17. STRAIGHT LINES & PAIR OF STRAIGHT LINES

1. DISTANCE FORMULA :

The distance between the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is $\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$.

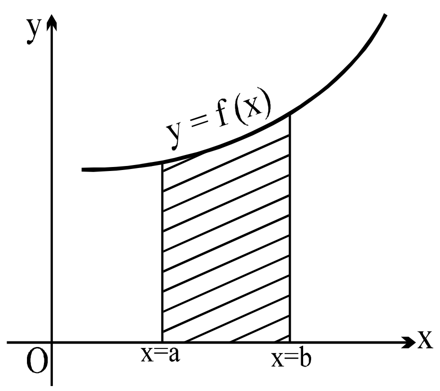
2. SECTION FORMULA :

If $P(x , y)$ divides the line joining $A(x_1 , y_1)$ & $B(x_2 , y_2)$ in the ratio m : n, then ;

$x = \dfrac{mx_2+nx_1}{m+n}$; $y = \dfrac{my_2+ny_1}{m+n}$ If $\dfrac{m}{n}$ is positive, the division is internal, but if $\dfrac{m}{n}$ is negative, the division is external .

Note : If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically ; $\dfrac{2}{AB}=\dfrac{1}{AP}+\dfrac{1}{AQ}$ i.e. AP, AB & AQ are in H.P.

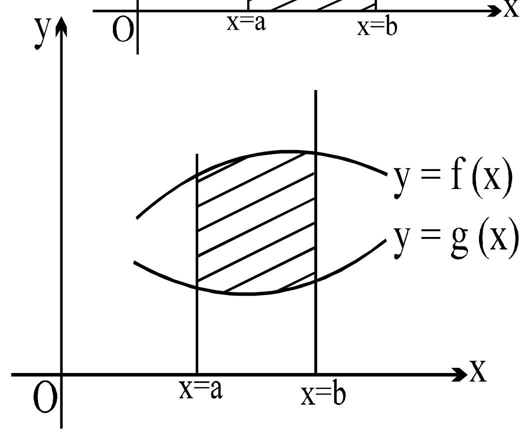


3. CENTROID AND INCENTRE :

If $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ are the vertices of triangle ABC, whose sides BC, CA, AB are of lengths a, b, c respectively, then the coordinates of the centroid are : $\left(\dfrac{x_1+x_2+x_3}{3}, \dfrac{y_1+y_2+y_3}{3}\right)$ & the coordinates of the incentre are : $\left(\dfrac{ax_1+bx_2+cx_3}{a+b+c}, \dfrac{ay_1+by_2+cy_3}{a+b+c}\right)$ Note that incentre divides the angle bisectors in the ratio $(b+c) : a$; $(c+a) : b$ & $(a+b) : c$.

REMEMBER :(i) Orthocentre , Centroid & circumcentre are always collinear & centroid divides the line joining orthocentre & cercumcentre in the ratio 2 : 1 .

(ii) In an isosceles triangle G, O, I & C lie on the same line .



4. SLOPE FORMULA :

If θ is the angle at which a straight line is inclined to the positive direction of x–axis, & $0° \le \theta < 180°$, $\theta \ne 90°$, then the slope of the line, denoted by m, is defined by $m = \tan\theta$. If θ is 90°, m does not exist, but the line is parallel to the y–axis.If θ = 0, then m = 0 & the line is parallel to the x–axis. If $A (x_1, y_1)$ & $B (x_2, y_2)$, $x_1 \ne x_2$, are points on a straight line, then the slope m of the line is given by: $m = \left(\dfrac{y_1-y_2}{x_1-x_2}\right)$.

5. CONDITION OF COLLINEARITY OF THREE POINTS – (SLOPE FORM) : Points $A (x_1, y_1)$, $B (x_2, y_2)$, $C(x_3, y_3)$ are collinear if $\left(\dfrac{y_1-y_2}{x_1-x_2}\right) = \left(\dfrac{y_2-y_3}{x_2-x_3}\right)$.

6. EQUATION OF A STRAIGHT LINE IN VARIOUS FORMS :

(i) Slope – intercept form: $y = mx + c$ is the equation of a straight line whose slope is m & which makes an intercept c on the y–axis .

(ii) Slope one point form: $y - y_1 = m (x - x_1)$ is the equation of a straight line whose slope is m & which passes through the point (x_1, y_1).

