3 Differentiation

Let's Study

1. Derivatives of composite functions.
2. Derivatives of inverse functions.
3. Derivatives of logarithmic functions.
4. Derivatives of implicit function.
5. Derivatives of parametric functions.
6. Derivative of second order.

Let's Recall

1. Concept of continuity
2. Concept of Differentiability.
3. Derivatives of some standard functions.

	$y = f(x)$	$\frac{dy}{dx} = f'(x)$
1	K(constant)	0
2	x	1
3	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
4	$\frac{1}{x}$	$\frac{-1}{x^2}$
5	x^n	$n.x^{n-1}$
6	a^x	$a^x.\log a$
7	e^x	e^x
8	$\log x$	$\frac{1}{x}$

4. Rules of Differentiation:

If u and v are differentiable functions of x and if

1. $y = u + v$ then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

2. $y = u - v$ then $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

3. $y = u.v$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

4. $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ $v \neq 0$

5. $y = k\,.\,u$ then $\frac{dy}{dx} = k.\frac{du}{dx}$, k constant.

Introduction:

In Standard XI, we have studied the concept of differentiation. We have used this concept in calculating marginal demand and marginal cost of a commodity.

Let's Learn

3.1 Derivative of a Composite Function:

Sometimes complex looking functions can be greatly simplified by expressing them as compositions of two or more different functions. It is then not possible to differentiate them directly is possible with simple functions.

Now, we discuss differentiation of such composite functions using the chain rule.

Result 1: If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

(*This is called Chain Rule*)

Generalisation:

If y is a differentiable function of u_1, u_i is a differentiable function of u_{i+1}, for $i = 1, 2, 3, \ldots\ldots (n-1)$ and u_n is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{du_1} \times \frac{du_1}{du_2} \times \frac{du_2}{du_3} \times \ldots\ldots\ldots \times \frac{du_n}{dx}$$

SOLVED EXAMPLES

1) $y = (4x^3 + 3x^2 - 2x)^6$. Find $\frac{dy}{dx}$

Solution: Given $y = (4x^3 + 3x^2 - 2x)^6$

Let $u = (4x^3 + 3x^2 - 2x)$

$\therefore \quad y = u^6$

$\therefore \quad \frac{dy}{du} = 6u^5$

$\therefore \quad \frac{dy}{du} = 6(4x^3 + 3x^2 - 2x)^5$

and $\frac{du}{dx} = 12x^2 + 6x - 2$

By chain Rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\therefore \quad \frac{dy}{dx} = 6\,(4x^3 + 3x^2 - 2x)^5\,(12x^2 + 6x - 2)$

2) $y = \log(4x^2 + 3x - 1)$. Find $\frac{dy}{dx}$

Solution: Given $y = \log(4x^2 + 3x - 1)$

Let $u = (4x^2 + 3x - 1)$

$\therefore \quad y = \log(u)$

$\therefore \quad \frac{dy}{du} = \frac{1}{u}$

$\therefore \quad \frac{dy}{du} = \frac{1}{(4x^2 + 3x - 1)}$

and $\frac{du}{dx} = (8x + 3)$

By chain Rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\therefore \quad \frac{dy}{du} = \frac{1}{(4x^2 + 3x - 1)}(8x + 3)$

$\therefore \quad \frac{dy}{dx} = \frac{8x + 3}{(4x^2 + 3x - 1)}$

3) If $y = \sqrt[3]{\left(3x^2 + 8x - 7\right)^5}$, find $\frac{dy}{dx}$

Solution: Given $y = \sqrt[3]{\left(3x^2 + 8x - 7\right)^5}$

$\therefore \quad y = \left(3x^2 + 8x - 7\right)^{\frac{5}{3}}$

Let $u = (3x^2 + 8x - 7)$

$\therefore \quad y = u^{\frac{5}{3}}$

$\therefore \quad \frac{dy}{du} = \frac{5}{3}u^{\frac{2}{3}}$

$\therefore \quad \frac{dy}{du} = \frac{5}{3}(3x^2 + 8x - 7)^{\frac{2}{3}}$

and $\frac{du}{dx} = (6x + 8)$

By chain Rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\therefore \quad \frac{dy}{du} = \frac{5}{3}(3x^2 + 8x - 7)^{\frac{2}{3}}\ (6x + 8)$

4) If $y = e^{(\log x + 6)}$, find $\frac{dy}{dx}$

Solution: Given $y = e^{(\log x + 6)}$

Let $u = \log x + 6$

$\therefore \quad y = e^u$

$\therefore \quad \frac{dy}{du} = e^u$

$\therefore \quad \frac{dy}{du} = e^{(\log x + 6)}$ and $\frac{du}{dx} = \frac{1}{x}$

By chain Rule $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\therefore \quad \frac{dy}{dx} = e^{(\log x + 6)}\,\frac{1}{x}$

EXERCISE 3.1

Q.1 Find $\frac{dy}{dx}$ if,

1) $y = \sqrt{x + \frac{1}{x}}$

2) $y = \sqrt[3]{a^2 + x^2}$

3) $y = (5x^3 - 4x^2 - 8x)^9$

Q.2 Find $\frac{dy}{dx}$ if,

1) $y = \log(\log x)$

2) $y = \log(10x^4 + 5x^3 - 3x^2 + 2)$

3) $y = \log(ax^2 + bx + c)$

Q.3 Find $\frac{dy}{dx}$ if,

1) $y = e^{5x^2 - 2x + 4}$

2) $y = a^{(1+\log x)}$

3) $y = 5^{(x+\log x)}$

3.2 Derivative of an Inverse Function:

Let $y = f(x)$ be a real valued function defined on an appropriate domain. The inverse of this function exists if and only if the function is one-one and onto.

