Rational and Irrational Numbers

Concepts Related To Rational Numbers

You have studied fractional numbers in your earlier classes. Some examples of fractional numbers are $\frac{1}{2}, \frac{-4}{7}, \frac{22}{27}$.

These numbers are also known as **rational numbers.**

What comes first to your mind when you hear the word **rational?**

Yes, you are right. It is something related to the ratios.

The ratio 4:5 can be written as 4/5, which is a rational number. In ratios, the numerator and denominator both are positive numbers while in rational numbers, they can be negative also.

Thus, rational numbers can be defined as follows.

"**Any number which can be expressed in the form $\frac{p}{q}$, where *p* and *q* are integers and $q \neq 0$*,* is called a rational number.**"

For example, 15/19 is a rational number in which the numerator is 15 and the denominator is 19.

Now, is −34 a rational number?

Yes, it is a rational number. −34 can be written as -34/1. It is in the form of p/q and $q \neq 0$.

Thus, we can say that **every integer is a rational number.**

Now, consider the following decimal numbers.

1.6, 3.49, and 2.5

These decimal numbers are also rational numbers as these can be written as $\frac{16}{10}, \frac{349}{100}$, and $\frac{25}{10}$

If in a rational number, either the numerator or the denominator is a negative integer, then the rational number is negative.

For example, $\frac{-5}{12}$ and $\frac{6}{-7}$ are **negative rational numbers.**

If the numerator and the denominator both are either positive integers or negative integers, then the rational number is positive.

For example, $\frac{-9}{-4}$ and $\frac{56}{5}$ **are positive rational numbers.**

Conventions used for writing a rational number:

We know that in a rational number, the numerator and denominator both can be positive or negative.

Conventionally, rational numbers are written with positive denominators.

For example, –9 can be represented in the form of a rational number as $\frac{-9}{1}$ or $-\frac{9}{1}$ or $\frac{9}{-1}$, but generally we do not write the denominator negative and thus, $\frac{9}{-1}$ is eliminated. So, according to the convention, –9 can be represented in the form of a rational number as $\frac{-9}{1}$ or $-\frac{9}{1}$.

Equality relation for rational numbers:

For any four non-zero integers *p*, *q*, *r* and *s*, we have

$\frac{p}{q} = \frac{r}{s}$ if $ps = qr$

Order relation for rational numbers:

If $\frac{p}{q}$ and $\frac{r}{s}$ are two rational numbers such that *q* > 0 and *s* > 0 then it can be said that $\frac{p}{q} > \frac{r}{s}$ if *ps* > *qr*.

Absolute Value of a Rational Number:

The absolute value of a rational number is its numerical value regardless of its sign. The absolute value of a rational number p/q is denoted as |p/q|.

Therefore, $\left|-\frac{3}{2}\right| = \frac{3}{2}$, $\left|\frac{12}{-7}\right| = \frac{12}{7}$ etc.

Note: The absolute value of any rational number is always non-negative.

Now, let us go through the given example.

Example:

Write each of the following rational numbers according to the convention.

i) $\frac{8}{-15}$

ii) $\frac{-1131}{-729}$

Solution:

According to the convention used in rational numbers, the denominator must be a positive number.

Let us now write the given numbers according to the convention.

i)

In the number $\frac{8}{-15}$, denominator is negative.

We have,

$$\frac{8}{-15} = \frac{-8}{15} = -\frac{8}{15}$$

According to convention, the given number should be written as $\frac{-8}{15}$ or $-\frac{8}{15}$.

ii)

In the number $\frac{-1131}{-729}$, denominator is negative.

We have,

$$\frac{-1131}{-729} = \frac{1131}{729}$$

According to convention, the given number should be written as $\frac{1131}{729}$.

Example:

Find the absolute value of the following:

(i) ||−12171||-12171

(ii) ||1219||1219

Solution:

(i) Absolute value = $\left|-\frac{121}{71}\right| = \frac{121}{71}$

(ii) Absolute value = $\left|\frac{12}{19}\right| = \frac{12}{19}$

Finding Rational Numbers between Given Rational Numbers

Let's summarize.

We know that each point on the number line represents a number. Thus, between any two rational numbers, there are infinitely many numbers on the number line.

Let us try to find some rational numbers between $\frac{1}{6}$ and $\frac{7}{8}$.

To find the rational numbers between $\frac{1}{6}$ and $\frac{7}{8}$, firstly we have to make their denominators same.

2	6,	8
2	3,	4
2	3,	2
3	3,	1
	1,	1

The L.C.M. of 6 and 8 is 2 × 2 × 2 × 3 = 24

Now, we can write

$$\frac{1}{6} = \frac{1 \times 4}{6 \times 4} = \frac{4}{24}$$
$$\frac{7}{8} = \frac{7 \times 3}{8 \times 3} = \frac{21}{24}$$

Therefore, between $\frac{4}{24}\left(\frac{1}{6}\right)$ and $\frac{21}{24}\left(\frac{7}{8}\right)$, we can find many rational numbers.

Some of them are

$$\frac{5}{24},\frac{6}{24},\frac{7}{24},\frac{8}{24},\frac{9}{24},\frac{10}{24},\frac{11}{24},\frac{12}{24},\frac{13}{24},\frac{14}{24},\frac{15}{24},\frac{16}{24},\frac{17}{24},\frac{18}{24},\frac{19}{24},\frac{20}{24}$$

Let us solve some more examples to understand the concept better.

Example 1:

Find three rational numbers between $\frac{-1}{15}$ and $\frac{1}{9}$.

Solution:

The first step is to find the L.C.M. of 15 and 9.

3	15,	9
3	5,	3
5	5,	1
	1**,**	1

The L.C.M. of 15 and 9 is 3 × 3 × 5 = 45

Now, we can write

$$\frac{-1}{15}=\frac{(-1)\times 3}{15\times 3}=\frac{-3}{45}$$
$$\frac{1}{9}=\frac{1\times 5}{9\times 5}=\frac{5}{45}$$

Therefore, three rational numbers between $\frac{-1}{15}$ and $\frac{1}{9}$ are $\frac{-2}{45},\frac{0}{45}(=0),\text{ and }\frac{1}{45}$.

Example 2:

Find 10 rational numbers between $\frac{2}{5}$ and $\frac{5}{7}$.

Solution:

The first step is to find the L.C.M. of 5 and 7.

5	5,	7
7	1,	7
	1**,**	1

The L.C.M. of 5 and 7 is 5 × 7 = 35

Now, we can write

$$\frac{2}{5} = \frac{2\times7}{5\times7} = \frac{14}{35}$$
$$\frac{5}{7} = \frac{5\times5}{7\times5} = \frac{25}{35}$$

Therefore, 10 rational numbers between

$\frac{2}{5}$ and $\frac{5}{7}$ are $\frac{15}{35}\left(\frac{3}{7}\right), \frac{16}{35}, \frac{17}{35}, \frac{18}{35}, \frac{19}{35}, \frac{20}{35}\left(\frac{4}{7}\right), \frac{21}{35}\left(\frac{3}{5}\right), \frac{22}{35}, \frac{23}{35} \text{ and } \frac{24}{35}$.

Closure Properties Of Rational Numbers

Consider the two rational numbers as $\frac{5}{6}$ and $\frac{1}{4}$.

What would we get if we add these two rational numbers, i.e. what is the value of $\frac{5}{6}+\frac{1}{4}$?

$$\frac{5}{6}+\frac{1}{4}=\frac{10+3}{12}$$
$$=\frac{13}{12}, \text{ which is again a rational number.}$$

This means that the sum of two rational numbers $\frac{5}{6}$ and $\frac{1}{4}$ is a rational number. In other words, we can say that rational numbers are closed under addition.

Is this true for all rational numbers?

