16. DIFFERENTIAL EQUATION

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

DEFINITIONS :

1. An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **DIFFERENTIAL EQUATION**.www.MathsBySuhag.com , www.TekoClasses.com

2. A differential equation is said to be ordinary , if the differential coefficients have reference to a single independent variable only and it is said to be **PARTIAL** if there are two or more independent variables . We are concerned with ordinary differential equations only. eg. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ is a partial differential equation.

3. Finding the unknown function is called **SOLVING OR INTEGRATING** the differential equation . The solution of the differential equation is also called its **PRIMITIVE**, because the differential equation can be regarded as a relation derived from it.

4. The order of a differential equation is the order of the highest differential coefficient occuring in it.

5. The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occuring in it , after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned, thus the differential equation :

$f(x , y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x , y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + ... = 0$ is order m & degree p. Note that in the differential equation $e^{y'''} - xy'' + y = 0$ order is three but degree doesn't apply.

6. **FORMATION OF A DIFFERENTIAL EQUATION :**

If an equation in independent and dependent variables having some arbitrary constant is given , then a differential equation is obtained as follows :

☞ Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it .

☞ Eliminate the arbitrary constants .

The eliminant is the required differential equation . Consider forming a differential equation for $y^2 = 4a(x + b)$ where a and b are arbitary constant .

Note : A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

7. **GENERAL AND PARTICULAR SOLUTIONS :**

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the **GENERAL SOLUTION (OR COMPLETE INTEGRAL OR COMPLETE PRIMITIVE)** . A solution obtainable from the general solution by giving particular values to the constants is called a **PARTICULAR SOLUTION**.

Note that the general solution of a differential equation of the n^{th} order contains 'n' & only 'n' independent arbitrary constants. The arbitrary constants in the solution of a differential equation are said to be independent, when it is impossible to deduce from the solution an equivalent relation containing fewer arbitrary constants. Thus the two arbitrary constants A, B in the equation $y = A e^{x + B}$ are not independent since the equation can be written as $y = A e^B . e^x = C e^x$. Similarly the solution $y = A \sin x + B \cos (x + C)$ appears to contain three arbitrary constants, but they are really equivalent to two only.

8. **Elementary Types Of First Order & First Degree Differential Equations .**

TYPE–1. VARIABLES SEPARABLE : If the differential equation can be expressed as ; $f(x)dx + g(y)dy = 0$ then this is said to be variable – separable type.

A general solution of this is given by $\int f(x)\, dx + \int g(y)\, dy = c$;

where c is the arbitrary constant . consider the example $(dy/dx) = e^{x-y} + x^2 . e^{-y}$.

Note : Sometimes transformation to the polar co–ordinates facilitates separation of variables.

In this connection it is convenient to remember the following differentials. If $x = r \cos\theta$; $y = r \sin\theta$ then,

(i) $x\, dx + y\, dy = r\, dr$ **(ii)** $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$ **(iii)** $x\, dy - y\, dx = r^2 d\theta$

If $x = r \sec\theta$ & $y = r \tan\theta$ then $x\, dx - y\, dy = r\, dr$ and $x\, dy - y\, dx = r^2 \sec\theta\, d\theta$.

TYPE–2 : $\frac{dy}{dx} = f(ax + by + c)$, $b \neq 0$. To solve this , substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved. Consider the example $(x + y)^2 \frac{dy}{dx} = a^2$.

TYPE–3. HOMOGENEOUS EQUATIONS :www.MathsBySuhag.com , www.TekoClasses.com

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$ where $f(x , y)$ & $\phi(x , y)$ are homogeneous functions of x & y , and of the same degree , is called **HOMOGENEOUS** . This equation may also be reduced to the form $\frac{dy}{dx} = g\left(\frac{x}{y}\right)$ & is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variables separable. Consider $\frac{dy}{dx} + \frac{y(x + y)}{x^2} = 0$.

TYPE–4. EQUATIONS REDUCIBLE TO THE HOMOGENEOUS FORM :

If $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$; where $a_1 b_2 - a_2 b_1 \neq 0$, i.e. $\frac{a_1}{b_1} \neq \frac{a_2}{b_2}$

then the substitution $x = u + h$, $y = v + k$ transform this equation to a homogeneous type in the new variables u and v where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in Type – 3. If

(i) $a_1 b_2 - a_2 b_1 = 0$, then a substitution $u = a_1 x + b_1 y$ transforms the differential equation to an equation with variables separable. and

(ii) $b_1 + a_2 = 0$, then a simple cross multiplication and substituting $d(xy)$ for $x\, dy + y\, dx$ & integrating term by term yields the result easily.

