^{n+1} b + 2ab^{n+1}$$

$$\Rightarrow \quad a^{n+2} + b^{n+2} = a^{n+1} b + ab^{n+1}$$

$$\Rightarrow \quad a^{n+2} - ba^{n+1} = ab^{n+1} - b^{n+2}$$

$$\Rightarrow \quad a^{n+1}(a - b) = b^{n+1}(a - b)$$

$$\Rightarrow \quad a^{n+1} = b^{n+1} \quad \Rightarrow \quad \left(\frac{a}{b}\right)^{n+1} = 1 \quad \Rightarrow \quad \left(\frac{a}{b}\right)^{n+1} = \left(\frac{a}{b}\right)^0$$

$\Rightarrow \quad n + 1 = 0 \quad \Rightarrow \quad$ $\boldsymbol{n = -1.}$

23. $\frac{1}{b-a} + \frac{1}{b-c} = \frac{1}{a} + \frac{1}{c} \quad \Rightarrow \quad \frac{b-c+b-a}{(b-a)(b-c)} = \frac{c+a}{ac}$

$$\Rightarrow \quad \frac{2b-(c+a)}{b^2 - b(a+c) + ac} = \frac{a+c}{ac}$$

$$\Rightarrow \quad 2bac - ac(a+c) = b^2(a+c) - b(a+c)^2 + ac(a+c)$$

$$\Rightarrow \quad b^2(a+c) - b(a+c)^2 + 2ac(a+c) - 2abc = 0$$

$$\Rightarrow \quad b(a+c)(b-(a+c)) + 2ac(a+c-b) = 0$$

$$\Rightarrow \quad b(a+c)(b-(a+c)) - 2ac(b-(a+c)) = 0$$

$$\Rightarrow \quad [b(a+c) - 2ac](b-(a+c)) = 0$$

$$\Rightarrow \quad b(a+c) - 2ac = 0 \ (\because b \neq a+c \ \Rightarrow \ b-(a+c) \neq 0)$$

$\Rightarrow \quad b = \frac{2ac}{a+c} \quad \Rightarrow$ **a, b, c are in H.P.**

24. Let the three numbers in H.P. be a, b, c.

$$\therefore \quad b = \frac{2ac}{a+c}$$

The new numbers after subtracting half the middle term from each term are:

$$a - \frac{b}{2} = a - \frac{ac}{a+c} = \frac{a^2}{a+c}$$

$$b - \frac{b}{2} = \frac{b}{2} = \frac{ac}{a+c}$$

$$c - \frac{b}{2} = c - \frac{ac}{a+c} = \frac{c^2}{a+c}$$

Now, $\left(a - \frac{b}{2}\right)\left(c - \frac{b}{2}\right) = \left(\frac{a^2}{a+c}\right)\left(\frac{c^2}{a+c}\right)$

$$= \frac{a^2c^2}{(a+c)^2} = \left(\frac{b}{2}\right)^2$$

$\Rightarrow \quad \left(a - \frac{b}{2}\right), \frac{b}{2}, \left(c - \frac{b}{2}\right)$ are in G.P.

25. p, q, r are in H.P. $\Rightarrow \quad q = \frac{2pr}{p+r}$ ...(i)

Let a and d be the first term and common difference respectively of the A.P.

$$\therefore \quad T_{p+1} = a + pd$$

$$T_{q+1} = a + qd$$

$$T_{r+1} = a + rd$$

Given, $T_{p+1}, T_{q+1}, T_{r+1}$ are in G.P.

$$\Rightarrow \quad (T_{q+1})^2 = T_{p+1} \cdot T_{r+1}$$

$$(a + qd)^2 = (a + pd)(a + rd)$$

$$\Rightarrow \quad a^2 + 2aqd + q^2d^2 = a^2 + ad(p + r) + prd^2$$

$$\Rightarrow \quad ad(2q - p - r) = d^2(pr - q^2)$$

$$\Rightarrow \quad \frac{a}{d} = \frac{pr - q^2}{2q - p - r} = \frac{\frac{1}{2}(p+r)q - q^2}{2q - (p+r)} \quad \text{(Using } (i))$$

$$= \frac{-\frac{q}{2}[2q - (p+r)]}{2q - (p+r)} = -\frac{\mathbf{q}}{\mathbf{2}}.$$

ARITHMETICO–GEOMETRIC SERIES

KEY FACTS

1. A series in which each term is the product of corresponding terms of an A.P. and a G.P. is called an **Arithmetico-Geometric series.**

The **general** or **standard** form of such a series is

$a + (a + d)r + (a + 2d)r^2 + (a + 3d)r^3 + ... + \{a + (n - 1) d\}r^{n-1} + ...$ where each term is formed by multiplying the corresponding terms of the two series.

A.P. : $a + (a + d) + (a + 2d) + ... + (a + (n - 1)d) + ...$ and

G.P. : $1 + r + r^2 + ... + r^{n-1} + ...$

2. nth term of an Arithmetico-Geometric series is

$$T_n = \{a + (n - 1) d\} r^{n-1}.$$

3. Sum of n terms of an Arithmetico-Geometric Series

Let S_n be the sum of the n terms of the series $a + (a + d)r + (a + 2d)r^2 + ... + (a + (n - 1) d)r^{n-1}$

Then, $S_n = a + (a + d)r + (a + 2d)r^2 + ... + (a + (n - 1) d)r^{n-1}$ ...(*i*)

$rS_n = \quad a r + (a + d)r^2 + ... + (a + (n - 2) d)r^{n-1} + (a + (n - 1) d)r^n$ ...(*ii*)

Subtracting eqn (*ii*) from eqn (*i*), we get

$$(1 - r) S_n = (a + dr + dr^2 + ... + dr^{n-1}) - (a + (n - 1) d)r^n$$

$$= a + \frac{dr\,(1 - r^{n-1})}{(1 - r)} - (a + (n - 1)d)r^n$$

$$\Rightarrow \quad S_n = \frac{a}{1 - r} + \frac{dr\,(1 - r^{n-1})}{(1 - r)^2} - \frac{(a + (n - 1)\, d)\, r^n}{(1 - r)}.$$

4. Sum of an infinite Arithmetico-Geometric Series

Let the infinite Arithmetico-Geometric series be

$a + (a + d)\, r + (a + 2d)\, r^2 + ... + (a + (n - 1)\, d)r^{n-1} + ... \infty$

If $|r| < 1$, then $r^n \to 0$ and $r^{n-1} \to 0$ as $n \to \infty$, then

$$S_\infty = \frac{a}{1 - r} + \frac{dr}{(1 - r)^2}.$$

SOLVED EXAMPLES

Ex. 1. Find the nth term of the given arithmetico-geometric series:

(*i*) $\frac{1}{3} + \frac{3}{9} + \frac{5}{27} + \frac{7}{81} + ...$ **(*ii*)** $1 - 2x + 3x^2 - 4x^3 + ...$

Sol. (*i*) The A.P. and G.P. corresponding to the given series

$\frac{1}{3} + \frac{3}{9} + \frac{5}{27} + \frac{7}{81} + ...$ are respectively

$1, 3, 5, 7, ...$ and $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, ...$

nth term of A.P. $= (1 + (n - 1)2) = (2n - 1)$

nth term of G.P. $= \frac{1}{3} \cdot \left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^n$

$\therefore$ nth term of the given series $= (2n - 1)\left(\frac{1}{3}\right)^n.$

(*ii*) The given arithmetico-geometric series is

$1 - 2x + 3x^2 - 4x^3 + ...$

where corresponding A.P. and G.P. are respectively 1, 2, 3, 4, ... and $1, -x, (-x)^2, (-x)^3, ...$

nth term of A.P $= (1 + (n-1)\,1) = n$

nth term of G.P $= 1 \,.\, (-x)^{n-1} = \mathbf{(-1)^{n-1}\,(x)^{n-1}}$.

