Cubes and Cube Roots

Chapter 7

7. 1 Introduction

This is a story about one of India's great mathematical geniuses, S. Ramanujan. Once another famous mathematician Prof. G.H. Hardy came to visit him in a taxi whose number was 1729. While talking to Ramanujan, Hardy described this number "a dull number". Ramanujan quickly pointed out that 1729 was indeed interesting. He said it is the smallest number that can be expressed as a sum of two cubes in two different ways:

$$1729 = 1728 + 1 = 12^3 + 1^3$$
$$1729 = 1000 + 729 = 10^3 + 9^3$$

1729 has since been known as the Hardy - Ramanujan Number, even though this feature of 1729 was known more than 300 years before Ramanujan.

Hardy - Ramanujan Number
1729 is the smallest Hardy - Ramanujan Number. There are an infinitely many such numbers. Few are 4104 (2, 16; 9, 15), 13832 (18, 20; 2, 24), Check it with the numbers given in the brackets.

How did Ramanujan know this? Well, he loved numbers. All through his life, he experimented with numbers. He probably found numbers that were expressed as the sum of two squares and sum of two cubes also.

There are many other interesting patterns of cubes. Let us learn about cubes, cube roots and many other interesting facts related to them.

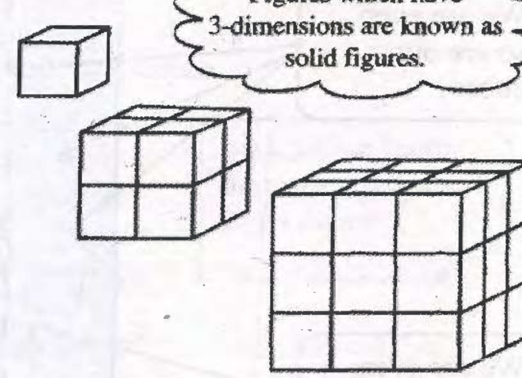
7.2 Cubes

You know that the word 'cube' is used in geometry. A cube is a solid figure which has all its sides equal.

How many cubes of side 1 cm will make a cube of side 2 cm?

How many cubes of side 1 cm will make a cube of side 3 cm?

Figures which have 3-dimensions are known as solid figures.



Consider the numbers 1, 8, 27, ...

These are called **perfect cubes or cube numbers**. Can you say why they are named so? Each of them is obtained when a number is multiplied by itself three times.

We know that $1 = 1 \times 1 \times 1 = 1^3$; $8 = 2 \times 2 \times 2 = 2^3$; $27 = 3 \times 3 \times 3 = 3^3$.

Since $5^3 = 5 \times 5 \times 5 = 125$, therefore 125 is a cube number.

Is 9 a cube number? No, as 9 =3 x 3 and there is no natural number which multiplied by itself three times gives 9. We can see also that $2 \times 2 \times 2 = 8$ and $3 \times 3 \times 3 = 27$. This shows that 9 is not a perfect cube.

The following are the cubes of numbers from 1 to 10.

Table 1

Number	Cube
1	$1^3 = 1$
2	$2^3 = 8$
3	$3^3 = 27$
4	$4^3 = 64$
5	$5^3 =$ ____
6	$6^3 =$ ____
7	$7^3 =$ ____
8	$8^3 =$ ____
9	$9^3 =$ ____
10	$10^3 =$ ____

The numbers 729, 1000, 1728 are also perfect cubes

Complete it.

There are only ten perfect cubes from 1 to 1000. (Check this). How many perfect cubes are there from 1 to 100?

Observe the cubes of even numbers. Are they all even? What can you say about the cubes of odd numbers?

Following are the cubes of the numbers from 11 to 20.

Table 2

Number	Cube
11	1331
12	1728
13	2197
14	2744
15	3375
16	4096
17	4913
18	5832
19	6859
20	8000

We are even, so are our cubes

We are even, so are our cubes

Consider a few numbers having 1 as the one's digit (or unit's). Find the cube of each of them. What can you say about the one's digit of the cube of a number having 1 as the one's digit?