(iii) Parametric form : The equation of the line in parametric form is given by $\dfrac{x-x_1}{\cos\theta} = \dfrac{y-y_1}{\sin\theta} = r$ (say). Where 'r' is the distance of any point (x , y) on the line from the fixed point (x_1, y_1) on the line. r is positive if the point (x, y) is on the right of (x_1, y_1) and negative if (x, y) lies on the left of (x_1, y_1) .www.MathsBySuhag.com , www.TekoClasses.com

(iv) Two point form : $y - y_1 = \dfrac{y_2-y_1}{x_2-x_1}(x - x_1)$ is the equation of a straight line which passes through the points (x_1, y_1) & (x_2, y_2) .

(v) Intercept form : $\dfrac{x}{a}+\dfrac{y}{b} = 1$ is the equation of a straight line which makes intercepts a & b on OX & OY respectively .

(vi) Perpendicular form : $x\cos\alpha + y\sin\alpha = p$ is the equation of the straight line where the length of the perpendicular from the origin O on the line is p and this perpendicular makes angle α with positive side of x–axis .

(vii) General Form : $ax + by + c = 0$ is the equation of a straight line in the general form

7. POSITION OF THE POINT (x_1, y_1) RELATIVE TO THE LINE $ax + by + c = 0$: If $ax_1 + by_1 + c$ is of the same sign as c, then the point (x_1, y_1) lie on the origin side of $ax + by + c = 0$. But if the sign of $ax_1 + by_1 + c$ is opposite to that of c, the point (x_1, y_1) will lie on the non-origin side of $ax + by + c = 0$.

8. THE RATIO IN WHICH A GIVEN LINE DIVIDES THE LINE SEGMENT JOINING TWO POINTS :

Let the given line $ax + by + c = 0$ divide the line segment joining $A(x_1, y_1)$ & $B(x_2, y_2)$ in the ratio m : n, then $\dfrac{m}{n} = -\dfrac{ax_1 + by_1 + c}{ax_2 + by_2 + c}$. If A & B are on the same side of the given line then $\dfrac{m}{n}$ is negative but if A & B are on opposite sides of the given line , then $\dfrac{m}{n}$ is positive

9. LENGTH OF PERPENDICULAR FROM A POINT ON A LINE :

The length of perpendicular from $P(x_1, y_1)$ on $ax + by + c = 0$ is $\left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$.

10. ANGLE BETWEEN TWO STRAIGHT LINES IN TERMS OF THEIR SLOPES :

If m_1 & m_2 are the slopes of two intersecting straight lines $(m_1 m_2 \ne -1)$ & θ is the acute angle between them, then $\tan\theta = \left|\dfrac{m_1 - m_2}{1+m_1 m_2}\right|$.

Note : Let m_1, m_2, m_3 are the slopes of three lines $L_1 = 0$; $L_2 = 0$; $L_3 = 0$ where $m_1 > m_2 > m_3$ then the interior angles of the Δ ABC found by these lines are given by,

$\tan A = \dfrac{m_1 - m_2}{1 + m_1 m_2}$; $\tan B = \dfrac{m_2 - m_3}{1 + m_2 m_3}$ & $\tan C = \dfrac{m_3 - m_1}{1 + m_3 m_1}$

11. PARALLEL LINES :

(i) When two straight lines are parallel their slopes are equal. Thus any line parallel to $ax + by + c = 0$ is of the type $ax + by + k = 0$. Where k is a parameter.

(ii) The distance between two parallel lines with equations $ax + by + c_1 = 0$ & $ax + by + c_2 = 0$ is $\left|\frac{c_1 - c_2}{\sqrt{a^2+b^2}}\right|$ Note that the coefficients of x & y in both the equations must be same.

(iii) The area of the parallelogram $= \frac{p_1 p_2}{\sin\theta}$, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides. Note that area of the parallelogram bounded by the lines $y = m_1x + c_1$, $y = m_1x + c_2$ and $y = m_2x + d_1$, $y = m_2x + d_2$ is given by $\left|\frac{(c_1 - c_2)(d_1 - d_2)}{m_1 - m_2}\right|$.

12. PERPENDICULAR LINES :

(i) When two lines of slopes m_1 & m_2 are at right angles, the product of their slopes is -1, i.e. $m_1 m_2 = -1$. Thus any line perpendicular to $ax + by + c = 0$ is of the form $bx - ay + k = 0$, where k is any parameter.www.MathsBySuhag.com , www.TekoClasses.com

(ii) St. lines $ax + by + c = 0$ & $a'x + b'y + c' = 0$ are right angles if & only if $aa' + bb' = 0$.