$\therefore$ nth term of the given series $= n \,.\, (-1)^{n-1}\, x^{n-1} = \mathbf{(-1)^{n-1}\, nx^{n-1}}$.

Ex. 2. Find the sum to *n* terms of the series $\mathbf{1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + ...}$

Sol. $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + ...$ is an arithmetico-geometric series with corresponding A.P. and G.P as:

A.P. : $1 + 3 + 5 + 7 + ...$

G.P. : $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$

nth term of A.P. $= (1 + (n-1)\,2) = (2n-1)$

nth term of G.P. $= 1.\left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^{n-1}}$

$\therefore$ nth term of the given A.G.P. $= \frac{2n-1}{2^{n-1}}$

$$\therefore \quad S_n = 1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} + ... + \frac{2n-1}{2^{n-1}}$$

$$\Rightarrow \quad \frac{1}{2}S_n = \frac{1}{2} + \frac{3}{4} + \frac{5}{8} + ... + ... + \frac{2n-3}{2^{n-1}} + \frac{2n-1}{2^n}$$

On subtraction, we get

$$\left(1 - \frac{1}{2}\right)S_n = 1 + \frac{2}{2} + \frac{2}{4} + \frac{2}{8} + ... + \frac{2}{2^{n-1}} - \frac{2n-1}{2^n}$$

$$\Rightarrow \quad \frac{1}{2}S_n = 1 + 2\left\{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^{n-1}}\right\} - \frac{2n-1}{2^n}$$

$$\Rightarrow \quad \frac{1}{2}S_n = 1 + 2\left\{\frac{\left(\frac{1}{2}\left(1 - \left(\frac{1}{2}\right)^n\right)\right)}{1 - \frac{1}{2}}\right\} - \frac{2n-1}{2^n} = 1 + 2 - \frac{4}{2^n} - \frac{2n-1}{2^n} = 3 - \frac{2n+3}{2^n}$$

$$\therefore \quad S_n = \mathbf{6 - \frac{2n+3}{2^{n-1}}}.$$

Ex. 3. Find the sum of the series $\mathbf{1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + ...\infty}$

Sol. Let

$$S_\infty = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + ...\infty \quad ...(i)$$

$$\frac{1}{5}S_\infty = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + ...\infty \quad ...(ii)$$

Subtracting eqn (*ii*) from eqn (*i*), we get

$$\left(1 - \frac{1}{5}\right)S_\infty = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + ...\infty$$

$$\frac{4}{5}S_\infty = 1+3\left(\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+....\infty\right)$$

$$= 1+3\left(\frac{\frac{1}{5}}{1-\frac{1}{5}}\right) \quad (\because \text{ Sum of infinite G.P} = \frac{a}{1-r}) = 1+\frac{3/5}{4/5} = 1+\frac{3}{4} = \frac{7}{4}$$

$$\therefore \quad S_\infty = \frac{7}{4}\times\frac{5}{4} = \mathbf{\frac{35}{16}}.$$

Ex. 4. If the sum to infinity of the series $3 + 5r + 7r^2 + ... \infty$ is $\frac{44}{9}$, find the value of r.

Sol. $3 + 5r + 7r^2 + ... \infty$ is an infinite arithmetico-geometric series, where

$a = 3$, $d = 2$, common ratio $(r) = r$.

Sum to infinity of an A.G.P., with first term of A.P, as a, common difference d and common ratio r is

$$S_\infty = \frac{a}{1-r}+\frac{dr}{(1-r)^2}$$

$$\therefore \quad \frac{44}{9} = \frac{3}{1-r}+\frac{2r}{(1-r)^2} \quad \Rightarrow \quad \frac{44}{9} = \frac{3(1-r)+2r}{(1-r)^2}$$

$$\Rightarrow \quad 44(1-r)^2 = 9(3-r) \quad \Rightarrow \quad 44(1-2r+r^2) = 27-9r$$

$$\Rightarrow \quad 44-88r+44r^2 = 27-9r \quad \Rightarrow \quad 44r^2-79r+17 = 0$$

$$\Rightarrow \quad (4r-1)(11r-17) = 0 \quad \Rightarrow \quad r = \frac{1}{4} \text{ or } \frac{17}{11}$$

$r \neq \frac{17}{11}$ as it is not possible to find the sum of an infinite G.P. with $|r| > 1$. So $r = \mathbf{\frac{1}{4}}$.

Ex. 5. Show that $2^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times 8^{\frac{1}{16}} \times 16^{\frac{1}{32}} \times ...... \infty = 2$

Sol. Let

$$x = 2^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times 8^{\frac{1}{16}} \times 16^{\frac{1}{32}} \times ..... \infty$$

$$\therefore \quad \log x = \frac{1}{4}\log 2+\frac{1}{8}\log 4+\frac{1}{16}\log 8+\frac{1}{32}\log 16+.....\infty$$

$$= \frac{1}{4}\log 2+\frac{1}{8}\log 2^2+\frac{1}{16}\log 2^3+\frac{1}{32}\log 2^4+.....\infty$$

$$= \frac{1}{4}\log 2+\frac{2}{8}\log 2+\frac{3}{16}\log 2+\frac{4}{32}\log 2+.....\infty$$

$$= \left(\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+.....\infty\right)\log 2 \qquad ...(i)$$

Now, $\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+.....\infty$ is an Arithmetico-Geometric series.

Let

$$S = \frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+.....\infty \qquad ...(ii)$$

$$\frac{1}{2}S = \frac{1}{8}+\frac{2}{16}+\frac{3}{32}+.....\infty \qquad ...(iii)$$

On subtracting eqn (*iii*) from eqn (*ii*), we get

$$\frac{1}{2}S = \frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+.....\infty = \frac{\frac{1}{4}}{1-\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2} \quad \Rightarrow \quad S = 1$$

$$\therefore \quad \log x = 1 \times \log 2 \qquad \text{(From } (i))$$

$$\Rightarrow \quad \log x = \log 2 \quad \Rightarrow \quad x = \mathbf{2}.$$

PRACTICE SHEET

1. The sum to n terms of the series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+.....$ is

(*a*) $\left(\frac{25}{14}-\frac{7n+10}{14\times 5^{n-1}}\right)$ (*b*) $\left(\frac{17}{12}-\frac{4n+7}{12\times 5^{n-1}}\right)$

(*c*) $\left(\frac{35}{16}-\frac{12n+7}{16\times 5^{n-1}}\right)$ (*d*) $\left(\frac{15}{11}-\frac{10n+2}{11\times 5^{n-1}}\right)$

2. Sum the series: $1 + 2.2 + 3.2^2 + ... + 100.2^{99}$

(*a*) 99.2^{100} (*b*) 100.2^{100}

(*c*) $99.2^{100} + 1$ (*d*) 1000.2^{100}

(*Kerala PET 2001*)

3. The sum to infinity of the series $1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+.....$ is

(*a*) 6 (*b*) 2 (*c*) 3 (*d*) 4

(*AIEEE 2009*)

4. Given $\cot\theta = 2\sqrt{2}$, the sum of the infinite series $1 + 2(1 - \sin\theta) + 3(1 - \sin\theta)^2 + 4(1 - \sin\theta)^3 + ...$ is

(*a*) $6\sqrt{2}$ (*b*) 8 (*c*) 9 (*d*) $8\sqrt{2}$

(*AMU 2006*)

5. If $4+\frac{4+d}{5}+\frac{4+2d}{5^2}+.....\infty=10,$ then d is equal to

(*a*) 5 (*b*) 8 (*c*) 10 (*d*) 16

(*DCE 2007*)

ANSWERS

1. (*c*) **2.** (*c*) **3.** (*b*) **4.** (*c*) **5.** (*d*)

HINTS AND SOLUTIONS

1. $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+$ .... is an A.G.P. with

A.P. : 1 + 4 + 7 + 10 + ...

