Integration (Indefinite Integrals)

→	**Type: Integration By Parts :**
Q.1)	(a) $I = \int x^2 \sin x \, dx$ (b) $I = \int x \sin^2 x dx$
Sol.1)	(a) $I = \int x^2 \sin x \, dx$ $= x^2(-\cos x) - \int 2x.(-\cos x)dx$ $= -x^2 \cos x - 2\int x \cos x \, dx$ $= -x^2 \cos x - 2[x(\sin x) - \int (1).\sin x \; dx]$ $= -x^2 \cos x - 2[x \sin x + \cos x] + c$ ans. (b) $I = \int x \sin^2 x \, dx$ $= \int x \left(\frac{1-\cos(2x)}{2}\right) dx$ $= \frac{1}{2}\int x - x \cos(2x) dx$ $= \frac{1}{2}\int x dx - \frac{1}{2}\int x \cos(2x) dx$ $= \frac{1}{2}.\frac{x^2}{2} - \frac{1}{2}\left[x \frac{\sin(2x)}{2} - \int (1).\frac{\sin(2x)}{2} dx\right]$ $= \frac{x^2}{4} - \frac{1}{2}\left[\frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x) dx\right]$ $= I = \frac{x^2}{4} - \frac{1}{2}\left[\frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x)\right] + c$ ans.
Q.2)	(a) $I = \int \log x \, dx$ (b) $I = \int (\log x)^2 dx$ (c) $I = \int \frac{\log x}{x^2} dx$
Sol.2)	(a) $I = \int \log x \, dx$ $= \int \log x.1dx$ $= \log x.(x) - \int \frac{1}{x}.(x)dx$ $= x \log x - \int 1.dx$ $I = x \log x - x + c$ ans. (b) $I = \int (\log x)^2 dx$ $= \int (\log x)^2.1dx$ $= (\log x)^2.x - \int \frac{2\log x}{x}.xdx$ $= x(\log x)^2 - 2\int \log x.1dx$ $= x(\log x)^2 - 2\left[\log x.(x) - \int \frac{1}{x}.xdx\right]$ $= x(\log x)^2 - 2[x\log x - x] + c$ ans. (c) $I = \int \frac{\log x}{x^2} dx$ $= \int \frac{1}{x^2}.\log x \; dx$ $= \log x\left(\frac{-1}{x}\right) - \int \frac{1}{x}\left(\frac{-1}{x}\right) dx$ $= \frac{-1}{x}\log x + \int \frac{1}{x^2} dx$

	$= \frac{-1}{x}\log x - \frac{1}{x} + c$ ans.
Q.3)	(a)$I = \int \sin^{-1}x\ dx$ (b)$I = \int (\sin^{-1}x)^2 dx$
Sol.3)	(a)$I = \int \sin^{-1}x\ dx$ $= \int \sin^{-1}x.1dx$ $= \sin^{-1}.(x) - \int \frac{1}{\sqrt{1-x^2}}.x\ dx$ put $1 - x^2 = t$ $-2x\ dx = dt$ $x\ dx = -\frac{dt}{2}$ $\therefore I = x\sin^{-1}x + \frac{1}{2}\int \frac{dt}{\sqrt{t}}$ $= x\sin^{-1}x + \frac{1}{2} \times 2\sqrt{t} + c$ $I = x\sin^{-1}x + \sqrt{1-x^2} + c$ ans. (b)$I = \int (\sin^{-1}x)^2 dx$ $= \int (\sin^{-1}x)^2.1dx$ $= (\sin^{-1}x)^2.x - \int 2\frac{\sin^{-1}x}{\sqrt{1-x^2}}.xdx$ $= x(\sin^{-1}x)^2 - 2\int x\frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$ put $\sin^{-1}x = t$ ⍰ $(x = \sin t)$ $\frac{1}{\sqrt{1-x^2}}dx = dt$ $\therefore I = x(\sin^{-1}x)^2 - 2\int \sin t.tdt$ $= x(\sin^{-1}x)^2 - 2[t(-\cos t) - \int (1)(-\cos t)dt]$ $= x(\sin^{-1}x)^2 - 2[-t\cos t + \sin t] + c$ $= x(\sin^{-1}x)^2 - 2[-\sin^{-1}x\sqrt{1-x^2} + x] + c$ ans. .... $\left\{\sin t = x \text{ then } \cos t = \sqrt{1-\sin^2 t} = \sqrt{1-x^2}\right\}$
Q.4)	(a)$I = \int \frac{x\tan^{-1}x}{(1+x^2)^{\frac{3}{2}}}dx$ (b)$I = x\frac{x^2\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}dx$
Sol.4)	(a)$I = \int \frac{x\tan^{-1}x}{(1+x^2)^{\frac{3}{2}}}dx$ $= \int \frac{x\tan^{-1}x}{\sqrt{1+x^2}(1+x^2)}dx$ put $\tan^{-1}x = t$ $\therefore \frac{1}{1+x^2}dx = dt$ also $x = \tan t$ $\therefore I = \int \frac{xt}{\sqrt{1+x^2}}dt$ put $x = \tan t$ $I = \int \frac{\tan t\,.\,t}{\sqrt{1+\tan^2 t}}dt$ $= \int t\frac{\tan t}{\sec t}dt$ $= \int t\sin t\ dt$ $= t(-\cos t) - \int (1)(-\cos t)dt$

