Differentiation

DEFINITION

If $f(x)$ is a function and a and $a+h$ belongs to the domains of f, then the limit given by $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, if it finitely exist, is called the derivative of $f(x)$ with respect to (or w.r.t) x at $x=a$ and is denoted by $f'(a)$,

$$\therefore\ f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

Note : $f'(a)$ is the derivates of $f(x)$ w.r.t x at $x=a$.

DERIVATIVE OF *f* (*x*) FROM THE FIRST PRINCIPLES (*i.e.* definition or ab-initio)

Let $y = f(x)$ ...(1)

be a given function defined in some domain.

Let δx be small change in x, and δy be the corresponding change in *y*.

$\therefore\ y+\delta y = f(x+\delta x)$ ...(2)

On subtracting (1) from (2), we have

$\therefore\ \delta y = f(x+\mathrm{d}x)-f(x)$

Dividing by $\delta x \neq 0$, $\frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x}$

Taking limits on both side as $\delta x \to 0$, we get

$$\lim_{\delta\to 0}\frac{\delta y}{\delta x} = \lim_{\delta\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} = f'(x)$$

if it finitely exist (i.e. if f is derivable at *x*) is called the differential coefficient (d.c.) or the derivative of $f(x)$ w.r.t. x or derived function

Denoting L.H.S by $\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x}$, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$ and it may be denoted by anyone of the following symbol :

$$f'(x),\ \frac{dy}{dx},\ \frac{d}{dx}(y),\ \frac{d}{dx}(f(x)),\ y',\ y_1,\ D_x(y).$$

The **general derivative** of f w.r.t. *x* is given by

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

The denominator '*h*' represents the change (increment) in the value of the *x* whenever it changes from *x* to $x+h$. The numerator represents the corresponding change (increment) in the value of $f(x)$. Hence we can write $f'(x) = \lim_{h\to 0}\frac{\text{change in } f(x) \text{ or } y}{\text{increment in } x}.$

DIFFERENTIABILITY

(i) A function f is said to have left hand derivative at $x=a$ iff f is defined in some (undeleted) left neighbourhood of a and $\lim\limits_{h\to 0^-}\dfrac{f(a+h)-f(a)}{h}$ exists finitely and its value is called the left hand derivative at a and is denoted by $f'(a^-)$.

(ii) A function f is said to have right-hand derivative at $x=a$ iff f is defined in some (undeleted) right neighbourhood of a and $\lim\limits_{h\to 0^+}\dfrac{f(a+h)-f(a)}{h}$ exists finitely and its value is called the right hand derivative at a and is denoted by $f'(a^+)$.

(iii) A function f is said to have a derivative (or is differentiable) at a if f is defined in some (undeleted) neighbourhood of a and $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ exists finitely and its value is called the derivative or differential coefficient of f at a and is denoted by $f'(a)$ or $\left.\dfrac{df(x)}{dx}\right|_{x=a}$

(iv) If a function f is differentiable at a point 'a' then it is also continuous at the point 'a'. But, converse may not be true. For example, $f(x)=|x|$ is continuous at $x=0$ but is not differentiable at $x=0$.

(v) A function f is differentiable at a point $x=a$ and $P(a,\ f(a))$ is the corresponding point on the graph of $y=f(x)$ iff the curve does not have P as a corner point.

Note : From (iv) and (v) it is clear that if a function f is not differentiable at a point $x=a$ then either the function f is not continuous at $x=a$ or the curve represented by $y=f(x)$ has a corner at the point $(a,\ f(a))$ (i.e. the curve suddenly changes the direction)

(vi) A function f is differentiable (or derivable) on $[a,\ b]$ if

(a) f is continuous at every point of $(a,\ b)$

(b) $\lim\limits_{h\to 0^+}\dfrac{f(a+h)-f(a)}{h}$ and $\lim\limits_{h\to 0^-}\dfrac{f(b+h)-f(b)}{h}$ both exist.

A function f is said to be differentiable if it is differentiable at every point of the domain.