G.P. : $1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+.....$

nth term of A.P. $= 1 + 3(n - 1) = 3n - 2$

nth term of G.P. $=1\times\left(\frac{1}{5}\right)^{n-1}=\frac{1}{5^{n-1}}$

$\therefore$ nth term of the A.G.P. $=\frac{3n-2}{5^{n-1}}$

$(n-1)$th term of the A.G.P. $=\frac{3(n-1)-2}{5^{n-2}}=\frac{3n-5}{5^{n-2}}$

Now let $S_n=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+.....+\frac{3n-2}{5^{n-1}}$

$\therefore\quad \frac{1}{5}S_n = \frac{1}{5}+\frac{4}{5^2}+\frac{7}{5^3}+.....+\frac{3n-5}{5^{n-1}}+\frac{3n-2}{5^n}$

$\therefore\quad S_n-\frac{1}{5}S_n=1+\frac{3}{5}+\frac{3}{5^2}+\frac{3}{5^3}+.....+\frac{3}{5^{n-1}}-\frac{3n-2}{5^n}$

$\Rightarrow \left(1-\frac{1}{5}\right)S_n=1+\frac{3}{5}\left(1+\frac{1}{5}+\frac{1}{5^2}+.....+\frac{1}{5^{n-2}}\right)-\frac{3n-2}{5^n}$

$\Rightarrow\quad \frac{4}{5}S_n=1+\frac{3}{5}\left\{\frac{1\left(1-\frac{1}{5^{n-1}}\right)}{1-\frac{1}{5}}\right\}-\frac{3n-2}{5^n}$

$\because S_n=\frac{a(1-r^n)}{1-r}, r<1$

$=1+\frac{3}{5}\times\frac{5}{4}\left[1-\frac{1}{5^{n-1}}\right]-\frac{3n-2}{5^n}$

$=1+\frac{3}{4}-\frac{3}{4\cdot 5^{n-1}}-\frac{3n-2}{5^n}=\frac{7}{4}-\frac{3}{4.5^{n-1}}-\frac{3n-2}{5^n}$

$\Rightarrow\quad S_n=\frac{35}{16}-\left(\frac{15}{16\times 5^{n-1}}+\frac{3n-2}{4.5^{n-1}}\right)$

$=\frac{35}{16}-\left(\frac{15+12n-8}{16\times 5^{n-1}}\right)=\mathbf{\frac{35}{16}-\frac{12n-7}{16\times 5^{n-1}}}.$

2. $1 + 2.2 + 3.2^2 + 4.2^3 + ... + 100.2^{99}$ is clearly an AGP with A.P. : 1 + 2 + 3 + ... 100

and G.P. : $1 + 2 + 2^2 + ... + 2^{99}$.

Let $S = 1 + 2.2 + 3.2^2 + 4.2^3 + ... + 100.2^{99}$

$\therefore\quad 2S = 2 + 2.2^2 + 3.2^3 + ... + 99.2^{99} + 100.2^{100}$

$\therefore\quad S - 2S = (1 + 2 + 2^2 + 2^3 + ... 2^{99}) - 100.2^{100}$

$\Rightarrow\quad -S=\frac{(2^{100}-1)}{2-1}-100.2^{100}$

$\left(\because S_n=\frac{a(r^n-1)}{r-1}\right.$ and this is a G.P. with 100 terms, $\left.a=1, r=2\right)$

$= 2^{100} - 1 - 100.2^{100}$

$S = 2^{100} . 100 - 2^{100} + 1 = 2^{100} . (100 - 1) + 1$

$=\mathbf{99.2^{100}+1.}$

3. Let $S=1+\frac{2}{3}+\frac{6}{3^2}+\frac{10}{3^3}+\frac{14}{3^4}+.....$ upto ∞

$\frac{1}{3}S = \frac{1}{3}+\frac{2}{3^2}+\frac{6}{3^3}+\frac{10}{3^4}+.....$ upto ∞

$$\Rightarrow \quad S - \frac{1}{3}S = 1 + \frac{1}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \text{..... upto } \infty$$

$$= \frac{4}{3} + 4\left[\frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \text{.....}\infty\right]$$

$$= \frac{4}{3} + 4\left[\frac{\frac{1}{3^2}}{1 - \frac{1}{3}}\right] \qquad \left(\because S_\infty = \frac{a}{1-r}\right)$$

$$= \frac{4}{3} + 4\left[\frac{\frac{1}{9}}{\frac{2}{3}}\right] = \frac{4}{3} + \frac{4}{6} = \frac{12}{6} = \mathbf{2.}$$

4. Let $S_\infty = 1 + 2(1 - \sin\theta) + 3(1 - \sin\theta)^2 + 4(1 - \sin\theta)^3 + \text{.....}\infty$

$\Rightarrow \quad S_\infty = 1 + 2a + 3a^2 + 4a^3 + \text{.....}\infty$ (where $a = 1 - \sin\theta$)

$\therefore \quad a S_\infty = a + 2a^2 + 3a^3 + \text{.....}\infty$

$\Rightarrow S_\infty - a S_\infty = 1 + a + a^2 + a^3 + \text{.....}\infty$

$$(1 - a) S_\infty = \frac{1}{1-a}$$

$$\Rightarrow \quad S_\infty = \frac{1}{(1-a)^2} = \frac{1}{(1 - 1 + \sin\theta)^2}$$

$$= \frac{1}{\sin^2\theta} = \text{cosec}^2\theta$$

$$\Rightarrow \quad S_\infty = \text{cosec}^2\theta = 1 + \cot^2\theta = 1 + (2\sqrt{2})^2 = 1 + 8 = \mathbf{9.}$$

5. Let $S = 4 + \dfrac{4+d}{5} + \dfrac{4+2d}{5^2} + \dfrac{4+3d}{5^3} + \text{.....}\infty$

$$\therefore \quad \frac{1}{5}S = \frac{4}{5} + \frac{4+d}{5^2} + \frac{4+2d}{5^3} + \text{.....}\infty$$

$$\Rightarrow \quad S - \frac{1}{5}S = 4 + \frac{d}{5} + \frac{d}{5^2} + \frac{d}{5^3} + \text{.....}\infty$$

$$\Rightarrow \quad \frac{4}{5}S = 4 + \frac{d}{5}\left[1 + \frac{1}{5} + \frac{1}{5^2} + \text{.....}\infty\right]$$

$$\Rightarrow \quad \frac{4}{5}S = 4 + \frac{d}{5} \times \frac{1}{1 - \frac{1}{5}} \quad \Rightarrow \quad \frac{4}{5}S = 4 + \frac{d}{5} \times \frac{5}{4}$$

$$\Rightarrow \quad \frac{4}{5}S = 4 + \frac{d}{4} \quad \Rightarrow \quad S = 5 + \frac{5d}{16}$$

Given $S = 10$

$$\therefore \quad 5 + \frac{5d}{16} = 10 \quad \Rightarrow \quad \frac{5d}{16} = 5 \quad \Rightarrow \quad \boldsymbol{d = 16.}$$

SOME SPECIAL SERIES

KEY FACTS

1. Sum of first *n* natural numbers

Let $S = 1 + 2 + 3 + 4 + \text{.....} + n$

Clearly, this is an A.P. with $a = 1$, $l = n$, $n = n$.