13. Equations of straight lines through (x_1, y_1) making angle α with $y = mx + c$ are:
$(y - y_1) = \tan(\theta - \alpha)(x - x_1)$ & $(y - y_1) = \tan(\theta + \alpha)(x - x_1)$, where $\tan\theta = m$.

14. CONDITION OF CONCURRENCY :

Three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ & $a_3x + b_3y + c_3 = 0$ are concurrent if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$. **Alternatively :** If three constants A, B & C can be found such that $A(a_1x + b_1y + c_1) + B(a_2x + b_2y + c_2) + C(a_3x + b_3y + c_3) \equiv 0$, then the three straight lines are concurrent.

15. AREA OF A TRIANGLE : If (x_i, y_i), i = 1, 2, 3 are the vertices of a triangle, then its area is equal to $\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$, provided the vertices are considered in the counter clockwise sense. The above formula will give a (–) ve area if the vertices (x_i, y_i), i = 1, 2, 3 are placed in the clockwise sense.

16. CONDITION OF COLLINEARITY OF THREE POINTS–(AREA FORM):

The points (x_i, y_i), i = 1 , 2 , 3 are collinear if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$.

17. THE EQUATION OF A FAMILY OF STRAIGHT LINES PASSING THROUGH THE POINTS OF INTERSECTION OF TWO GIVEN LINES:

The equation of a family of lines passing through the point of intersection of $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ is given by $(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$, where k is an arbitrary real number.

Note: If $u_1 = ax + by + c$, $u_2 = a'x + b'y + d$, $u_3 = ax + by + c'$, $u_4 = a'x + b'y + d'$ then, $u_1 = 0$; $u_2 = 0$; $u_3 = 0$; $u_4 = 0$ form a parallelogram.
$u_2 u_3 - u_1 u_4 = 0$ represents the diagonal BD.

Proof : Since it is the first degree equation in x & y it is a straight line. Secondly point B satisfies the equation because the co–ordinates of B satisfy $u_2 = 0$ and $u_1 = 0$.
Similarly for the point D. Hence the result.
On the similar lines $u_1u_2 - u_3u_4 = 0$ represents the diagonal AC.

Note: The diagonal AC is also given by $u_1 + \lambda u_4 = 0$ and $u_2 + \mu u_3 = 0$, if the two equations are identical for some λ and μ.
[For getting the values of λ & μ compare the coefficients of x, y & the constant terms]

18. BISECTORS OF THE ANGLES BETWEEN TWO LINES :

(i) Equations of the bisectors of angles between the lines $ax + by + c = 0$ & $a'x + b'y + c' = 0$ $(ab' \neq a'b)$ are : $\frac{ax + by + c}{\sqrt{a^2+b^2}} = \pm \frac{a'x + b'y + c'}{\sqrt{a'^2+b'^2}}$

(ii) To discriminate between the acute angle bisector & the obtuse angle bisector
If θ be the angle between one of the lines & one of the bisectors, find $\tan\theta$.
If $|\tan\theta| < 1$, then $2\theta < 90°$ so that this bisector is the acute angle bisector.
If $|\tan\theta| > 1$, then we get the bisector to be the obtuse angle bisector.

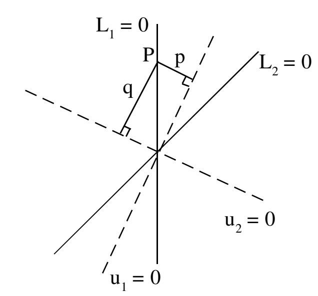
(iii) To discriminate between the bisector of the angle containing the origin & that of the angle not containing the origin. Rewrite the equations, $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that the constant terms c, c' are positive. Then; $\frac{ax + by + c}{\sqrt{a^2+b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle containing the origin & $\frac{ax + by + c}{\sqrt{a^2+b^2}} = - \frac{a'x + b'y + c'}{\sqrt{a'^2+b'^2}}$ gives the equation of the bisector of the angle not containing the origin.www.MathsBySuhag.com , www.TekoClasses.com

(iv) To discriminate between acute angle bisector & obtuse angle bisector proceed as follows
Write $ax + by + c = 0$ & $a'x + b'y + c' = 0$ such that constant terms are positive.
If $aa' + bb' < 0$, then the angle between the lines that contains the origin is acute and the equation of the bisector of this acute angle is $\frac{ax+by+c}{\sqrt{a^2+b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2+b'^2}}$

therefore $\frac{ax + by + c}{\sqrt{a^2+b^2}} = - \frac{a'x + b'y + c'}{\sqrt{a'^2+b'^2}}$ is the equation of other bisector.