For example: Let f : R → R be such that $f(x) = x + 10$ then inverse of f is

f^{-1} : R → R such that $f^{-1}(y) = y - 10$

That is, if $y = x + 10$ then $x = y - 10$

Result 2 : If $y = f(x)$ is a differentiable function of x such that inverse function $x = f^{-1}(y)$ exists, then x is a differentiable function of y

and $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, $\frac{dy}{dx} \neq 0$

SOLVED EXAMPLES

1) Find rate of change of demand (x) of a commodity with respect to its price (y) if

$y = 20 + 15x + x^2$

Solution: Let $y = 20 + 15x + x^2$

Differentiating both sides with respect to x, we get

$\therefore \frac{dy}{dx} = 15 + 2x$

By derivative of the inverse function

$\therefore \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, $\frac{dy}{dx} \neq 0$

$\therefore$ rate of change of demand with respect to price $= \frac{dx}{dy} = \frac{1}{15 + 2x}$

2) Find rate of change of demand (x) of a commodity with respect to its price (y) if

$y = 5 + x^2 e^{-x} + 2x$

Solution: Let $y = 5 + x^2 e^{-x} + 2x$

Differentiating both sides with respect to x, we get

$\therefore \frac{dy}{dx} = (-x^2 e^{-x} + 2xe^{-x} + 2)$

By derivative of the inverse function

$\therefore \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$, $\frac{dy}{dx} \neq 0$

$\therefore$ Rate of change of demand with respect to price $= \frac{dx}{dy} = \frac{1}{(-x^2 e^{-x} + 2xe^{-x} + 2)}$

3) Find rate of change of demand (x) of a commodity with respect to its price (y) if

$y = \frac{3x + 7}{2x^2 + 5}$

Solution: Let $y = \frac{3x + 7}{2x^2 + 5}$

Differentiating both sides with respect to x, we get

$$\therefore \quad \frac{dy}{dx} = \frac{(2x^2+5)(3)-(3x+7)(4x)}{(2x^2+5)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(6x^2+15)-(12x^2+28x)}{(2x^2+5)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(6x^2+15-12x^2-28x)}{(2x^2+5)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{(-6x^2-28x+15)}{(2x^2+5)^2}$$

By derivative of the inverse function

$$\therefore \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \quad \frac{dy}{dx} \neq 0$$

$\therefore$ Rate of change of demand with respect to

$$\text{price} = \frac{dx}{dy} = \frac{(2x^2+5)^2}{(-6x^2-28x+15)}$$

EXERCISE 3.2

Q.1 Find the rate of change of demand (x) of a commodity with respect to its price (y) if

1) $y = 12 + 10x + 25x^2$

2) $y = 18x + \log(x-4)$

3) $y = 25x + \log(1+x^2)$

Q.2 Find the marginal demand of a commodity where demand is x and price is y

1) $y = xe^{-x} + 7$

2) $y = \dfrac{x+2}{x^2+1}$

3) $y = \dfrac{5x+9}{2x-10}$

3.3 Derivative of a Logarithmic Function:

Sometimes we have to differentiate a function involving complicated expressions like $f(x).g(x)$, $\dfrac{f(x)}{g(x)}$ and $[f(x)]^{g(x)}$. In this case, we first transform the expression to a logarithmic form and then find its derivative. Hence the method is called logarithmic differentiation. That is,

$$\frac{d(\log y)}{dx} = \frac{1}{y}\frac{dy}{dx}$$

Examples of Logarithmic Functions.

1) $y = \dfrac{(6x+5)^5}{(3x^2-1)\sqrt{8+2x}}$

2) $y = (e^x+1)^x \times (x+1)^{(x+2)}$

Note : 1) The log function to the base "e" is called Natural log and the log function to the base 10 is called common log.

2) In $(a)^{b^c}$, a is the base and b^c is the index.