$\therefore \quad S = \dfrac{n}{2}(a + l) = \dfrac{\mathbf{n(n+1)}}{\mathbf{2}}$ or $\boldsymbol{\Sigma n = \dfrac{n(n+1)}{2}}$, where **Σ stands for summation.**

2. Sum of the squares of first *n* natural numbers

Let $S = 1^2 + 2^2 + 3^2 + 4^2 + \text{.....} + n^2$.

Then, $\boldsymbol{S = \Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}}$

3. Sum of the cubes of first *n* natural numbers

Let $S = 1^3 + 2^3 + 3^3 + 4^3 + \text{.....} + n^3$

Then, $\boldsymbol{S = \Sigma n^3 = \left\{\dfrac{n(n+1)}{2}\right\}^2 = (\Sigma n)^2}$

4. If $T_n = an^3 + bn^2 + cn + d$, then

$$S_n = a\,\Sigma n^3 + b\,\Sigma n^2 + c\,\Sigma n + dn$$

$$= a\left[\frac{n(n+1)}{2}\right]^2 + b\left[\frac{n(n+1)(2n+1)}{6}\right] + c\left[\frac{n(n+1)}{2}\right] + dn.$$

SOLVED EXAMPLES

Ex. 1. Find the *n*th term and then sum to *n* terms of the following series.

(*i*) $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + ...$ (*ii*) $3.5 + 4.7 + 5.9 + ...$

Sol. (*i*) *n*th term of the given series

$$T_n = 1^2 + 2^2 + 3^2 + ... + n^2 = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\therefore S_n = \frac{1}{3}\cdot\Sigma n^3 + \frac{1}{2}\cdot\Sigma n^2 + \frac{1}{6}\cdot\Sigma n = \frac{1}{3}\cdot\frac{n^2(n+1)^2}{4} + \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{1}{6}\cdot\frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{12}\{n(n+1) + (2n+1) + 1\} = \frac{n(n+1)(n^2+3n+2)}{12} = \frac{n(n+1)(n+1)(n+2)}{12} = \mathbf{\frac{n(n+1)^2(n+2)}{12}}.$$

(*ii*) In the given series each term is the product of two factors. The factors 3, 4, 5, ... are in A.P. having 3 as the first term and 1 as the common difference, therefore the *n*th term of this A.P. = $3 + (n - 1)1 = 2 + n$. Also, the factors 5, 7, 9, ... are in A.P. having 5 as the first term and 2 as the common difference, therefore the *n*th term of this A.P. = $5 + (n - 1)^2 = 2n + 3$

$\therefore$ *n*th term of the series = $(2 + n)(2n + 3)$

$\Rightarrow$ $T_n = 2n^2 + 7n + 6$

$$\therefore S_n = 2\cdot\Sigma n^2 + 7\Sigma n + 6n = \frac{2n(n+1)(2n+1)}{6} + \frac{7n(n+1)}{2} + 6n$$

$$= \frac{2n(n+1)(2n+1) + 21n(n+1) + 36n}{6} = \mathbf{\frac{4n^3 + 27n^2 + 59n}{6}}.$$

Ex. 2. What is the sum of the series $15^2 + 16^2 + 17^2 + ... + 30^2$ equal to? (*Kerala PET 2004*)

Sol. $15^2 + 16^2 + 17^2 + ... + 30^2 = (1^2 + 2^2 + 3^2 + ... + 30^2) - (1^2 + 2^2 + 3^2 + ... + 14^2)$

$$= \sum_{k=1}^{30} k^2 - \sum_{k=1}^{14} k^2 = \frac{1}{6} \times 30 \times 31 \times (2 \times 30 + 1) - \frac{1}{6} \times 14 \times 15 \times (2 \times 14 + 1)$$

$$\left(\because \sum_{k=1}^{n} k^2 = \frac{1}{6} \times n(n+1)(2n+1)\right)$$

$$= \frac{30\times31\times61}{6} - \frac{14\times15\times29}{6} = 9455 - 1015 = \mathbf{8440.}$$

Ex. 3. What is the sum of the series $1^3 - 2^3 + 3^3 - 4^3 + ... + 9^3$ equal to? (*AIEEE 2002*)

Sol. Reqd. Sum $= (1^3 + 2^3 + 3^3 + 4^3 + ... + 9^3) - 2(2^3 + 4^3 + 6^3 + 8^3)$

$$= (1^3 + 2^3 + 3^3 + ... + 9^3) - 2^4(1^3 + 2^3 + 3^3 + 4^3)$$

$$= \sum_{k=1}^{9} k^3 - 2^4 \sum_{k=1}^{4} k^3 = \frac{9^2(9+1)^2}{4} - 2^4 \times \frac{4^2 \times (4+1)^2}{4} \quad \left(\because \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}\right)$$

$$= \frac{81\times100}{4} - \frac{16\times16\times25}{4} = 2025 - 1600 = \mathbf{425.}$$

Ex. 4. Find the sum of *n* terms of the series whose *n*th term is $n^2 + 3^n$

Sol. $t_n = n^2 + 3^n$

$\Rightarrow$ $t_1 = 1^2 + 3^1$

$t_2 = 2^2 + 3^2$

$t_3 = 3^2 + 3^3$

$t_4 = 4^2 + 3^4$

$\vdots \quad \vdots \quad \vdots$

$t_n = n^2 + 3^n$

$\therefore$ Adding columnwise, we get

$$t_1 + t_2 + t_3 + ... + t_n = (1^2 + 2^2 + 3^2 + ... + n^2) + (3^1 + 3^2 + ... + 3^n)$$

$$= \sum_{k=1}^{n} k^2 + \frac{3.(3^n - 1)}{(3-1)} = \mathbf{\frac{n(n+1)(2n+1)}{6} + \frac{3}{2}(3^n - 1).}$$

Ex. 5. If S_1, S_2 and S_3 are the sums of the first n natural numbers, their squares, their cubes respectively, then show that $9S_2^2 = S_3(1 + 8S_1)$

Sol. $S_1 = \frac{n(n+1)}{2},\ S_2 = \frac{n(n+1)(2n+1)}{6},\ S_3 = \frac{n^2(n+1)^2}{4}$

$$\therefore \quad 9S_2^2 = 9\left\{\frac{1}{6}n(n+1)(2n+1)\right\}^2 = 9\left\{\frac{1}{36}n^2(n+1)^2(2n+1)^2\right\}$$

$$= \frac{1}{4}n^2(n+1)^2(4n^2+4n+1) = \frac{1}{4}n^2(n+1)^2(4n(n+1)+1)$$

$$= \frac{1}{4}n^2(n+1)^2\left(1+8\left(\frac{1}{2}n(n+1)\right)\right) = \mathbf{S_3(1+8S_1)}.$$

PRACTICE SHEET

1. What is the nth term of the series $1 + \frac{(1+2)}{2} + \frac{(1+2+3)}{3} + ...?$

(*a*) $\frac{n+1}{2}$ (*b*) $\frac{n(n+1)}{2}$

(*c*) $n^2 - (n+1)$ (*d*) $\frac{(n+1)(2n+3)}{2}$

2. For $n \in N$, the sum of the series 2.3 + 3.5 + 4.7 + ... + $(n+1)(2n+1)$ is equal to

(*a*) $\frac{n}{3}(2n^2+3n+1)$ (*b*) $\frac{1}{6}n(n^2+n-1)$

(*c*) $\frac{n}{3}(3n^2+5n+11)$ (*d*) $\frac{n}{6}(4n^2+15n+17)$

(*AMU 2004*)