A function f is said to be everywhere differentiable if it is differentiable for each $x\in R$.

SOME STANDARD RESULTS ON DIFFERENTIABLITY

(i) Every polynomial function, every exponential function $a^x\ (a>0)$ and every constant function are differentiable at each $x\in R$.

(ii) The logarithmic functions, trigonometrical functions and inverse – trigonometrical functions are always differentiable in their domains.

(iii) The sum, difference, product and quotient (under condition) of two differentiable functions is differentiable.

(iv) The composition of differentiable functions (under condition) is a differentiable function.

DERIVATIVES OF SOME STANDARD FUNCTIONS

(i) $\dfrac{d}{dx}(c)=0$ if c is a constant and conversely also.

Test of constancy. If at all points of a certain interval $f'(x)=0,$ then the function f is constant in that interval.

(ii) $\frac{d}{dx}(x^n)=nx^{n-1}$

(iii) $\frac{d}{dx}(ax+b)^n=n(ax+b)^{n-1}.a$

(iv) $\frac{d}{dx}(\sin x)=\cos x$

(v) $\frac{d}{dx}(\cos x)=-\sin x$

(vi) $\frac{d}{dx}(\tan x)=\sec^2 x$

(vii) $\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$

(viii) $\frac{d}{dx}(\sec x)=\sec x\tan x$

(ix) $\frac{d}{dx}(\text{cosec}\, x)=-\text{cosec}\, x\cot x$

(x) $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$

(xi) $\frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}$

(xii) $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$

(xiii) $\frac{d}{dx}(\cot^{-1}x)=\frac{-1}{1+x^2}$

(xiv) $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{|x|\sqrt{x^2-1}}$

(xv) $\frac{d}{dx}(\text{cosec}^{-1}x)=\frac{-1}{|x|\sqrt{x^2-1}}$

(xvi) $\frac{d}{dx}(a^x)=a^x\log_e a$

(xvii) $\frac{d}{dx}(e^x)=e^x$

(xviii) $\frac{d}{dx}(\log_a x)=\frac{1}{x\log_e a}$

$\frac{d}{dx}(\log_e x)=\frac{1}{x}$

(xix) $\frac{d}{dx}(|x|)=\frac{x}{|x|},\ x\neq 0,\quad y=|x|$ is not differentiable at $x=0$

(xx) $\frac{d}{dx}([x])=\begin{cases}0 \text{ for } x\in R-I\\ \text{does not exist for } x\in I\end{cases}$

(xxi) $\frac{d}{dx}(\{x\})=\begin{cases}1 \text{ if } x\in R-I\\ \text{does not exist if } x\in I\end{cases}$

SOME RULES FOR DIFFERENTIATION

1. The derivative of a constant function is zero , i.e. $\frac{d}{dx}(c)=0.$

2. The derivative of constant times a function is constant times the derivative of the function, i.e.

$$\frac{d}{dx}\{c.f(x)\}=c\cdot\frac{d}{dx}\{f(x)\}.$$

3. The derivative of the sum or difference of two function is the sum or difference of their derivatives, i.e.,

$$\frac{d}{dx}\{f(x)\pm g(x)\}=\frac{d}{dx}\{f(x)\}\pm\frac{d}{dx}\{g(x)\}.$$

4. **PRODUCT RULE OF DIFFERENTIATION**

The derivative of the product of two functions = (first function) × (derivative of second function)
+ (second function) × (derivative of first function)

i.e. $\frac{d}{dx}\{f(x).g(x)\}=f(x)\cdot\frac{d}{dx}\{g(x)\}+g(x)\cdot\frac{d}{dx}\{f(x)\}$

5. **QUOTIENT RULE OF DIFFRENTIATION**

The derivative of the quotient of two functions

$$= \frac{(\text{denom.} \times \text{ derivative of num.}) - (\text{num.} \times \text{ derivative of denom.})}{(\text{denominator})^2}$$

i.e. $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\cdot\frac{d}{dx}\{f(x)\} - f(x)\cdot\frac{d}{dx}\{g(x)\}}{\{g(x)\}^2}$

6. **DERIVATIVE OF A FUNCTION OF A FUNCTION (CHAIN RULE)**

If y is a differentiable function of t and t is a differentiable function of x i.e. $y = f(t)$ and $t = g(x)$, then $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx}$.

