Continuity and Differentiability

QUICK RECAP

CONTINUITY

- A real valued function f is said to be continuous at a point $x = c$, if the function is defined at $x = c$ and $\lim_{x \to c} f(x) = f(c)$ or we say f is continuous at $x = c$ iff $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)$

- **Discontinuity of a Function :** A real function f is said to be discontinuous at $x = c$, if it is not continuous at $x = c$.
  i.e., f is discontinuous if any of the following reasons arise:
  (i) $\lim_{x \to c^-} f(x)$ or $\lim_{x \to c^+} f(x)$ or both does not exist.

(ii) $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$

(iii) $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \neq f(c)$

- A function f is said to be continuous in an interval (a, b) iff f is continuous at every point in the interval (a, b) ; and f is said to be continuous in the interval $[a, b]$ iff f is continuous in the interval (a, b) and it is continuous at a from the right and at b from the left.
- A function f is said to be discontinuous in the interval (a, b) if it is not continuous at atleast one point in the given interval.
- **Algebra of Continuous Functions :** If f and g be two real valued functions, continuous at $x = c$, then
  (i) $f + g$ is continuous at $x = c$.
  (ii) $f - g$ is continuous at $x = c$.
  (iii) $f \cdot g$ is continuous at $x = c$.
  (iv) $\left(\frac{f}{g}\right)$ is continuous at $x = c$, (provided $g(c) \neq 0$).
- Composition of two continuous functions is continuous *i.e.*, if f and g are two real valued functions and g is continuous at c and f is continuous at $g(c)$, then fog is continuous at c.
- The following functions are continuous everywhere.
  (i) Constant function
  (ii) Identity function
  (iii) Polynomial function
  (iv) Modulus function
  (v) Sine and cosine functions
  (vi) Exponential function

DIFFERENTIABILITY

- Let $f(x)$ be a real function and a be any real number. Then, we define
  (i) **Right-hand derivative :**
  $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$, if it exists, is called the right-hand derivative of $f(x)$ at $x = a$ and is denoted by $Rf'(a)$.
  (ii) **Left-hand derivative :**
  $\lim_{h \to 0^-} \frac{f(a-h) - f(a)}{-h}$, if it exists, is called the left-hand derivative of $f(x)$ at $x = a$ and is denoted by $Lf'(a)$.
  A function $f(x)$ is said to be differentiable at $x = a$, if $Rf'(a) = Lf'(a)$.
  The common value of $Rf'(a)$ and $Lf'(a)$ is denoted by $f'(a)$ and is known as the derivative of $f(x)$ at $x = a$. If, however, $Rf'(a) \neq Lf'(a)$ we say that $f(x)$ is not differentiable at $x = a$.
- A function is said to be differentiable in (a, b), if it is differentiable at every point of (a, b).
- Every differentiable function is continuous but the converse is not necessarily true.

SOME GENERAL DERIVATIVES

Function	Derivative	Function	Derivative	Function	Derivative
x^n	nx^{n-1}	$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\operatorname{cosec}^2 x$	$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	e^{ax}	ae^{ax}	e^x	e^x
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$; $x \in (-1,1)$	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$; $x \in (-1,1)$	$\tan^{-1} x$	$\frac{1}{1+x^2}$; $x \in R$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$; $x \in R$	$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$; $x \in R-[-1,1]$	$\operatorname{cosec}^{-1} x$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$; $x \in R-[-1,1]$
$\log_e x$	$\frac{1}{x}$; $x > 0$	a^x	$a^x \log_e a$; $a > 0$	$\log_a x$	$\frac{1}{x \log_e a}$; $x > 0$ and $a > 0$

EXPONENTIAL FUNCTION

- If a is any positive real number, then the function f defined by $f(x) = a^x$ is called the exponential function.

LOGARITHMIC FUNCTION

- Let $a > 1$ be a real number. The logarithmic function of x to the base a is the function $y = f(x) = \log_a x$ *i.e.*, $\log_a x = b$, if $x = a^b$
- The logarithm function, with base $a = 10$, is called common logarithm and with base $a = e$, is called natural logarithm.
- The function $\log_a x$ $(a > 0, \neq 1)$ has the following properties :
  - (i) $\log_a(mn) = \log_a m + \log_a n\ ;\ m, n > 0$
  - (ii) $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n\ ;\ m, n > 0$
  - (iii) $\log_a m^n = n\log_a m\ ;\ m > 0$
  - (iv) $\log_a x = \frac{\log x}{\log a}\ ;\ x > 0$
  - (v) $\log_a a = 1, \log_a 1 = 0$

SOME PROPERTIES OF DERIVATIVES

1.	Sum or Difference	$(u \pm v)' = u' \pm v'$
2.	Product Rule	$(uv)' = u'v + uv'$
3.	Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}, v \neq 0$
4.	Composite Function (Chain Rule)	(a) Let $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ (b) Let $y = f(t)$, $t = g(u)$ and $u = m(x)$, then $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{du} \times \frac{du}{dx}$
5.	Implicit Function	Here, we differentiate the function of type $f(x, y) = 0$.
6.	Logarithmic Function	If $y = u^v$, where u and v are the functions of x, then $\log y = v \log u$. Differentiating w.r.t. x, we get $\frac{d}{dx}(u^v) = u^v\left[\frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx}\right]$
7.	Parametric Function	If $x = f(t)$ and $y = g(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}, f'(t) \neq 0$
8.	Second Order Derivative	Let $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ If $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = f''(x)$ or $\frac{d^2y}{dx^2} = f''(x)$

ROLLE'S THEOREM

- Let $f : [a, b] \to R$ be a continuous function on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, then there exists some $c \in (a, b)$ such that $f'(c) = 0$

MEAN VALUE THEOREM

- Let $f : [a, b] \to R$ be a continuous function on $[a, b]$ and differentiable on (a, b), then there exists some $c \in (a, b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Previous Years' CBSE Board Questions

5.2 Continuity

VSA (1 mark)

1. The value of λ so that the function f defined by $f(x) = \begin{cases} \lambda x, \text{ if } x \le \pi \\ \cos x, \text{ if } x > \pi \end{cases}$

is continuous at $x = \pi$ is ________. *(2020)*

2. Determine the value of the constant 'k' so that the function $f(x)=\begin{cases} \dfrac{kx}{|x|}, \text{ if } x<0 \\ 3 \quad, \text{ if } x \ge 0 \end{cases}$ is continuous at $x = 0$. *(Delhi 2017)*

3. Determine the value of 'k' for which the following function is continuous at $x = 3$.

$$f(x)=\begin{cases} \dfrac{(x+3)^2-36}{x-3} &, x \ne 3 \\ k &, x=3 \end{cases}$$ *(AI 2017)*

LA 1 (4 marks)

4. Find the values of p and q, for which

$$f(x)=\begin{cases} \dfrac{1-\sin^3 x}{3\cos^2 x}, & \text{if } x<\pi/2 \\ p, & \text{if } x=\pi/2 \\ \dfrac{q(1-\sin x)}{(\pi-2x)^2}, & \text{if } x>\pi/2 \end{cases}$$

is continuous at $x = \pi/2$. *(Delhi 2016)*

5. Find the value of the constant k so that the function f, defined below, is continuous at $x = 0$, where

$$f(x)=\begin{cases} \left(\dfrac{1-\cos 4x}{8x^2}\right), & \text{if } x \ne 0 \\ k &, \text{ if } x=0 \end{cases}$$ *(AI 2014C)*

6. Find the value of k, for which

$$f(x)=\begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x<0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x<1 \end{cases}$$

is continuous at $x = 0$. *(AI 2013)*

7. If $f(x)=\begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{when } x<0 \\ a, & \text{when } x=0 \\ \dfrac{\sqrt{x}}{\left(\sqrt{16+\sqrt{x}}\right)-4}, & \text{when } x>0 \end{cases}$

and f is continuous at $x = 0$, find the value of a. *(Delhi 2013C, AI 2012C)*

8. If $f(x)=\begin{cases} 1, & \text{if } x \le 3 \\ ax+b, & \text{if } 3<x<5, \\ 7, & \text{if } x \ge 5 \end{cases}$

find the values of a and b so that $f(x)$ is a continuous function. *(AI 2013C, Delhi 2012C)*

9. Find the value of k so that the following function is continuous at $x = 2$.

$$f(x)=\begin{cases} \dfrac{x^3+x^2-16x+20}{(x-2)^2}; & x \ne 2 \\ k; & x=2 \end{cases}$$ *(Delhi 2012C)*

10. Find the value of k so that the following function is continuous at $x = \dfrac{\pi}{2}$:

$$f(x)=\begin{cases} \dfrac{k\cos x}{\pi-2x}, & \text{if } x \ne \dfrac{\pi}{2} \\ 5, & \text{if } x=\dfrac{\pi}{2} \end{cases}$$ *(Delhi 2012C)*

11. If the function $f(x)$ given by

$$f(x)=\begin{cases} 3ax+b, & \text{if } x>1 \\ 11, & \text{if } x=1 \\ 5ax-2b, & \text{if } x<1 \end{cases}$$

is continuous at $x = 1$, find the values of a and b. *(Delhi 2012C, 2011)*

12. Find the values of a and b such that the following function $f(x)$ is a continuous function :

$$f(x)=\begin{cases} 5, & x\le 1 \\ ax+b, & 2<x<10 \\ 21, & x\ge 10 \end{cases}$$

(Delhi 2011)

13. For what value of a is the function f defined by

$$f(x)=\begin{cases} a\sin\frac{\pi}{2}(x+1), & x\le 0 \\ \frac{\tan x-\sin x}{x^3}, & x>0 \end{cases}$$

continuous at $x = 0$? *(Delhi 2011)*

14. Find the relationship between a and b so that the function 'f' defined by

$$f(x)=\begin{cases} ax+1, & \text{if} \quad x\le 3 \\ bx+3, & \text{if} \quad x>3 \end{cases}$$

is continuous at $x = 3$. *(AI 2011)*

15. Discuss the continuity of the function $f(x)$ at $x=\frac{1}{2}$, when $f(x)$ is defined as follows:

$$f(x)=\begin{cases} \frac{1}{2}+x, & 0\le x<\frac{1}{2} \\ 1, & x=\frac{1}{2} \\ \frac{3}{2}+x, & \frac{1}{2}<x\le 1 \end{cases}$$

(Delhi 2011C)

16. Find the value of 'a' if the function $f(x)$ defined by

$$f(x)=\begin{cases} 2x-1, & x<2 \\ a, & x=2 \\ x+1, & x>2 \end{cases}$$

is continuous at $x = 2$. *(AI 2011C)*

5.3 Differentiability

VSA (1 mark)

17. The greatest integer function defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at $x =$ ________. *(2020)*

18. If $y = \tan^{-1} x + \cot^{-1} x$, $x \in R$, then $\frac{dy}{dx}$ is equal to ________. *(2020)*

19. If $\cos(xy) = k$, where k is a constant and $xy \ne n\pi$, $n \in Z$, then $\frac{dy}{dx}$ is equal to ________. *(2020)*

20. Differentiate $\sin^2(\sqrt{x})$ with respect to x. *(2020)*

21. Let $f(x) = x|x|$, for all $x \in R$ check its differentiability at $x = 0$. *(2020)*

22. If $f(x)=x+1$, find $\frac{d}{dx}(fof)(x)$. *(Delhi 2019)*

SA (2 marks)

23. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x. *(2018)*

24. Find $\frac{dy}{dx}$ at $x = 1$, $y=\frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$. *(Delhi 2017)*

LA 1 (4 marks)

25. Find the values of a and b, if the function f defined by $f(x)=\begin{cases} x^2+3x+a, & x\le 1 \\ bx+2, & x>1 \end{cases}$ is differentiable at $x = 1$. *(Foreign 2016)*

26. If $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$, $x^2\le 1$ then find $\frac{dy}{dx}$. *(Delhi 2015)*

27. If $f(x)=\sqrt{x^2+1}$; $g(x)=\frac{x+1}{x^2+1}$ and $h(x) = 2x–3$, then find $f'[h'(g'(x))]$. *(AI 2015)*

28. Show that the function $f(x) = |x - 1| + |x + 1|$, for all $x \in R$, is not differentiable at the points $x = -1$ and $x = 1$. *(AI 2015)*