3. The value of $1^2 - 2^2 + 3^2 - 4^2 + ... + 11^2$ is equal to

(*a*) 55 (*b*) 66 (*c*) 77 (*d*) 88

(*Kerala PET 2011*)

4. Sum of n terms of the series $1^3 + 3^3 + 5^3 + 7^3 + ...$ is

(*a*) $n^2(2n^2 - 1)$ (*b*) $2n^2 + 3n^2$

(*c*) $n^3(n-1)$ (*d*) $n^3 + 8n + 4$

(*WBJEE 2010*)

5. If $S_n = 1^3 + 2^3 + ... + n^3$ and $T_n = 1 + 2 + 3 + ... + n$, then

(*a*) $S_n = T_n^2$ (*b*) $S_n = T_n^3$ (*c*) $S_n^2 = T_n$ (*d*) $S_3^2 = T_n$

(*EAMCET 2007*)

ANSWERS

1. (*a*) **2.** (*d*) **3.** (*b*) **4.** (*a*) **5.** (*a*)

HINTS AND SOLUTIONS

1. Reqd. nth term $= \frac{1+2+3+.....+n}{n}$

$$= \frac{n(n+1)}{2} \times \frac{1}{n} = \mathbf{\frac{n+1}{2}}.$$

2. Let $S = 2\cdot3 + 3\cdot5 + 4\cdot7 + ... + (n+1)(2n+1)$

$$= \sum_{k=1}^{n}(k+1)(2k+1) = \sum_{k=1}^{n}(2k^2+3k+1)$$

$$= 2\sum_{k=1}^{n}k^2 + 3\sum_{k=1}^{n}k + n$$

$= 2\times\dfrac{n(n+1)(2n+1)}{6} + 3\times\dfrac{n(n+1)}{2} + n$

$= \dfrac{n}{6}[2(n+1)(2n+1) + 9(n+1) + 6]$

$= \dfrac{n}{6}[4n^2 + 6n + 2 + 9n + 9 + 6]$

$= \boldsymbol{\dfrac{n}{6}(4n^2 + 15n + 17)}$.

3. Let $S = 1^2 - 2^2 + 3^2 - 4^2 + ... + 11^2$

$= (1^2 + 2^2 + 3^2 + 4^2 + ... + 11^2) - 2(2^2 + 4^2 + 6^2 + 8^2 + 11^2)$

$= (1^2 + 2^2 + 3^2 + 4^2 + ... + 11^2) - 2^3(1^2 + 2^2 + 3^2 + 4^2 + 5^2)$

$= \dfrac{11\times(11+1)\times(2\times11+1)}{6} - \dfrac{8\times5\times(5+1)\times(2\times5+1)}{6}$

$= \dfrac{11\times12\times23}{6} - \dfrac{8\times5\times6\times11}{6} = 22\times23 - 40\times11$

$= 506 - 440 =$ **66.**

4. Let $S_n = 1^3 + 3^3 + 5^3 + 7^3 + ...$ upto n terms

nth term of the series $= (2n-1)^3$

$\therefore\ S_n = \sum_{k=1}^{n}(2k-1)^3 = \sum_{k=1}^{n}[8k^3 - 1 - 12k^2 + 6k]$

$= 8\sum_{k=i}^{n}k^3 - 12\sum_{k=1}^{n}k^2 + 6\sum_{k=1}^{n}k - n$

$= 8\,\dfrac{n^2(n+1)^2}{4} - 12\times\dfrac{n(n+1)(2n+1)}{6} + 6\times\dfrac{n(n+1)}{2} - n$

$= 2n^2(n^2 + 2n + 1) - 2n(2n^2 + 3n + 1) + 3n(n+1) - n$

$= 2n^4 + 4n^3 + 2n^2 - 4n^3 - 6n^2 - 2n + 3n^2 + 3n - n$

$= 2n^4 + n^2 = \boldsymbol{n^2(2n^2 - 1)}$.

5. $S_n = \Sigma n^3$

$= \dfrac{n^2(n+1)^2}{4} = \left(\dfrac{n(n+1)}{2}\right)^2$ ...(*i*)

$T_n = \dfrac{n(n+1)}{2}$ ...(*ii*)

$\therefore$ From (*i*) and (*ii*)

$\boldsymbol{S_n = T_n^2}$

SELF ASSESSMENT SHEET

1. If the first term of an A.P. be 'a', second be 'b', and nth be '$2a$', then the sum of n terms is

(*a*) $\dfrac{3ab}{2(b-a)}$ (*b*) $\dfrac{2ab}{5(b-a)}$ (*c*) $\dfrac{ab}{2(b-a)}$ (*d*) $\dfrac{3ab}{(b-a)}$

(*MPPET 2010*)

2. If S_n denotes the sum of n terms of an A.P., then $S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n$ is equal to

(*a*) 0 (*b*) 1 (*c*) 2 (*d*) 3

(*Kerala PET 2000*)

3. The interior angles of a polygon are in A.P. The smallest angles is 120° and the common difference is 5°. The number of sides of the polygon is

(*a*) 8 (*b*) 9 (*c*) 12 (*d*) 19

(*AMU 2010*)

4. Divide 20 into four parts which are in A.P. and such that the product of the first and fourth is to the product of the second and third in the ratio 2 : 3. Find the product of the first and fourth term of the A.P.

(*a*) 12 (*b*) 16 (*c*) 20 (*d*) 25

5. The sum of the first p terms of an A.P. is q and the sum of the first q terms is p. Find the sum of the first $(p + q)$ terms.

(*a*) pq (*b*) $p - q$ (*c*) $-(p + q)$ (*d*) 0

6. In *a* G.P, the ratio of the sum of first 3 terms to that of the first 6 terms is 125 : 152. Find the common ratio of the G.P.

(*a*) $\dfrac{2}{3}$ (*b*) $\dfrac{3}{4}$ (*c*) $\dfrac{4}{3}$ (*d*) $\dfrac{3}{5}$

(*J&K CET 2007*)

7. The sum of n terms of the following series $1 + (1 + x) + (1 + x + x^2) + ...$ will be

(*a*) $\dfrac{1-x^n}{1-x}$ (*b*) $\dfrac{x(1-x^n)}{1-x}$

(*c*) $\dfrac{n(1-x) - x(1-x^n)}{(1-x)^2}$ (*d*) None of the above

(*AMU 2003*)

8. If $a_1, a_2, ..., a_{50}$ are in G.P., then $\dfrac{a_1 - a_3 + a_5 - ..... + a_{49}}{a_2 - a_4 + a_5 - ..... + a_{50}}$ is equal to

(*a*) 0 (*b*) 1 (*c*) $\dfrac{a_1}{a_2}$ (*d*) $\dfrac{a_1}{a_{50}}$

(*Kerala PET 2006*)

9. If a, b, c are in G.P and x, y are the arithmetic means of a, b and b, c respectively, then $\dfrac{1}{x} + \dfrac{1}{y}$ is equal to

(*a*) $\dfrac{b}{2}$ (*b*) $\dfrac{b}{3}$ (*c*) $\dfrac{2}{b}$ (*d*) $\dfrac{3}{b}$

10. If S is the sum to infinity of a G.P. whose first term is 1, then the sum of its first n terms is

(*a*) $S\left(1-\dfrac{1}{S}\right)^{n-1}$ (*b*) $S\left(1-\dfrac{1}{S}\right)^{n}$