Yes. We can try for different rational numbers and see that this property is true for all rational numbers. Thus, we can say that the sum of two rational numbers is again a rational number. In other words, we can say that **rational numbers are closed under addition.** This property of rational numbers is known as the closure property for rational numbers and it can be stated as follows.

"If *a* and *b* are any two rational numbers and *a* + *b* = *c*, then *c* will always be a rational number".

Are rational numbers closed under subtraction also?

Let us find out.

Consider two rational numbers $\frac{-11}{12}$ and $\frac{7}{8}$.

$$\text{Now, } \left(\frac{-11}{12}\right)-\frac{7}{8}=\frac{-22-21}{24}$$

$$=\frac{-43}{24}, \text{ which is a rational number.}$$

Thus, **rational numbers are closed under subtraction also.**

Closure property of rational numbers under subtraction can be stated as follows.

"If a and b are any two rational numbers and a − b = c, then c will always be a rational number".

Now, let us check whether rational numbers are closed under multiplication also. For this, consider two rational numbers $\frac{3}{7}$ and $\frac{-4}{11}$.

Now, $\frac{3}{7}\times\frac{-4}{11}=\frac{-12}{77}$, which is a rational number.

Thus, **rational numbers are closed under multiplication also.**

Closure property of rational numbers under multiplication can be defined as follows.

"If *a* and *b* are any two rational numbers, then *a* × *b* = *c*, then *c* will always be a rational number".

But rational numbers are not closed under division. If we consider the division of $\frac{2}{5}$ by 0, then we will not obtain a rational number.

$\frac{2}{5} \div 0$ is not a rational number because division of a rational number by zero is not defined.

Thus, we can say that ***rational numbers are not closed under division.***

We can summarize the above discussed facts as follows.

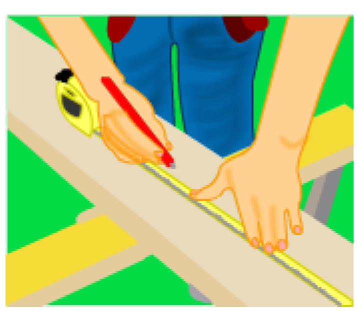
Rational numbers are closed under addition, subtraction and multiplication.

Rational numbers are not closed under division.

Decimal Expansions of Rational Numbers

The Need for Converting Rational Numbers into Decimals

A carpenter wishes to make a point on the **edge** of a wooden plank at 95 mm from any end. He has a centimeter tape, but how can he use that to mark the required point?



Simple! He should convert 95 mm into its corresponding centimeter value, i.e., 9.5 cm and then measure and mark the required length on the wooden plank.

This is just one of the many situations in life when we face the need to convert numbers into decimals. In this lesson, we will learn to convert rational numbers into decimals, observe the types of decimal numbers, and solve a few examples based on this concept.

Know More

Two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if *ad* = *bc*.

Take, for example, the rational numbers $\frac{2}{4}$ and $\frac{3}{6}$. Let us see if they are equal or not.

Here, $a = 2$, $b = 4$, $c = 3$ and $d = 6$

Now, we have:

$ad = 2 \times 6 = 12$

$bc = 4 \times 3 = 12$

Since $ad = bc$, we obtain 2/4 = 3/6.

Rational Numbers as Division of Integers

We know that the form p/q represents the division of integer *p* by the integer *q*. By solving this division, we can find the decimal equivalent of the rational number $\frac{p}{q}$. Now, let us convert the numbers $\frac{5}{8}$, $\frac{4}{3}$ and $\frac{2}{7}$ into decimals using the long division method.

$$
\begin{array}{r}
0.625 \\
8\overline{)5.000} \\
\underline{48} \\
20 \\
\underline{16} \\
40 \\
\underline{40} \\
0
\end{array}
\qquad
\begin{array}{r}
1.33... \\
3\overline{)4.00} \\
\underline{3} \\
10 \\
\underline{9} \\
10 \\
\underline{9} \\
1
\end{array}
\qquad
\begin{array}{r}
0.285714... \\
7\overline{)2.000000} \\
\underline{14} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
50 \\
\underline{49} \\
10 \\
\underline{7} \\
30 \\
\underline{28} \\
2
\end{array}
$$

While the remainder is zero in the division of 5 by 8, it is not so in case of the other two divisions. Thus, we can get two different cases in the decimal expansions of rational numbers.

Observing the Decimal Expansions of Rational Numbers

We can get the following two cases in the decimal expansions of rational numbers.

Case I: When the remainder is zero

In this case, the remainder becomes zero and the quotient or decimal expansion terminates after a finite number of digits after the decimal point. For example, in the decimal expansion of 5/8, we get the remainder as zero and the quotient as 0.625.

Case II: When the remainder is never zero

In this case, the remainder never becomes zero and the corresponding decimal expansion is non-terminating. For example, in the decimal expansions of $\frac{4}{3}$ and $\frac{2}{7}$, we see that the remainder never becomes zero and their corresponding quotients are **non-terminating decimals**.

When we divide 4 by 3 and 2 by 7, we get 1.3333… and 0.285714285714… as the respective quotients. In these decimal numbers, the digit ‘3’ and the group of digits ‘285714’ get repeated. Therefore, we can write $\frac{4}{3} = 1.3333... = 1.\bar{3}$ and

$\frac{2}{7} = 0.285714285714... = 0.\overline{285714}$. Here, the symbol $\overline{\quad}$ indicates the digit or group of digits that gets repeated.

Solved Examples

Easy

Example 1:

Write the decimal expansion of $\mathbf{\frac{1237}{25}}$ and find if it is terminating or non-terminating and repeating.

Solution:

Here is the long division method to find the decimal expansion of $\frac{1237}{25}$.

$$
\begin{array}{r}
\,\underline{49.48} \\
25\overline{)1237.00} \\
\underline{100} \\
237 \\
\underline{225} \\
120 \\
\underline{100} \\
200 \\
\underline{200} \\
0
\end{array}
$$

Hence, the decimal expansion of $\frac{1237}{25}$ is 49.48. Since the remainder is obtained as zero, the decimal numberis terminating.

Example 2:

Write the decimal expansion of $\mathbf{\frac{2358}{27}}$ and find if it is terminating or non-terminating and repeating.

4.

Solution:

Here is the long division method to find the decimal expansion of $\frac{2358}{27}$.

$$
\begin{array}{r}
87.33\ldots \\
27\overline{\smash{)}2358.00} \\
\underline{216} \\
198 \\
\underline{189} \\
90 \\
\underline{81} \\
90 \\
\underline{81} \\
9\ldots
\end{array}
$$

Hence, the decimal expansion of $\frac{2358}{27}$ is 87.33.... Since the remainder 9 is obtained again and again, the decimal numberis non-terminating and repeating. The decimal number can also be written as $87.\overline{3}$.

Medium

Example 1:

Find the decimal expansion of each of the following rational numbers and write the nature of the same.