Some Basic Laws of logarithms:

1) $\log_a mn = \log_a m + \log_a n$

2) $\log_a \dfrac{m}{n} = \log_a m - \log_a n$

3) $\log_a m^n = n \log_a m$

4) $\log_n m = \dfrac{\log_a m}{\log_a n}$

5) $\log e = 1$ $(= \log_a a)$

6) $\log_a a^x = x$

SOLVED EXAMPLES

1) Find $\dfrac{dy}{dx}$, if $y = (3+x)^x$

Solution: let $y = (3+x)^x$

Taking logarithm of both sides, we get

$\therefore \log y = \log(3+x)^x$

$\therefore \log y = x \log(3+x)$

Differentiating both sides with respect to x, we get

$$\therefore \frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{3+x}\right) + \log(3+x) \times 1$$

$$\therefore \frac{dy}{dx} = y\left[x\left(\frac{1}{3+x}\right) + \log(3+x)\right]$$

$$\therefore \frac{dy}{dx} = (3+x)^x\left[\frac{x}{3+x} + \log(3+x)\right]$$

2) Find $\frac{dy}{dx}$, if $y = x^{x^x}$

Solution: Let $y = x^{x^x}$

Taking logarithm of both sides, we get

$$\therefore \log y = \log x^{x^x}$$

$$\therefore \log y = x^x \log(x)$$

Differentiating both sides with respect to x, we get

$$\therefore \frac{1}{y}\frac{dy}{dx} = x^x\frac{1}{x} + \log(x).\frac{dx^x}{dx}$$

$$\therefore \frac{dy}{dx} = y\left[x^x\frac{1}{x} + \log(x).\frac{dx^x}{dx}\right] \quad \text{.......... (I)}$$

Let $u = x^x$

Taking logarithm of both sides, we get

$$\therefore \log u = x.\log(x)$$

Differentiating both sides with respect to x, we get

$$\therefore \frac{1}{u}\frac{du}{dx} = x.\frac{1}{x} + \log(x).1$$

$$\therefore \frac{du}{dx} = \text{u}\,[1 + \log x]$$

$$\therefore \frac{du}{dx} = x^x(1 + \log x) \quad \text{.......... (II)}$$

Substituting eq^n (II) in eq^n (I), we get

$$\therefore \frac{dy}{dx} = y\left[x^x\frac{1}{x} + \log(x).x^x(1+\log x)\right]$$

$$\therefore \frac{dy}{dx} = x^{x^x}.x^x\left[\frac{1}{x} + \log(x).(1+\log x)\right]$$

3) Find $\frac{dy}{dx}$, if $y = \sqrt{\frac{(2x+3)^5}{(3x-1)^3(5x-2)}}$

Solution: Let $y = \sqrt{\frac{(2x+3)^5}{(3x-1)^3(5x-2)}}$

$$y = \left(\frac{(2x+3)^5}{(3x-1)^3(5x-2)}\right)^{\frac{1}{2}}$$

Taking logarithm of both sides, we get

$$\therefore \log y = \frac{1}{2}\left\{\log\left(\frac{(2x+3)^5}{(3x-1)^3(5x-2)}\right)\right\}$$

$$\therefore \log y = \frac{1}{2}\,[5\log(2x+3) - 3\log(3x-1) - \log(5x-2)]$$

Differentiating both sides with respect to x, we get

$$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[5\frac{2}{(2x+3)} - 3\frac{3}{(3x-1)} - \frac{5}{(5x-2)}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\,y\left[\frac{10}{(2x+3)} - \frac{9}{(3x-1)} - \frac{5}{(5x-2)}\right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(2x+3)^5}{(2x-1)^3(5x-2)}}\left[\frac{10}{(2x+3)} - \frac{9}{(3x-1)} - \frac{5}{(5x-2)}\right]$$

4) Find $\frac{dy}{dx}$, if $y = x^x + (\log x)^x$

Solution: Let $y = x^x + (\log x)^x$

Let $u = x^x$ and $v = (\log x)^x$

$$\therefore y = u + v$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

Now, $u = x^x$

Taking logarithm of both sides, we get

$$\therefore \log u = x\log x$$

Differentiating both sides with respect to x, we get,

$$\therefore \quad \frac{1}{u}\frac{du}{dx} = x\frac{1}{x} + \log x.1$$

$$\therefore \quad \frac{du}{dx} = u\,(1 + \log x)$$

$$\therefore \quad \frac{du}{dx} = x^x\,(1 + \log x)............. \text{(II)}$$

Now, $v = (\log x)^x$

Taking logarithm of both sides, we get

$\therefore \quad \log v = x\log(\log x)$

Differentiating both sides with respect to x, we get,

$$\therefore \quad \frac{1}{v}\frac{dv}{dx} = x\frac{1}{x.\log x} + \log(\log x).1$$

$$\therefore \quad \frac{dv}{dx} = v\left[\frac{1}{\log x} + \log(\log x)\right]$$

$$\therefore \quad \frac{dv}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] \quad ..... \text{(III)}$$

Substituting eq^n (II) and eq^n (III) in eq^n (I), we get

$$\therefore \frac{dy}{dx} = x^x(1+\log x) + (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right]$$

EXERCISE 3.3

Q.1 Find $\frac{dy}{dx}$ if,

1) $y = x^{x^{2x}}$

2) $y = x^{e^x}$

3) $y = e^{x^x}$

Q.2 Find $\frac{dy}{dx}$ if,

1) $y = \left(1+\frac{1}{x}\right)^x$

2) $y = (2x+5)^x$

3) $y = \sqrt[3]{\frac{(3x-1)}{(2x+3)(5-x)^2}}$

Q.3 Find $\frac{dy}{dx}$ if,

1) $y = (\log x)^x + x^{\log x}$

2) $y = (x)^x + (a)^x$

3) $y = 10^{x^x} + 10^{x^{10}} + 10^{10^x}$

3.4 Derivative of an Implicit Function:

If the variable y can be expressed as a function of the variable x. that is, $y = f(x)$ then the function $f(x)$ is called an explicit function of x.