29. Find whether the following function is differentiable at $x = 1$ and $x = 2$ or not.

$$f(x)=\begin{cases} x, & x<1 \\ 2-x, & 1\le x\le 2 \\ -2+3x-x^2, & x>2 \end{cases}$$

(Foreign 2015)

30. For what value λ of the function defined by $f(x)=\begin{cases}\lambda(x^2+2), & \text{if } x\le 0\\ 4x+6, & \text{if } x>0\end{cases}$ is continuous at $x = 0$? Hence check the differentiability of $f(x)$ at $x = 0$. *(AI 2015C)*

31. If $\cos y = x\cos(a + y)$, where $\cos a \neq \pm 1$, prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$. *(Foreign 2014)*

32. If $y=\sin^{-1}\left\{x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right\}$ and $0 < x < 1$, then find $\frac{dy}{dx}$. *(AI 2014C)*

33. Show that the function $f(x) = |x - 3|, x \in R$, is continuous but not differentiable at $x = 3$. *(Delhi 2013, AI 2012C)*

34. If $\sin y = x \sin(a + y)$, then prove that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$. *(Delhi 2012, 2011C)*

35. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ with respect to x. *(AI 2012)*

36. If $x\sqrt{1+y}+y\sqrt{1+x}=0$ for $x \neq y$. Prove the following: $\frac{dy}{dx}=\frac{-1}{(1+x)^2}$. *(Delhi 2011C)*

37. If $y = a \sin x + b \cos x$, prove that $y^2+\left(\frac{dy}{dx}\right)^2=a^2+b^2$. *(AI 2011C)*

5.4 Exponential and Logarithmic Functions

VSA (1 mark)

38. If $y = \log(\cos e^x)$, then find $\frac{dy}{dx}$. *(AI 2019)*

LA 1 (4 marks)

39. If $y = e^{x^2\cos x} + (\cos x)^x$, then find $\frac{dy}{dx}$. *(2020)*

40. If $\log(x^2+y^2)=2\tan^{-1}\left(\frac{y}{x}\right)$, show that $\frac{dy}{dx}=\frac{x+y}{x-y}$. *(Delhi 2019)*

41. If $y=\frac{x\cos^{-1}x}{\sqrt{1-x^2}}-\log\sqrt{1-x^2}$, then prove that $\frac{dy}{dx}=\frac{\cos^{-1}x}{(1-x^2)^{3/2}}$. *(Delhi 2015C)*

42. If $e^x + e^y = e^{x+y}$, prove that $\frac{dy}{dx}+e^{y-x}=0$ *(Foreign 2014)*

43. If $y=\tan^{-1}\left(\frac{a}{x}\right)+\log\sqrt{\frac{x-a}{x+a}}$, prove that $\frac{dy}{dx}=\frac{2a^3}{x^4-a^4}$. *(AI 2014C)*

44. If $\log(\sqrt{1+x^2}-x)=y\sqrt{1+x^2}$, show that $(1+x^2)\frac{dy}{dx}+xy+1=0$ *(AI 2011C)*

5.5 Logarithmic Differentiation

LA 1 (4 marks)

45. If $y = x^3(\cos x)^x + \sin^{-1}\sqrt{x}$, find $\frac{dy}{dx}$. *(2020)*

46. If $y = (\log x)^x + x^{\log x}$, then find $\frac{dy}{dx}$. *(2020, Delhi 2013)*

47. If $x^y - y^x = a^b$, find $\frac{dy}{dx}$. *(Delhi 2019)*

48. If $y = (x)^{\cos x} + (\cos x)^{\sin x}$, then find $\frac{dy}{dx}$. *(AI 2019)*

49. Differentiate the function $(\sin x)^x + \sin^{-1}\sqrt{x}$ with respect to x. *(Delhi 2017)*

50. If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$. *(AI 2017)*

51. Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x. *(AI 2016)*

52. If $y=(\sin x)^x+\sin^{-1}\sqrt{x}$, then find $\frac{dy}{dx}$. *(Delhi 2015C, 2013C)*

53. If $x^m y^n = (x + y)^{m+n}$, prove that $\frac{dy}{dx}=\frac{y}{x}$. *(Foreign 2014)*

54. If $(x-y)\cdot e^{\frac{x}{x-y}}=a$, prove that $y\frac{dy}{dx}+x=2y$.

(Delhi 2014C)

55. If $(\tan^{-1}x)^y + y^{\cot x} = 1$, then find $\frac{dy}{dx}$.

(AI 2014C)

56. If $y^x = e^{y-x}$, prove that $\frac{dy}{dx}=\frac{(1+\log y)^2}{\log y}$.

(AI 2013)

57. Differentiate the following with respect to x:

$\sin^{-1}\left(\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right)$ *(AI 2013)*

58. If $x^y = e^{x-y}$, prove that $\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$.

(AI 2013)

59. Find $\frac{dy}{dx}$, if $y=\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$. *(AI 2013C)*

60. If $(\cos x)^y = (\cos y)^x$, find $\frac{dy}{dx}$. *(Delhi 2012)*

61. If $y=x^{\sin x-\cos x}+\frac{x^2-1}{x^2+1}$, find $\frac{dy}{dx}$.

(Delhi 2012C)

62. Find $\frac{dy}{dx}$ when $y=x^{\cot x}+\frac{2x^2-3}{x^2+x+2}$.

(AI 2012C)

63. Differentiate $x^{x\cos x}+\frac{x^2+1}{x^2-1}$ w.r.t. x.

(Delhi 2011)

64. If $x^y = e^{x-y}$, show that $\frac{dy}{dx}=\frac{\log x}{\{\log(xe)\}^2}$.

(AI 2011)

5.6 Derivatives of Functions in Parametric Forms

SA (2 marks)

65. If $x = a\sec\theta$, $y = b\tan\theta$, then find $\frac{dy}{dx}$ at $\theta=\frac{\pi}{3}$. *(2020)*

66. Find the differential of $\sin^2 x$ w.r.t. $e^{\cos x}$.

(2020)

LA 1 (4 marks)

67. Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$

with respect to $\cos^{-1}x^2$. *(AI 2019)*

68. If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta=\frac{\pi}{3}$. *(2018)*

69. If $x = a\sin 2t(1 + \cos 2t)$ and $y = b\cos 2t(1 - \cos 2t)$, find the values of $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$ and $t=\frac{\pi}{3}$.

(Delhi 2016, AI 2016)

70. Differentiate

$\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\sin^{-1}\frac{2x}{1+x^2}$, if $x\in(-1,1)$

(Foreign 2016, Delhi 2014)

71. If $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$, prove that $\frac{dy}{dx}=\frac{x+y}{x-y}$. *(AI 2015C)*

72. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$, when $x\neq 0$. *(Delhi 2014)*

73. Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\sin^{-1}(2x\sqrt{1-x^2})$. *(Delhi 2014)*

74. Find the value of $\frac{dy}{dx}$ at $\theta=\frac{\pi}{4}$, if $x = ae^\theta(\sin\theta - \cos\theta)$ and $y = ae^\theta(\sin\theta + \cos\theta)$.

(AI 2014)

75. If $x = a\sin 2t(1+\cos 2t)$ and $y = b\cos 2t(1-\cos 2t)$, show that at $t=\frac{\pi}{4}$, $\left(\frac{dy}{dx}\right)=\frac{b}{a}$. *(AI 2014)*

76. If $x = \cos t(3 - 2\cos^2 t)$ and $y = \sin t(3 - 2\sin^2 t)$, find the value of $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$. *(AI 2014)*

77. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$, then prove that $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$. *(Delhi 2013C)*

78. If $x = \sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$. *(AI 2012)*

79. If $x = a(\theta - \sin\theta)$ and $y = a(1 + \cos\theta)$, find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$. *(Delhi 2011C)*

80. If $x = a(\cos t + \log\tan\frac{t}{2})$ and $y = a\sin t$, find $\frac{dy}{dx}$. *(Delhi 2011C)*

5.7 Second Order Derivative

VSA (1 mark)

81. If $y = \log_e\left(\frac{x^2}{e^2}\right)$, then $\frac{d^2y}{dx^2}$ equals

(a) $-\frac{1}{x}$ (b) $-\frac{1}{x^2}$

(c) $\frac{2}{x^2}$ (d) $-\frac{2}{x^2}$ *(2020)*

SA (2 marks)

82. If $x = at^2$, $y = 2at$, then find $\frac{d^2y}{dx^2}$. *(2020)*

83. If $x = a\cos\theta$; $y = b\sin\theta$, then find $\frac{d^2y}{dx^2}$. *(2020)*

LA 1 (4 marks)

84. If $y = (\sin^{-1}x)^2$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$. *(Delhi 2019)*

85. If $x = \sin t$, $y = \sin pt$, prove that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$. *(AI 2019, Foreign 2016)*

86. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2x = 0$. *(2018)*

87. If $x^m y^n = (x + y)^{m+n}$, prove that $\frac{d^2y}{dx^2} = 0$. *(Delhi 2017)*

88. If $e^y(x + 1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$. *(AI 2017)*

89. If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$. *(Delhi 2016, 2014)*

90. If $y = 2\cos(\log x) + 3\sin(\log x)$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ *(AI 2016)*

91. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$, show that $y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$ *(Delhi 2015, Foreign 2014, AI 2013C)*

92. If $y = e^{m\sin^{-1}x}, -1 \le x \le 1$, then show that $(1-x^2)\frac{d^2y}{dx^2} - \frac{xdy}{dx} - m^2y = 0$. *(AI 2015)*

93. If $y = (x + \sqrt{1+x^2})^n$, then show that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = n^2y$. *(Foreign 2015, Delhi 2013C)*

94. If $x = a\sec^3\theta$, $y = a\tan^3\theta$, find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$. *(Delhi 2015C)*

95. If $y = Ae^{mx} + Be^{nx}$, show that $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$. *(AI 2015C, 2014)*

96. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, then find the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. *(Delhi 2014C)*

97. If $x = a\left(\cos t + \log\tan\frac{t}{2}\right), y = a\sin t$, evaluate $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{3}$. *(Delhi 2014C)*

98. If $x = a \sin t$ and $y = a(\cos t + \log \tan(t/2))$, find $\frac{d^2y}{dx^2}$. *(Delhi 2013)*

99. If $y = \log\left[x + \sqrt{x^2 + a^2}\right]$, show that $(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$ *(Delhi 2013, 2013C)*

100. If $x = a \cos^3\theta$ and $y = a \sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{6}$. *(AI 2013)*

101. If $y = x\log\left(\frac{x}{a+bx}\right)$, then prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$. *(Delhi 2013C)*

102. If $x = \tan\left(\frac{1}{a}\log y\right)$, then show that $(1+x^2)\frac{d^2y}{dx^2} + (2x - a)\frac{dy}{dx} = 0.$ *(AI 2013C, 2011)*

103. If $x = \cos\theta$ and $y = \sin^3\theta$, then prove that $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 3\sin^2\theta(5\cos^2\theta - 1).$ *(AI 2013C)*

104. If $y = \sin^{-1} x$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$ *(Delhi 2012)*

105. If $y = (\tan^{-1}x)^2$, show that $(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$ *(Delhi 2012, AI 2012)*

106. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$ *(Delhi 2012)*

107. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, $0 < t < \frac{\pi}{2}$, find $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$. *(AI 2012, 2011C, Delhi 2012C)*

108. If $x = a\left(\cos t + \log\tan\frac{t}{2}\right), y = a\sin t,$ find $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$. *(AI 2012)*

109. If $x = \cos t + \log\tan\frac{t}{2}, y = \sin t,$ then find the value of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. *(AI 2012C)*

110. If $x = a(\theta - \sin\theta), y = a(1 + \cos\theta)$, find $\frac{d^2y}{dx^2}$. *(Delhi 2011)*

111. If $x = a(\theta + \sin\theta)$ and $y = a(1 - \cos\theta)$, find $\frac{d^2y}{dx^2}$. *(AI 2011C)*

5.8 Mean Value Theorem

SA (2 marks)

112. Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $[-\sqrt{3}, 0]$. *(AI 2017)*

Detailed Solutions

1. $\because f(x)$ is continuous at $x = \pi$

$\therefore f(\pi) = \lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^+} f(x)$ ...(i)

Here, $f(\pi) = \lambda\pi$ ...(ii)

And $\lim_{x\to\pi^+} f(x) = \lim_{h\to0} f(\pi + h)$

$= \lim_{h\to0}\cos(\pi + h) = -1$ ...(iii)

From (i), (ii) and (iii), we get

$\lambda\pi = -1 \Rightarrow \lambda = -\frac{1}{\pi}$

2. We have, $f(x) = \begin{cases} \frac{kx}{|x|}, & x < 0 \\ 3, & x \ge 0 \end{cases}$

L.H.L. $= \lim_{x\to0^-} f(x) = \lim_{x\to0}\frac{kx}{-x} = -k$

R.H.L. = $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} 3 = 3$

Since, $f(x)$ is continuous at $x = 0$.