1. $\frac{65}{101}$
2. $\frac{923}{400}$
3. $\frac{37}{99}$
4. $\frac{67}{100}$

Solution:

$$
\begin{array}{l}
\qquad\quad\ \ 0.643564\ldots \\
\text{i)}\quad 101\overline{)65.000000} \\
\qquad\quad\ \ \underline{606} \\
\qquad\qquad\ 440 \\
\qquad\qquad\ \underline{404} \\
\qquad\qquad\ \ 360 \\
\qquad\qquad\ \ \underline{303} \\
\qquad\qquad\quad 570 \\
\qquad\qquad\quad \underline{505} \\
\qquad\qquad\qquad 650 \\
\qquad\qquad\qquad \underline{606} \\
\qquad\qquad\qquad\ 440 \\
\qquad\qquad\qquad\ \underline{404} \\
\qquad\qquad\qquad\quad 36
\end{array}
$$

We have $\frac{65}{101}$ = 0.64356435... = $0.\overline{6435}$

The group of digits ‘6435’ repeats after the decimal point. Hence, the decimal expansion of the given rational number is non-terminating and repeating.

$$
\begin{array}{l}
\qquad\qquad\quad 2.3075 \\
\text{ii)}\quad 400\overline{)923.0000} \\
\qquad\quad\ \ \underline{800} \\
\qquad\quad\ \ 1230 \\
\qquad\quad\ \ \underline{1200} \\
\qquad\qquad\ \ 300 \\
\qquad\qquad\quad \underline{0} \\
\qquad\qquad\ \ 3000 \\
\qquad\qquad\ \ \underline{2800} \\
\qquad\qquad\quad 2000 \\
\qquad\qquad\quad \underline{2000} \\
\qquad\qquad\qquad 0
\end{array}
$$

We have $\frac{923}{400}$ = 2.3075

Hence, the given rational number has a terminating decimal expansion.

iii)
$$
\begin{array}{r|l}
 & 0.3737... \\
99 & 37.0000 \\
 & \underline{297} \\
 & \;\;730 \\
 & \;\;\underline{693} \\
 & \;\;\;\;370 \\
 & \;\;\;\;\underline{297} \\
 & \;\;\;\;\;\;730 \\
 & \;\;\;\;\;\;\underline{693} \\
 & \;\;\;\;\;\;\;\;37
\end{array}
$$

We have $\frac{37}{99}$ = 0.3737... = $0.\overline{37}$

The pair of digits ‘37’ repeats after the decimal point. Hence, the decimal expansion of the given rational number is non-terminating and repeating.

iv)
$$
\begin{array}{r|l}
 & 0.67 \\
100 & 67.00 \\
 & \underline{600} \\
 & \;\;700 \\
 & \;\;\underline{700} \\
 & \;\;\;\;0
\end{array}
$$

We have $\frac{67}{100}$ = 0.67

Hence, the given rational number has a terminating decimal expansion.

Irrational Numbers

We know that a number which cannot be written in the form of p/q, where p and q are integers and q ≠ 0, is known as an irrational number.

For example: all numbers of the form $\sqrt{p}$, where *p* is a prime number such as $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}$ etc., are irrational numbers.

How can we prove that these are irrational numbers?

We can prove this by making use of a theorem which can be stated as follows.

"If *p* divides *a*^{2}, then *p* divides *a* (where *p* is a prime number and *a* is a positive integer)".

Proof of theorem: Let $a = p_1 \times p_2 \times \ldots \times p_n$ ...(1)
where $p_1, p_2, \ldots p_n$ are the prime factors of a.
Squaring (1) we get
$a^2 = (p_1 \times p_2 \times \ldots\ldots \times p_n)^2$
But p divides a^2
p is a factor of ${p_1}^2 \times {p_2}^2 \times \ldots\ldots \times {p_n}^2$
By the fundamental theorem of arithmetic, the primes in the factorisation of ${p_1}^2 \times {p_2}^2 \times \ldots\ldots \times {p_n}^2$ are unique.
So, *p* is one out of the primes $p_1, p_2, \ldots p_n$.
If $p = p_k$ where *k* has value from 1 to *n*, p_k divides $p_1 \times p_2 \times \ldots\ldots p_n$
So, p_k divides *a*.
Thus, *p* divides *a* as $p_k = p$.

So go through the given video to understand the application of the above stated property.

Similarly, we can prove that square roots of other prime numbers like $\sqrt{3}, \sqrt{5}, \sqrt{11}$, etc. are irrational numbers.

Besides these irrational numbers, there are some other irrational numbers like $9\sqrt{2}, 5\sqrt{7}$ etc.

We can also prove why these numbers are irrational. Before this, let us first see what happens to irrational numbers, when we apply certain mathematical operations on them.

Addition or subtraction of two irrational numbers gives a rational or an irrational number.
Addition or subtraction of a rational and an irrational number gives an irrational number.

Multiplication of a non-zero rational number and an irrational number gives an irrational number.
Multiplication of two irrational numbers gives a rational or an irrational number.

We will now prove that $9\sqrt{2}$ **is irrational.**

We know that $\sqrt{2}$ is irrational (as proved before).

Now, the multiplication of a rational and an irrational number gives an irrational number.

Therefore, $9\sqrt{2}$ is an irrational number.

Let us now try to understand the concept further through some more examples.

Example 1:

Prove that $5+2\sqrt{7}$ **is irrational.**

Solution:

Let us assume that $5+2\sqrt{7}$ is not irrational, i.e. $5+2\sqrt{7}$ is a rational number.

Then we can write $5+2\sqrt{7}=\frac{a}{b}$, where *a* and *b* are integers and*b* ≠ 0.

Let *a* and *b* have a common factor other than 1.

After dividing by the common factor, we obtain

$5+2\sqrt{7}=\frac{c}{d}$, where *c* and *d* are co-prime numbers.

$$\Rightarrow 5-\frac{c}{d}=-2\sqrt{7}$$

$$\Rightarrow 2\sqrt{7}=\frac{c}{d}-5$$

$\Rightarrow \sqrt{7} = \frac{c}{2d} - \frac{5}{2}$

As *c*, *d* and 2 are integers, $\frac{c}{2d}$ and $\frac{5}{2}$ are rational numbers.

Thus, $\frac{c}{2d} - \frac{5}{2}$ is rational.

$\Rightarrow \sqrt{7}$ is rational as the difference of two rational numbers is again a rational number.

This is a contradiction as $\sqrt{7}$ is irrational.

Therefore, our assumption that $5+2\sqrt{7}$ is rational is wrong.

Hence, $5+2\sqrt{7}$ is irrational.

Example 2:

Prove that $3-\sqrt{5}$ is irrational.

Solution:

Let us assume $3-\sqrt{5}$ is rational. Then, we can write

$3-\sqrt{5} = \frac{a}{b}$,

where *a* and *b* are co-prime and *b* ≠ 0.

$\Rightarrow \sqrt{5} = 3 - \frac{a}{b}$

Now, as *a* and *b* are integers, $\frac{a}{b}$ is rational or $3-\frac{a}{b}$ is a rational number.

This means that $\sqrt{5}$ is rational.

This is a contradiction as $\sqrt{5}$ is irrational.

Therefore, our assumption that $3-\sqrt{5}$ is rational is wrong.

Hence, $3-\sqrt{5}$ is an irrational number.

Operations on Irrational Numbers

Mathematical Operations and Irrational Numbers

We have learnt to perform addition, subtraction, multiplication and division on integers, decimals and fractions. We can also perform these operations on irrational numbers of the form $\sqrt{n}$, where *n* is a positive real number.

Performing mathematical operations on irrational numbers is similar to performing these operations on **algebraic expressions**. For example, to add the algebraic expressions $2xy + 3y^2$ and $x^2y - 4xy$, we first check and add the like terms and then write the unlike terms as they are. In our example, $2xy$ and $-4xy$ are like terms as they have the common algebraic part xy.

So, $(2xy + 3y^2) + (x^2y - 4xy) = (2xy - 4xy) + 3y^2 + x^2y = -2xy + 3y^2 + x^2y$

Irrational numbers are also categorized as like and unlike irrational numbers. We can add or subtract like irrational numbers only.

In this lesson, we will learn how to perform the four mathematical operations on irrational numbers.

Like and Unlike Irrational Terms

Like terms: The terms or numbers whose irrational parts are the same are known as like terms. For example, $\frac{2}{7}\sqrt{3}$ and $\frac{3}{5}\sqrt{3}$ are like terms because the irrational parts in these numbers are the same, i.e., $\sqrt{3}$.