For Example: $f(x) = x^2 + x^{-3}$, $y = \log x + e$

If it is not possible to express y as a function of x or x as a function of y then the function is called an implicit function.

For Example: $ax^2 + 2hxy + by^2 = 0$;

$x^m + y^n = (x+y)^{m+n}$

The general form of an implicit function of two variables x and y is $f(x,y)=0$

Solved Examples:

1) Find $\frac{dy}{dx}$ if $y^3 - 3y^2x = x^3 + 3x^2y$

Solution: Given $y^3 - 3y^2x = x^3 + 3x^2y$

Differentiating both sides with respect to x, we get

$$\therefore \quad 3y^2\frac{dy}{dx} - 3y^2 - 3x(2y)\frac{dy}{dx}$$

$$= 3x^2 + 3x^2\frac{dy}{dx} + 3y(2x)$$

$$\therefore \quad 3y^2\frac{dy}{dx} - 6xy\frac{dy}{dx} - 3x^2\frac{dy}{dx}$$

$$= 3x^2 + 6xy + 3y^2$$

$$\therefore \quad (3y^2 - 6xy - 3x^2)\frac{dy}{dx} = (3x^2 + 6xy + 3y^2)$$

$$\therefore \quad (y^2 - 2xy - x^2)\frac{dy}{dx} = (x^2 + 2xy + y^2)$$

$$\therefore \quad \frac{dy}{dx}=\frac{(x^2+2xy+y^2)}{(y^2-2xy-x^2)}$$

2) Find $\frac{dy}{dx}$ if $x^y = y^x$

Solution: Given $x^y = y^x$

Taking logarithm of both sides, we get

$\therefore \quad y \log x = x \log y$

Differentiating both sides with respect to x, we get

$$\therefore \quad y\frac{1}{x}+\log x\frac{dy}{dx}=x\frac{1}{y}\frac{dy}{dx}+\log y.1$$

$$\therefore \quad \log x\frac{dy}{dx}-\frac{x}{y}\frac{dy}{dx}=\log y-\frac{y}{x}$$

$$\therefore \quad \left(\log x-\frac{x}{y}\right)\frac{dy}{dx}=\left(\log y-\frac{y}{x}\right)$$

$$\therefore \quad \left(\frac{y.\log x-x}{y}\right)\frac{dy}{dx}=\left(\frac{x.\log y-y}{x}\right)$$

$$\therefore \quad \frac{dy}{dx}=\left(\frac{x.\log y-y}{x}\right)\left(\frac{y}{y.\log x-x}\right)$$

$$\therefore \quad \frac{dy}{dx}=\frac{y}{x}\left(\frac{x.\log y-y}{y.\log x-x}\right)$$

3) If $x^m.y^n = (x+y)^{(m+n)}$ then show that,

$$\frac{dy}{dx}=\frac{y}{x}$$

Solution: Given $x^m.y^n = (x+y)^{(m+n)}$

Taking logarithm of both sides, we get

$\therefore \quad m.\log x + n.\log y = (m+n)\log(x+y)$

Differentiating both sides with respect to x, we get

$$\therefore \quad m\frac{1}{x}+n\frac{1}{y}\frac{dy}{dx}=(m+n)\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\therefore \quad \frac{m}{x}+\frac{n}{y}\frac{dy}{dx}=\frac{(m+n)}{(x+y)}\left(1+\frac{dy}{dx}\right)$$

$$\therefore \quad \frac{n}{y}\frac{dy}{dx}-\frac{(m+n)}{(x+y)}\frac{dy}{dx}=\frac{(m+n)}{(x+y)}-\frac{m}{x}$$

$$\therefore \quad \left(\frac{n}{y}-\frac{(m+n)}{(x+y)}\right)\frac{dy}{dx}=\left(\frac{(m+n)}{(x+y)}-\frac{m}{x}\right)$$

$$\therefore \quad \left[\frac{nx+ny-my-ny}{y(x+y)}\right]\frac{dy}{dx}=\left[\frac{mx+nx-mx-my}{x(x+y)}\right]$$

$$\therefore \quad \left[\frac{nx-my}{y}\right]\frac{dy}{dx}=\left[\frac{nx-my}{x}\right]$$

$$\therefore \quad \frac{dy}{dx}=\left[\frac{nx-my}{x}\right]\left[\frac{y}{nx-my}\right]$$

$$\therefore \quad \frac{dy}{dx}=\frac{y}{x}$$

EXERCISE 3.4

Q.1 Find $\frac{dy}{dx}$ if,

1) $\sqrt{x}+\sqrt{y}=\sqrt{a}$

2) $x^3+y^3+4x^3y=0$

3) $x^3+x^2y+xy^2+y^3=81$

Q.2 Find $\frac{dy}{dx}$ if,

1) $y.e^x + x.e^y = 1$

2) $x^y = e^{(x-y)}$

3) $xy = \log(xy)$

Q.3 Solve the following.

