

Continuity and Differentiability

- Suppose f is a real function on a subset of the real numbers and c be a point in the domain of f . Then, f is continuous at c , if $\lim_{x \rightarrow c} f(x) = f(c)$

More elaborately, we can say that f is continuous at c , if

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$$

- If f is not continuous at c , then we say that f is discontinuous at c and c is called the point of discontinuity.
- A real function f is said to be continuous, if it is continuous at every point in the domain of f .
- If f and g are two continuous real functions, then
 - $(f + g), (f - g), f \cdot g$ are continuous
 - $\frac{f}{g}$ is continuous provided g assumes non zero value.
- If f and g are two continuous functions, then $f \circ g$ is also continuous.

- Suppose f is a real function and c is a point in its domain. Then, the derivative of f at c is defined by, $f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$
- Derivative of a function $f(x)$, denoted by $\frac{d}{dx}(f(x))$ or $f'(x)$, is defined by $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Example:

Find derivative of $\sin 2x$.

Solution:

Let $f(x) = \sin 2x$

$$\begin{aligned}\therefore f'(x) &= \lim_{h \rightarrow 0} \frac{\sin 2(x+h) - \sin 2x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2\cos(2x+h) \cdot \sin h}{h} \\ &= 2 \lim_{h \rightarrow 0} \cos(2x+h) \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= 2 \times \cos 2x \times 1 \\ &= 2\cos 2x\end{aligned}$$

- For two functions f and g , the rules of algebra of derivatives are as follows:

- $(f + g)' = f' + g'$
 - $(f - g)' = f' - g'$
 - $(fg)' = f'g + fg'$ [Leibnitz or product rule]

- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, where $g \neq 0$ [Quotient rule]

- Every differentiable function is continuous, but the converse is not true.

Example:

$f(x) = |x|$ is continuous at all points on real line, but it is not differentiable at $x = 0$.

Since L.H.S $= \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \frac{-h}{h} = -1$

$= \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h}{h} = 1$

R.H.S $= \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h}{h} = 1$

$\therefore$ L.H.S $\neq$ R.H.S.

Therefore, $f'(x)$ does not exist at $x = 0$; i.e., f is not differentiable at $x = 0$.

The derivatives of some useful functions are as follows:

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$$\circ \quad \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

- **Chain rule:** This rule is used to find the derivative of a composite function. Let

$f = v \circ u$. Suppose $t = u(x)$; and if both $\frac{dt}{dx}$ and $\frac{dv}{dt}$ exist, then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$

Similarly, if $f = (w \circ u) \circ v$, and if $t = v(x)$, $s = u(t)$, then

$$\frac{df}{dx} = \frac{d(w \circ u)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

Example: Find the derivative of $\sin^2(\log x + \cos^2 x)$.

Solution:

$$\begin{aligned} \frac{d}{dx} \left[\sin^2(\log x + \cos^2 x) \right] &= 2\sin(\log x + \cos^2 x) \times \frac{d}{dx} \left[\sin(\log x + \cos^2 x) \right] \\ &= 2\sin(\log x + \cos^2 x) \cdot \cos(\log x + \cos^2 x) \times \frac{d}{dx} (\log x + \cos^2 x) \\ &= \sin 2(\log x + \cos^2 x) \cdot \left[\frac{1}{x} + 2\cos x \times \frac{d}{dx} (\cos x) \right] \\ &= \sin(\log x^2 + 2\cos^2 x) \times \left(\frac{1}{x} - 2\sin x \cos x \right) \\ &= \left(\frac{1}{x} - \sin 2x \right) \sin(\log x^2 + 2\cos^2 x) \end{aligned}$$

The derivatives of exponential functions are as follows:

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^{ax}) = ae^{ax}$

- **Mean value theorem:**

If $f: [a, b] \rightarrow \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b) , then there

exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Example: Verify Mean Value Theorem for the function:

$f(x) = 2x^2 - 17x + 30$ in the interval $\left[\frac{5}{2}, 6\right]$.

Solution:

$$f(x) = 2x^2 - 17x + 30$$

$$\therefore f'(x) = 4x - 17$$

The function $f(x)$ being a polynomial, is continuous on $\left[\frac{5}{2}, 6\right]$ and is differentiable on $\left(\frac{5}{2}, 6\right)$.