Unlike terms: The terms or numbers whose irrational parts are not the same are known as unlike terms. For example, $\frac{31}{8}\sqrt{7}$ and $\frac{11}{13}\sqrt{5}$ are unlike terms because the irrational parts in these numbers are different, i.e., $\sqrt{7}$ and $\sqrt{5}$.

Sometimes, two numbers may appear to have different irrational parts, but on simplification they are found to be the same. For example, $5\sqrt{2}$ and $3\sqrt{8}$ seem to have different irrational parts, i.e., $\sqrt{2}$ and $\sqrt{8}$.

However, on simplifying $3\sqrt{8}$, we get $3\sqrt{8} = 3\left(2\sqrt{2}\right) = 6\sqrt{2}$. Thus, we see that the two numbers have the same irrational part, i.e., $\sqrt{2}$.

Arithmetic Operations between Rational and Irrational Numbers

We have learnt to perform operations between fractions and integers, decimals and whole numbers and different types of numbers. Now, let us try to perform the same between rational and irrational numbers.

Let us take the rational number 4 and the irrational number $\sqrt{7}$. On applying the four operations on these numbers, we get $4 + \sqrt{7}$, $4 - \sqrt{7}$, $4 \times \sqrt{7}$ and $\frac{4}{\sqrt{7}}$.

Since $\sqrt{7}$ has a non-terminating and non-repeating decimal expansion, the decimal expansions of $4 + \sqrt{7}$, $4 - \sqrt{7}$, $4 \times \sqrt{7}$ and $\frac{4}{\sqrt{7}}$ will also be non-terminating and non-repeating. Hence, these numbers will also be irrational.

So, we can conclude that:

- The sum or difference of a rational and an irrational number is always irrational.
- The product or quotient of a non-zero rational number and an irrational number is always irrational.

Solved Examples

Medium

Example 1:

Check whether π + 8 and $\frac{4\sqrt{3}-6}{5}$ are irrational numbers or not.

Solution:

We know that π is an irrational number and 8 is a rational number. The sum of a rational and an irrational number is always irrational. Hence, π + 8 is an irrational number. It can be proved as follows:

π = 3.1415…

⇒ π + 8 = 3.1415… + 8 = 11.1415…

11.1415… is a non-terminating and non-repeating decimal number, so it is irrational.

We know that $\sqrt{3}$ is an irrational number and 4 is rational number. The product of a non-zero rational number and an irrational number is always irrational. Thus, $4\sqrt{3} = 4 \times \sqrt{3}$ is an irrational number. Similarly, $4\sqrt{3} - 6$ is an irrational number as it is the difference between a rational and an irrational number. Finally, $\frac{4\sqrt{3}-6}{5}$ is an irrational number as it is the quotient of a non-zero rational number and an irrational number.

Performing Operations on Irrational Numbers

The decimal expansion of an irrational number is non-terminating and non-repeating. For this reason, unlike irrational terms cannot be added or subtracted.

Suppose $\sqrt{x}$ and $\sqrt{y}$ are two unlike irrational numbers. The arithmetic operations between them are shown as follows:

- Addition = $\sqrt{x} + \sqrt{y}$
- Subtraction = $\sqrt{x} - \sqrt{y}$ or $\sqrt{y} - \sqrt{x}$
- Multiplication = $\sqrt{x} \times \sqrt{y} = \sqrt{x \times y} = \sqrt{xy}$
- Division = $\sqrt{x} \div \sqrt{y} = \frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$

Suppose $a\sqrt{x}$ and $b\sqrt{x}$ are two like irrational numbers. The arithmetic operations between them are shown as follows:

- Addition = $a\sqrt{x} + b\sqrt{x} = a \times \sqrt{x} + b \times \sqrt{x} = (a+b) \times \sqrt{x} = (a+b)\sqrt{x}$
- Subtraction = $a\sqrt{x} - b\sqrt{x} = a \times \sqrt{x} - b \times \sqrt{x} = (a-b) \times \sqrt{x} = (a-b)\sqrt{x}$
- Multiplication = $a\sqrt{x} \times b\sqrt{x} = ab\sqrt{x \times x} = ab\sqrt{x^2} = abx$
- Division = $\frac{a\sqrt{x}}{b\sqrt{x}} = \frac{a \times \sqrt{x}}{b \times \sqrt{x}} = \frac{a}{b} \times \frac{\sqrt{x}}{\sqrt{x}} = \frac{a}{b}$

Solved Examples

Easy

Example 1:

1. **Divide** $8\sqrt{21}$ **by** $2\sqrt{7}$ **.**

2. **Multiply** $8\sqrt{21}$ **with** $2\sqrt{7}$ **.**

Solution:

i) $$\frac{8\sqrt{21}}{2\sqrt{7}}$$
$$=\frac{8\sqrt{7\times 3}}{2\sqrt{7}}$$
$$=\frac{8\sqrt{7}\times\sqrt{3}}{2\sqrt{7}}$$
$$=4\sqrt{3}$$

ii) $$8\sqrt{21}\times 2\sqrt{7}=8\times\sqrt{3\times 7}\times 2\times\sqrt{7}$$
$$=8\times 2\times\sqrt{7}\times\sqrt{7}\times\sqrt{3}$$
$$=16\times\sqrt{7\times 7}\times\sqrt{3}$$
$$=16\times 7\sqrt{3}$$
$$=112\sqrt{3}$$

Example 2:

1. **Prove that** $\sqrt{24}+\sqrt{54}=\sqrt{150}$ **.**

- **Subtract** $\sqrt{125}$ **from** $2\sqrt{5}$ **.**

Solution:

i) We have to prove that $\sqrt{24}+\sqrt{54}=\sqrt{150}$.

$$\begin{aligned}
\text{LHS} &= \sqrt{24}+\sqrt{54} \\
&= \sqrt{2\times 2\times 2\times 3}+\sqrt{2\times 3\times 3\times 3} \\
&= 2\sqrt{2\times 3}+3\sqrt{2\times 3} \\
&= 2\sqrt{6}+3\sqrt{6} \\
&= 5\sqrt{6}
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \sqrt{150} \\
&= \sqrt{2\times 3\times 5\times 5} \\
&= 5\sqrt{6}
\end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

ii) $2\sqrt{5}-\sqrt{125}$

$$\begin{aligned}
&= 2\sqrt{5}-\sqrt{5\times 5\times 5} \\
&= 2\sqrt{5}-5\sqrt{5} \\
&= (2-5)\sqrt{5} \\
&= -3\sqrt{5}
\end{aligned}$$

Medium

Example 1:

Simplify $\dfrac{\sqrt{20}\times 2\sqrt{12}}{6\sqrt{15}}$ **.**

Solution:

$$\frac{\sqrt{20}\times 2\sqrt{12}}{6\sqrt{15}}$$
$$=\frac{\sqrt{2\times 2\times 5}\times 2\sqrt{2\times 2\times 3}}{6\sqrt{3\times 5}}$$
$$=\frac{2\sqrt{5}\times 2\times 2\sqrt{3}}{6\sqrt{3}\times\sqrt{5}}$$
$$=\frac{8}{6}\times\frac{\sqrt{5}\times\sqrt{3}}{\sqrt{5}\times\sqrt{3}}$$
$$=\frac{4}{3}$$

Solving Expressions Having Irrational Numbers

Concept Builder

There is an order in which calculations should be performed while simplifying an expression. This order of performing operations is called BODMAS, with each letter in this word standing for a particular operation.

B	**O**	**D**	**M**	**A**	**S**
Brackets	Of	Division	Multiplication	Addition	Subtraction

While simplifying an expression, we should first remove the '**brackets'**. Next, we should perform operations involving '**of',** e.g., one fourth of 16, 20% of 100, etc. Then, we should carry out 'division**', 'multiplication',** '**addition'** and '**subtraction',** in that order.