1) If $x^5.y^7 = (x+y)^{12}$ then show that, $\frac{dy}{dx}=\frac{y}{x}$

2) If $\log(x+y) = \log(xy) + a$ then show that, $\frac{dy}{dx}=\frac{-y^2}{x^2}$

3) If $e^x+e^y = e^{(x+y)}$ then show that, $\frac{dy}{dx}=-e^{y-x}$

3.5 Derivative of a Parametric Function:

Now we consider y as a function of x where both x and y are functions of a variable 't'. Here 't' is called a parameter.

Result 3: If $x = f(t)$ and $y = g(t)$ are differentiable functions of a parameter 't', then y is a differential function of x and

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \frac{dx}{dt} \neq 0$$

SOLVED EXAMPLES

1) Find $\frac{dy}{dx}$, if $x = 2at$, $y = 2at^2$

Solution: Given $x = 2at$, $y = 2at^2$

Now, $y = 2at^2$

Differentiate with respect to t

$$\therefore \frac{dy}{dt} = 2a.2t = 4at \quad .... \text{(I)}$$

$x = 2at$

Differentiate with respect to t

$$\therefore \frac{dx}{dt} = 2a \quad .... \text{(II)}$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\therefore \frac{dx}{dt} = \frac{4at}{2a}$$

$$\therefore \frac{dx}{dt} = 2t$$

2) Find $\frac{dy}{dx}$, if $x = e^{2t}$, $y = e^{\sqrt{t}}$

Solution: Given $x = e^{2t}$, $y = e^{\sqrt{t}}$

Now, $y = e^{\sqrt{t}}$

Differentiate y with respect to t

$$\therefore \frac{dy}{dt} = e^{\sqrt{t}} \frac{d}{dt}\sqrt{t} \quad .... \text{(I)}$$

Differentiate with respect to t

$$\therefore \frac{dx}{dt} = 2\,e^{2t} \quad .... \text{(II)}$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\therefore \frac{dy}{dx} = \frac{e^{\sqrt{t}} \frac{1}{2\sqrt{t}}}{2e^{2t}}$$

$$\therefore \frac{dy}{dx} = \frac{e^{\sqrt{t}}}{4\sqrt{t}\,e^{2t}}$$

3) Differentiate $\log(t)$ with respect to $\log(1+t^2)$

Solution: let $y = \log(t)$ and $x = \log(1+t^2)$

Now, $y = \log(t)$

Differentiate with respect to t

$$\therefore \frac{dy}{dt} = \frac{1}{t} \quad .... \text{(I)}$$

Now, $x = \log(1+t^2)$

Differentiate with respect to t

$$\therefore \frac{dx}{dt} = \frac{2t}{1+t^2} \quad .... \text{(II)}$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$$\therefore \frac{dy}{dx} = \frac{\frac{1}{t}}{\frac{2t}{1+t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{1+t^2}{2t^2}$$

EXERCISE 3.5

Q.1 Find $\frac{dy}{dx}$ if,

1) $x = at^2, y = 2at$

2) $x = 2at^2, y = at^4$

3) $x = e^{3t}, y = e^{(4t+5)}$

Q.2 Find $\frac{dy}{dx}$ if,

1) $x = \left(u+\frac{1}{u}\right)^2, y = (2)^{\left(u+\frac{1}{u}\right)}$

2) $x = \sqrt{1+u^2}$, $y = \log(1+u^2)$

3) Differentiate 5^x with respect to $\log x$

Q.3 Solve the following.

1) If $x = a\left(1-\frac{1}{t}\right), y = a\left(1+\frac{1}{t}\right)$ then,

show that $\frac{dy}{dx} = -1$

2) If $x = \frac{4t}{1+t^2}, y = 3\left(\frac{1-t^2}{1+t^2}\right)$ then,

show that $\frac{dy}{dx} = \frac{-9x}{4y}$

3) If $x = t.\log t, y = t^t$ then, show that

$\frac{dy}{dx} - y = 0$

3.6 Second Order Derivative:

Consider a differentiable function $y = f(x)$

then $\frac{dy}{dx} = f'(x)$is the first order derivative of y with respect to x. It is also denoted by y' or y_1

If $f'(x)$ is a differentiable function of x

then $\frac{d\left(\frac{dy}{dx}\right)}{dx}$ denoted by $\frac{d^2y}{dx^2}$ or $f''(x)$ is called the second order derivative of y with respect to x. It is also denoted by y'' or y_2

If $f''(x)$ is a differential function of x

then $\frac{d\left(\frac{d^2y}{dx^2}\right)}{dx}$ denoted by $\frac{d^3y}{dx^3}$ or $f'''(x)$ is called the third order derivative of y with respect to x. It is also denoted by y''' or y_3.