	$= -t\cos t + \sin t + c \quad \begin{bmatrix} \because \tan t = x \\ P = x, B = 1 \\ H = \sqrt{x^2+1} \\ cost = \frac{B}{H}, sint = \frac{B}{H} \end{bmatrix}$ $= -\tan^{-1}x.\frac{1}{\sqrt{x^2+1}} + \frac{x}{\sqrt{1+x^2}} + c$ ans. (b)$I = \frac{x^2\sin^{-1}x}{(1-x^2)^{\frac{3}{2}}}dx$ $= \int \frac{x^2\sin^{-1}x}{(1-x^2)\sqrt{1-x^2}}dx$ put $\sin^{-1}x = t \qquad (\sin t = x)$ $\therefore \frac{1}{\sqrt{1-x^2}}dx = dt$ $\therefore\ I = \int \frac{x^2t}{(1-x^2)}dt$ put sin$t = x$ $I = \int \frac{\sin^2 t.t}{1-\sin^2 t}dt$ $= \int \tan^2 t.tdt$ $= \int (sec^2 t - 1)tdt$ $= \int tsec^2 t - tdt$ $= \int tsec^2 tdt - \int tdt$ $= t(\tan t) - \int (1).\tan t\,dt - \frac{t^2}{2} \quad \begin{bmatrix} \because \sin t = x \\ P = x, H = 1 \\ B = \sqrt{1-x^2} \\ \tan t = \frac{P}{B} \\ sect = \frac{H}{B} \end{bmatrix}$ $= t\tan t - \log \mid \sec t \mid - \frac{t^2}{2} + c$ $= \sin^{-1}x.\frac{x}{\sqrt{1-x^2}} - \log\left\|\frac{1}{\sqrt{1-x^2}}\right\| - \frac{(\sin^{-1}x)^2}{2} + c$ $= \frac{x\sin^{-1}x}{\sqrt{1-x^2}} + \frac{1}{2}\log \mid 1-x^2 \mid - \frac{(\sin^{-1}x)^2}{2} + c$ ans.
Q.5)	$I = \int \frac{x-\sin x}{1-\cos x}dx$
Sol.5)	(a)$I = \int \frac{x-\sin x}{1-\cos x}dx$ $= \int \frac{x-2\sin\frac{x}{2}.\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}dx$ Separate $= \frac{1}{2}\int xcosec^2\left(\frac{x}{2}\right)dx - \int \cot\left(\frac{x}{2}\right)dx$ $= \frac{1}{2}\left[x\left(-2\cot\left(\frac{x}{2}\right)\right) - \int 2\left(-\cot\left(\frac{x}{2}\right)dx\right)\right] - 2\log\left\|\sin\left(\frac{x}{2}\right)\right\|$ $= \frac{1}{2}\left[-2x\cot\left(\frac{x}{2}\right) + 2\int \cot\left(\frac{x}{2}\right)dx\right] - 2\log\left\|\sin\frac{x}{2}\right\|$ $= \frac{1}{2}\left[-2x\cot\left(\frac{x}{2}\right) + 2\times 2\log\left\|\sin\left(\frac{x}{2}\right)\right\|\right] - 2\log\left\|\sin\frac{x}{2}\right\|$