All expressions are solved using the **BODMAS rule**. Take, for example, the expression 36 ÷ 12 + 7 × 2 − 7. We simplify this expression as follows:

36 ÷ 12 + 7 × 2 − 7

= **3** + 7 × 2 − 7 (Division)

= 3 + **14** − 7 (Multiplication)

= **17** − 7 (Addition)

= **10** (Subtraction)

Solved Examples

Easy

Example 1:

Multiply $\left(3\sqrt{7}+2\sqrt{13}\right)$ **with** $\sqrt{5}$ **.**

Solution:

$$\begin{aligned}
&\left(3\sqrt{7}+2\sqrt{13}\right)\times\sqrt{5} \\
&=3\sqrt{7}\times\sqrt{5}+2\sqrt{13}\times\sqrt{5} \\
&=3\sqrt{7\times5}+2\sqrt{13\times5} \\
&=3\sqrt{35}+2\sqrt{65}
\end{aligned}$$

Example 2:

Simplify $\dfrac{2\sqrt{150}-9}{\sqrt{3}}$ **.**

Solution:

$$\begin{aligned}
&\frac{2\sqrt{150}-9}{\sqrt{3}} \\
&=\frac{2\sqrt{2\times3\times5\times5}-3\times3}{\sqrt{3}} \\
&=\frac{2\times5\times\sqrt{2}\times\sqrt{3}-3\times\sqrt{3}\times\sqrt{3}}{\sqrt{3}} \\
&=\frac{\sqrt{3}\left(10\sqrt{2}-3\sqrt{3}\right)}{\sqrt{3}} \\
&=10\sqrt{2}-3\sqrt{3}
\end{aligned}$$

Medium

Example 1:

Simplify $\left(5+\sqrt{7}\right)\left(\sqrt{7}-2\right)+\left(3+\sqrt{3}\right)^2$ **.**

Solution:

$$
\begin{aligned}
&\left(5+\sqrt{7}\right)\left(\sqrt{7}-2\right)+\left(3+\sqrt{3}\right)^{2}\\
&=\left(5+\sqrt{7}\right)\left(\sqrt{7}-2\right)+\left(3+\sqrt{3}\right)\left(3+\sqrt{3}\right)\\
&=5\times\sqrt{7}-5\times2+\sqrt{7}\times\sqrt{7}-\sqrt{7}\times2+3\times3+3\times\sqrt{3}+\sqrt{3}\times3+\sqrt{3}\times\sqrt{3}\\
&=5\sqrt{7}-10+7-2\sqrt{7}+9+6\sqrt{3}+3\\
&=3\sqrt{7}+6\sqrt{3}+9
\end{aligned}
$$

Example 2:

Simplify $\left(\sqrt{486}\div 2\right)-\left(\sqrt{27}\div\sqrt{2}\right)$**.**

Solution:

$$\left(\sqrt{486} \div 2\right) - \left(\sqrt{27} \div \sqrt{2}\right)$$
$$= \frac{\sqrt{486}}{2} - \frac{\sqrt{27}}{\sqrt{2}}$$
$$= \frac{\sqrt{486}}{2} - \sqrt{\frac{27}{2}}$$
$$= \frac{9\sqrt{6}}{2} - 3\sqrt{\frac{3}{2}}$$
$$= \frac{9\sqrt{6}}{\sqrt{4}} - 3\sqrt{\frac{3}{2}}$$
$$= 9\sqrt{\frac{6}{4}} - 3\sqrt{\frac{3}{2}}$$
$$= 9\sqrt{\frac{3}{2}} - 3\sqrt{\frac{3}{2}}$$
$$= 6\sqrt{\frac{3}{2}}$$
$$= 3 \times 2\sqrt{\frac{3}{2}}$$
$$= 3 \times 2\frac{\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= 3 \times \frac{2\sqrt{6}}{2}$$
$$= 3\sqrt{6}$$

Concepts Related to Surds

Look at the following numbers.

$\sqrt{4}, \sqrt{25}$ and $\sqrt{49}$

All these are rational numbers as $\sqrt{4} = \pm 2, \sqrt{25} = \pm 5$ and $\sqrt{49} = \pm 7$.

Now, observe the numbers $\sqrt{2}, \sqrt{5}$ and $\sqrt{7}$. These numbers are irrational.

Roots of rational numbers:

Suppose 5 is the square of a rational number, then

$x^2 = 5$

$\Rightarrow x = \pm\sqrt{5}$

Here, 5 is a rational number, but $\sqrt{5}$ is not a rational number. Thus, x can not be a rational number.

Now, let us assume that 10 is the cube of a rational number, therefore $y^3 = 10$.

$\Rightarrow y = \sqrt[3]{10}$

Here, 10 is a rational number. Since cube root of 10 is not a rational number, y cannot be a rational number.

Similarly, there are many rational numbers that are not square, cube, etc. of any rational number. In other words, we can say that there are many rational numbers whose roots are irrational.

Irrational root of a positive rational number is called surd.

For example:

$\sqrt[3]{10}$, $\sqrt[4]{15}$, $\sqrt[8]{12}$, $\sqrt[7]{50}$, etc.

It can be generally defined in the following way:

If $\sqrt[a]{x}$ is an irrational number such that x is a positive rational number and a ($a \neq 1$) is a natural number, then $\sqrt[a]{x}$ is known as a surd. Here, $\sqrt{}$ is the **radical sign**, a is the **order** of the surd and x is the **radicand**.

When $a = 2$, the surd is called a quadratic surd.

Now, consider the number $\sqrt{-4}$.

Is it a surd?

No, it is not.

Since $\sqrt{-4}$ is the root of the negative rational number −4, it cannot be called as surd. Similarly, $\sqrt{\pi}$ is the root of an irrational number π, so it is not a surd.

Now, observe the number $\sqrt{\sqrt{7}}$.

Is it a surd?

Yes, it is.

By just looking at the number, it seems that $\sqrt{\sqrt{7}}$ is not a surd, but it can be reduced to the surd form in the following way:

$$\sqrt{\sqrt{7}} = \sqrt[4]{7}$$

$\sqrt[4]{7}$ is a surd.

Rules for surds:

Let Q and N be the sets of rational numbers and natural numbers respectively.

If *x*, *y* ∈ Q, *x*, *y* > 0 and *a*, *b*, *c* ∈ N, then

(1) $\sqrt[a]{x} = (x)^{\frac{1}{a}}$ (2) $\left(\sqrt[a]{x}\right)^a = x$

(3) $\sqrt[a]{x}.\sqrt[a]{y} = \sqrt[a]{xy}$ (4) $\dfrac{\sqrt[a]{x}}{\sqrt[a]{y}} = \sqrt[a]{\dfrac{x}{y}}$

(5) $\sqrt[a]{\sqrt[b]{x}} = \sqrt[b]{\sqrt[a]{x}} = \sqrt[ab]{x}$ (6) $\sqrt[a]{x^b} = \sqrt[ac]{x^{bc}}$

(7) $\sqrt[a]{x^b} = \left(\sqrt[a]{x}\right)^b$

These rules are very useful to solve the problems related to surds.

Let us go through a few examples to understand the concept better.

Example 1:

Identify the surds among the given numbers and observe their orders.

(1) $\sqrt{15}$ (2) $\sqrt[3]{5}$ (3) $\sqrt[5]{32}$ (4) $\sqrt{\sqrt{12}}$

Solution:

(1) $\sqrt{15}$ is the square root of a positve rational number 15, so $\sqrt{15}$ is a surd.

Order of $\sqrt{15}$ is 2, so it can also be called a quadratic surd.