Similarly, explore the one's digit of cubes of numbers ending in 2, 3, 4, ..., etc.

TRY THESE

Find the one's digit of the cube of each of the following numbers.

(i) 3331 (ii) 8888 (iii) 149 (iv) 1005
(v) 1024 (vi) 77 (vii) 5022 (viii) 53

7.2.1 Some interesting patterns

1. Adding consecutive odd numbers

Observe the following pattern of sums of odd numbers.

$$1 = 1 = 1^3$$
$$3 + 5 = 8 = 2^3$$
$$7 + 9 + 11 = 27 = 3^3$$
$$13 + 15 + 17 + 19 = 64 = 4^3$$
$$21 + 23 + 25 + 27 + 29 = 125 = 5^3$$

Is it not interesting? How many consecutive odd numbers will be needed to obtain the sum as 10^3?

TRY THESE

Express the following numbers as the sum of odd numbers using the above pattern?

(a) 6^3 (b) 8^3 (c) 7^3

Consider the following pattern.

$$2^3 - 1^3 = 1 + 2 \times 1 \times 3$$
$$3^3 - 2^3 = 1 + 3 \times 2 \times 3$$
$$4^3 - 3^3 = 1 + 4 \times 3 \times 3$$

Using the above pattern, find the value of the following.

(i) $7^3 - 6^3$ (ii) $12^3 - 11^3$ (iii) $20^3 - 19^3$ (iv) $51^3 - 50^3$

2. Cubes and their prime factors

Consider the following prime factorisation of the numbers and their cubes.

Prime factorisation of a number	**Prime factorisation of its cube**
$4 = 2 \times 2$	$4^3 = 64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^3 \times 2^3$
$6 = 2 \times 3$	$6^3 = 216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^3 \times 3^3$
$15 = 3 \times 5$	$15^3 = 3375 = 3 \times 3 \times 3 \times 5 \times 5 \times 5 = 3^3 \times 5^3$
$12 = 2 \times 2 \times 3$	$12^3 = 1728 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 = 2^3 \times 2^3 \times 3^3$

each prime factor appears three times in its cubes

$$\begin{array}{r|l} 2 & 216 \\ \hline 2 & 108 \\ \hline 2 & 54 \\ \hline 3 & 27 \\ \hline 3 & 9 \\ \hline 3 & 3 \\ \hline & 1 \end{array}$$

Observe that each prime factor of a number appears three times in the prime factorisation of its cube.

In the prime factorisation of any number, if each factor appears three times, then, is the number a perfect cube?

Think about it. Is 216 a perfect cube?

By prime factorisation, $216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3$

Each factor appears 3 times. $216 = 2^3 \times 3^3\ (2 \times 3)^3$

$= 6^3$ which is a perfect cube!

Do you remember that $a^m \times b^m = (a \times b)^m$

Is 729 a perfect cube? $729 = \underline{3 \times 3 \times 3} \times \underline{3 \times 3 \times 3}$

Factors can be grouped in triples

Yes, 729 is a perfect cube.

Now let us check for 500.

Prime factorisation of 500 is $2 \times 2 \times \underline{5 \times 5 \times 5}$.

There are three 5's in the product but only 2's

Example 1: Is 243 a perfect cube?

Solution: $243 = \underline{3 \times 3 \times 3} \times 3 \times 3$

In the above factorisation 3×3 remains after grouping the 3's in triplets. Therefore, 243 is not a perfect cube.

TRY THESE

Which of the following are perfect cubes?

1. 400	2. 3375	3. 8000	4. 15625
5. 9000	6. 6859	7. 2025	8. 10648

7.22 Smallest multiple that is a perfect cube

Raju made a cuboid of plasticine. Length, breadth and height of the cuboid are 15 cm, 30 cm, 15 cm respectively.

Asifa asks how many such cuboids will she need to make a perfect cube? Can you tell?