If, however, $aa' + bb' > 0$, then the angle between the lines that contains the origin is obtuse & the equation of the bisector of this obtuse angle is:
$\frac{ax + by + c}{\sqrt{a^2+b^2}} = + \frac{a'x + b'y + c'}{\sqrt{a'^2+b'^2}}$; therefore $\frac{ax + by + c}{\sqrt{a^2+b^2}} = - \frac{a'x + b'y + c'}{\sqrt{a'^2+b'^2}}$
is the equation of other bisector.

(v) Another way of identifying an acute and obtuse angle bisector is as follows :
Let $L_1 = 0$ & $L_2 = 0$ are the given lines & $u_1 = 0$ and $u_2 = 0$ are the bisectors between $L_1 = 0$ & $L_2 = 0$. Take a point P on any one of the lines $L_1 = 0$ or $L_2 = 0$ and drop perpendicular on $u_1 = 0$ & $u_2 = 0$ as shown. If ,



$|p| < |q| \Rightarrow u_1$ is the acute angle bisector.
$|p| > |q| \Rightarrow u_1$ is the obtuse angle bisector.
$|p| = |q| \Rightarrow$ the lines L_1 & L_2 are perpendicular.

Note : Equation of straight lines passing through $P(x_1, y_1)$ & equally inclined with the lines $a_1x + b_1y + c_1 = 0$ & $a_2x + b_2y + c_2 = 0$ are those which are parallel to the bisectors between these two lines & passing through the point P.

19. A PAIR OF STRAIGHT LINES THROUGH ORIGIN :

(i) A homogeneous equation of degree two of the type $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines passing through the origin & if :

(a) $h^2 > ab \Rightarrow$ lines are real & distinct .

(b) $h^2 = ab \Rightarrow$ lines are coincident .

(c) $h^2 < ab \Rightarrow$ lines are imaginary with real point of intersection i.e. (0, 0)

(ii) If $y = m_1x$ & $y = m_2x$ be the two equations represented by $ax^2 + 2hxy + by^2 = 0$, then;

$m_1 + m_2 = -\frac{2h}{b}$ & $m_1 m_2 = \frac{a}{b}$.

(iii) If θ is the acute angle between the pair of straight lines represented by, $ax^2 + 2hxy + by^2 = 0$, then;

$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$. The condition that these lines are:

(a) At right angles to each other is $a + b = 0$. i.e. co–efficient of x^2 + coefficient of $y^2 = 0$.

(b) Coincident is $h^2 = ab$.

(c) Equally inclined to the axis of x is $h = 0$. i.e. coeff. of $xy = 0$.

Note: A homogeneous equation of degree n represents n straight lines passing through origin.

20. GENERAL EQUATION OF SECOND DEGREE REPRESENTING A PAIR OF STRAIGHT LINES:

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if:

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, i.e. if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only .

21. The joint equation of a pair of straight lines joining origin to the points of intersection of the line given by $lx + my + n = 0$ .... **(i)** &
the 2nd degree curve : $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ .... **(ii)**

is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{lx+my}{-n}\right) + 2fy\left(\frac{lx+my}{-n}\right) + c\left(\frac{lx+my}{-n}\right)^2 = 0$ ...... **(iii)**

(iii) is obtained by homogenizing (ii) with the help of **(i)**, by writing **(i)** in the form: $\left(\frac{lx+my}{-n}\right) = 1$.

22. The equation to the straight lines bisecting the angle between the straight lines,

$ax^2 + 2hxy + by^2 = 0$ is $\frac{x^2 - y^2}{a-b} = \frac{xy}{h}$.

23. The product of the perpendiculars, dropped from (x_1, y_1) to the pair of lines represented by the equation,

$ax^2 + 2hxy + by^2 = 0$ is $\frac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}$.

24. Any second degree curve through the four point of intersection of $f(x\,y) = 0$ & $xy = 0$ is given by $f(x\,y) + \lambda\, xy = 0$ where $f(xy) = 0$ is also a second degree curve.