(*c*) $S\left\{1-\left(1-\dfrac{1}{S}\right)^{n-1}\right\}$ (*d*) $S\left\{1-\left(1-\dfrac{1}{S}\right)^{n}\right\}$

(*AMU 2004*)

11. If H is the harmonic mean between P and Q, then $\frac{H}{P}+\frac{H}{Q}=$

(*a*) 2 (*b*) $\frac{1}{2}$ (*c*) $\frac{PQ}{P+Q}$ (*d*) $\frac{P+Q}{PQ}$

(*VITEEE 2007*)

12. If a, b, c are in G.P, then $\log_a 10$, $\log_b 10$ and $\log_c 10$ are in

(*a*) A.P (*b*) G.P (*c*) H.P. (*d*) None of these

13. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ be equal to the sum of the reciprocals of their squares, then bc^2, ca^2, ab^2 will be in

(*a*) A.P. (*b*) G.P. (*c*) H.P. (*d*) None of these

14. Let $a_1, a_2, \ldots, a_{10}$ be in A.P. and $h_1, h_2, \ldots, h_{10}$ be in H.P. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then a_4h_7 is

(*a*) 2 (*b*) 3 (*c*) 5 (*d*) 6 **(*IIT*)**

15. If a, b, c are in H.P, then $\frac{a}{b+c}, \frac{b}{c+a}+\frac{c}{a+b}$ will be in

(*a*) A.P. (*b*) G.P. (*c*) H.P. (*d*) None of these

(*VITEEE 2011*)

16. If $|x| < 1$, then the square root of the sum $1 + 2x + 3x^2 + 4x^3 + \ldots \infty$

(*a*) $(1 - x)$ (*b*) $(1 + x)$ (*c*) $(1 - x)^{-1}$ (*d*) $(1 + x)^{-1}$

(*Rajasthan PET 2007*)

17. The sum to n terms of the series $1+2\left(1+\frac{1}{n}\right)+3\left(1+\frac{1}{n}\right)^2 + \ldots$ is

(*a*) n^2 (*b*) $n(n-1)$ (*c*) $(n+1)^2$ (*d*) $n(n+1)$

(*EAMCET 2000*)

18. Find the sum of n terms of the series whose nth term is $2n^2 + 3n$

(*a*) $\frac{1}{2}n(n+1)(2n+5)$ (*b*) $\frac{1}{6}n(n+1)(4n+1)$

(*c*) $\frac{1}{3}(n^2-6n+3)$ (*d*) $\frac{1}{6}n(n-1)(n^2+1)$

19. Find the sum of the series $1.2^2 + 3.3^2 + 5.4^2 + \ldots$ to n terms

(*a*) $\frac{n}{2}(n^3+4n^2+4n-1)$ (*b*) $\frac{n}{2}(n+1)^2(2n+1)$

(*c*) $\frac{n^2}{3}(n+1)^2(2n+1)$ (*d*) $\frac{n}{3}(n^3+4n^2-2n+1)$

20. If the sum of first n terms of an A.P. is cn^2, then the sum of the squares of these n terms is

(*a*) $\frac{n(4n^2-1)c^2}{6}$ (*b*) $\frac{n(4n^2+1)c^2}{6}$

(*c*) $\frac{n(4n^2-1)c^2}{3}$ (*d*) $\frac{n(4n^2+1)c^2}{3}$

(*IIT 2009*)

ANSWERS

1. (*a*)	**2.** (*a*)	**3.** (*b*)	**4.** (*b*)	**5.** (*c*)	**6.** (*d*)	**7.** (*c*)	**8.** (*c*)	**9.** (*c*)	**10.** (*d*)
11. (*a*)	**12.** (*c*)	**13.** (*a*)	**14.** (*d*)	**15.** (*c*)	**16.** (*c*)	**17.** (*a*)	**18.** (*b*)	**19.** (*a*)	**20.** (*c*)

HINTS AND SOLUTIONS

1. Let the common difference of the A.P. be 'd'. Then

$$d = b - a \quad \ldots(i)$$

and $T_n = 2a = a + (n-1)d$

$\Rightarrow 2a = a + (n-1)(b-a)$ (Using (*i*))

$\Rightarrow 2a = a + bn - b - na + a$

$\Rightarrow b = n(b-a) \Rightarrow n = \frac{b}{b-a}$

$\therefore S_n = \frac{n}{2}\{a + T_n\} = \frac{b}{2(b-a)}(a+2a) = \mathbf{\frac{3ab}{2(b-a)}}$.

2. $S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n$

$= (S_{n+3} - S_{n+2}) - 2(S_{n+2} - S_{n+1}) + (S_{n+1} - S_n)$

$= t_{n+3} - 2t_{n+2} + t_{n+1}$ ... (*i*)

($\because t_n = S_n - S_{n-1}$)

$\because t_{n+1}, t_{n+2}, t_{n+3}$ are consecutive terms of an A.P,

$2t_{n+2} = t_{n+1} + t_{n+3}$ ...(*ii*)

$\therefore$ From (*i*)

Reqd. Sum $= (t_{n+3} + t_{n+1}) - 2t_{n+2}$

$= 2t_{n+2} - 2t_{n+2} = 0$ (Using (*ii*))

3. Let the polygon have n sides.

Then, sum of its interior $\angle s$ $(S_n) = (2n-4)$ rt. $\angle s$

$= (n-2) \times 180°$ ...(*i*)

Number of sides of the polygon = number of interior angles of the polygon = n.

The interior angles form an A.P. with first term = 120° and common difference 5°.

$\therefore S_n = \frac{n}{2}[2 \times 120° + (n-1) \times 5°]$

$= \frac{n}{2}[240° + 5n - 5°]$ ...(*ii*)

From (*i*) and (*ii*),

$(n-2) \times 180° = \frac{n}{2}[240° + 5n - 5°]$

$\Rightarrow$ $(n-2) \times 360 = 5n^2 + 235n$

$\Rightarrow$ $5n^2 + 235n - 360n + 720 = 0$

$\Rightarrow$ $5n^2 - 125n + 720 = 0 \Rightarrow n^2 - 25n + 144 = 0$

$\Rightarrow$ $(n-16)(n-9) = 0 \quad \Rightarrow n = 16$ or 9.

when $n = 16$, the last angle $a_n = a + (n-1)d$

$= 120° + (16-1) \times 5° = 195°$

which is not possible.

Hence, $\boldsymbol{n = 9.}$

4. Let the required four parts be $(a - 3d)$, $(a - d)$, $(a + d)$, $(a + 3d)$

Given, $a - 3d + a - d + a + d + a + 3d = 20$

$\Rightarrow$ $4a = 20 \Rightarrow a = 5$

Also, $\dfrac{(a-3d)(a+3d)}{(a-d)(a+d)} = \dfrac{2}{3}$

$\Rightarrow$ $3(a^2 - 9d^2) = 2(a^2 - d^2)$

$\Rightarrow$ $3(25 - 9d^2) = 2(25 - d^2)$

$\Rightarrow$ $75 - 27d^2 = 50 - 2d^2 \quad \Rightarrow \quad 25 = 25d^2$

$\Rightarrow$ $d = \pm 1$

Taking $d = 1$, the numbers are $5 - 3, 5 - 1, 5 + 1, 5 + 3$, *i.e.*, 2, 4, 6, 8

Taking $d = -1$, the numbers are $5 + 3, 5 + 1, 5 - 1, 5 - 3$, *i.e.*, 8, 6, 4, 2

$\therefore$ Required product in either case $= 2 \times 8 =$ **16.**

5. Since $S_n = \dfrac{n}{2}[2a + (n-1)d]$ for an A.P. whose first term $= a$, common difference $= d$, number of terms $= n$.