(2) $\sqrt[3]{5}$ is the cube root of a positve rational number 5, so $\sqrt[3]{5}$ is a surd.

Order of $\sqrt[3]{5}$ is 3.

(3) $\sqrt[5]{32} = \sqrt[5]{2^5}$

$$= \left(\sqrt[5]{2}\right)^5 \qquad \left[\sqrt[a]{x^a} = \left(\sqrt[a]{x}\right)^a\right]$$

$$= 2 \qquad \left[\left(\sqrt[a]{x}\right)^a = a\right]$$

Thus, $\sqrt[5]{32}$ is not a surd.

(4) $\sqrt{\sqrt{12}} = \sqrt[4]{12}$

$\sqrt[4]{12}$ is the root of a positve rational number 12, so $\sqrt[4]{12}$ is a surd and thus, $\sqrt{\sqrt{12}}$ is a surd.

Order of $\sqrt{\sqrt{12}} = \sqrt[4]{12}$ is 4.

Example 2: Simplify the following using the rules of surds.

(1) $\sqrt[3]{5}.\sqrt[3]{25}$ (2) $\dfrac{\sqrt[5]{3}}{\sqrt[5]{7}}$

(3) $\sqrt[4]{\sqrt{8}}$ (4) $\sqrt[2]{\sqrt[3]{10}}$

(5) $\sqrt[6]{4}$

Solution:

(1) $\sqrt[3]{5}.\sqrt[3]{25} = \sqrt[3]{5 \times 25}$ $\left[\sqrt[a]{x}.\sqrt[a]{y} = \sqrt[a]{xy}\right]$

$= \sqrt[3]{125}$

$= \sqrt[3]{5^3}$

$= 5$ $\left[\sqrt[a]{x^a} = x\right]$

(2) $\dfrac{\sqrt[5]{3}}{\sqrt[5]{7}} = \sqrt[5]{\dfrac{3}{7}}$ $\left[\dfrac{\sqrt[a]{x}}{\sqrt[a]{y}} = \sqrt[a]{\dfrac{x}{y}}\right]$

(3) $\sqrt[4]{\sqrt{8}} = \sqrt[4\times 2]{8}$ $\left[\sqrt[a]{\sqrt[b]{x}} = \sqrt[ab]{x}\right]$

$= \sqrt[8]{8}$

(4) $\sqrt[2]{\sqrt[3]{10}} = \sqrt[2\times 3]{10}$ $\left[\sqrt[a]{\sqrt[b]{x}} = \sqrt[ab]{x}\right]$

$= \sqrt[6]{10}$

(5) $\sqrt[6]{4} = \sqrt[6]{2^2}$

$= \sqrt[3\times 2]{2^{1\times 2}}$

$= \sqrt[3]{2^1}$ $\left[\sqrt[a\times c]{x^{b\times c}} = \sqrt[a]{x^b}\right]$

$= \sqrt[3]{2}$

Various Forms of Surds and Their Conversions

Surds can be represented in two main forms, which are pure and mixed form.

Pure form: A surd of the form $k\sqrt[a]{x}$ is said to be in **pure form when** *k* ∈ Q, such that *k* = ±1.

For example, $\sqrt[3]{7}, -\sqrt{11}, \sqrt[4]{3}, -\sqrt[5]{15}$ are pure surds.

Mixed form:

A surd of the form $k\sqrt[a]{x}$ is said to be in **mixed form** when *k* ∈ Q, such that *k* ≠ 0 and *k* ≠ ±1.

For example, $3\sqrt[3]{5}, -4\sqrt{16}, 2\sqrt[4]{3}$ are mixed surds.

We can easily convert mixed surds to pure surds and vice versa. Let us study a few examples to understand the conversion.

$$-5\sqrt{3} = -\sqrt{25}\sqrt{3} = -\sqrt{25 \times 3} = -\sqrt{75}$$

Therefore, mixed surd $-5\sqrt{3}$ can be written in pure surd form as $-\sqrt{75}$.

Let us take another example.

$$\sqrt{27} = \sqrt{9 \times 3} = \sqrt{9} \times \sqrt{3} = \sqrt{3^2} \times \sqrt{3} = 3\sqrt{3}$$

Therefore, pure surd $\sqrt{27}$ can be written in mixed surd form as $3\sqrt{3}$.

It should be noted that it is not possible to express every pure surd as mixed surd.

For example:

$\sqrt{11}, \sqrt{21}$, etc.

In such cases, the radicand is a prime number or it has the factors whose roots are irrational.

Let us go through a few examples to get more familiar with the concept.

Example 1: Convert the following mixed surds to pure surds.

$$\begin{array}{ll} (1) -2\sqrt{5} & (2)\, 5\sqrt[3]{9} \\ (3)\, 4\sqrt{3} & (4) -2\sqrt[4]{5} \end{array}$$

Solution:

$$\begin{aligned} (1) -2\sqrt{5} &= -\sqrt{4}\sqrt{5} \\ &= -\sqrt{4 \times 5} \qquad \left[\sqrt[q]{x}.\sqrt[q]{y} = \sqrt[q]{xy}\right] \\ &= -\sqrt{20} \end{aligned}$$

$$(2)\, 5\sqrt[3]{9} = \sqrt[3]{5^3}.\sqrt[3]{9} \qquad \left[\left(\sqrt[a]{x}\right)^a = x\right]$$

$$= \sqrt[3]{125}.\sqrt[3]{9}$$

$$= \sqrt[3]{125\times 9} \qquad \left[\sqrt[a]{x}.\sqrt[a]{y} = \sqrt[a]{xy}\right]$$

$$= \sqrt[3]{1125}$$

$$(3)\, 4\sqrt{3} = \sqrt{16}\sqrt{3}$$

$$= \sqrt{16\times 3} \qquad \left[\sqrt[a]{x}.\sqrt[a]{y} = \sqrt[a]{xy}\right]$$

$$= \sqrt{48}$$

$$(4) - 2\sqrt[4]{5} = -\sqrt[4]{2^4}.\sqrt[4]{5} \qquad \left[\left(\sqrt[a]{x}\right)^a = x\right]$$

$$= -\sqrt[4]{16}.\sqrt[4]{5}$$

$$= -\sqrt[4]{16\times 5} \qquad \left[\sqrt[a]{x}.\sqrt[a]{y} = \sqrt[a]{xy}\right]$$

$$= -\sqrt[4]{80}$$

Example 2: Convert the following pure surds to mixed surds.

$(1)\sqrt{125}$ $\qquad$ $(2) - \sqrt[3]{81}$

$(3) - \sqrt{108}$ $\qquad$ $(4)\sqrt[4]{112}$

Solution:

$$(1)\sqrt{125} = \sqrt{5\times 5\times 5}$$

$$= \sqrt{5^2\times 5}$$

$$= \sqrt{5^2}\,\sqrt{5} \qquad \left[\sqrt[a]{x}.\sqrt[a]{y} = \sqrt[a]{xy}\right]$$

$$= 5\sqrt{5} \qquad \left[\left(\sqrt[a]{x}\right)^a = x\right]$$

$$(2) - \sqrt[3]{81} = -\sqrt[3]{27\times 3}$$

$$= -\sqrt[3]{27}\times\sqrt[3]{3} \qquad \left[\sqrt[a]{x}.\sqrt[a]{y} = \sqrt[a]{xy}\right]$$

$$= -\sqrt[3]{3^3}\times\sqrt[3]{3}$$

$$= -3\sqrt[3]{3} \qquad \left[\left(\sqrt[a]{x}\right)^a = x\right]$$

$$(3) -\sqrt{108} = -\sqrt{36\times 3}$$

$$= -\sqrt{36}\times\sqrt{3} \qquad \left[\sqrt[a]{x}.\sqrt[a]{y} = \sqrt[a]{xy}\right]$$

$$= -\sqrt{6^2}\times\sqrt{3}$$

$$= -6\sqrt{3} \qquad \left[\left(\sqrt[a]{x}\right)^a = x\right]$$

$$(4) \sqrt[4]{112} = \sqrt[4]{16\times 7}$$

$$= \sqrt[4]{16}\times\sqrt[4]{7} \qquad \left[\sqrt[a]{x}.\sqrt[a]{y} = \sqrt[a]{xy}\right]$$

$$= \sqrt[4]{2^4}\times\sqrt[4]{7}$$

$$= 2\sqrt[4]{7} \qquad \left[\left(\sqrt[a]{x}\right)^a = x\right]$$

Rationalization of Surds

When two or more surds are multiplied, the product can either be a rational number ora surd. If a rational number is obtained after multiplying two surds, then each surd is called the rationalizing factor of the other.