SOLVED EXAMPLES

1) Find $\frac{d^2y}{dx^2}$, if $y = x^2$

Solution: Given $y = x^2$

Differentiate with respect to x

$\therefore \frac{dy}{dx} = 2x$

Differentiate with respect to x, again

$\therefore \frac{d^2y}{dx^2} = 2$

2) Find $\frac{d^2y}{dx^2}$, if $y = x^6$

Solution: Given $y = x^6$

Differentiate with respect to x

$\therefore \frac{dy}{dx} = 6x^5$

Differentiate with respect to x, again

$\therefore \frac{d^2y}{dx^2} = 6(5x^4)$

$\therefore \frac{d^2y}{dx^2} = 30x^4$

3) Find $\frac{d^2y}{dx^2}$, if $y = \log x$

Solution: Given $y = \log x$

Differentiate with respect to x

$\therefore \frac{dy}{dx} = \frac{1}{x}$

Differentiate with respect to x, again

$\therefore \frac{d^2y}{dx^2} = \frac{-1}{x^2}$

4) Find $\frac{d^2y}{dx^2}$, if $y = e^{4x}$

Solution: Given $y = e^{4x}$

Differentiate with respect to x

$\therefore \quad \frac{dy}{dx} = 4e^{4x}$

$\therefore \quad \frac{d^2y}{dx^2} = 4(4e^{4x})$

$\therefore \quad \frac{d^2y}{dx^2} = 16e^{4x}$

EXERCISE 3.6

Q.1 Find $\frac{d^2y}{dx^2}$ if,

1) $y = \sqrt{x}$

2) $y = x^5$

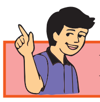
3) $y = x^{-7}$

Q.2 Find $\frac{d^2y}{dx^2}$ if,

1) $y = e^x$

2) $y = e^{(2x+1)}$

3) $y = e^{\log x}$



Let's Remember

Derivative of some standard functions.

	$y = f(x)$	$\frac{dy}{dx} = f'(x)$
1	K (constant)	0
2	x	1
3	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
4	$\frac{1}{x}$	$\frac{-1}{x^2}$
5	x^n	$n.x^{n-1}$
6	a^x	$a^x. \log a$
7	e^x	e^x
8	$\log x$	$\frac{1}{x}$

Rules of Differentiation:

If u and v differentiable function of x and if

1. $y = u + v$ then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

2. $y = u - v$ then $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

3. $y = u.v$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

4. $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, $v \neq 0$

Derivative of a Composite Function:

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Derivative of an Inverse Function :

If $y = f(x)$ is a differentiable function of x such that the inverse function $x = f^{-1}(y)$ exists, then x is a differentiable function of y and

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}, \quad \frac{dy}{dx} \neq 0$$

Derivative of a Parametric Function:

If $x = f(t)$ and $y = g(t)$ are differential functions of parameter 't' then y is a differential function of x and

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}},$$

$$\frac{dx}{dt} \neq 0$$

MISCELLANEOUS EXERCISE - 3

Q.I] Choose the correct alternative.

1) If $y = (5x^3 - 4x^2 - 8x)^9$ then $\frac{dy}{dx}$ = -----------

a) $9(5x^3 - 4x^2 - 8x)^8 (15x^2 - 8x - 8)$

b) $9(5x^3 - 4x^2 - 8x)^9 (15x^2 - 8x - 8)$

c) $9(5x^3 - 4x^2 - 8x)^8 (5x^2 - 8x - 8)$

d) $9(5x^3 - 4x^2 - 8x)^9 (5x^2 - 8x - 8)$

2) If $y = \sqrt{x + \frac{1}{x}}$ then $\frac{dy}{dx} = ?$

a) $\frac{x^2 - 1}{2x^2\sqrt{x^2+1}}$ b) $\frac{1 - x^2}{2x^2\sqrt{x^2+1}}$

c) $\frac{x^2 - 1}{2x\sqrt{x}\sqrt{x^2+1}}$ d) $\frac{1 - x^2}{2x\sqrt{x}\sqrt{x^2+1}}$

3) If $y = e^{\log x}$ then $\frac{dy}{dx} = ?$

a) $\frac{e^{\log x}}{x}$ b) $\frac{1}{x}$ c) 0 d) $\frac{1}{2}$

4) If $y = 2x^2 + 2^2 + a^2$ then $\frac{dy}{dx} = ?$

a) x b) $4x$ c) $2x$ d) $-2x$

5) If $y = 5^x.x^5$ then $\frac{dy}{dx} = ?$

a) $5^x.x^4 (5 + \log 5)$ b) $5^x.x^5 (5 + \log 5)$

c) $5^x.x^4 (5 + x\log 5)$ d) $5^x.x^5 (5 + x\log 5)$

6) If $y = \log\left(\frac{e^x}{x^2}\right)$ then $\frac{dy}{dx} = ?$

a) $\frac{2 - x}{x}$ b) $\frac{x - 2}{x}$

c) $\frac{e - x}{ex}$ d) $\frac{x - e}{ex}$

7) If $ax^2+2hxy+by^2 = 0$ then $\frac{dy}{dx} = ?$

a) $\frac{(ax+hy)}{(hx+by)}$ b) $\frac{-(ax+hy)}{(hx+by)}$

c) $\frac{(ax-hy)}{(hx+by)}$ d) $\frac{(2ax+hy)}{(hx+3by)}$

8) If $x^4.y^5 = (x+y)^{(m+1)}$

and $\frac{dy}{dx} = \frac{y}{x}$ then $m = ?$

a) 8 b) 4 c) 5 d) 20

9) If $x = \frac{e^t + e^{-t}}{2}$, $y = \frac{e^t - e^{-t}}{2}$ then $\frac{dy}{dx} = ?$