$\therefore$ $S_p = q = \dfrac{p}{2}(2a + (p-1)d)$

$\Rightarrow$ $2q = 2ap + p(p-1)d$ ...(*i*)

$S_q = p = \dfrac{q}{2}(2a + (q-1)d)$

$\Rightarrow$ $2p = 2aq + q(q-1)d$ ...(*ii*)

Subtracting eqn (*ii*) from eqn (*i*), we get

$2(q-p) = 2a(p-q) + (p^2 - q^2)d - (p-q)d$

$\Rightarrow -2(p-q) = 2a(p-q) + (p-q)(p+q)d - (p-q)d$

$-2 = 2a + [(p+q) - 1]d$ ...(*iii*)

Now $S_{p+q} = \dfrac{p+q}{2}[2a + (p+q-1)d]$

$= \dfrac{p+q}{2} \times -2$ (From (*iii*))

$\boldsymbol{= -(p+q).}$

6. Let a be the first term and r the common ratio of the G.P.

Then, $\dfrac{S_3}{S_6} = \dfrac{125}{152} \Rightarrow \dfrac{\dfrac{a(r^3-1)}{(r-1)}}{\dfrac{a(r^6-1)}{(r-1)}} = \dfrac{125}{152}$

$\dfrac{(r^3-1)}{(r^6-1)} = \dfrac{125}{152} \Rightarrow \dfrac{(r^3-1)}{(r^3+1)(r^3-1)} = \dfrac{125}{152}$

$\Rightarrow$ $r^3 + 1 = \dfrac{152}{125} \Rightarrow r^3 = \dfrac{152}{125} - 1 = \dfrac{27}{125}$

$\Rightarrow$ $r^3 = \left(\dfrac{3}{5}\right)^3 \Rightarrow r = \boldsymbol{\dfrac{3}{5}.}$

7. $S_n = 1 + (1 + x) + (1 + x + x^2) + \ldots.\ n$ terms

$\Rightarrow S_n = \dfrac{1}{1-x}[(1-x) + (1-x)(1+x) + (1-x)(1+x+x^2)$ $+ \ldots.$ to n terms]

$= \dfrac{1}{1-x}[(1-x) + (1-x^2) + (1-x^3) + \ldots.$ to n terms]

$= \dfrac{1}{1-x}[n - (x + x^2 + x^4 + \ldots.$ to n terms]

$= \dfrac{1}{1-x}\left[n - \dfrac{x(1-x^n)}{1-x}\right] \qquad \left(\because S_n = \dfrac{a(1-r^n)}{1-r}\right)$

$= \boldsymbol{\dfrac{n(1-x) - x(1-x^n)}{(1-x)^2}}.$

8. Let the first term of the G.P. be a and common ratio r. Then,

$\dfrac{a_1 - a_3 + a_5 - \ldots. + a_{49}}{a_2 - a_4 + a_6 - \ldots. + a_{50}} = \dfrac{a - ar^2 + ar^4 - \ldots. + ar^{48}}{ar - ar^3 + ar^5 - \ldots. + ar^{49}}$

$= \dfrac{a(1 - r^2 + r^4 - \ldots. + r^{48})}{ar(1 - r^2 + r^4 - \ldots. + r^{48})}$

$= \boldsymbol{\dfrac{a}{ar} = \dfrac{a_1}{a_2}}.$

9. a, b, c are in G.P $\Rightarrow$ $b^2 = ac$ ...(*i*)

x is the A.M. of a, b $\Rightarrow$ $x = \dfrac{a+b}{2}$ ...(*ii*)

y is the A.M. of b, c $\Rightarrow$ $y = \dfrac{b+c}{2}$ ...(*iii*)

Now $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{2}{a+b} + \dfrac{2}{b+c} = \dfrac{2(b+c) + 2(a+b)}{(a+b)(b+c)}$

$= \dfrac{2(a+2b+c)}{ab + b^2 + ac + bc} = \dfrac{2(a+2b+c)}{ab + b^2 + b^2 + bc}$

$= \dfrac{2(a+2b+c)}{b(a+2b+c)} = \boldsymbol{\dfrac{2}{b}}.$

10. Let the first term of the G.P. be a and common ratio r.

Given $S = \dfrac{a}{1-r} \Rightarrow S = \dfrac{1}{1-r}$

$\Rightarrow$ $1 - r = \dfrac{1}{S} \Rightarrow r = 1 - \dfrac{1}{S}$

Let S_n be the sum of first n terms of the series. Then,

$S_n = \dfrac{1(1-r^n)}{1-r}$ $(\because |r| < 1)$

$= \dfrac{1\left(1 - \left(1 - \dfrac{1}{S}\right)^n\right)}{\left(1 - \left(1 - \dfrac{1}{S}\right)\right)} = \boldsymbol{S\left\{1 - \left(1 - \dfrac{1}{S}\right)^n\right\}}.$

11. H being the harmonic mean between P and Q.

$$H = \frac{2PQ}{P+Q}$$

$$\therefore \quad \frac{H}{P}+\frac{H}{Q} = \frac{2Q}{P+Q}+\frac{2P}{P+Q}=\frac{2(P+Q)}{P+Q}=\mathbf{2}.$$

12. a, b, c are in G.P $\Rightarrow b^2 = ac$

$$\Rightarrow \frac{1}{\log_a 10}+\frac{1}{\log_c 10} = \log_{10} a + \log_{10} c = \log_{10}(ac)$$

$$= \log_{10} b^2 = 2\log_{10} b = \frac{2}{\log_b 10}$$

$$\Rightarrow \frac{1}{\log_a 10}, \frac{1}{\log_b 10}, \frac{1}{\log_c 10} \text{ are in A.P.}$$

$\Rightarrow \log_a 10, \log_b 10, \log_c 10$ are in H.P.

13. Let α, β, be the roots of the equation $ax^2 + bx + c = 0$. Then

$$\text{Sum of roots} = \alpha+\beta = -\frac{b}{a} \quad ...(i)$$

$$\text{Product of roots} = \alpha\beta = \frac{c}{a} \quad ...(ii)$$

Given, $\alpha+\beta = \frac{1}{\alpha^2}+\frac{1}{\beta^2}$

$$\Rightarrow \alpha+\beta = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2} = \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}$$

$$\left(-\frac{b}{a}\right) = \frac{\left(-\frac{b}{a}\right)^2-\frac{2c}{a}}{\left(\frac{c}{a}\right)^2} \quad \text{(From (i) and (ii))}$$

$$-\frac{b}{a} = \frac{\frac{b^2}{a^2}-\frac{2c}{a}}{\frac{c^2}{a^2}} \Rightarrow \frac{-bc^2}{a^3} = \frac{b^2-2ac}{a^2}$$

$$\Rightarrow -\frac{bc^2}{a} = b^2 - 2ac \Rightarrow -bc^2 = ab^2 - 2ac$$

$$\Rightarrow 2a^2c = ab^2 + bc^2$$

$\Rightarrow ab^2, a^2c, bc^2$ are in A.P.

14. Let d be the common difference of the given A.P, $a_1, a_2, ..., a_{10}$.

Then, given, $a_1 = 2$ and $a_{10} = a_1 + 9d = 3$

$$\Rightarrow 2 + 9d = 3 \Rightarrow d = \frac{1}{9}$$

$$\Rightarrow a_4 = a_1 + 3d = 2 + 3\times\frac{1}{9} = 2\frac{1}{3} = \frac{7}{3} \quad ...(i)$$

Now, $h_1, h_2, ..., h_{10}$ are in H.P.

$$\Rightarrow \frac{1}{h_1}, \frac{1}{h_2}, ..., \frac{1}{h_{10}} \text{ are in A.P.}$$

Let d_1 be the common difference of this A.P.