$\therefore$ L.H.L. = R.H.L. = $f(0)$

$\Rightarrow$ $-k = 3 \Rightarrow k = -3$

3. Given, $f(x)$ is continuous at $x = 3$.

So, $\lim_{x\to 3} f(x) = f(3) \Rightarrow \lim_{x\to 3} \frac{(x+3)^2 - 36}{x-3} = k$

$$\Rightarrow \lim_{x\to 3} \frac{(x+3)^2 - 6^2}{x-3} = k$$

$$\Rightarrow \lim_{x\to 3} \frac{(x+3+6)\,(x+3-6)}{x-3} = k$$

$\Rightarrow$ $3 + 3 + 6 = k$

$\Rightarrow$ $k = 12$

4. $\because$ $f(x)$ is continuous at $\pi/2$.

$$\therefore \lim_{x\to \pi/2^-} f(x) = \lim_{x\to \pi/2^+} f(x) = f(\pi/2) \quad ...(1)$$

Now, $\lim_{x\to \pi/2^-} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2} - h\right)$

$$= \lim_{h\to 0} \frac{1-\sin^3\left(\frac{\pi}{2}-h\right)}{3\cos^2\left(\frac{\pi}{2}-h\right)} = \lim_{0} \frac{1-\cos^3 h}{3\sin^2 h}$$

$$= \lim_{h\to 0} \frac{(1-\cos h)(1+\cos^2 h+\cos h)}{3(1-\cos h)(1+\cos h)}$$

$$= \lim_{h\to 0} \frac{(1+\cos^2 h+\cos h)}{3(1+\cos h)} = \frac{1+1+1}{3(1+1)} = \frac{1}{2}$$

and $\lim_{x\to \pi/2^+} f(x) = \lim_{h\to 0} f\left(\frac{\pi}{2}+h\right)$

$$= \lim_{h\to 0} \frac{q\left[1-\sin\left(\frac{\pi}{2}+h\right)\right]}{\left[\pi - 2\left(\frac{\pi}{2}+h\right)\right]^2} = \lim_{h\to 0} \frac{q(1-\cos h)}{4h^2}$$

$$= \frac{q}{4} \times \lim_{h\to 0} \frac{2\sin^2 \frac{h}{2}}{\frac{h^2}{4}\times 4} = \frac{q}{4}\times\frac{2}{4} = \frac{q}{8}$$

and $f(\pi/2) = p \;\therefore\; \frac{1}{2} = \frac{q}{8} = p$

$\Rightarrow$ $p = \frac{1}{2}$ and $q = 4$ [From (1)]

5. $\because f(x)$ is continuous at $x = 0$.

$\therefore$ $f(0) = k$

and $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos 4x}{8x^2}$

$$= \lim_{x\to 0} \frac{2\cdot\sin^2 2x}{8x^2} = \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 1$$

$\because$ f is continuous at $x = 0$

$\therefore$ $f(0) = \lim_{x\to 0} f(x) \Rightarrow k = 1$

6. $\because$ $f(x)$ is continuous at $x = 0$

$$\therefore \lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-} f(x) \quad ...(1)$$

Now $f(0) = \frac{2\times 0+1}{0-1} = -1$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{2h+1}{h-1} = -1$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h}$$

$$= \lim_{h\to 0} \frac{\sqrt{1-kh}-\sqrt{1+kh}}{-h} \times \frac{\sqrt{1-kh}+\sqrt{1+kh}}{\sqrt{1-kh}+\sqrt{1+kh}}$$

$$= \lim_{h\to 0} \frac{(1-kh)-(1+kh)}{-h[\sqrt{1-kh}+\sqrt{1+kh}]}$$

$$= \lim_{h\to 0} \frac{2k}{\sqrt{1-kh}+\sqrt{1-kh}} = \frac{2k}{2} = k$$

$\therefore$ From (1), we get $k = -1$

7. $\because$ $f(x)$ is continuous at $x = 0$.

$$\therefore \lim_{x\to 0^+} f(x) = f(0) = \lim_{x\to 0^-} f(x) \quad ...(1)$$

Now $f(0) = a$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4}$$

$$= \lim_{h\to 0} \frac{\sqrt{h}}{\sqrt{16+\sqrt{h}}-4} \times \frac{\sqrt{16+\sqrt{h}}+4}{\sqrt{16+\sqrt{h}}+4}$$

$$= \lim_{h\to 0} \frac{\sqrt{h}\left(\sqrt{16+\sqrt{h}}+4\right)}{16+\sqrt{h}-4^2} = \lim_{h\to 0} \sqrt{16+\sqrt{h}}+4 = 8$$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{1-\cos 4(-h)}{(-h)^2}$$

$$= \lim_{h\to 0} \frac{1-\cos 4h}{h^2} = \lim_{h\to 0} \frac{2\sin^2 2h}{h^2}$$

$$= 8.\lim_{h\to 0}\left(\frac{\sin 2h}{2h}\right)^2 = 8$$

$\therefore$ From (1), we get $a = 8$

8. Continuity at $x = 3$

$\because$ $f(x)$ is continuous at $x = 3$

$\therefore$ $f(3) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$ ...(1)

Now, $f(3) = 1$

$\lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h) = \lim_{h\to 0} 1 = 1$

$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h)$

$= \lim_{h\to 0} [a(3+h)+b] = 3a+b$

$\therefore$ From (1), $3a + b = 1$ ...(2)

Continuity at $x = 5$

$\because$ $f(x)$ is continuous at $x = 5$

$\therefore$ $f(5) = \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x)$ ...(3)

Now $f(5) = 7$

$\lim_{x\to 5^+} f(x) = \lim_{h\to 0} f(5+h) = \lim_{h\to 0} 7 = 7$

$\lim_{x\to 5^-} f(x) = \lim_{h\to 0} f(5-h)$

$= \lim_{h\to 0} [a(5-h)+b] = 5a+b$

$\therefore$ From (3), $5a + b = 7$ ...(4)

Solving (2) and (4) we get

$a = 3, b = -8.$

9. $\because$ $f(x)$ is continuous at $x = 2$

$\therefore$ $\lim_{x\to 2} f(x) = f(2)$ ...(1)

Here $f(2) = k$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3+x^2-16x+20}{(x-2)^2}$$

$$= \lim_{x\to 2} \frac{(x-2)^2(x+5)}{(x-2)^2}$$

$$= \lim_{x\to 2} (x+5) = 7$$

$\therefore$ From (1), we get $k = 7$

10. $\because$ $f(x)$ is continuous at $x = \frac{\pi}{2}$,

$\therefore$ $f\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}} f(x)$ ...(1)

Here $f\left(\frac{\pi}{2}\right) = 5$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \frac{k\cos x}{\pi-2x} = \lim_{h\to 0} \frac{k\cos\left(\frac{\pi}{2}+h\right)}{\pi-2\left(\frac{\pi}{2}+h\right)}$$

$$= \lim_{h\to 0} \frac{-k\sin h}{-2h} = \frac{k}{2}\lim_{h\to 0}\frac{\sin h}{h} = \frac{k}{2}.1 = \frac{k}{2}$$

$\therefore$ From (1), we get $k = 10$

11. $\because$ $f(x)$ is continuous at $x = 1$

$\therefore$ $\lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$ ...(1)

Here $f(1) = 11$

$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} [5a(1-h)-2b]$

$= 5a - 2b$

$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0} [3a(1+h)+b] = 3a+b$

$\therefore$ From (1), we get

$5a - 2b = 11$ and $3a + b = 11$

Solving these, we get

$a = 3, b = 2.$

12. *Refer to answer 8.*

13. $\because$ $f(x)$ is continuous at $x = 0$,

$\therefore$ $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$ ...(1)

Here, $f(0) = a\sin\frac{\pi}{2} = a$

$\therefore$ $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} a\sin\left(\frac{\pi}{2}(-h+1)\right)$

$= \lim_{h\to 0} a\sin\left(\frac{\pi}{2} - h\frac{\pi}{2}\right) = a$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} \frac{\tan h - \sin h}{h^3}$$

$$= \lim_{h\to 0} \frac{\frac{\sin h}{\cos h} - \sin h}{h^3} = \lim_{h\to 0} \frac{\sin h\left(\frac{1}{\cos h} - 1\right)}{h^3}$$

$$= \lim_{h\to 0} \frac{\sin h}{h} \times \lim_{h\to 0} \frac{1-\cos h}{\cos h \cdot h^2}$$

$$= 1 \times \lim_{h\to 0} \frac{1}{\cos h} \times \lim_{x\to 0} \frac{2\sin^2\left(\frac{h}{2}\right)}{4\times\left(\frac{h}{2}\right)^2} = 1\times\frac{1}{2} = \frac{1}{2}$$

$\Rightarrow$ $a = \frac{1}{2}$

Hence, $f(x)$ is continuous at $x = 0$, if $a = \frac{1}{2}$.

14. $\because$ $f(x)$ is continuous at $x = 3$

$\therefore$ $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$ ...(1)

$$\lim_{x\to 3^-} f(x) = \lim_{h\to 0}(a(3-h)+1) = 3a+1$$

$$\lim_{x\to 3^+} f(x) = \lim_{h\to 0}(b(3+h)+3) = 3b+3$$

Also, $f(3) = 3a + 1$

From (1), $3a + 1 = 3b + 3 \Rightarrow a - b = \frac{2}{3}$,

which is the required relation between a and b.

15. Here, $f\left(\frac{1}{2}\right) = 1.$

$$\lim_{x\to\left(\frac{1}{2}\right)^-} f(x) = \lim_{h\to 0} f\left(\frac{1}{2}-h\right)$$

$$= \lim_{h\to 0}\left[\frac{1}{2}+\left(\frac{1}{2}-h\right)\right] = 1$$

$$\lim_{x\to\left(\frac{1}{2}\right)^+} f(x) = \lim_{h\to 0} f\left(\frac{1}{2}+h\right) = \lim_{h\to 0}\left[\frac{3}{2}+\left(\frac{1}{2}+h\right)\right] = 2$$

Since $\lim_{x\to\left(\frac{1}{2}\right)^+} f(x) \neq \lim_{x\to\left(\frac{1}{2}\right)^-} f(x)$

$\Rightarrow$ f is not continuous at $x = \frac{1}{2}$.

16. For f to be continuous at $x = 2$, we must have

$$\lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x) \quad \text{... (1)}$$

Now $f(2) = a$

$$\lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h) = \lim_{h\to 0}[2(2-h)-1] = 3$$

$$\lim_{x\to 2^+} f(x) = \lim_{h\to 0} f(2+h) = \lim_{h\to 0}[(2+h)+1] = 3$$

$\therefore$ From (1), we get $a = 3$

17. The greatest integer function defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at $x = 1$.

18. We have, $y = \tan^{-1} x + \cot^{-1} x$

$\Rightarrow \; y = \frac{\pi}{2} \qquad \left(\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right)$

$$\Rightarrow \frac{dy}{dx} = 0$$

19. We have, $\cos(xy) = k$

$$\Rightarrow -\sin(xy)\left(y + x\frac{dy}{dx}\right) = 0$$

$\Rightarrow \; y + x\frac{dy}{dx} = 0 \qquad [\because xy \neq n\pi]$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

20. Let $y = \sin^2(\sqrt{x})$

$$\therefore \frac{dy}{dx} = 2\sin(\sqrt{x})\cdot\cos(\sqrt{x})\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}\sin\left(2\sqrt{x}\right)$$

21. To check the differentiability of $f(x) = x|x|$ at $x = 0$.

Consider, $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h|h|-0}{h} = \lim_{h\to 0}|h| = 0$$

Hence, $f'(0)$ exists, so $f(x) = x|x|$ is differentiable at $x = 0$.