Raju said, Volume of cuboid is $15 \times 30 \times 15 = 3 \times 5 \times 2 \times 3 \times 5 \times 3 \times 5$

$= 2 \times \underline{3 \times 3 \times 3} \times \underline{5 \times 5 \times 5}$

Since there is only one 2 in the prime factorisation. So we need 2×2, i.e., 4 to make it a perfect cube. Therefore, we need 4 s .ch cuboids to make a cube.

Example 2: Is 392 a perfect ıbe? If not, find the smallest natural number by which 392 must be multiplied so that the product ıs a perfect cube.

Solution: $392 = \underline{2 \times 2 \times 2} \times 7 \times 7$

The prime factor 7 does not appear in a group of three. Therefore, 392 is not a perfect cube. To make it a cube, we need one more 7. In that case

$392 \times 7 = \underline{2 \times 2 \times 2} \times \underline{7 \times 7 \times 7} = 2744$ which is a perfect cube.

Hence the smallest natural number by which 392 should be multiplied to make a perfect cube is 7.

Example 3: Is 53240 a perfect cube? If not, then by which smallest natural number should 53240 be divided so that the quotient is a perfect cube?

Solution: $53240 = \underline{2\times2\times2} \times \underline{11\times11\times11} \times 5$

The prime factor 5 does not appear in a group of three. So, 53240 is not a perfect cube. In the factorisation 5 appears only one time. If we divide the number by 5, then the prime factorisation of the quotient will not contain 5.

So, $\quad 53240 \div 5 = \underline{2\times2\times2} \times \underline{11\times11\times11}$

Hence the smallest number by which 53240 should be divided to make it a perfect cube is 5. The perfect cube in that case is = 10648.

Example 4: Is 1188 a perfect cube? If not, by which smallest natural number should 1188 be divided so that the quotient is a perfect cube?

Solution: $1188 = 2 \times 2 \times \underline{3\times3\times3} \times 11$

The primes 2 and 11 do not appear in groups of three. So, 1188 is not a perfect cube. In the factorisation of 1188 the prime 2 appears only two times and the prime 11 appears once. So, if we divide 1188 by $2 \times 2 \times 11 = 44$, then the prime factorisation of the quotient will not contain 2 and 11.

Hence the smallest natural number by which 1188 should be divided to make it a perfect cube is 44.

And the resulting perfect cube is $1188 \div 44 = 27\ (= 3^3)$.

Example 5: Is 68600 a perfect cube? If not, find the smallest number by which 68600 must be multiplied to get a perfect cube.

Solution: We have, $68600 = 2 \times 2 \times 2 \times 5 \times 5 \times 7 \times 7$. In this factorisation, we find that there is not triplet of 5.

So, 68600 is not a perfect cube. To make it a perfect cube we multiply it by 5.

Thus, $\quad 68600 \times 5 = 2 \times 2 \times 2 \times 5 \times 5 \times 5 \times 7 \times 7 \times 7$

$= 343000$, which is a perfect cube.

Observe that 343 is a perfect cube. From example 5 we know that 343000 is also perfect cube.

THINK, DISCUSS AND WRITE

Check which of the following are perfect cubes.

(i) 2700 (ii) 16000 (iii) 64000 (iv) 900

(v) 125000 (vi) 36000 (vii) 21600 (viii) 10,000

(ix) 27000000 (x) 1000.

What pattern do you observe in these perfect cubes?

Exercise 7.1

1. Which of the following numbers are not perfect cubes?
   (i) 216 (ii) 128 (iii) 1000 (iv) 100
   (v) 46656
2. Find the smallest number by which each of the following numbers must be multiplied to obtain a perfect cube.
   (i) 243 (ii) 256 (iii) 72 (iv) 675
   (v) 100
3. Find the smallest number by which each of the following numbers must be divided to obtain a perfect cube.
   (i) 81 (ii) 128 (iii) 135 (iv) 192
   (v) 704
4. Imtiyaz makes a cuboid of plasticine of sides 5 cm, 2cm, 5 cm. How many such cuboids will he need to form a cube?