Similarly, if $y = f(u)$, where $u = g(v)$ and $v = h(x)$, then, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$.

7. **DERIVATIVE OF PARAMETRIC FUNCTIONS**

Sometimes x and y are separately given as functions of a single variable t (called a parameter) i.e. $x = f(t)$ and $y = g(t)$.

In this case,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)}. \text{ and } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\times\frac{dt}{dx} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\Big/\frac{dx}{dt}.$$

8. **DIFFERENTIATION OF IMPLICIT FUNCTIONS**

If in an equation, *x* and *y* both occurs together i.e. $f(x, y) = 0$ and this equation can not be solved either for *y* or *x*, then *y* (or *x*) is called the implicit function of *x* (or *y*).

For example $x^3 + y^3 + 3axy + c = 0$, $x^y + y^x = a$ etc.

Working rule for finding the derivative

First Method :

(i) Differentiate every term of $f(x, y) = 0$ with respect to *x*.

(ii) Collect the coefficients of $\frac{dy}{dx}$ and obtain the value of $\frac{dy}{dx}$.

Second Method :

If $f(x, y) = \text{constant}$, then $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$, where $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are partial differential coefficients of $f(x, y)$ with respect of *x* and *y* respectively.

9. **DIFFERENTIATION OF LOGARITHMIC FUNCTIONS**

When base and power both are the functions of *x i.e.*, the function is of the form $[f(x)]^{g(x)}$.

$$y = [f(x)]^{g(x)}$$

$$\log y = g(x)\log[f(x)]$$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{d}{dx}g(x).\log[f(x)]$$

$$\frac{dy}{dx} = [f(x)]^{g(x)}.\left\{\frac{d}{dx}[g(x)\log f(x)]\right\}.$$

10. DIFFERENTIATION BY TRIGONOMETRICAL SUBSTITUTIONS

Some times before differentiation, we reduce the given function in a simple form using suitable trigonometrical or algebraic transformations.

	Function	Substitution	
(i)	$\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $a\cos\theta$	where $\theta\in[0,\pi/2]$
(ii)	$\sqrt{x^2+a^2}$	$x=a\tan\theta$ or $a\cot\theta$	where $\theta\in(0,\pi/2)$
(iii)	$\sqrt{x^2-a^2}$	$x=a\sec\theta$ or $a\operatorname{cosec}\theta$	where $\theta\in(0,\pi/2)$
(iv)	$\sqrt{\frac{a-x}{a+x}}$	$x=a\cos 2\theta$	where $\theta\in[0,\pi/2]$
(v)	$\sqrt{\frac{a^2-x^2}{a^2+x^2}}$	$x^2=a^2\cos 2\theta$	where $\theta\in[0,\pi/2]$
(vi)	$\sqrt{ax-x^2}$	$x=a\sin^2\theta$	where $\theta\in[0,\pi/2]$
(vii)	$\sqrt{\frac{x}{a+x}}$	$x=a\tan^2\theta$	where $\theta\in(0,\pi/2)$
(viii)	$\sqrt{\frac{x}{a-x}}$	$x=a\sin^2\theta$	where $\theta\in[0,\pi/2]$
(ix)	$\sqrt{(x-a)(x-b)}$	$x=a\sec^2\theta-b\tan^2\theta$	where $\theta\in(0,\pi/2)$
(x)	$\sqrt{(x-a)(b-x)}$	$x=a\cos^2\theta+b\sin^2\theta$	where $\theta\in[0,\pi/2]$