22. Given, $f(x) = x + 1$

Now, $(fof)(x) = f(f(x)) = f(x + 1)$

$= (x + 1) + 1 = x + 2$

$$\therefore \frac{d}{dx}(fof)(x) = \frac{d}{dx}(x+2) = 1$$

23. Let, $y = \tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\tan^{-1}\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\tan^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{x}{2}\right)\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2}-\frac{x}{2}\right) = -\frac{1}{2}$$

24. We have, $\sin^2 y + \cos xy = K$

Differentiating w.r.t. x on both sides, we get

$$2\sin y\cos y\frac{dy}{dx} + (-\sin xy)\left(x\frac{dy}{dx}+y\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy}$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{\left(1,\frac{\pi}{4}\right)} = \frac{\frac{\pi}{4}\sin\frac{\pi}{4}}{\sin\frac{\pi}{2}-\sin\frac{\pi}{4}} = \frac{\pi}{4(\sqrt{2}-1)}$$

25. Given that $f(x)$ is differentiable at $x = 1$. Therefore, $f(x)$ is continuous at $x = 1$

$$\Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1}(x^2+3x+a) = \lim_{x\to 1}(bx+2)$$

$\Rightarrow$ $1 + 3 + a = b + 2$

$\Rightarrow$ $a - b + 2 = 0$ ...(1)

Again, $f(x)$ is differentiable at $x = 1$. So,

(L.H.D. at $x = 1$) = (R.H.D. at $x = 1$)

$$\Rightarrow \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{(x^2+3x+a)-(4+a)}{x-1} = \lim_{x\to 1}\frac{(bx+2)-(4+a)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{x^2+3x-4}{x-1} = \lim_{x\to 1}\frac{bx-2-a}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{(x+4)(x-1)}{x-1} = \lim_{x\to 1}\frac{bx-b}{x-1} \quad \text{[From (1)]}$$

$$\Rightarrow \lim_{x\to 1}(x+4) = \lim_{x\to 1}\frac{b(x-1)}{x-1} \Rightarrow 5 = b$$

Putting $b = 5$ in (1), we get $a = 3$

Hence, $a = 3$ and $b = 5$

26. We have,

$$y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2 \le 1$$

Putting $x^2 = \cos\theta \Rightarrow \theta = \cos^{-1}(x^2)$ we get

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right) = \tan^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right) = \frac{\pi}{4}+\frac{\theta}{2}$$

$$\Rightarrow y = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}(x^2)$$

Differentiating w.r.t x on both sides, we get

$$\frac{dy}{dx} = -\frac{1\times 2x}{2\sqrt{1-x^4}} = \frac{-x}{\sqrt{1-x^4}}$$

27. Here $f(x) = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$

$$\Rightarrow f'(x) = \frac{1}{2}\cdot(x^2+1)^{-\frac{1}{2}}\cdot 2x = \frac{x}{\sqrt{x^2+1}}, \quad \text{...(1)}$$

$$g(x) = \frac{x+1}{x^2+1}$$

$$\Rightarrow g'(x) = \frac{(x^2+1)\cdot 1-(x+1)\cdot 2x}{(x^2+1)^2} = \frac{-x^2-2x+1}{(x^2+1)^2} \quad \text{...(2)}$$

and $h(x) = 2x - 3$

$\Rightarrow$ $h'(x) = 2$ ...(3)

$$\therefore \quad f'[h'(g'(x))] = f'\left[h'\left(\frac{-x^2-2x+1}{(x^2+1)^2}\right)\right] \quad \text{[Using (2)]}$$

$$= f'(2) \quad \text{[Using (3)]}$$

$$= \frac{2}{\sqrt{2^2+1}} = \frac{2}{\sqrt{5}} \quad \text{[Using (1)]}$$

28. The given function is $f(x) = |x - 1| + |x + 1|$

$$= \begin{cases} -(x-1)-(x+1), & x < -1 \\ -(x-1)+x+1, & -1 \le x \le 1 \\ x-1+x+1, & x > 1 \end{cases} = \begin{cases} -2x, & x < -1 \\ 2, & -1 \le x \le 1 \\ 2x, & x > 1 \end{cases}$$

At $x = 1$,

$$f'(1^-) = \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{2-2}{-h} = 0$$

$$f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{2(1+h)-2}{h} = \lim_{h\to 0}\frac{2h}{h} = 2$$

$\therefore$ $f'(1^-) \ne f'(1^+)$

$\Rightarrow$ f is not differentiable at $x = 1$.

At $x = -1$,

$$f'(-1^-) = \lim_{h\to 0}\frac{f(-1-h)-f(-1)}{-h}$$

$$= \lim_{h\to 0}\frac{-2(-1-h)-(2)}{-h} = \lim_{h\to 0}\frac{2h}{-h} = -2$$

$$f'(-1^+) = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0}\frac{2-2}{h} = 0$$

$\therefore$ $f'(-1^-) \ne f'(-1^+)$

$\Rightarrow$ f is not differentiable at $x = -1$.

29. At $x = 1$:

$$f'(1^-) = \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{x-1}{x-1} = 1$$

$$f'(1^+) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$$

$$= \lim_{x\to 1} \frac{2-x-1}{x-1} = \lim_{x\to 1} \frac{1-x}{x-1} = -1$$

Since, $f'(1^-) \neq f'(1^+)$

$\therefore$ $f(x)$ is not differentiable at $x = 1$.

At $x = 2$:

$$f'(2^-) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2} \frac{2-x-0}{x-2} = -1$$

$$f'(2^+) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2} \frac{-2+3x-x^2-0}{x-2} = \lim_{x\to 2} \frac{(1-x)(x-2)}{x-2} = -1$$

Since, $f'(2^-) = f'(2^+)$

$\therefore$ $f(x)$ is differentiable at $x = 2$.

30. Here $f(x) = \begin{cases} \lambda(x^2+2), & \text{if } x \le 0 \\ 4x+6, & \text{if } x > 0 \end{cases}$

At $x = 0$, $f(0) = \lambda(0^2 + 2) = 2\lambda$

$$\text{L.H.L.} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \lambda[(0-h)^2+2] = 2\lambda$$

$$\text{R.H.L.} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} [4(0+h)+6] = 6$$

$\therefore$ For f to be continuous at $x = 0$

$2\lambda = 6 \Rightarrow \lambda = 3$.

Hence the function becomes

$$f(x) = \begin{cases} 3(x^2+2), & \text{if } x \le 0 \\ 4x+6 & \text{if } x > 0 \end{cases}$$

$$f'(0^-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h}$$

$$= \lim_{h\to 0} \frac{3(h^2+2)-6}{-h} = \lim_{h\to 0} (-3h) = 0$$

and $f'(0^+) = \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{0+h} = \lim_{h\to 0} \dfrac{4h+6-6}{h} = 4$

$\Rightarrow$ $f'(0^-) \neq f'(0^+)$

$\therefore$ f is not differentiable at $x = 0$.

31. We have $\cos y = x\cos(a + y)$

$$\Rightarrow \quad x = \frac{\cos y}{\cos(a+y)}$$

Differentiating w.r.t. y on both sides, we get

$$\frac{dx}{dy} = \frac{\cos(a+y)\left(\frac{d}{dy}\cos y\right) - \cos y\left(\frac{d}{dy}\cos(a+y)\right)}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cos(a+y)(-\sin y) + \cos y \sin(a+y)}{\cos^2(a+y)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{\cos y \sin(a+y) - \cos(a+y)\sin y}{\cos^2(a+y)}$$

$$= \frac{\sin[(a+y)-y]}{\cos^2(a+y)} = \frac{\sin a}{\cos^2(a+y)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

32. We have, $y = \sin^{-1}\left(x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}\right)$

$$\Rightarrow \quad y = \sin^{-1}\left(x\sqrt{1-(\sqrt{x})^2} - \sqrt{x}\sqrt{1-x^2}\right)$$

$$\Rightarrow \ y = \sin^{-1} x - \sin^{-1}\sqrt{x}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx}(\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{x-x^2}}$$

33. $f(x) = |x - 3| = \begin{cases} x-3 & , \text{ if } x \ge 3 \\ -(x-3), & \text{if } x < 3 \end{cases}$

We have $f(3) = |3 - 3| = 0$

$$\lim_{x\to 3^+} f(x) = \lim_{h\to 0} f(3+h) = \lim_{h\to 0} (3+h)-3 = \lim_{h\to 0} h = 0$$

$$\lim_{x\to 3^-} f(x) = \lim_{h\to 0} f(3-h) = \lim_{h\to 0} [-(3-h-3)] = \lim_{h\to 0} h = 0$$

$$\because \quad \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3) = 0$$

So, $f(x)$ is continuous at $x = 3$.

Now, $Rf'(3) = \lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$

$$= \lim_{h\to 0} \frac{(3+h-3)-0}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

And $Lf'(3) = \lim_{h\to 0} \dfrac{f(3-h)-f(3)}{-h}$

$$= \lim_{h\to 0} \frac{[-(3-h-3)]-0}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

Thus, $Rf'(3) \neq Lf'(3)$

$\therefore$ $f(x)$ is not differentiable at $x = 3$.

34. *Refer to answer 31.*

35. Let $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1} x$

$\therefore \quad y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$\Rightarrow \quad y = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$\Rightarrow \quad y = \frac{1}{2}\theta = \frac{1}{2}\tan^{-1} x \quad \therefore \quad \frac{dy}{dx} = \frac{1}{2}\cdot\left(\frac{1}{1+x^2}\right)$

36. We have, $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow \quad x\sqrt{1+y} = -y\sqrt{1+x} \Rightarrow x^2(1+y) = y^2(1+x)$

$\Rightarrow \quad (x^2 - y^2) + xy(x-y) = 0 \Rightarrow y = \frac{-x}{x+1}$

$\therefore \quad \frac{dy}{dx} = \frac{(x+1)(-1)-(-x)(1)}{(x+1)^2} = \frac{-x-1+x}{(x+1)^2} = \frac{-1}{(x+1)^2}$

37. Here, $y = a\sin x + b\cos x$

$\Rightarrow \quad \frac{dy}{dx} = a\cos x - b\sin x$

Now, L.H.S. $= y^2 + \left(\frac{dy}{dx}\right)^2$

$= (a\sin x + b\cos x)^2 + (a\cos x - b\sin x)^2$

$= a^2\sin^2 x + b^2\cos^2 x + 2ab\sin x\cos x + a^2\cos^2 x + b^2\sin^2 x - 2ab\sin x\cos x$

$= a^2(\sin^2 x + \cos^2 x) + b^2(\cos^2 x + \sin^2 x)$

$= a^2 + b^2 =$ R.H.S.

38. Given $y = \log(\cos e^x)$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\cos e^x}(-\sin e^x \cdot e^x) = -e^x \tan e^x$$

39. We have, $y = e^{x^2\cos x} + (\cos x)^x$

$= e^{x^2\cos x} + e^{x(\ln\cos x)}$

$\therefore \quad \frac{dy}{dx} = e^{x^2\cos x}\frac{d}{dx}(x^2\cos x) + e^{x\ln\cos x}\frac{d}{dx}(x\ln\cos x)$

$= e^{x^2\cos x}(2x\cos x - x^2\sin x) + e^{x\ln\cos x}\left(\ln\cos x - \frac{x}{\cos x}\sin x\right)$

$= e^{x^2\cos x}(2x\cos x - x^2\sin x) + (\cos x)^x(\ln\cos x - x\tan x)$

40. Given, $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$

On differentiating w.r.t. x on both sides, we get

$$\frac{1}{x^2+y^2}\left(2x + 2y\frac{dy}{dx}\right) = 2\times\frac{1}{1+\frac{y^2}{x^2}}\frac{d}{dx}\left(\frac{y}{x}\right)$$

$\Rightarrow \quad \frac{2x}{x^2+y^2} + \frac{2y}{x^2+y^2}\frac{dy}{dx} = \frac{2x^2}{x^2+y^2}\left(\frac{1}{x}\frac{dy}{dx} + y\left(\frac{-1}{x^2}\right)\right)$

$\Rightarrow \quad \frac{dy}{dx}\left[\frac{2y}{x^2+y^2} - \frac{2x}{x^2+y^2}\right] = \frac{2x^2}{x^2+y^2}\left[\frac{-y}{x^2} - \frac{1}{x}\right]$

$\Rightarrow \quad \frac{2(y-x)}{x^2+y^2}\frac{dy}{dx} = \frac{-2x^2}{x^2+y^2}\left(\frac{y+x}{x^2}\right)$