	$= -x\cot\left(\frac{x}{2}\right) + 2\log\left\|\sin\left(\frac{x}{2}\right)\right\| - 2\log\left\|\sin\left(\frac{x}{2}\right)\right\| + c$ $= -x\cot\left(\frac{x}{2}\right) + c$ ans.
Q.6)	(a)$I = \int x\sin^{-1}x\,dx$ (b)$I = \int \sin^{-1}\sqrt{x}dx$
Sol.6)	(a)$I = \int x\sin^{-1}x\,dx$ $= \sin^{-1}x.\frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}}.\frac{x^2}{2}dx$ $= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$ $= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}}dx$ $= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \frac{1-x^2-1}{\sqrt{1-x^2}}dx$ $= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\int \sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}dx$ $= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x) - \sin^{-1}(x)\right] + c$ ......$\left[\because \int \sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$ $= \frac{x^2}{2}\sin^{-1}x + \frac{1}{2}\left[\frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x\right] + c$ ans. (b)$I = \int \sin^{-1}\sqrt{x}dx$ put $\sqrt{x} = t$ $x = t^2$ $dx = 2\text{t}\,dt$ $\therefore I = 2\int \sin^{-1}t.t\,dt$ Proceed as above Ques. $x\sin^{-1}x + \frac{1}{2}\sqrt{x-x^2} - \frac{1}{2}\sin^{-1}\sqrt{x} + c$ ans.
Q.7)	(a)$I = \int \frac{\sin^{-1}x}{x^2}dx$ (b)$I = \int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx$
Sol.7)	(a)$I = \int \frac{\sin^{-1}x}{x^2}dx$ put $\sin^{-1}x = t$ $x = \sin t$ $\therefore\ dx = \cos t\,dt$ $\therefore I = \int \frac{t\cos t}{\sin^2 t}dt$ $= \int t\,cosec\,t\,.\cot dt$ $= t(-\text{cosec}\,t) - \int (1)(-\text{cosec}\,t)dt$ $= \frac{-t}{\sin t} + \int cosec\,t\,dt$ $= \frac{-t}{\sin t} + \log \mid cosec\,t - \cot t \mid +c$ $= \frac{-\sin^{-1}x}{x} + \log\left\|\frac{1}{\sin t} - \frac{\cot t}{\sin t}\right\| + c$ $= \frac{-\sin^{-1}x}{x} + \log\left\|\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right\| + c$ ans. (b)$I = \int \frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}dx$

	Property : $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2} \therefore . \sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$ $I = \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}} dx$ $= \frac{2}{\pi}\int 2\sin^{-1}\sqrt{x} - \frac{\pi}{2} dx$ $= \frac{4}{\pi}\int \sin^{-1}\sqrt{x}dx - \int 1.dx$ put $\sqrt{x} = t$ $x = t^2$ $dx = 2\text{t}\, dt$ $\therefore I = \frac{8}{\pi}\int \sin^{-1}t.tdt - x$ $= \frac{8}{\pi}\left[\sin^{-1}t.\frac{t^2}{2} - \frac{1}{2}\int \frac{1}{\sqrt{1-t^2}}.t^2dt\right] - x$ $= \frac{8}{\pi}\left[\frac{t^2}{2}.\sin^{-1}t + \frac{1}{2}\int \frac{-t^2}{\sqrt{1-t^2}}dt\right] - x$ $= \frac{8}{\pi}\left[\frac{t^2}{2}\sin^{-1}t + \frac{1}{2}\int \frac{1-t^2-1}{\sqrt{1-t^2}}dt\right] - x$ $= \frac{8}{\pi}\left[\frac{t^2}{2}\sin^{-1}t + \frac{1}{2}\int \sqrt{1-t^2} - \frac{1}{\sqrt{1-t^2}}dt\right] - x$ $= \frac{8}{\pi}\left[\frac{t^2}{2}\sin^{-1}t + \frac{1}{2}\left\{\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1}(t) - \sin^{-1}(t)\right\}\right] - x + c$ $= \frac{8}{\pi}\left[\frac{t^2}{2}\sin^{-1}t + \frac{1}{2}\left\{\frac{t}{2}\sqrt{1-t^2} - \frac{1}{2}\sin^{-1}t\right\}\right] - x + c$ $= \frac{8}{\pi}\left[\frac{t^2}{2}\sin^{-1}t + \frac{t}{4}\sqrt{1+t^2} - \frac{1}{4}\sin^{-1}t\right] - x + c$ replace t by$\sqrt{x}$ $= \frac{8}{\pi}\left[\frac{x}{2}\sin^{-1}\sqrt{x} + \frac{\sqrt{x}\sqrt{1-x}}{4} - \frac{\sin^{-1}\sqrt{x}}{4}\right] - x + c$ ans.
Q.8)	(a)$I = \int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx$ (b)$I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$
Sol.8)	(a)$I = \int \frac{\sqrt{x^2+1}[\log(x^2+1)-2\log x]}{x^4}dx$ $= \int \frac{\sqrt{x^2+1}(\log(x^2+1)-\log(x^2))}{x^4}dx$ $= \int \frac{\sqrt{x^2+1}\left[\log\frac{(x^2+1)}{x^2}\right]}{x^4}dx$ take x^2common $= \int \frac{x\sqrt{1+\frac{1}{x^2}}.\log\left(1+\frac{1}{x^2}\right)}{x^4}dx$ $= \int \frac{\sqrt{1+\frac{1}{x^2}}.\log\left(1+\frac{1}{x^2}\right)}{x^3}dx$ put $1 + \frac{1}{x^2} = t$ $\frac{-2}{x^3}dx = dt \quad \Rightarrow \frac{1}{x^3}dx = \frac{-dt}{2}$ $\therefore I = \frac{-1}{2}\int \sqrt{t}.\log t dt$ $= \frac{-1}{2}\left[\log t.\frac{2}{3}(t)^{\frac{3}{2}} - \frac{2}{3}\int \frac{1}{t}.t^{\frac{3}{2}}dt\right]$