a) $\frac{-y}{x}$ b) $\frac{y}{x}$ c) $\frac{-x}{y}$ d) $\frac{x}{y}$

10) If $x = 2at^2$, $y = 4at$ then $\frac{dy}{dx} = ?$

a) $-\frac{1}{2at^2}$ b) $\frac{1}{2at^3}$

c) $\frac{1}{t}$ d) $\frac{1}{4at^3}$

Q.II] Fill in the blanks:

1) If $3x^2y + 3xy^2 = 0$

then $\frac{dy}{dx} =$

2) If $x^m.y^n = (x+y)^{(m+n)}$ then $\frac{dy}{dx} = \frac{\ldots\ldots}{x}$

3) If $0 = \log(xy) + a$ then $\frac{dy}{dx} = \frac{-y}{\ldots\ldots}$

4) If $x = t\log t$ and $y = t^t$ then $\frac{dy}{dx} = \ldots\ldots\ldots$

5) If $y = x.\log x$ then $\frac{d^2y}{dx^2} = \ldots\ldots\ldots$

6) If $y = \left[\log(x)\right]^2$ then $\dfrac{d^2y}{dx^2}$ = ...

7) If $x = y + \dfrac{1}{y}$ then $\dfrac{dy}{dx}$ = ...

8) If $y = e^{ax}$, then $x.\dfrac{dy}{dx}$ = ...

9) If $x = t.\log t$, $y = t^t$ then $\dfrac{dy}{dx}$ = ...

10) If $y = \left(x + \sqrt{x^2 - 1}\right)^m$

then $\sqrt{(x^2 - 1)}\ \dfrac{dy}{dx}$ = ...

Q.III] State whether each of the following is True or False:

1) If f' is the derivative of f, then the derivative of the inverse of f is the inverse of f'

2) The derivative of $\log_a x$, where a is constant is $\dfrac{1}{x.\log a}$.

3) The derivative of $f(x) = a^x$, where a is constant is $x.a^{x-1}$

4) The derivative of polynomial is polynomial.

5) $\dfrac{d}{dx}(10^x) = x.10^{x-1}$

6) If $y = \log x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$

7) If $y = e^2$ then $\dfrac{dy}{dx} = 2e$

8) The derivative of a^x is $a^x.\log a$

9) The derivative of $x^m.y^n = (x+y)^{(m+n)}$ is $\dfrac{x}{y}$

Q.IV] Solve the following:

1) If $y = (6x^3 - 3x^2 - 9x)^{10}$, find $\dfrac{dy}{dx}$

2) If $y = \sqrt[5]{\left(3x^2 + 8x + 5\right)^4}$, find $\dfrac{dy}{dx}$

3) If $y = [\log(\log(\log x))]^2$, find $\dfrac{dy}{dx}$

4) Find the rate of change of demand (x) of a commodity with respect to its price (y)

if $y = 25+30x - x^2$.

5) Find the rate of change of demand (x) of a commodity with respect to its price (y)