$\Rightarrow \quad \frac{dy}{dx} = \frac{x+y}{x-y}$

41. Here $y = \frac{x\cos^{-1}x}{\sqrt{1-x^2}} - \log\sqrt{1-x^2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\sqrt{1-x^2}\cdot\frac{d}{dx}(x\cos^{-1}x) - x\cos^{-1}x\cdot\frac{d}{dx}\sqrt{1-x^2}}{1-x^2} - \frac{1}{\sqrt{1-x^2}}\cdot\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$= \frac{\sqrt{1-x^2}\cdot\left(1\cdot\cos^{-1}x - \frac{x}{\sqrt{1-x^2}}\right) - x\cos^{-1}x\left(\frac{-x}{\sqrt{1-x^2}}\right)}{1-x^2} - \frac{1}{\sqrt{1-x^2}}\left(\frac{-x}{\sqrt{1-x^2}}\right)$$

$$= \frac{\sqrt{1-x^2}\cos^{-1}x - x + \frac{x^2\cos^{-1}x}{\sqrt{1-x^2}}}{1-x^2} + \frac{x}{1-x^2}$$

$$= \frac{(1-x^2)\cos^{-1}x + x^2\cos^{-1}x}{(1-x^2)\sqrt{1-x^2}} = \frac{\cos^{-1}x}{(1-x^2)^{3/2}}$$

42. Given $e^x + e^y = e^{x+y} \Rightarrow 1 + e^{y-x} = e^y \quad$ ...(1)

Differentiating (1) w.r.t. x, we get

$e^{y-x}\cdot\frac{d}{dx}(y-x) = e^y\frac{dy}{dx}$

$\Rightarrow \quad e^{y-x}\left(\frac{dy}{dx} - 1\right) = e^y\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx}(e^{y-x}-e^{y})=e^{y-x}$$

$$\Rightarrow \frac{dy}{dx}(-1)=e^{y-x} \quad \text{[Using (1)]}$$

$$\Rightarrow \frac{dy}{dx}+e^{y-x}=0$$

43. Here, $y=\tan^{-1}\left(\frac{a}{x}\right)+\log\sqrt{\frac{x-a}{x+a}}$

$$=\tan^{-1}\left(\frac{a}{x}\right)+\frac{1}{2}\log\left(\frac{x-a}{x+a}\right)$$

$$=\tan^{-1}\left(\frac{a}{x}\right)+\frac{1}{2}[\log(x-a)-\log(x+a)]$$

Differentiating w.r.t x, we get

$$\frac{dy}{dx}=\frac{1}{1+\frac{a^2}{x^2}}\cdot\frac{d}{dx}\left(\frac{a}{x}\right)+\frac{1}{2}\left[\frac{1}{x-a}-\frac{1}{x+a}\right]$$

$$=\frac{x^2}{x^2+a^2}\cdot a\cdot\left(-\frac{1}{x^2}\right)+\frac{1}{2}\left[\frac{(x+a)-(x-a)}{x^2-a^2}\right]$$

$$=\frac{-a}{x^2+a^2}+\frac{a}{x^2-a^2}=\frac{-a(x^2-a^2)+a(x^2+a^2)}{x^4-a^4}$$

$$=\frac{2a^3}{x^4-a^4}$$

44. We have, $\log(\sqrt{1+x^2}-x)=y\sqrt{1+x^2}$

Differentiating w.r.t. x, we get

$$\frac{1}{\sqrt{1+x^2}-x}\cdot\left[\frac{1}{\sqrt{1+x^2}}\cdot x-1\right]=\frac{dy}{dx}\sqrt{1+x^2}+y\cdot\frac{x}{\sqrt{1+x^2}}$$

$$\Rightarrow \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}-x}\cdot\frac{x-\sqrt{1+x^2}}{\sqrt{1+x^2}}=(1+x^2)\frac{dy}{dx}+xy$$

$$\Rightarrow (1+x^2)\frac{dy}{dx}+xy+1=0$$

45. We have $y = x^3(\cos x)^x + \sin^{-1}\sqrt{x}$ ...(i)

Let $z = (\cos x)^x = e^{x\log\cos x}$

$$\therefore \frac{dz}{dx}=e^{x\log\cos x}\left[\frac{x(-\sin x)}{\cos x}+\log\cos x\right]$$

$$= (\cos x)^x \times [-x\tan x + \log\cos x] \quad \text{...(ii)}$$

Now, differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 3x^2(\cos x)^x + x^3(\cos x)^x[-x\tan x + \log\cos x] + \frac{1}{\sqrt{1-x}}\left(\frac{1}{2\sqrt{x}}\right) \quad \text{[Using (ii)]}$$

$$= x^2(\cos x)^x[3 - x^2\tan x + x\log\cos x] + \frac{1}{2\sqrt{x}}\left(\frac{1}{\sqrt{1-x}}\right)$$

46. Let $y = (\log x)^x + x^{\log x}$

$\therefore\ y = e^{x\log(\log x)} + e^{(\log x)^2}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=(\log x)^x\frac{d}{dx}\{x\log(\log x)\}+x^{\log x}\frac{d}{dx}\{(\log x)^2\}$$

$$=(\log x)^x\left\{x\left(\frac{1}{\log x}\right)\frac{1}{x}+\log(\log x)\right\}+x^{\log x}\left(2(\log x)\frac{1}{x}\right)$$

$$=(\log x)^x\left\{\frac{1}{\log x}+\log(\log x)\right\}+2\left(\frac{\log x}{x}\right)x^{\log x}$$

47. We have, $x^y - y^x = a^b$

Taking log on both sides, we get

$y\log x - x\log y = b\log a$ ...(i)

Now, differentiating (i) w.r.t. x on both sides, we get

$$\frac{y}{x}+(\log x)\frac{dy}{dx}-\log y-\frac{x\cdot 1}{y}\frac{dy}{dx}=0$$

$$\Rightarrow \left(\frac{y}{x}-\log y\right)=\frac{dy}{dx}\left(\frac{x}{y}-\log x\right)$$

$$\Rightarrow \frac{1}{x}(y-x\log y)=\frac{dy}{dx}\left(\frac{x-y\log x}{y}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{y\,(y-x\log y)}{x\,(x-y\log x)}$$

48. Given $y = (x)^{\cos x} + (\cos x)^{\sin x}$

Let $u = (x)^{\cos x}$, $v = (\cos x)^{\sin x}$

$\therefore\ y = u + v$

$$\Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx} \quad \text{...(i)}$$

Now, $u = x^{\cos x} \Rightarrow \log u = \cos x \log x$

Differentiating with respect to x, we get

$$\frac{1}{u}\frac{du}{dx}=\cos x\cdot\frac{1}{x}+\log x(-\sin x)$$

$$\Rightarrow \frac{du}{dx}=x^{\cos x}\left[\frac{\cos x}{x}-\sin x\log x\right] \quad \text{...(ii)}$$

Now, $v = (\cos x)^{\sin x}$ $\Rightarrow$ $\log v = \sin x \log\cos x$

Differentiating with respect to x, we get

$$\frac{1}{v}\frac{dv}{dx} = \sin x \cdot \frac{1}{\cos x}(-\sin x) + \log\cos x \cdot \cos x$$

$$\Rightarrow \frac{dv}{dx} = (\cos x)^{\sin x}\left[\frac{-\sin^2 x + \cos^2 x \log\cos x}{\cos x}\right] \quad \text{... (iii)}$$

Putting value of (ii) and (iii) into (i), we get

$$\frac{dy}{dx} = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \log x\right] + (\cos x)^{\sin x}\left[\frac{-\sin^2 x + \cos^2 x \log\cos x}{\cos x}\right]$$

49. Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

$\Rightarrow$ $y = u + v$ [where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$]

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{...(i)}$$

Now, $u = (\sin x)^x$

Taking logarithm on both sides, we get

$\log u = x \log \sin x$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\frac{1}{\sin x}(\cos x) + \log\ \sin$$

$$\Rightarrow \frac{du}{dx} = (\sin x)^x\ (x \cot x + \log \sin x) \quad \text{...(ii)}$$

$v = \sin^{-1}\sqrt{x}$

$$\Rightarrow \frac{dv}{dx} = \left(\frac{1}{\sqrt{1-x}}\right)\cdot\frac{1}{2\sqrt{x}} \quad \text{...(iii)}$$

From (i), (ii) and (iii), we get

$$\frac{dy}{dx} = (\sin x)^x (x\cot x + \log\sin x) + \frac{1}{2\sqrt{x}}\left(\frac{1}{\sqrt{1-x}}\right)$$

50. *Refer to answer 47.*

51. *Refer to answer 48.*

52. *Refer to answer 49.*

53. Given $x^m y^n = (x + y)^{m+n}$

Taking log on both the sides, we get

$\log x^m + \log y^n = (m + n)\log (x + y)$

$\Rightarrow$ $m \log x + n \log y = (m + n) \log (x + y)$

Differentiating w.r.t. x, we get

$$m\cdot\frac{1}{x} + n\cdot\frac{1}{y}\cdot\frac{dy}{dx} = (m+n)\cdot\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx+ny-my-ny}{y(x+y)}\right) = \frac{mx+nx-mx-my}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx-my}{y(x+y)}\right) = \frac{nx-my}{x(x+y)} \quad \therefore \quad \frac{dy}{dx} = \frac{y}{x}.$$

54. Here, $(x-y)\cdot e^{\frac{x}{x-y}} = a$

Taking log on both sides, we get

$$\Rightarrow \log\left\{(x-y)\cdot e^{\frac{x}{x-y}}\right\} = \log a$$

$$\Rightarrow \log(x-y) + \frac{x}{x-y} = \log a$$

Differentiating w.r.t. x, we get

$$\frac{1}{x-y}\cdot\left(1-\frac{dy}{dx}\right) + \frac{(x-y)\cdot 1 - x\left(1-\frac{dy}{dx}\right)}{(x-y)^2} = 0$$

$$\Rightarrow (x-y)\left(1-\frac{dy}{dx}\right) + x - y - x\left(1-\frac{dy}{dx}\right) = 0$$

$$\Rightarrow -y\left(1-\frac{dy}{dx}\right) + x - y = 0 \Rightarrow y\frac{dy}{dx} + x = 2y$$

55. Here, $(\tan^{-1}x)^y + y^{\cot x} = 1$

$\Rightarrow$ $u + v = 1$ where $u = (\tan^{-1}x)^y$ and $v = y^{\cot x}$

Differentiating w.r.t. x, we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad \text{...(1)}$$

Now, $u = (\tan^{-1}x)^y$

$\Rightarrow$ $\log u = y \log (\tan^{-1}x)$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = \frac{dy}{dx}\log(\tan^{-1}x) + y\cdot\frac{1}{\tan^{-1}x}\cdot\frac{1}{1+x^2}$$

$$\Rightarrow \frac{du}{dx} = (\tan^{-1}x)^y \times \left[\frac{dy}{dx}\log(\tan^{-1}x) + \frac{y}{(1+x^2)\tan^{-1}x}\right] \quad \text{...(2)}$$

And $v = y^{\cot x}$

$\Rightarrow$ $\log v = \cot x \cdot \log y$

Differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \cot x\cdot\frac{1}{y}\cdot\frac{dy}{dx} - \operatorname{cosec}^2 x\cdot\log y$$

$$\Rightarrow \frac{dv}{dx} = y^{\cot x}\left[\frac{\cot x}{y}\cdot\frac{dy}{dx} - \operatorname{cosec}^2 x\cdot\log y\right] \quad \text{...(3)}$$

From (1), (2) and (3), we get

$$(\tan^{-1} x)^y \left[\frac{dy}{dx}\log(\tan^{-1} x) + \frac{y}{(1+x^2)\tan^{-1} x}\right] + y^{\cot x}\left[\frac{\cot x}{y}\cdot\frac{dy}{dx} - \operatorname{cosec}^2 x \cdot \log y\right] = 0$$

$$\Rightarrow \frac{dy}{dx}[(\tan^{-1} x)^y \cdot \log(\tan^{-1} x) + y^{\cot x - 1}\cdot \cot x]$$

$$= y^{\cot x}\cdot \operatorname{cosec}^2 x \cdot \log y - (\tan^{-1} x)^{y-1}\cdot\frac{y}{(1+x^2)}$$

$$\therefore \frac{dy}{dx} = \frac{y^{\cot x}\cdot\operatorname{cosec}^2 x\cdot\log y - (\tan^{-1} x)^{y-1}\cdot\frac{y}{(1+x^2)}}{(\tan^{-1} x)^y.\log(\tan^{-1} x) + y^{\cot x - 1}.\cot x}$$