$= \frac{-1}{2}\left[\frac{2}{3}\log t.t^{\frac{3}{2}} - \frac{2}{3}\int t^{\frac{1}{2}}dt\right]$

$= \frac{-1}{2}\left[\frac{2}{3}\log t.t^{\frac{3}{2}} - 4t^{\frac{3}{2}}\right] + c$

$= \frac{-1}{2} \times \frac{2}{3} \times t^{\frac{3}{2}}\left[\log t - \frac{2}{3}\right] + c$

replacing t

$= \frac{-1}{3}\left(1+\frac{1}{x^2}\right)^{\frac{3}{2}}.\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + c$ ans.

(b)$I = \int \tan^{-1}\sqrt{\frac{1-x}{1+x}}dx$

put $x = \cos(2\theta)$

$\therefore I = \int \tan^{-1}\sqrt{\frac{1-\cos(2\theta)}{1+\cos(2\theta)}}dx$

$= \int \tan^{-1}\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}dx$

$= \int \tan^{-1}(\tan\theta)dx$

$= \int \theta.dx$

replacing θ

$= \frac{1}{2}\int \cos^{-1}xdx$

$= \frac{1}{2}\int \cos^{-1}x.1dx$

$= \frac{1}{2}\left[\cos^{-1}x.x - \int \frac{-1}{\sqrt{1-x^2}}.xdx\right]$

$= \frac{1}{2}\left[x\cos^{-1}x + \int \frac{x}{\sqrt{1-x^2}}dx\right]$

put $1 - x^2 = t$ ⍰$xdx = \frac{-dt}{2}$

$\therefore\ I = \frac{1}{2}\left[x\cos^{-1}x - \frac{1}{2}\int \frac{dt}{\sqrt{t}}\right]$

$= \frac{1}{2}\left[x\cos^{-1}x - \frac{1}{2}\times 2\sqrt{t}\right] + c$

$= \frac{1}{2}\left[x\cos^{-1}x - \sqrt{1-x^2}\right] + c$ ans.

Q.9) (a)$I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$ (b)$I = \int \frac{x^2}{(x\sin x+\cos x)^2}dx$

Sol.9) (a)$I = \int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$

put $x = a\tan^2\theta$

$dx = 2a\tan\theta.sec^2\theta d\theta$

$\therefore I = 2a\int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}.\tan\theta.sec^2\theta d\theta$

$= 2a\int \sin^{-1}\sqrt{\frac{\tan^2\theta}{sec^2\theta}}.\tan\theta.sec^2\theta.d\theta$