7.3 Cube Roots

If the volume of a cube is 125 cm^3, what would be the length of its side? To get the length of the side of the cube, we need to know a number whose cube is 125.
Finding the square root, as you know, is the inverse operation of squaring. Similarly, finding the cube root is the inverse operation of finding cube.
We know that $2^3 = 8$; so we say that the cube root of 8 is 2.
We write $\sqrt[3]{8} = 2$. **The symbol $\sqrt[3]{\ }$ denotes 'cube-root.'**
Consider the following:

Consider the following:

Statement	Inference
$1^3 = 1$	$\sqrt[3]{1} = 1$
$2^3 = 8$	$\sqrt[3]{8} = \sqrt[3]{2^3} = 2$
$3^3 = 27$	$\sqrt[3]{27} = \sqrt[3]{3^3} = 3$
$4^3 = 64$	$\sqrt[3]{64} = 4$
$5^3 = 125$	$\sqrt[3]{125} = 5$

Statement	Inference
$6^3 = 216$	$\sqrt[3]{216} = 6$
$7^3 = 343$	$\sqrt[3]{343} = 7$
$8^3 = 512$	$\sqrt[3]{512} = 8$
$9^3 = 729$	$\sqrt[3]{729} = 9$
$10^3 = 1000$	$\sqrt[3]{1000} = 10$

7.3.1 Cube root through prime factorisation method

Consider 3375. We find its cube root by prime factorisation:

$$3375 = \underline{3 \times 3 \times 3} \times \underline{5 \times 5 \times 5} = 3^3 \times 5^3 = (3 \times 5)^3$$

Therefore, cube root of 3375 = $\sqrt[3]{3375} = 3 \times 5 = 15$
Similarly, to find $\sqrt[3]{74088}$, we have,

$$74088 = \underline{2 \times 2 \times 2} \times \underline{3 \times 3 \times 3} \times \underline{7 \times 7 \times 7} = 2^3 \times 3^3 \times 7^3 = (2 \times 3 \times 7)^3$$

Therefore, $\sqrt[3]{74088} = 2 \times 3 \times 7 = 42$

Example 6: Find the cube root of 8000.

Solution: Prime factorisation of 8000 is $\underline{2 \times 2 \times 2} \times \underline{2 \times 2 \times 2} \times \underline{5 \times 5 \times 5}$

So, $\sqrt[3]{8000} = 2 \times 2 \times 5 = 20$

Example 7: Find the cube root of 13825 by prime factorisation method.

Solution:

$$13824 = \underline{2 \times 2 \times 2} \times \underline{2 \times 2 \times 2} \times \underline{2 \times 2 \times 2} \times \underline{3 \times 3 \times 3} = 2^3 \times 2^3 \times 2^3 \times 3^3.$$

Therefore, $\sqrt[3]{13824} = 2 \times 2 \times 2 \times 3 = 24$

THINK, DISCUSS AND WRITE

State true or false: for any integer m, $m^2 < m^3$. why?

7.3.2 Cube root of a cube number

If you know that the given number is a cube number then following method can be used.

Step 1 Take any cube number say 857375 and start making groups of three digits starting from the right most digit of the number.

$$\underset{\text{Second group}}{\underset{\downarrow}{\underline{857}}} \qquad \underset{\text{First group}}{\underset{\downarrow}{\underline{375}}}$$

We can estimate the cube root of a given cube number through a step by step process.

We get 375 and 857 as two groups of three digits each.

Step 2 First group,. i.e., 375 will give you the one's (or unit's) digit of the required cube root.

The number 375 ends with 5. We know that 5 comes at the unit's place of a number only when it's cube root ends in 5.

So, we get 5 at the unit's place of the cube root.

Step 3 Now take another group, i.e., 857.

We know that 93 = 729 and 10^3= 1000. Also, 729 < 857 < 1000. We take the one's place, of the smaller number 729 as the ten's place of the required cube root. So, we get $\sqrt[3]{857375} = 95$.

Example 8: Find the cube root of 17576 through estimation.

Solution: The given number is 17576.