56. Here $y^x = e^{y-x}$

Taking log on both sides, we get

$x\log y = (y - x)\log e = y - x$

$\Rightarrow x(1 + \log y) = y$

$$\Rightarrow x = \frac{y}{1+\log y}$$

Differentiating w.r.t. y, we get

$$\frac{dx}{dy} = \frac{(1+\log y)\cdot 1 - y\cdot\frac{1}{y}}{(1+\log y)^2} = \frac{\log y}{(\ +\log y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$$

57. Let $y = \sin^{-1}\left[\frac{2^{x+1}\cdot 3^x}{1+(36)^x}\right]$

$$\Rightarrow y = \sin^{-1}\left[\frac{2\cdot 2^x\cdot 3^x}{1+(36)^x}\right] = \sin^{-1}\left[\frac{2\cdot 6^x}{1+(6^x)^2}\right]$$

Put $6^x = \tan\theta \Rightarrow \theta = \tan^{-1}6^x$

$$\therefore y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta)$$

$\Rightarrow y = 2\theta = 2\tan^{-1}(6^x)$

Now differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\cdot\frac{1}{1+(6^x)^2}\cdot\frac{d}{dx}(6^x)$$

$$= \frac{2}{1+(36)^x}\cdot 6^x\log 6 = \frac{2\log 6\cdot 6^x}{1+(36)^x}$$

58. *Refer to answer 56.*

59. *Refer to answer 57.*

60. We have, $(\cos x)^y = (\cos y)^x$

Taking log on both sides, we get

$y\log(\cos x) = x\log(\cos y)$ ... (i)

Differentiating w.r.t. x, we get

$$y\cdot\frac{1}{\cos x}\cdot(-\sin x) + \log(\cos x)\cdot\frac{dy}{dx}$$

$$= x\cdot\frac{1}{\cos y}\cdot(-\sin y)\cdot\frac{dy}{dx} + \log(\cos y)\cdot 1$$

$$\Rightarrow \frac{dy}{dx}(x\tan y + \log(\cos x)) = y\tan x + \log(\cos y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y\tan x + \log(\cos y)}{x\tan y + \log(\cos x)}$$

61. Here $y = x^{\sin x - \cos x} + \frac{x^2-1}{x^2+1}$

Let $u = x^{\sin x - \cos x}$ and $v = \frac{x^2-1}{x^2+1}$

$\therefore y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$$

Now, $u = x^{\sin x - \cos x}$

$\Rightarrow \log u = (\sin x - \cos x)\log x$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = (\sin x - \cos x)\cdot\frac{1}{x} + (\cos x + \sin x)\cdot\log x$$

$$\Rightarrow \frac{du}{dx} = x^{\sin x - \cos x} \times\left[\frac{\sin x - \cos x}{x} + (\cos x + \sin x)\cdot\log x\right] \quad ...(2)$$

$$\text{Now, } v = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$$

$$\Rightarrow \frac{dv}{dx} = 0 - 2\cdot(-1)(x^2+1)^{-2}\cdot 2x = \frac{4x}{(x^2+1)^2} \quad ...(3)$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = x^{\sin x - \cos x}\cdot\left[\frac{\sin x - \cos x}{x} + (\sin x + \cos x)\cdot\log x\right] + \frac{4x}{(x^2+1)^2}$$

62. Here, $y = x^{\cot x} + \frac{2x^2-3}{x^2+x+2}$

Let $u = x^{\cot x}, v = \frac{2x^2-3}{x^2+x+2}$

$$\Rightarrow y = u + v \Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$$

Now, $u = x^{\cot x} \Rightarrow \log u = \cot x \cdot \log x$
Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = \cot x \cdot \frac{1}{x} - \operatorname{cosec}^2 x \cdot \log x$$

$$\therefore \quad \frac{du}{dx} = x^{\cot x}\left(\frac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x\right) \quad ...(2)$$

$$\text{Also, } v = \frac{2x^2-3}{x^2+x+2}$$

$$\Rightarrow \frac{dv}{dx} = \frac{(x^2+x+2)\cdot 4x - (2x^2-3)\cdot(2x+1)}{(x^2+x+2)^2}$$

$$\Rightarrow \frac{dv}{dx} = \frac{(4x^3+4x^2+8x)-(4x^3+2x^2-6x-3)}{(x^2+x+2)^2}$$

$$\Rightarrow \frac{dv}{dx} = \frac{2x^2+14x+3}{(x^2+x+2)^2} \quad ...(3)$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = x^{\cot x}\left(\frac{\cot x}{x} - \operatorname{cosec}^2 x \cdot \log x\right) + \frac{2x^2+14x+3}{(x^2+x+2)^2}$$

63. Let $y = x^{x\cos x} + \dfrac{x^2+1}{x^2-1}$

Let $u = x^{x\cos x}$ and $v = \dfrac{x^2+1}{x^2-1}$

$\therefore \quad y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$$

Now, $u = x^{x\cos x}$
Taking log on both sides, we get
$\log u = x\cos x \cdot \log x$
Differentiating w.r.t. x, we get

$$\frac{1}{u}\cdot\frac{du}{dx} = x\cos x\frac{d}{dx}(\log x) + x\log x\frac{d}{dx}(\cos x) + \cos x\log x\frac{d}{dx}(x)$$

$$\Rightarrow \frac{1}{u}\cdot\frac{du}{dx} = x\cos x\cdot\frac{1}{x} + x\log x(-\sin x) + \cos x\log x$$

$$= \cos x - x\sin x\log x + \cos x\log x$$

$$\therefore \frac{du}{dx} = x^{x\cos x}[\cos x - x\sin x\log x + \cos x\log x] \quad ...(2)$$

$$\text{Also, } v = \frac{x^2+1}{x^2-1}$$

Differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}$$

$$= \frac{2x[x^2-1-x^2-1]}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2} \quad ...(3)$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = x^{x\cos x}[\cos x - x\sin x\log x + \cos x\log x] - \frac{4x}{(x^2-1)^2}$$

$$= x^{x\cos x}[\cos x(1+\log x) - x\sin x\log x] - \frac{4x}{(x^2-1)^2}$$

64. *Refer to answer 58.*
We have,

$$\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2} = \frac{\log x}{(\log e + x)^2} = \frac{\log x}{(\log(xe))^2}$$

65. We have, $x = a\sec\theta$, $y = b\tan\theta$

$$\therefore \quad \frac{dx}{d\theta} = a\sec\theta\tan\theta \text{ and } \frac{dy}{d\theta} = b\sec^2\theta$$

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b}{a}\cdot\frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}$$

$$= \frac{b}{a}\cdot\frac{1}{\sin\theta}$$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{b}{a}\cdot\frac{1}{\sin\frac{\pi}{3}} = \frac{b}{a}\cdot\frac{1}{\frac{\sqrt{3}}{2}} = \frac{2b}{a\sqrt{3}}$$

66. Let $u = \sin^2 x$, $v = e^{\cos x}$

$$\therefore \quad \frac{du}{dx} = 2\sin x\cos x \text{ and } \frac{dv}{dx} = e^{\cos x}(-\sin x)$$

$$\text{Now, } \frac{du}{dv} = \frac{du}{dx}\cdot\frac{dx}{dv} = \frac{2\sin x\cos x}{e^{\cos x}(-\sin x)} = \frac{-2\cos x}{e^{\cos x}}$$

67. Let $u = \tan^{-1}\left[\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ ...(i)

Putting $x^2 = \cos 2\theta$ in (i), we get

$$u = \tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right] = \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \theta\right)\right] = \frac{\pi}{4} - \theta$$

$$= \frac{\pi}{4} - \frac{\cos^{-1}(x^2)}{2}$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{2}\frac{1}{\sqrt{1-x^4}}2x = \frac{x}{\sqrt{1-x^4}} \quad \text{... (ii)}$$

Let $v = \cos^{-1}(x^2) \Rightarrow \frac{dv}{dx} = \frac{-2x}{\sqrt{1-x^4}}$ ... (iii)

Dividing (ii) by (iii), we get $\frac{du}{dv} = -\frac{1}{2}$.

68. We have, $x = a(2\theta - \sin 2\theta)$ ...(i)
and $y = a(1 - \cos 2\theta)$ ...(ii)
Differentiating (i) w.r.t. θ, we get

$\frac{dx}{d\theta} = a(2 - 2\cos 2\theta)$ ...(iii)

Differentiating (ii) w.r.t. θ, we get

$\frac{dy}{d\theta} = 2a \sin 2\theta$ ...(iv)

Now, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2a\sin 2\theta}{a(2-2\cos 2\theta)} = \frac{\sin 2\theta}{1-\cos 2\theta}$

$$\therefore \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = \frac{\sin 2\left(\frac{\pi}{3}\right)}{1-\cos\frac{2\pi}{3}} = \frac{\sin\left(\pi - \frac{\pi}{3}\right)}{1-\cos\left(\pi - \frac{\pi}{3}\right)}$$

$$= \frac{\sin\left(\frac{\pi}{3}\right)}{1+\cos\left(\frac{\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

69. $x = a \sin 2t\,(1 + \cos 2t)$, $y = b \cos 2t\,(1 - \cos 2t)$

Now, $\frac{dx}{dt} = 2a\cos 2t(1+\cos 2t) + a\sin 2t(-2\sin 2t)$

$= 2a \cos 2t + 2a[\cos^2 2t - \sin^2 2t]$

$= 2a \cos 2t + 2a \cos 4t$

Also, $\frac{dy}{dt} = -2b\sin 2t(1-\cos 2t) + b\cos 2t(2\sin 2t)$

$= -2b \sin 2t + 4b (\sin 2t \cos 2t)$

$= -2b \sin 2t + 2b \sin 4t$

So, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2b(\sin 4t - \sin 2t)}{2a(\cos 4t + \cos 2t)}$

$$\therefore \left(\frac{dy}{dx}\right)_{\text{at } t=\pi/4} = \frac{b}{a}\left[\frac{\sin\pi - \sin(\pi/2)}{\cos\pi + \cos(\pi/2)}\right]$$

$$= \frac{b}{a}\left[\frac{0-1}{-1+0}\right] = \frac{b}{a}$$

$$\left(\frac{dy}{dx}\right)_{\text{at } t=\pi/3} = \frac{b}{a}\left[\frac{\sin(4\pi/3) - \sin(2\pi/3)}{\cos(4\pi/3) + \cos(2\pi/3)}\right]$$

$$= \frac{b}{a}\left[\frac{\frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2}}{\frac{-1}{2} - \frac{1}{2}}\right] = \frac{\sqrt{3}b}{a}$$

70. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\therefore u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) \Rightarrow u = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) \Rightarrow u = \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\therefore u = \frac{\theta}{2} \Rightarrow u = \frac{1}{2}\tan^{-1}x$$

Differentiating w.r.t. x, we get

$$\frac{du}{dx} = \frac{1}{2(1+x^2)}$$

Also, let $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right) \Rightarrow v = 2\tan^{-1}x$

Differentiating w.r.t. x, we get

$$\frac{dv}{dx} = \frac{2}{1+x^2}$$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}} \Rightarrow \frac{du}{dv} = \frac{1}{4}$$

71. We have $x = ae^t(\sin t + \cos t)$

$$\Rightarrow \frac{dx}{dt} = ae^t(\sin t + \cos t) + ae^t(\cos t - \sin t) = 2ae^t\cos t$$

and $y = ae^t(\sin t - \cos t)$

$$\Rightarrow \frac{dy}{dt} = ae^t(\sin t - \cos t) + ae^t(\cos t + \sin t) = 2ae^t\sin t$$

$$\therefore \text{L.H.S.} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2ae^t\sin t}{2ae^t\cos t} = \tan t$$

Also, R.H.S$=\dfrac{x+y}{x-y}$

$$=\frac{ae^t(\sin t+\cos t)+ae^t(\sin t-\cos t)}{ae^t(\sin t+\cos t)-ae^t(\sin t-\cos t)}$$

$$=\frac{2ae^t\sin t}{2ae^t\cos t}=\tan t=\text{L.H.S.}$$

72. Let $u=\tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

Put $x = \cos\theta$

$$\therefore\ u=\tan^{-1}\left[\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right]=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$= \tan^{-1}(\tan\theta) = \theta \Rightarrow \frac{du}{d\theta}=1$$