if $y = \dfrac{5x+7}{2x-13}$

6) Find $\dfrac{dy}{dx}$, if $y = x^x$

7) Find $\dfrac{dy}{dx}$, if $y = 2^{x^x}$

8) Find $\dfrac{dy}{dx}$, if $y = \sqrt{\dfrac{(3x-4)^3}{(x+1)^4(x+2)}}$

9) Find $\dfrac{dy}{dx}$, if $y = x^x + (7x - 1)^x$

10) If $y = x^3+3xy^2+3x^2y$ Find $\dfrac{dy}{dx}$

11) If $x^3+y^2+xy = 7$ Find $\dfrac{dy}{dx}$

12) If $x^3y^3 = x^2-y^2$ Find $\dfrac{dy}{dx}$

13) If $x^7.y^9 = (x+y)^{16}$ then show that

Find $\dfrac{dy}{dx} = \dfrac{y}{x}$

14) If $x^a.y^b = (x+y)^{(a+b)}$ then show that

Find $\dfrac{dy}{dx} = \dfrac{y}{x}$

15) Find $\dfrac{dy}{dx}$ if , $x = 5t^2$, $y = 10t$

16) Find $\dfrac{dy}{dx}$ if , $x = e^{3t}$, $y = e^{\sqrt{t}}$

17) Differentiate $\log(1+x^2)$ with respective to a^x

18) Differentiate $e^{(4x+5)}$ with respective to 10^{4x}

19) Find $\dfrac{d^2y}{dx^2}$, if $y = \log(x)$

20) Find $\dfrac{d^2y}{dx^2}$, if $y = 2at$, $x = at^2$

21) Find $\dfrac{d^2y}{dx^2}$, if $y = x^2.e^x$

22) If $x^2+6xy+y^2= 10$ then show

that $\dfrac{d^2y}{dx^2} = \dfrac{80}{(3x+y)^3}$

23) If $ax^2+2hxy+by^2 = 0$ then show

that $\dfrac{d^2y}{dx^2} = 0$

Activities

(1): $y = (6x^4 - 5x^3 + 2x + 3)^5$ find $\dfrac{dy}{dx}$

Solution:- Given

$y = (6x^4 - 5x^3 + 2x + 3)^5$

Let $u = [6x^4 - 5x^3 + \square + 3]$

$\therefore y = u^{\square}$

$\therefore \dfrac{dy}{du} = 5u^4$

And $\dfrac{du}{dx} = 24x^3 - 15(\square)+2$

By chain rule

$\dfrac{dy}{dx} = \dfrac{dy}{\square} \times \dfrac{\square}{dx}$

$\therefore \dfrac{dy}{dx} = 5(6x^4 - 5x^3 + 2x + 3)^{\square}$

$\times (24x^3 - 15x^2 + \square)$

(2): The rate of change of demand (x) of a commodity with respect to its price (y).

If $y = 30 + 25x + x^2$

Solution : Let $y = 30 + 25x + x^2$

Diff. w.r.to x, we get

$\therefore \dfrac{dy}{dx} = \square + \square + \square$

$\therefore \dfrac{dy}{dx} = 25 + 2x$

$\therefore$ By derivation of the inverse function

$\dfrac{dx}{dy} = \dfrac{1}{\square}, \quad \dfrac{dy}{dx} \neq 0$

Rate of change of demand with respect to

price $= \dfrac{1}{\square + \square}$

(3): find $\dfrac{dy}{dx}$, if $y = x^{(\log x)} + 10^x$

Solution:- Let $y = x^{(\log x)} + 10^x$

Let $u = x^{\log x}$, $v = 10^x$

$y = u + v$

Now, $u = x^{\log x}$

Taking log on both sides, we get

$\log u = \log x^{\log x}$

$\log u = \log x.^{\log x . \log x}$

$\log u = (\log x)^2$

Diff. w.r.to x, we get

$\therefore \dfrac{1}{u}\dfrac{du}{dx} = 2(\log x) \times \dfrac{d\square}{dx}$

$\therefore \dfrac{du}{dx} = u\left[2\log x \times \dfrac{1}{x}\right]$

$\therefore \dfrac{du}{dx} = x^{\log x}\left[2\square \times \dfrac{1}{\square}\right]$.........(II)

Now, $v = 10^x$

Diff.w.r.to x, we get

$\therefore \dfrac{dv}{dt} = 10^x \square$

Substitution equation (II) & (III) in equation (I), we get

$$\therefore \quad \frac{dy}{dx} = x^{\log x}\left[2\log x + \frac{1}{x}\right] + 10^x.\log(10)$$

(4): Find $\frac{dy}{dx}$, if $y^x = e^{x+y}$

Solution:- Given $y^x = e^{x+y}$

Taking log on both side, we get,

$\therefore$ log $(y)^x$ = log $(e)^{x+y}$

$\therefore$ $x.\square = \square$.log e

$\therefore$ x. log y = (x + $\square$). 1

$\therefore$ x. log y = x + $\square$

Diff. w.r.to x, we get

$$\therefore x\ \frac{1}{y}\ \frac{d\square}{dx} + \log y.\ 1 = \square + \frac{dy}{dx}$$

$$\therefore x\frac{1}{y}\frac{dy}{dx} + \log y = 1 + \frac{dy}{dx}$$

$$\therefore \frac{x}{y}\frac{dy}{dx} - \frac{dy}{dx} = 1 - \square$$

$$\therefore \frac{dy}{dx}\left(\frac{x}{y} - 1\right) = \square - \log y$$

$$\therefore \frac{dy}{dx} = \frac{(\square - \log y)\,(y)}{x - y}$$

5: Find $\frac{dy}{dx}$ if $x = e^t$, $y = e^{\sqrt{t}}$

Solution:- given, $x = e^t$, $y = e^{\sqrt{t}}$

Now, $y = e^{\sqrt{t}}$

Diff. w.r.to t

$$\therefore \frac{dy}{dt} = e^{\sqrt{t}}\ \frac{d\square}{dt}$$

$$\therefore \frac{dx}{dt} = e^{\sqrt{t}}.\frac{1}{2\sqrt{t}} \qquad \text{......... (I)}$$

Now, $x = e^t$

Diff.w.r.to t

$$\therefore \frac{dx}{dt} = \square \qquad \text{....... (II)}$$

$$\text{Now,} \quad \frac{dy}{dx} = \frac{dy/dt}{\square}$$

$$\therefore \quad = \frac{\frac{e^{\sqrt{t}}}{\square}}{e^t}$$

$$\therefore \quad \frac{dy}{dx} = \frac{e^{\sqrt{t}}}{2\sqrt{t}\,e^t}$$

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