Step 1 Form groups of three starting from the rightmost digit of 17576.
17 576 In this case one group i.e., 576 has three digits whereas 17 has only two digits.

Step2 Take 576.
The digit 6 is at its one's place.
We take the one's place of the required cube root as 6.

Step 3 Take the other group, i.e., 17.
Cube of 2 is 8 and cube of 3 is 27. 17 lies between 8 and 27.
The smaller number among 2 and 3 is 2.
The one's place of 2 is 2 itself. Take 2 as ten's place of the cube root of 17576.
Thus, $\sqrt[3]{17576} = 26$ (Check it!)

EXERCISE 7.2

1. Find the cube root of each of the following numbers by prime factorisation method.
   (i) 64 (ii) 512 (iii) 10648 (iv) 27000
   (v) 15625 (vi) 13824 (vii) 110592 (viii) 46656
   (ix) 175616 (x) 91125
2. State true or false.
   (i) Cube of any odd number is even.
   (ii) A perfect cube does not end with two zeros.
   (iii) If square of a number ends with 5, then its cube ends with 25.
   (iv) There is no perfect cube which ends with 8.
   (v) The cube of a two digit number may be a three digit number.
   (vi) The cube of a two digit number may have seven or more digits.
   (vii) The cube of a single digit number may be a single digit number.
3. You are told that 1,331 is a perfect cube. Can you guess without factorisation what is its cube root? Similarly, guess the cube roots of 4913, 12167, 32768.

MISCELLANEOUS EXAMPLE: 7

Example 1: Express 32832 as a sum of two cubes in two different ways.

Example 2: product of three consecutive integers added by middle integer is always a perfect cube of middle integer.

e.g., $(12 \times 13 \times 14) + 13 = 13^3$

Observe the pattern and fill in the blanks:

(i) $(0 \times 1 \times 2) + 1 = 1 = 1^3$

(ii) $(1 \times 2 \times 3) + 2 = 8 = 2^3$

(iii) $(2 \times 3 \times 4) + 3 =$ ____ $=$ ____

(iv) $(3 \times 4 \times 5) +$ ____ $=$ ____ $= 4^3$

(v) $(4 \times$ ____ $\times$ ____$) + 5 = 125 =$ ____

(vi) (____ $\times$ ____ $\times$ ____$) + 6 =$ ____ $=$ ____

Example 3: How many cuboids of dimension 3 meters, 4 meters and 5 meters are required to form a smallest perfect cube?

Example 4: Find the smallest perfect cube of five digits.

Example 5: Guess the cube roots of following perfect cubes:

(i) 729 (ii) 512 (iii) 10648 (iv) 79507 (v) 185193

Example 6: We can guess the cube root of a six digit number. We can use this guess to discard non-perfect cubes.

e.g., Let us guess cube root of 10658 as 22. Verification using pattern of example 2 above (by reverse process). $10658 - 22 = 10636$ which should be divisible by 22. But it is not divisible by 2. So it is not a perfect cube. Now guess and verify the following as being perfect or non perfect cubes.

(i) 739 (ii) 1331 (iii) 3365 (iv) 9261 (v) 19683

Example 7: State True or False.

(i) There are triplets in the factorization of perfect cube.

(ii) The cube of number is never equal to the number itself again.

(iii) The cube of a three digit number can have seven or eight or nine digits.

(iv) Some perfect cubes can also be perfect squares.

(v) None of the perfect squares can be perfect cube.

Example 8: Is there a number which is equal to its cube but not equal to its square if yes find it.

What Have We Discussed

1. Numbers like 1729, 4104, 13832, are known as Hardy - Ramanujan Numbers. They can be expressed as sum of two cubes in two different ways.

2. Numbers obtained when a number is multiplied by itself three times are known as cube numbers. For example 1, 8, 27, ... etc.

3. If in the prime factorisation of any number each factor appears three times, then the number is a perfect cube.

4. The symbol $\sqrt[3]{\ }$ denotes cube root. For example $\sqrt[3]{27} = 3$.