Given, $h_1 = 2 \Rightarrow$ First term of the A.P $= \frac{1}{h_1} = \frac{1}{2}$

Also, $h_{10} = 3 \Rightarrow \frac{1}{h_{10}} = \frac{1}{h_1} + 9d_1$

$$\Rightarrow \frac{1}{3} = \frac{1}{2}+9d_1 \Rightarrow 9d_1 = \frac{1}{3}-\frac{1}{2} = -\frac{1}{6}$$

$$\Rightarrow d_1 = -\frac{1}{54}$$

$$\therefore \frac{1}{h_7} = \frac{1}{h_1}+6d_1 = \frac{1}{2}+6\times-\frac{1}{54} = \frac{1}{2}-\frac{1}{9} = \frac{7}{18}$$

$$\Rightarrow h_7 = \frac{18}{7} \quad ...(ii)$$

$\therefore$ From (i) and (ii) $a_4h_7 = \frac{7}{3}\times\frac{18}{7} = \mathbf{6}.$

15. a, b, c are in H.P.

$$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in A.P}$$

$$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c} \text{ are in A.P}$$

(Multiplying each term by $(a + b + c)$)

$$\Rightarrow 1+\frac{b+c}{a}, 1+\frac{a+c}{b}, 1+\frac{a+b}{c} \text{ are in A.P}$$

$$\Rightarrow \frac{b+c}{a}, \frac{a+c}{b}, \frac{a+b}{c} \text{ are in A.P.}$$

(Subtracting 1 from each term)

$$\Rightarrow \frac{a}{b+c}, \frac{b}{a+c}, \frac{c}{a+b} \text{ are in H.P.}$$

16. Let $S_\infty = 1 + 2x + 3x^2 + 4x^3 + ... \infty$ ...(i)

S_∞ is an infinite A.G.P. with A.P. : $1 + 2 + 3 + ... \infty$

and G.P. : $1 + x + x^2 + x^3 + ... \infty$

The common ratio of the A.G.P. is x

$\therefore \quad xS_\infty = x + 2x^2 + 3x^2 + ... \infty$ ...(iii)

Eq (i) – Eq (ii)

$\Rightarrow \quad (1-x)S_\infty = 1 + x + x^2 + x^3 + ... \infty$

$$\Rightarrow (1-x)S_\infty = \frac{1}{1-x} \quad \left(\because S_\infty = \frac{a}{1-r}\right)$$

$$\Rightarrow S_\infty = \frac{1}{(1-x)^2} = (1-x)^{-2}$$

$\therefore$ Square root of $S_\infty = \mathbf{(1-x)^{-1}}$.

17.
$$S_n = 1+2\left(1+\frac{1}{n}\right)+3\left(1+\frac{1}{n}\right)^2+....+n\left(1+\frac{1}{n}\right)^{n-1}$$

$$\therefore \left(1+\frac{1}{n}\right)S_n = \left(1+\frac{1}{n}\right)+2\left(1+\frac{1}{n}\right)^2+....$$

$$+(n-1)\left(1+\frac{1}{n}\right)^{n-1}+n\left(1+\frac{1}{n}\right)^n$$

$$\left(\because S_n \text{ is an A.G.P with common ratio} \left(1+\frac{1}{n}\right)\right)$$

$$\Rightarrow \quad S_n\left[1-\left(1+\frac{1}{n}\right)\right]=1+\left(1+\frac{1}{n}\right)+\left(1+\frac{1}{n}\right)^2+\ldots.\left(1+\frac{1}{n}\right)^{n-1}-n\left(1+\frac{1}{n}\right)^n$$

$$\Rightarrow \quad -\frac{1}{n}S_n=\frac{1\left(\left(1+\frac{1}{n}\right)^n-1\right)}{\left(1+\frac{1}{n}\right)-1}-n\left(1+\frac{1}{n}\right)^n$$

$$\Rightarrow \quad -\frac{1}{n}S_n=n\left[\left(1+\frac{1}{n}\right)^n-1\right]-n\left(1+\frac{1}{n}\right)^n$$

$$\Rightarrow \quad -\frac{1}{n}S_n=-n \quad \Rightarrow \quad S_n=\boldsymbol{n^2}.$$

18. $T_n = 2n^2 + 3n \quad \therefore \quad S_n = 2\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k$

$$= 2\times\frac{1}{6}\times n\,(n+1)\,(2n+1)+3\times\frac{n\,(n+1)}{2}$$

$$= \frac{1}{3}\times n\,(n+1)\,(2n+1)+\frac{3}{2}\,n\,(n+1)$$

$$= n\,(n+1)\left[\frac{2n+1}{3}+\frac{3}{2}\right]=n\,(n+1)\left[\frac{4n+2+9}{6}\right]$$

$$= \frac{\boldsymbol{n\,(n+1)\,(4n+1)}}{\boldsymbol{6}}.$$

19. nth term (T_n) of the given series

$= (2n-1)\,(n+1)^2 = (2n-1)\,(n^2+2n+1)$

$= 2n^3 - n^2 + 4n^2 - 2n + 2n - 1 = 2n^3 + 3n^2 - 1$

$$\therefore \; S_n = 2\sum_{n=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 - n$$

$$= 2\times\frac{n^2\,(n+1)^2}{4}+3\times\frac{1}{6}\times n\,(n+1)\,(2n+1)-n$$

$$= \frac{1}{2}[n^2\,(n^2+2n+1)+(n^2+n)\,(2n+1)-2n]$$

$$= \frac{1}{2}\left[n^4+2n^3+n^2+2n^3+2n^2+n^2+n-2n\right]$$

$$= \frac{1}{2}[n^4+4n^3+4n^2-n]=\frac{\boldsymbol{n}}{\boldsymbol{2}}\left[\boldsymbol{n^3+4n^2+4n-1}\right].$$

20. Let the sum of first n terms of the A.P, $S_n = cn^2$

$\therefore \quad S_{n-1} = c\,(n-1)^2 = c\,(n^2 - 2n + 1)$

$\therefore$ nth term of the A.P. $= S_n - S_{n-1}$

$= cn^2 - c\,(n^2 - 2n + 1) = c\,(2n - 1)$

Let S_n^2 be the sum of the squares of these n terms. Then,

$$S_n^2 = \sum_{k=1}^{n} t_k^{\,2} \qquad \text{where } t_k = c\,(2k-1)$$

$$= \sum_{k=1}^{n}\left[c^2\,(2k-1)^2\right]=\sum_{k=1}^{n}\left[c^2\,.\,4k^2-c^2\,.\,4k+c^2\right]$$

$$= 4c^2\sum_{k=1}^{n}k^2-4c^2\sum_{k=1}^{n}k+c^2n$$

$$= 4c^2\times\frac{n\,(n+1)\,(2n+1)}{6}-4c^2\times\frac{n\,(n+1)}{2}+c^2n$$

$$= \frac{c^2n}{6}\left[4\,(n+1)\,(2n+1)-12\,(n+1)+6\right]$$

$$= \frac{c^2n}{6}\left[8n^2+12n+4-12n-12+6\right]$$

$$= \frac{c^2n}{6}[8n^2-2] = \frac{\boldsymbol{n\,(4n^2-1)\,c^2}}{\boldsymbol{3}}.$$