Also let,

$$v=\cos^{-1}(2x\sqrt{1-x^2})\Rightarrow v=\cos^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$$

$$= \cos^{-1}(2\cos\theta\sin\theta) = \cos^{-1}(\sin 2\theta)$$

$$=\cos^{-1}\left(\cos\left(\frac{\pi}{2}-2\theta\right)\right)=\frac{\pi}{2}-2\theta \Rightarrow \frac{dv}{d\theta}=-2$$

Now $\dfrac{du}{dv}=\dfrac{du/d\theta}{dv/d\theta}=\dfrac{-1}{2}$

73. *Refer to answer 72.*

74. *Refer to answer 71.*

$$\therefore\ \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{2ae^\theta\cos\theta}{2ae^\theta\sin\theta}=\cot\theta$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}}=\cot\frac{\pi}{4}=1$$

75. *Refer to answer 69.*

76. Here, $x = \cos t(3 - 2\cos^2 t)$, $y = \sin t(3 - 2\sin^2 t)$

$$\Rightarrow \frac{dx}{dt}=-\sin t(3-2\cos^2 t)+\cos t\,[2\cdot 2\cos t\sin t]$$

$$= -3\sin t + 6\cos^2 t\sin t$$

and $\dfrac{dy}{dt}=\cos t(3-2\sin^2 t)+\sin t(-2\cdot 2\sin t\cos t)$

$$= 3\cos t - 6\sin^2 t\cos t$$

$$\therefore\ \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3\cos t-6\sin^2 t\cos t}{-3\sin t+6\cos^2 t\sin t}$$

$$=\frac{3\cos t\cdot\cos 2t}{3\sin t\cdot\cos 2t}=\cot t$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{4}}=\cot\frac{\pi}{4}=1$$

77. Here, $x = 2\cos\theta - \cos 2\theta$, $y = 2\sin\theta - \sin 2\theta$

$$\Rightarrow \frac{dx}{d\theta}=-2\sin\theta+2\sin 2\theta$$

and $\dfrac{dy}{d\theta}=2\cos\theta-2\cos 2\theta$

$$\therefore\ \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{2(\cos\theta-\cos 2\theta)}{2(\sin 2\theta-\sin\theta)}$$

$$=\frac{2\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{3\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}=\tan\left(\frac{3\theta}{2}\right).$$

78. Here $x=\sqrt{a^{\sin^{-1}t}}$

$$\Rightarrow \log x=\frac{1}{2}\sin^{-1}t\cdot\log a$$

Differentiating w.r.t. t, we get

$$\frac{1}{x}\cdot\frac{dx}{dt}=\frac{1}{2}\log a\cdot\frac{1}{\sqrt{1-t^2}} \quad ...(1)$$

Also, $y=\sqrt{a^{\cos^{-1}t}}$

$$\Rightarrow \log y=\frac{1}{2}\cos^{-1}t\cdot\log a$$

Differentiating w.r.t. t, we get

$$\frac{1}{y}\cdot\frac{dy}{dt}=\frac{1}{2}\log a\cdot\frac{-1}{\sqrt{1-t^2}} \quad ...(2)$$

Now dividing (2) by (1), we get

$$\frac{\frac{1}{y}\cdot\frac{dy}{dt}}{\frac{1}{x}\cdot\frac{dx}{dt}}=-1\Rightarrow\frac{dy}{dx}=-\frac{y}{x}$$

79. Here, $x = a(\theta - \sin\theta) \Rightarrow \dfrac{dx}{d\theta}=a(1-\cos\theta)$

and $y=a(1+\cos\theta) \Rightarrow \dfrac{dy}{d\theta}=a(-\sin\theta)$

$$\therefore\ \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{-a\sin\theta}{a(1-\cos\theta)}=\frac{-\sin\theta}{(1-\cos\theta)}$$

$$\therefore\ \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{3}}=\frac{-\sin\frac{\pi}{3}}{1-\cos\frac{\pi}{3}}=\frac{-\sqrt{3}/2}{1-(1/2)}=-\sqrt{3}$$

80. Here, $x=a\,(\cos t+\log\tan\frac{t}{2})$

$$\Rightarrow \frac{dx}{dt}=a\left(-\sin t+\frac{1}{\tan\frac{t}{2}}\cdot\frac{1}{2}\sec^2\frac{t}{2}\right)$$

$$= a\left(-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}}\cdot\frac{1}{2\cos^2\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$= a\frac{(-\sin^2 t + 1)}{\sin t} = \frac{a\cos^2 t}{\sin t}$$

Also, $y = a\sin t \Rightarrow \frac{dy}{dt} = a\cos t$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = a\cos t\cdot\frac{\sin t}{a\cos^2 t} = \tan t.$$

81. (d) : We have, $y = \log_e\left(\frac{x^2}{e^2}\right)$

$$\therefore \quad \frac{dy}{dx} = \frac{e^2}{x^2}\cdot\frac{1}{e^2}\cdot 2x = \frac{2}{x} \Rightarrow \frac{d^2y}{dx^2} = -\frac{2}{x^2}$$

82. Given, $x = at^2$, $y = 2at$

$$\therefore \quad \frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

So, $\frac{d^2y}{dx^2} = -\frac{1}{t^2}\cdot\frac{dt}{dx} = -\frac{1}{t^2}\cdot\frac{1}{2at} = -\frac{1}{2at}$

83. We have, $x = a\cos\theta$, $y = b\sin\theta$

$$\therefore \quad \frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = b\cos\theta$$

So, $\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$

Now, $\frac{d^2y}{dx^2} = -\frac{b}{a}(-\text{cosec}^2\theta)\frac{d\theta}{dx}$

$$= \frac{b}{a}\text{cosec}^2\theta\left(-\frac{1}{a}\text{cosec}\,\theta\right) = -\frac{b}{a^2}\text{cosec}^3\theta$$

84. We have, $y = (\sin^{-1}x)^2$ ...(i)

$$\Rightarrow \frac{dy}{dx} = 2(\sin^{-1}x)\times\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{4(\sin^{-1}x)^2}{(1-x^2)}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{4y}{1-x^2} \qquad \text{[From (i)]}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2(1-x^2) = 4y \qquad \text{...(ii)}$$

Again, differentiating (ii) w.r.t. x on both sides, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}(1-x^2) + \left(\frac{dy}{dx}\right)^2(-2x) = 4\left(\frac{dy}{dx}\right)$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0$$

85. We have, $x = \sin t$ and $y = \sin pt$

$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = p\cos pt$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{p\cos pt}{\cos t}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{-p^2\sin pt\cos t + p\cos pt\sin t}{\cos^2 t}\times\frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-p^2\sin pt\cos t + p\cos pt\sin t}{\cos^3 t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{p^2\sin pt\cos t}{\cos^3 t} + \frac{p\cos pt\sin t}{\cos^3 t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-p^2y}{\cos^2 t} + \frac{x\frac{dy}{dx}}{\cos^2 t}$$

$$\Rightarrow \cos^2 t\frac{d^2y}{dx^2} = -p^2y + x\frac{dy}{dx}$$

$$\Rightarrow (1-\sin^2 t)\frac{d^2y}{dx^2} = -p^2y + x\frac{dy}{dx}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0$$

86. Here, $y = \sin(\sin x)$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \cos(\sin x)\cdot\cos x$$

Again differentiating w.r.t. x both sides, we get

$$\frac{d^2y}{dx^2} = -\sin(\sin x)\cdot\cos x\cdot\cos x + (-\sin x)\cos(\sin x)$$

$$= -\cos^2 x\cdot\sin(\sin x) - \sin x\cdot\cos(\sin x)$$

Now, L.H.S. $= \frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x$

$= -\cos^2 x\cdot\sin(\sin x) - \sin x\cdot\cos(\sin x)$
$\quad + \tan x(\cos x\cdot\cos(\sin x)) + \cos^2 x\cdot\sin(\sin x)$

$= -\sin x\cdot\cos(\sin x) + \sin x\cdot\cos(\sin x)$

$= 0 =$ R.H.S.

87. *Refer to answer 53.*

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{x\frac{dy}{dx}-y}{x^2}=\frac{x\left(\frac{y}{x}\right)-y}{x^2}=0 \quad \therefore \quad \frac{d^2y}{dx^2}=0$$

88. Given that $e^y\cdot(x+1)=1$ ...(i)

Differentiating (i) w.r.t. x, we get

$$e^y\frac{d}{dx}(x+1)+(x+1)\frac{d}{dx}e^y=\frac{d}{dx}(1)$$

$$\Rightarrow \quad e^y+(x+1)e^y\frac{dy}{dx}=0$$

$$\Rightarrow \quad e^y\left[1+(x+1)\frac{dy}{dx}\right]=0 \;\Rightarrow\; (x+1)\frac{dy}{dx}=-1$$

and $\dfrac{dy}{dx}=\dfrac{-1}{x+1}$ ...(ii)

or $\left(\dfrac{dy}{dx}\right)^2=\dfrac{1}{(x+1)^2}$ ...(iii)

Again differentiating (ii) w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{1}{(x+1)^2} \;\Rightarrow\; \frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2 \quad \text{[From (iii)]}$$

89. We have, $y=x^x \Rightarrow y=e^{x\log x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=e^{x\log x}\left(x\times\frac{1}{x}+\log x\right)$$

$$\Rightarrow \frac{dy}{dx}=x^x(1+\log x) \;\Rightarrow\; \frac{dy}{dx}=y(1+\log x) \quad ...(1)$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=(1+\log x)\cdot\frac{dy}{dx}+y\times\frac{1}{x}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{1}{y}\left(\frac{dy}{dx}\right)^2+\frac{y}{x} \quad \text{[From (1)]}$$

$$\Rightarrow \frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=0$$

90. We have, $y=2\cos(\log x)+3\sin(\log x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=-2\sin(\log x)\times\frac{1}{x}+3\cos(\log x)\times\frac{1}{x}$$

$$\Rightarrow x\frac{dy}{dx}=-2\sin(\log x)+3\cos(\log x) \quad ...(1)$$

Again differentiating w.r.t. x, we get

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=-2\cos(\log x)\times\frac{1}{x}-3\sin(\log x)\times\frac{1}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-[2\cos(\log x)+3\sin(\log x)]$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-y \;\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$$

91. Given, $x=a\cos\theta+b\sin\theta$, $y=a\sin\theta-b\cos\theta$

$\Rightarrow \quad x^2=a^2\cos^2\theta+b^2\sin^2\theta+2ab\cos\theta\sin\theta$

and $y^2=a^2\sin^2\theta+b^2\cos^2\theta-2ab\sin\theta\cos\theta$

Adding (1) and (2), we get $x^2+y^2=a^2+b^2$

Differentiating w.r.t x, we get

$$2x+2y\frac{dy}{dx}=0 \Rightarrow x+y\frac{dy}{dx}=0 \quad ...(1)$$

Again differentiating w.r.t. x, we get

$$1+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$$

Multiplying by y on both sides, we get

$$y^2\frac{d^2y}{dx^2}+\left(y\cdot\frac{dy}{dx}\right)\frac{dy}{dx}+y=0$$

$$\Rightarrow y^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0 \quad \text{[From (1)]}$$

92. We have, $y=e^{m\sin^{-1}x}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=e^{m\sin^{-1}x}\left(\frac{m}{\sqrt{1-x^2}}\right)=\frac{my}{\sqrt{1-x^2}} \quad ...(1)$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=m\left[\frac{\sqrt{1-x^2}\frac{dy}{dx}-\frac{y}{2\sqrt{1-x^2}}\cdot(-2x)}{(1-x^2)}\right]$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}=m\left[my+\frac{xy}{\sqrt{1-x^2}}\right] \quad \text{[From (1)]}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}=m\left[my+x\cdot\left(\frac{1}{m}\cdot\frac{dy}{dx}\right)\right]$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}=m^2y+x\frac{dy}{dx}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-m^2y=0$$

93. We have, $y=(x+\sqrt{1+x^2})^n$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=n(x+\sqrt{1+x^2})^{n-1}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{n(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}=\frac{ny}{\sqrt{1+x^2}} \quad ...(1)$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=n\left[\frac{\sqrt{1+x^2}\cdot\frac{dy}{dx}-\frac{2(xy)}{2\sqrt{1+x^2}}}{1+x^2}\right]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n\left[\sqrt{1+x^2}\times\frac{ny}{\sqrt{1+x^2}}-\frac{xy}{\sqrt{1+x^2}}\right]$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n^2y-\frac{nxy}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}=n^2y-x\frac{dy}{dx} \quad \text{[From (1)]}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=n^2y$$