Consider $\frac{dy}{dx} = \frac{x - 2y + 5}{2x + y - 1}$; $\frac{dy}{dx} = \frac{2x + 3y - 1}{4x + 6y - 5}$ & $\frac{dy}{dx} = \frac{2x - y + 1}{6x - 5y + 4}$

(iii) In an equation of the form : $y f(xy)\, dx + x g(xy) dy = 0$ the variables can be separated by the substitution $xy = v$.

IMPORTANT NOTE :

(a) The function $f(x , y)$ is said to be a homogeneous function of degree n if for any real number t $(\neq 0)$, we have $f(tx , ty) = t^n f(x , y)$.

For e.g. $f(x, y) = ax^{2/3} + hx^{1/3} . y^{1/3} + by^{2/3}$ is a homogeneous function of degree 2/3

(b) A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is homogeneous if $f(x, y)$ is a homogeneous function of degree zero i.e. $f(tx, ty) = t^{\circ} f(x, y) = f(x, y)$. The function f does not depend on x & y separately but only on their ratio $\frac{y}{x}$ or $\frac{x}{y}$.

LINEAR DIFERENTIAL EQUATIONS :www.MathsBySuhag.com , www.TekoClasses.com

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are not multiplied together The nth order linear differential equation is of the form ;www.MathsBySuhag.com , www.TekoClasses.com

$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + ... + a_n(x) . y = \phi(x)$. Where $a_0(x)$, $a_1(x)$ ..... $a_n(x)$ are called the coefficients of the differential equation. Note that a linear differential equation is always of the first degree but every differental equation of the first degree need not be

linear. e.g. the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$ is not linear, though its degree is 1.

TYPE – 5. LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER :

The most general form of a linear differential equations of first order is $\frac{dy}{dx} + Py = Q$, where

P& Q are functions of x . To solve such an equation multiply both sides by $e^{\int P dx}$.

NOTE : (1) The factor $e^{\int P dx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x & y , is called integrating factor of the differential equation popularly abbreviated as I. F.

(2) It is very important to remember that on multiplying by the integrating factor , the left hand side becomes the derivative of the product of y and the I. F.

(3) Some times a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation ;

$(x + y + 1)\frac{dy}{dx} = y^2 + 3$ can be written as $(y^2 + 3)\frac{dx}{dy} = x + y + 1$ which is a linear differential equation.

TYPE–6. EQUATIONS REDUCIBLE TO LINEAR FORM :

The equation $\frac{dy}{dx} + py = Q . y^n$ where P & Q functions of x , is reducible to the linear form by dividing it by y^n & then substituting $y^{-n+1} = Z$. Its solution can be obtained as in **Type–5**. Consider the example $(x^3 y^2 + xy)\, dx = dy$.

The equation $\frac{dy}{dx} + Py = Q . y^n$ is called **BERNOULI'S EQUATION**.

9. TRAJECTORIES :

Suppose we are given the family of plane curves. $\Phi(x, y, a) = 0$ depending on a single parameter a. A curve making at each of its points a fixed angle α with the curve of the family passing through that point is called an ***isogonal trajectory*** of that family ; if in particular $\alpha = \pi/2$, then it is called an ***orthogonal trajectory***.

Orthogonal trajectories : We set up the differential equation of the given family of curves. Let it be of the form $F(x, y, y') = 0$ The differential equation of the orthogonal trajectories is of the form $F\left(x, y, -\frac{1}{y'}\right) = 0$ The general integral of this equation $\Phi_1(x, y, C) = 0$ gives the family of orthogonal trajectories.

Note : Following exact differentials must be remembered :

(i) $x\,dy + y\,dx = d(xy)$

(ii) $\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$

(iii) $\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$

(iv) $\frac{x\,dy + y\,dx}{xy} = d(\ln xy)$

(v) $\frac{dx + dy}{x + y} = d(\ln(x+y))$

(vi) $\frac{x\,dy - y\,dx}{xy} = d\left(\ln\frac{y}{x}\right)$

(vii) $\frac{y\,dx - x\,dy}{xy} = d\left(\ln\frac{x}{y}\right)$

(viii) $\frac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$

(ix) $\frac{y\,dx - x\,dy}{x^2 + y^2} = d\left(\tan^{-1}\frac{x}{y}\right)$

(x) $\frac{x\,dx + y\,dy}{x^2 + y^2} = d\left[\ln\sqrt{x^2 + y^2}\right]$

(xi) $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2 y^2}$

(xii) $d\left(\frac{e^x}{y}\right) = \frac{y e^x\,dx - e^x\,dy}{y^2}$

(xiii) $d\left(\frac{e^y}{x}\right) = \frac{x e^y\,dy - e^y\,dx}{x^2}$ www.MathsBySuhag.com , www.TekoClasses.com