For example:

$$\sqrt{3}\times\sqrt{27} = \sqrt{3\times 27} = \sqrt{81} = \sqrt{9^2} = 9$$

Since the product,i.e. 9 is a rational number, $\sqrt{3}$ and $\sqrt{27}$ are rationalizing factors of each other.

So, if we have to find the rationalizing factor of a given surd, then we need to multiply it by another surd so that the product obtained is a rational number.

For example, let us find the rationalizing factor of $2\sqrt[3]{5}$.

It can be seen that the order of the given surd is 3 and the radicand cannot be factorized as it is a prime number. Thus, we need to find a surd of the same order, but having radicand which can make a perfect cube with 5. That radicand can be 5 × 5, i.e. 25. Thus, the required surd will be $\sqrt[3]{25}$.

Let us check the result.

$$\left(2\sqrt[3]{5}\right)\times\sqrt[3]{25} = 2\sqrt[3]{5}\times\sqrt[3]{5^2} = 2\sqrt[3]{5\times 5^2} = 2\sqrt[3]{5^3} = 2\times 5 = 10$$

The product is a rational number, so $\sqrt[3]{25}$ is the rationalizing factor of $2\sqrt[3]{5}$ or vice versa.

More rationalizing factors of a surd can be obtained by multiplying the original factor by a rational number.

For example, $2\sqrt[3]{25}, 3\sqrt[3]{25}, 63\sqrt[3]{25}$,etc. are also rationalizing factors of $2\sqrt[3]{5}$.

Let us have a look at a few more examples based on this concept.

Example 1: Find the rationalizing factors of the following surds.

(1) $3\sqrt{6}$ (2) $\sqrt[3]{24}$ (3) $\sqrt[3]{16}$

Solution:

(1) $3\sqrt{6}\times\sqrt{6}=3\sqrt{6^2}=3\times6=18$

Thus, $\sqrt{6}$ is the rationalizing factor of $3\sqrt{6}$.

(2) $\sqrt[3]{24}=\sqrt[3]{8\times3}=\sqrt[3]{2^3\times3}=\sqrt[3]{2^3}\times\sqrt[3]{3}=2\sqrt[3]{3}$

Now, $2\sqrt[3]{3}\times\sqrt[3]{9}=2\sqrt[3]{27}=2\sqrt[3]{3^3}=2\times3=6$

Thus, $\sqrt[3]{9}$ is the rationalizing factor of $\sqrt[3]{24}$.

(3) $\sqrt[3]{16}\times\sqrt[3]{4}=\sqrt[3]{64}=\sqrt[3]{4^3}=4$

Thus, $\sqrt[3]{4}$ is the rationalizing factor of $\sqrt[3]{16}$.

Example 2: Rationalize the denominator.

(1) $\frac{2}{\sqrt{3}}$ (2) $\frac{1}{\sqrt[3]{9}}$

Solution:

$$(1)\ \frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{\sqrt{3\times3}}=\frac{2\sqrt{3}}{\sqrt{3^2}}=\frac{2\sqrt{3}}{3}$$

$$(2)\ \frac{1}{\sqrt[3]{9}}=\frac{1}{\sqrt[3]{3\times3}}=\frac{1}{\sqrt[3]{3\times3}}\times\frac{\sqrt[3]{3}}{\sqrt[3]{3}}=\frac{\sqrt[3]{3}}{\sqrt[3]{3\times3\times3}}=\frac{\sqrt[3]{3}}{\sqrt[3]{3^3}}=\frac{\sqrt[3]{3}}{3}$$

Rationalizing the Denominators

Rationalization

So far we have studied different operations on rational and irrational numbers, such as addition and subtraction. Now, what if we have to add two irrational fractions whose denominators are irrational numbers, say of the form $\sqrt{2}+3$ and $\sqrt{3}+5$? In such cases, we have to first rationalize the denominators, i.e., we have to make the denominators rational quantities (even if the numerators remain in the irrational form).

What does rationalization mean? Rationalization is the process in which an irrational fraction having a **surd** in the denominator is rewritten to obtain a rational number in the denominator. The surd may be a monomial or a binomial having a square root.

In this lesson, we will learn to:

- Rationalize the denominator of an irrational fraction
- Solve expressions using the rationalization method

Rationalizing the Denominator

Let us understand the method of rationalizing the denominator of an irrational fraction by taking the example of $\frac{1}{\sqrt{7}}$.

We know that $\sqrt{x} \times \sqrt{x} = x$.

$\therefore \sqrt{7} \times \sqrt{7} = 7$, which is a rational number

So, we can write the expression $\frac{1}{\sqrt{7}}$ as:

$$\frac{1}{\sqrt{7}} \times 1$$

$$= \frac{1}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} \qquad \left(\because \frac{\sqrt{7}}{\sqrt{7}} = 1\right)$$

$$= \frac{\sqrt{7}}{7}$$

Thus, we get a rational number as the denominator.

Let us now understand what is meant by ‘conjugate of a number’.

When a number is represented as the sum or difference of a rational number and an irrational number or two irrational numbers, the conjugate of that number just differs by the sign in between.

For example, the conjugate of $\sqrt{2}+3$ is $\sqrt{2}-3$ and that of $\sqrt{2}+\sqrt{3}$ is $\sqrt{2}-\sqrt{3}$ or $\sqrt{3}-\sqrt{2}$. While solving irrational fractions having such denominators, we multiply and divide the fractions by the conjugates of the denominators.

Solved Examples

Easy

Example 1:

Rationalize the denominator of $\mathbf{\frac{2}{\sqrt{3}}}$.

Solution:

We know that $\sqrt{3}\times\sqrt{3}=3$, which is a rational number.

So, we multiply and divide $\frac{2}{\sqrt{3}}$ by $\sqrt{3}$.

$$\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$$

Thus, we obtain a rational number as the denominator.

Example 2:

Rationalize the denominator of $\mathbf{\frac{11\sqrt{3}-6}{\sqrt{3}+2}}$.

Solution:

To rationalize the denominator, we multiply and divide $\frac{11\sqrt{3}-6}{\sqrt{3}+2}$ by the conjugate of $\sqrt{3}+2$.

The conjugate of $\sqrt{3}+2$ is $\sqrt{3}-2$.

$$\begin{aligned}
\therefore \frac{11\sqrt{3}-6}{\sqrt{3}+2} &= \frac{11\sqrt{3}-6}{\left(\sqrt{3}+2\right)} \times \frac{\left(\sqrt{3}-2\right)}{\left(\sqrt{3}-2\right)} \\
&= \frac{\left(11\sqrt{3}\times\sqrt{3}\right)-\left(11\sqrt{3}\times 2\right)-\left(6\times\sqrt{3}\right)+\left(6\times 2\right)}{\left(\sqrt{3}\right)^2-(2)^2} \\
&= \frac{33-22\sqrt{3}-6\sqrt{3}+12}{3-4} \\
&= 28\sqrt{3}-45
\end{aligned}$$

Medium

Example 1:

Simplify the expression $\frac{\mathbf{6}}{\mathbf{2\sqrt{3}-\sqrt{6}}}+\frac{\mathbf{\sqrt{6}}}{\mathbf{\sqrt{3}+\sqrt{2}}}-\frac{\mathbf{4\sqrt{3}}}{\mathbf{\sqrt{6}-\sqrt{2}}}$.