11. DIFFERENTIATION OF INFINITE SERIES

(i) If $y=\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+.....\infty}}}$ then

$$\Rightarrow \quad y=\sqrt{f(x)+y} \quad \Rightarrow \quad y^2=f(x)+y$$

$$2y\frac{dy}{dx}=f'(x)+dy/dx$$

$$\therefore \frac{dy}{dx}=\frac{f'(x)}{2y-1}$$

(ii) If $y=f(x)^{f(x)^{f(x)....\infty}}$ then $y=f(x)^y$.

$$\therefore \log y = y\log[f(x)]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{y\cdot f'(x)}{f(x)}+\log f(x)\cdot\left(\frac{dy}{dx}\right)$$

$$\therefore \frac{dy}{dx}=\frac{y^2 f'(x)}{f(x)[1-y\log f(x)]}$$

(iii) If $y=f(x)+\cfrac{1}{f(x)+\cfrac{1}{f(x)+\cfrac{1}{f(x).....}}}$

Then $\dfrac{dy}{dx}=\dfrac{y\,f'(x)}{2y-f(x)}$.

12. $f''(\alpha)=\lim\limits_{h\to 0}\dfrac{f(a+2h)-2f(a+h)+f(a)}{h^2}$ and in general

$$f^{(n)}(\alpha)=\lim_{h\to 0}\frac{f(\alpha+nh)-{}^nC_1 f(\alpha+(n-1)h)+{}^nC_2 f(\alpha+(n-2)h)+....+(-1)^n f(\alpha)}{h^n}$$

13. $\left\{\dfrac{d}{dx}f^{-1}(x)\right\}_{x=f(\alpha)}=\dfrac{1}{\left\{\dfrac{d}{dx}f(x)\right\}_{x=\alpha}}$

ALGEBRA OF DIFFERENTIABLE FUNCTIONS

(i) **Logarithmic differentiation**

If $y=f_1(x)\,f_2(x)$ or $y=f_1(x)f_2(x)f_3(x)....$

or $y=\dfrac{f_1(x)f_2(x).....}{g_1(x)g_2(x).....}$, then first take log on both sides and then differentiate.

If u, v are functions of x, then $\dfrac{d}{dx}(u^v)=u^v\dfrac{d}{dx}(v\log u)$

In particular, $\dfrac{d}{dx}(x^x)=x^x(1+\log x)$

(ii) $\dfrac{d}{dx}(|u|)=\dfrac{u}{|u|}\dfrac{du}{dx}$ (iii) $\dfrac{d}{dx}(\log f(x))=\dfrac{1}{f(x)}\dfrac{d}{dx}f(x)$

(iv) $\dfrac{d}{dx}\left(a^{f(x)}\right)=a^{f(x)}\log a\,.\,f'(x)$.

LEIBNITZ THEOREM AND *n*TH DERIVATIVES

Let $f(x)$ and $g(x)$ be functions both possessing derivatives up to nth order. Then,

$$\frac{d^n}{dx^n}\left(f(x)g(x)\right)=f^n(x)g(x)+{}^nC_1 f^{n-1}(x)g^1(x)+{}^nC_2 f^{n-2}(x)g^2(x)+...+$$
$${}^nC_r f^{n-r}(x)g^r(x)+...+{}^nC_n f(x)g^n(x).$$

$$\frac{d^n}{dx^n}(x^n)=n!;\ \frac{d^n}{dx^n}\left(\frac{1}{x}\right)=\frac{(-1)^n n!}{x^{n+1}};\ \frac{d^n}{dx^n}(\sin x)=\sin\left(x+n\frac{\pi}{2}\right),$$

$$\frac{d^n}{dx^n}(\cos x)=\cos\left(x+n\frac{\pi}{2}\right);\ \frac{d^n}{dx^n}(e^{mx})=m^n e^{mx}.$$

SUCCESSIVE DIFFERENTIATION

(i) If $y=(ax+b)^m$, $m\notin N$, then $y_n=m(m-1)(m-2).....(m-n+1)(ax+b)^{m-n}.a^n$