94. Here $x = a\sec^3\theta$

$$\Rightarrow \frac{dx}{d\theta}=a\cdot 3\sec^2\theta\cdot\sec\theta\tan\theta = 3a\sec^3\theta\tan\theta$$

and $y = a\tan^3\theta$

$$\Rightarrow \frac{dy}{d\theta}=a\cdot 3\tan^2\theta\cdot\sec^2\theta$$

$$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3a\tan^2\theta\sec^2\theta}{3a\sec^3\theta\tan\theta}=\frac{\tan\theta}{\sec\theta}=\sin\theta$$

On differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\cos\theta\cdot\frac{d\theta}{dx}=\frac{\cos\theta}{3a\sec^3\theta\tan\theta}=\frac{1}{3a}\cos^4\theta\cdot\cot\theta$$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{4}}=\frac{1}{3a}\cos^4\frac{\pi}{4}\cdot\cot\frac{\pi}{4}=\frac{1}{3a}\cdot\left(\frac{1}{\sqrt{2}}\right)^4\cdot 1$$

$$=\frac{1}{3a}\cdot\frac{1}{4}=\frac{1}{12a}$$

95. Given $y = Ae^{mx} + Be^{nx}$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=Ae^{mx}\cdot m+Be^{nx}\cdot n$$

$$\Rightarrow \frac{d^2y}{dx^2}=m^2Ae^{mx}+n^2Be^{nx}$$

Now, $\text{L.H.S.}=\frac{d^2y}{dx^2}-(m+n)\frac{dy}{dx}+mny$

$= m^2Ae^{mx} + n^2Be^{nx} -(m + n)(mAe^{mx} + nBe^{nx})$
$+ mn(Ae^{mx} + Be^{nx})$

$=Ae^{mx}[m^2 - (m + n)m + mn]$
$+ Be^{nx}[n^2 - (m + n)n + mn]$

$= Ae^{mx} \times 0 + Be^{nx} \times 0 = 0 = \text{R.H.S.}$

96. Here, $x = a(\cos t + t\sin t)$

$$\Rightarrow \frac{dx}{dt}=a[-\sin t+1\cdot\sin t+t\cos t]=at\cos t \quad ...(1)$$

and $y = a(\sin t - t\cos t)$

$$\Rightarrow \frac{dy}{dt}=a[\cos t-(1\cdot\cos t-t\sin t)]=a\,t\sin t$$

$$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{a\,t\sin t}{a\,t\cos t}=\tan t$$

$$\Rightarrow \frac{d^2y}{dx^2}=\sec^2t\cdot\frac{dt}{dx}=\frac{\sec^2t}{a\,t\cos t} \quad \text{[Using (1)]}$$

$$=\frac{1}{a}\cdot\frac{1}{t\cos^3t}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{4}}=\frac{1}{a}\cdot\frac{1}{\frac{\pi}{4}\cos^3\frac{\pi}{4}}=\frac{4}{\pi a}\cdot(\sqrt{2})^3=\frac{8\sqrt{2}}{\pi a}$$

97. *Refer to answer 80.*

We get $\frac{dy}{dx}=\tan t$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\sec^2t\cdot\frac{dt}{dx}=\frac{\sec^2t}{a\cos^2t/\sin t}=\frac{\sin t}{a\cos^4t}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{3}}=\frac{\sin\frac{\pi}{3}}{a\cos^4\frac{\pi}{3}}=\frac{\sqrt{3}/2}{a(1/2)^4}=\frac{8\sqrt{3}}{a}$$

98. *Refer to answer 97.*

99. Given that $y=\log(x+\sqrt{x^2+a^2})$ ...(1)

Differentiating (1) w.r.t. 'x' on both sides, we get

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\frac{d}{dx}(x+\sqrt{x^2+a^2})$$

$$=\frac{1}{x+\sqrt{x^2+a^2}}\left(1+\frac{1}{2\sqrt{x^2+a^2}}\cdot 2x\right)$$

$$=\frac{1}{x+\sqrt{x^2+a^2}}\cdot\frac{(\sqrt{x^2+a^2}+x)}{\sqrt{x^2+a^2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{x^2+a^2}}\Rightarrow\sqrt{x^2+a^2}\,\frac{dy}{dx}=1 \quad ...(2)$$

Again differentiating (2) on both sides w.r.t. x, we get

$$\sqrt{x^2+a^2}\,\frac{d^2y}{dx^2}+\frac{2x}{2\sqrt{x^2+a^2}}\frac{dy}{dx}=0$$

$$\Rightarrow (x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$$

100. Here $x = a\cos^3\theta$, $y = a\sin^3\theta$

$$\Rightarrow \frac{dx}{d\theta}=3a\cos^2\theta\cdot(-\sin\theta) \text{ and } \frac{dy}{d\theta}=3a\sin^2\theta\cdot\cos\theta$$

$$\Rightarrow \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}=-\tan\theta$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=-\sec^2\theta\frac{d\theta}{dx}=-\frac{\sec^2\theta}{-3a\cos^2\theta\sin\theta}$$

$$=\frac{1}{3a}\cdot\frac{1}{\cos^4\theta\cdot\sin\theta}$$

$$\therefore \left.\frac{d^2y}{dx^2}\right|_{\theta=\frac{\pi}{6}}=\frac{1}{3a}\cdot\frac{1}{\cos^4\frac{\pi}{6}.\sin\frac{\pi}{6}}$$

$$=\frac{1}{3a}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)^4\cdot\frac{1}{2}}=\frac{32}{27a}$$

101. Here, $y=x\log\left(\frac{x}{a+bx}\right)$ ... (1)

$$\Rightarrow y=x[\log x-\log(a+bx)]=x\log x-x\log(a+bx)$$

$$\Rightarrow \frac{dy}{dx}=x\cdot\frac{1}{x}+1\cdot\log x-\left[1\cdot\log(a+bx)+x\cdot\frac{1}{a+bx}\cdot b\right]$$

$$=1-\frac{bx}{a+bx}+\log x-\log(a+bx)$$

$$\Rightarrow \frac{dy}{dx}=\frac{a}{a+bx}+\log\left(\frac{x}{a+bx}\right) \quad ...(2)$$

$$\Rightarrow \frac{dy}{dx}=\frac{a}{a+bx}+\frac{y}{x} \quad \text{[Using (1)]}$$

Again differentiating (2) w.r.t. x, we get

$$\frac{d^2y}{dx^2}=a\cdot(-1)(a+bx)^{-2}\cdot b+\frac{a(a+bx)}{x(a+bx)^2}$$

$$=\frac{-ab}{(a+bx)^2}+\frac{a}{x(a+bx)}=\frac{-abx+a(a+bx)}{x(a+bx)^2}$$

$$=\frac{a^2}{x(a+bx)^2}$$

Now, $\text{R.H.S.}=\left(x\frac{dy}{dx}-y\right)^2$

$$=\left\{x\cdot\left[\frac{a}{a+bx}+\frac{y}{x}\right]-y\right\}^2=\left(\frac{ax}{a+bx}\right)^2$$

and $\text{L.H.S.}=x^3\frac{d^2y}{dx^2}$

$$=\frac{a^2x^2}{(a+bx)^2}=\left[\frac{ax}{a+bx}\right]^2=\text{R.H.S.}$$

102. We have, $x=\tan\left(\frac{1}{a}\log y\right)$

$$\Rightarrow \frac{1}{a}\log y=\tan^{-1}x$$

Differentiating w.r.t. x, we get

$$\frac{1}{a}\cdot\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{1+x^2} \Rightarrow (1+x^2)\frac{dy}{dx}=ay$$

Again differentiating w.r.t. x, we get

$$(1+x^2)\frac{d^2y}{dx^2}+2x\cdot\frac{dy}{dx}=a\frac{dy}{dx}$$

$$\Rightarrow (1+x^2)\frac{d^2y}{dx^2}+(2x-a)\frac{dy}{dx}=0$$

103. Given $x = \cos\theta$ and $y = \sin^3\theta$

$$\Rightarrow \frac{dx}{d\theta}=-\sin\theta \text{ and } \frac{dy}{d\theta}=3\sin^2\theta\cos\theta$$

$$\therefore \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{3\sin^2\theta\cos\theta}{-\sin\theta}=-3\sin\theta\cos\theta$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2}=[-3\cos^2\theta-3\sin\theta(-\sin\theta)]\frac{d\theta}{dx}$$

$$=(-3\cos^2\theta+3\sin^2\theta)\cdot\frac{-1}{\sin\theta}$$

Now, $\text{L.H.S.}=y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2$

$$=\sin^3\theta\left(\frac{3\cos^2\theta-3\sin^2\theta}{\sin\theta}\right)+(-3\sin\theta\cos\theta)^2$$

$$=\sin^2\theta(3\cos^2\theta-3\sin^2\theta)+9\sin^2\theta\cos^2\theta$$

$= 3\sin^2\theta(\cos^2\theta - \sin^2\theta + 3\cos^2\theta)$
$= 3\sin^2\theta(4\cos^2\theta - \sin^2\theta)$
$= 3\sin^2\theta(4\cos^2\theta - 1 + \cos^2\theta)$
$= 3\sin^2\theta(5\cos^2\theta - 1) =$ R.H.S.

104. Given $y = \sin^{-1} x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Again differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\sqrt{1-x^2}\cdot 0 - 1\cdot\frac{1(-2x)}{2\sqrt{1-x^2}}}{1-x^2}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} = \frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\cdot\frac{1}{\sqrt{1-x^2}} = 0$$

$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\cdot\frac{dy}{dx} = 0$$

105. $y = (\tan^{-1}x)^2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\cdot\tan^{-1}x\cdot\frac{d}{dx}(\tan^{-1}x) = 2\tan^{-1}x\cdot\frac{1}{1+x^2}$$

$$\Rightarrow (x^2+1)\frac{dy}{dx} = 2\tan^{-1}x$$

Again differentiating. w.r.t. x, we get

$$(x^2+1)\frac{d^2y}{dx^2} + 2x\cdot\frac{dy}{dx} = 2\cdot\frac{1}{1+x^2}$$

$$\Rightarrow (x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2$$

106. *Refer to answer 90.*

107. *Refer to answer 96.*

Since, we have $\frac{dx}{dt} = at\cos t$

$$\therefore \frac{d^2x}{dt^2} = a[t(-\sin t) + \cos t] = -at\sin t + a\cos t$$

and $\frac{dy}{dt} = at\sin t \Rightarrow \frac{d^2y}{dt^2} = a[t\cos t + \sin t]$

108. *Refer to answer 97.*

$\because \quad y = a\sin t \quad \therefore \frac{dy}{dt} = a\cos t$

Again differentiating w.r.t. t, we get

$$\frac{d^2y}{dt^2} = -a\sin t$$

109. *Refer to answer 97.*

110. *Refer to answer 79.*

We have, $\frac{dy}{dx} = \frac{-\sin\theta}{(1-\cos\theta)} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$

$$\Rightarrow \frac{dy}{dx} = -\cot\frac{\theta}{2} \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\times\frac{d\theta}{dx}$$

$$= \frac{1}{2}\times\operatorname{cosec}^2\frac{\theta}{2}\times\frac{1}{2a\sin^2\frac{\theta}{2}} = \frac{\operatorname{cosec}^4\frac{\theta}{2}}{4a}$$

111. *Refer to answer 110.*

112. We have, $f(x) = x^3 - 3x$

$$f(-\sqrt{3}) = (-\sqrt{3})^3 - 3(-\sqrt{3}) = -3\sqrt{3} + 3\sqrt{3} = 0 = f(0).$$

Since, $f(x)$ is continuous in $[-\sqrt{3}, 0]$ and differentiable on $(-\sqrt{3}, 0)$ then $\exists$ some c in $(-\sqrt{3}, 0)$ such that $f'(c) = 0$

Now, $f'(x) = 3x^2 - 3$
$\Rightarrow \quad 3c^2 - 3 = 0 \Rightarrow c^2 = 1$
$\Rightarrow \quad c = \pm 1 \Rightarrow c = -1$
as $c \in (-\sqrt{3}, 0)$.