Solution:

Let $\frac{6}{2\sqrt{3}-\sqrt{6}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}=a+b-c$

Where,

$a=\frac{6}{2\sqrt{3}-\sqrt{6}}$, $b=\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$ and $c=\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$

To solve such an expression, we have to first rationalize the denominator of each term.

$$a=\frac{6}{2\sqrt{3}-\sqrt{6}}$$

$$\Rightarrow a=\frac{6}{2\sqrt{3}-\sqrt{6}}\times\frac{2\sqrt{3}+\sqrt{6}}{2\sqrt{3}+\sqrt{6}}=\frac{6\left(2\sqrt{3}+\sqrt{6}\right)}{\left(2\sqrt{3}\right)^2-\left(\sqrt{6}\right)^2}=\frac{6\left(2\sqrt{3}+\sqrt{6}\right)}{12-6}=2\sqrt{3}+\sqrt{6}$$

$$b=\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$$

$$\Rightarrow b=\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}\times\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}=\frac{\sqrt{6}\left(\sqrt{3}-\sqrt{2}\right)}{\left(\sqrt{3}\right)^2-\left(\sqrt{2}\right)^2}=\frac{3\sqrt{2}-2\sqrt{3}}{3-2}=3\sqrt{2}-2\sqrt{3}$$

$$c=\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$$

$$\Rightarrow c=\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}\times\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}=\frac{4\sqrt{3}\left(\sqrt{6}+\sqrt{2}\right)}{\left(\sqrt{6}\right)^2-\left(\sqrt{2}\right)^2}=\frac{4\left(3\sqrt{2}+\sqrt{6}\right)}{6-2}=3\sqrt{2}+\sqrt{6}$$

On substituting the values of *a*, *b* and *c*, we obtain:

$$\begin{aligned}\frac{6}{2\sqrt{3}-\sqrt{6}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}-\frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}&=\left(2\sqrt{3}+\sqrt{6}\right)+\left(3\sqrt{2}-2\sqrt{3}\right)-\left(3\sqrt{2}+\sqrt{6}\right)\\&=2\sqrt{3}+\sqrt{6}+3\sqrt{2}-2\sqrt{3}-3\sqrt{2}-\sqrt{6}\\&=0\end{aligned}$$

Example 2:

Evaluate $\mathbf{\dfrac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}}}$.

Solution:

Let us first simplify the denominator of the given expression.

$$\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}$$
$$=\sqrt{10}+\sqrt{2^2\times 5}+\sqrt{2^2\times 10}-\sqrt{5}-\sqrt{4^2\times 5}$$
$$=\sqrt{10}+2\sqrt{5}+2\sqrt{10}-\sqrt{5}-4\sqrt{5}$$
$$=3\sqrt{10}-3\sqrt{5}$$
$$=3\left(\sqrt{10}-\sqrt{5}\right)$$

$$\begin{aligned}\therefore \frac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}} &= \frac{15}{3\left(\sqrt{10}-\sqrt{5}\right)}\\ &= \frac{5}{\sqrt{10}-\sqrt{5}}\\ &= \frac{5}{\sqrt{10}-\sqrt{5}}\times\frac{\sqrt{10}+\sqrt{5}}{\sqrt{10}+\sqrt{5}}\\ &= \frac{5(\sqrt{10}+\sqrt{5})}{\left(\sqrt{10}\right)^2-\left(\sqrt{5}\right)^2}\\ &= \frac{5\left(\sqrt{10}+\sqrt{5}\right)}{10-5}\\ &= \sqrt{10}+\sqrt{5}\end{aligned}$$

Hard

Example 1:

Prove that $\mathbf{\frac{1}{3-\sqrt{8}}-\frac{1}{\sqrt{8}-\sqrt{7}}+\frac{1}{\sqrt{7}-\sqrt{6}}-\frac{1}{\sqrt{6}-\sqrt{5}}+\frac{1}{\sqrt{5}-2}=5}$.

Solution:

$$\text{LHS} = \frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2}$$

$$= \left(\frac{1}{3-\sqrt{8}} \times \frac{3+\sqrt{8}}{3+\sqrt{8}}\right) - \left(\frac{1}{\sqrt{8}-\sqrt{7}} \times \frac{\sqrt{8}+\sqrt{7}}{\sqrt{8}+\sqrt{7}}\right) + \left(\frac{1}{\sqrt{7}-\sqrt{6}} \times \frac{\sqrt{7}+\sqrt{6}}{\sqrt{7}+\sqrt{6}}\right)$$

$$- \left(\frac{1}{\sqrt{6}-\sqrt{5}} \times \frac{\sqrt{6}+\sqrt{5}}{\sqrt{6}+\sqrt{5}}\right) + \left(\frac{1}{\sqrt{5}-2} \times \frac{\sqrt{5}+2}{\sqrt{5}+2}\right)$$

$$= \frac{3+\sqrt{8}}{(3)^2 - \left(\sqrt{8}\right)^2} - \frac{\sqrt{8}+\sqrt{7}}{\left(\sqrt{8}\right)^2 - \left(\sqrt{7}\right)^2} + \frac{\sqrt{7}+\sqrt{6}}{\left(\sqrt{7}\right)^2 - \left(\sqrt{6}\right)^2}$$

$$- \frac{\sqrt{6}+\sqrt{5}}{(\sqrt{6})^2 - (\sqrt{5})^2} + \frac{\sqrt{5}+2}{(\sqrt{5})^2 - (2)^2}$$

$$= \frac{3+\sqrt{8}}{9-8} - \frac{\sqrt{8}+\sqrt{7}}{8-7} + \frac{\sqrt{7}+\sqrt{6}}{7-6} - \frac{\sqrt{6}+\sqrt{5}}{6-5} + \frac{\sqrt{5}+2}{5-4}$$

$$= \left(3+\sqrt{8}\right) - \left(\sqrt{8}+\sqrt{7}\right) + \left(\sqrt{7}+\sqrt{6}\right) - \left(\sqrt{6}+\sqrt{5}\right) + \left(\sqrt{5}+2\right)$$

$$= 3 + \sqrt{8} - \sqrt{8} - \sqrt{7} + \sqrt{7} + \sqrt{6} - \sqrt{6} - \sqrt{5} + \sqrt{5} + 2$$

$$= 3 + 2$$

$$= 5$$

$$= \text{RHS}$$

Example 2:

If $x = \frac{\sqrt{3}+1}{2}$, then find the value of $4x^3 + 2x^2 - 8x + 7$.

Solution:

$$x=\frac{\sqrt{3}+1}{2}$$

$$\Rightarrow 2x=\sqrt{3}+1$$

$$2x-1=\sqrt{3}$$

$$\Rightarrow (2x-1)^2=\left(\sqrt{3}\right)^2$$

$$\Rightarrow 4x^2-4x+1=3$$

$$\Rightarrow 4x^2-4x-2=0$$

$$\Rightarrow 2x^2-2x-1=0 \qquad \ldots(1)$$

$$\begin{aligned}\therefore 4x^3+2x^2-8x+7 &= 2x\left(2x^2-2x-1\right)+3\left(2x^2-2x-1\right)+10\\ &= 2x(0)+3(0)+10 \qquad (\text{From equation} 1)\\ &= 0+0+10\\ &= 10\end{aligned}$$