$= 2a\int \sin^{-1}\sqrt{\sin^2\theta}.\tan\theta.sec^2\theta d\theta$

$= 2a\int \sin^{-1}(\sin\theta).\tan\theta.sec^2\theta d\theta$

$= 2a\int \theta.\tan\theta.sec^2\theta d\theta$

$= 2a\left[\theta.\int \tan\theta sec^2\theta d\theta - \int\left(1.\int \tan\theta.sec^2\theta d\theta\right)d\theta\right]$

	put $\tan\theta = t$ in both integrals $\therefore sec^2\theta d\theta = dt$ $= 2a\left[\theta.\int tdt - \int\left(\int tdt\right)d\theta\right]$ $= 2a\left[\theta.\frac{t^2}{2} - \int\left(\frac{t^2}{2}\right)d\theta\right]$ replacing t $= 2a\left[\theta.\frac{\tan^2\theta}{2} - \frac{1}{2}\int \tan^2\theta.d\theta\right]$ $= 2a\left[\theta.\frac{\tan^2\theta}{2} - \frac{1}{2}\int (sec^2\theta - 1)d\theta\right]$ $= 2a\left[\theta.\frac{\tan^2\theta}{2} - \frac{1}{2}\{\tan\theta - \theta\}\right] + c$ replacing θ by $\tan^{-1}\sqrt{\frac{x}{a}}$ $= a\left[\tan^{-1}\sqrt{\frac{x}{a}}.\frac{x}{a} - \left\{\sqrt{\frac{x}{a}} - \tan^{-1}\sqrt{\frac{x}{a}}\right\}\right] + c$ $= a\left[\frac{x}{a}\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}} + \tan^{-1}\sqrt{\frac{x}{a}}\right] + c$ $= a\left[\tan^{-1}\sqrt{\frac{x}{a}}\left(\frac{x}{a} + 1\right) - \sqrt{\frac{x}{a}}\right] + c$ ans. (b)$I = \int \frac{x^2}{(x\sin x + \cos x)^2}dx$ adjustment $= \int \frac{x.x.\cos x.\sec x}{(x\sin x + \cos x)^2}dx$ $=(x\ sec\ x).\int \frac{x\cos x}{(x\sin x + \cos x)^2}dx$ Both parts $= x\sec x.\int \frac{x\cos x}{(x\sin x + \cos x)^2} - \int \frac{d}{dx}(x\sec x).\int \frac{x\cos x}{(x\sin x + \cos x)^2}dx$ put $x\sin x + \cos x = t$ in both integrals $(x\cos x + \sin x - \sin)dx = dt$ $x\cos x\ dx = dt$ $\therefore I = x\sec x\int \frac{dt}{t^2} - \int (x\ sec\ x\tan x + \sec x).\int \frac{dt}{t^2}$ $= x\ \sec x\left[\frac{-1}{t}\right] - \int (x\sec x\tan x + \sec x)\left(\frac{-1}{t}\right)dx$ replacing t by$x\sin x + \cos x = t$ $= \frac{-x\ \sec x}{x\sin x + \cos x} - \int (x\sec x\tan x + \sec x)\left(\frac{-1}{x\sin x + \cos x}\right)dx$ $= \frac{-x\ \sec x}{x\sin x + \cos x} + \int \sec x(x\tan x + 1)\frac{1}{x\sin x + \cos x}dx$ $= \frac{-x\sec x}{x\sin x + \cos x} + \int \sec x\left(\frac{x\sin x + \cos x}{\cos x}\right)\left(\frac{1}{x\sin x + \cos x}\right)dx$ $= \frac{-x\sec x}{x\sin x + \cos x} - \int sec^2x\ dx$ $= \frac{-x\sec x}{x\sin x + \cos x} + \tan x + c$ ans.
Q.10)	$I = \int sec^3x\ dx$
Sol.10)	$I = \int sec^3xdx$ $= \int sec\ x.sec^2x\ dx$

	$= \sec x \,.\tan x - \int secx.\tan x.\tan x dx$ $= \sec x \,.\tan x - \int secx(sec^2 x - 1)dx$ $= \sec x \,.\tan x - \int sec^3 x - \sec x \;\; dx$ $= \sec x \,.\tan x - \int sec^3 x \; dx + \int sec \; x \; dx$ $I = \sec x \,.\tan x - I + \log \mid \sec x + \tan x \mid$ $2I = \sec x \tan x + \log \mid \sec x + \tan x \mid + c$ $\therefore I = \frac{1}{2}[\sec x \tan x + \log \mid \sec x + \tan x \mid] + c$ ans.