(ii) If $y=(ax+b)^m$, $m\in N$, then

$$y_n=\begin{cases} m(m-1)(m-2).....(m-n+1)(ax+b)^{m-n}.a^n & \text{for } n<m \\ m!a^m, & \text{if } n=m \\ 0, & n>m \end{cases}$$

(iii) If $y = \frac{1}{ax+b}$, then $y_n = (-1)^n .n!(ax+b)^{-n-1} .a^n$

(iv) If $y = \log(ax+b)$, then $y_n = (-1)^{n-1}(n-1)!a^n(ax+b)^{-n}$

(v) If $y = \sin(ax+b)$, then $y_n = a^n \sin\left(ax+b+n\frac{\pi}{2}\right)$

(vi) If $y = \cos(ax+b)$, then $y_n = a^n \cos\left(ax+b+n\frac{\pi}{2}\right)$

(vii) If $y = a^x$ then $y_n = a^x(\log_e a)^n$.

PARTIAL DIFFERENTIATION

The partial differential coefficient of $f(x, y)$ with respect to x is the ordinary differential coefficient of $f(x, y)$ when y is regarded as a constant. It is written as $\frac{\partial f}{\partial x}$ or $D_x f$ or f_x.

Thus, $\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h, y)-f(x, y)}{h}$

Again, the partial differential coefficient $\frac{\partial f}{\partial y}$ of $f(x, y)$ with respect to y is the ordinary differential coefficient of $f(x, y)$ when x is regarded as a constant.

Thus, $\frac{\partial f}{\partial y} = \lim_{k\to 0}\frac{f(x, y+k)-f(x, y)}{k}$

e.g., If $z = f(x, y) = x^4 + y^4 + 3xy^2 + x^2y + x + 2y$

Then $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or $f_x = 4x^3 + 3y^2 + 2xy + 1$ (Here y is regarded as constant)

$\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or $f_x = 4y^3 + 6xy + x^2 + 2$ (Here x is regarded as constant)

HIGHER PARTIAL DERIVATIVES

Let $f(x, y)$ be a function of two variables such that $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ both exist.

(i) The partial derivative of $\frac{\partial f}{\partial x}$ w.r.t. 'x' is denoted by $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}.

(ii) The partial derivative of $\frac{\partial f}{\partial y}$ w.r.t. 'y' is denoted by $\frac{\partial^2 f}{\partial y^2}$ or f_{yy}.

(iii) The partial derivative of $\frac{\partial f}{\partial x}$ w.r.t. 'y' is denoted by $\frac{\partial^2 f}{\partial x\,\partial y}$ or f_{xy}.

(iv) The partial derivative of $\frac{\partial f}{\partial y}$ w.r.t. 'x' is denoted by $\frac{\partial^2 f}{\partial y\,\partial x}$ or f_{yx}.

These four are second order partial derivatives.

Note : If $f(x, y)$ possesses continuous partial derivatives then in all ordinary cases.

$$\frac{\partial^2 f}{\partial x\,\partial y}=\frac{\partial^2 f}{\partial y\,\partial x} \quad \text{or} \quad f_{xy} = f_{yx}.$$

EULER'S THEOREM ON HOMOGENEOUS FUNCTIONS

If $f(x,\ y)$ is a homogeneous function in $x,\ y$ of degree n, then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf$$

DEDUCTION OF EULER'S THEOREM

If $f(x,\ y)$ is a homogeneous function in x, y of degree n, then

(i) $x\frac{\partial^2 f}{\partial x^2}+y\frac{\partial^2 f}{\partial x\,\partial y}=(n-1)\frac{\partial f}{\partial x}$

(ii) $x\frac{\partial^2 f}{\partial y\,\partial x}+y\frac{\partial^2 f}{\partial y^2}=(n-1)\frac{\partial f}{\partial y}$

(iii) $x\frac{\partial^2 f}{\partial x^2}+2xy\frac{\partial^2 f}{\partial x\,\partial y}+y^2\frac{\partial^2 f}{\partial y^2}=n(n-1)f(x,\ y)$