$$\text{Also, } f\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right)^2 - 17\left(\frac{5}{2}\right) + 30 = 0$$

$$\text{and, } f(6) = 2(6)^2 - 17 \times 6 + 30 = 0$$

$$\therefore f\left(\frac{5}{2}\right) = f(6)$$

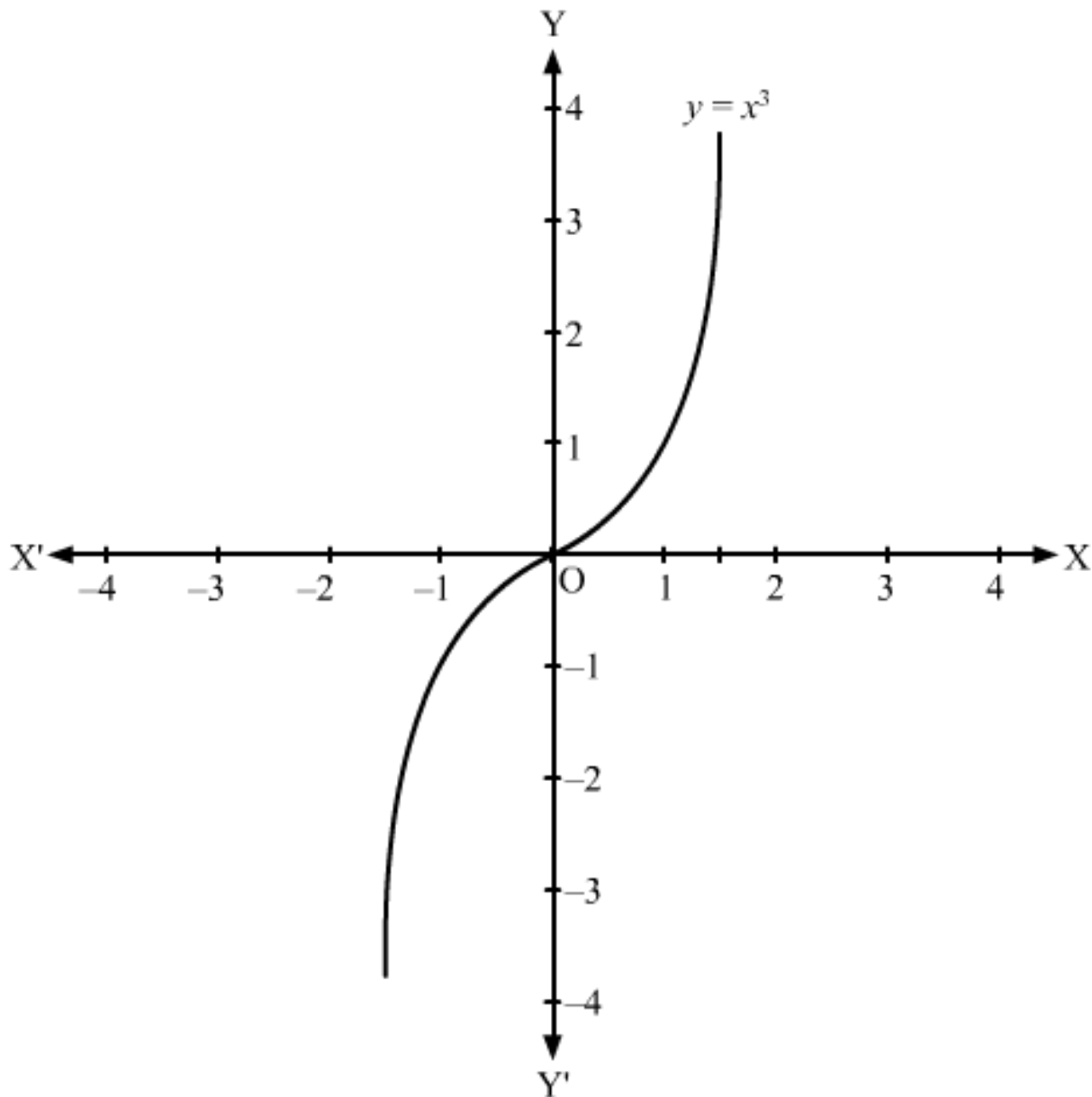
$$\frac{f(6) - f\left(\frac{5}{2}\right)}{6 - \frac{5}{2}} = 0$$

Now,

According to Mean Value Theorem (MVT), there exists $c \in \left(\frac{5}{2}, 6\right)$ such that $f'(c) = 0$.

$$\therefore 4c - 17 = 0$$

$$\Rightarrow c = \frac{17}{4} \in \left(\frac{5}{2}, 6\right)$$



Therefore, M.V.T is verified.

- Derivative of a function $f(x) = [u(x)]^{v(x)}$ can be calculated by taking logarithm on both the sides, i.e. $\log f(x) = v(x)\log [u(x)]$, and then differentiating both sides with respect to x .

Example: If $y = x^{x^x}$, find $\frac{dy}{dx}$

Solution:

$$\text{Let If } y = x^{x^x} = x^y$$

$$\therefore \log y = y \log x$$

$$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx}(y \log x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{dy}{dx} \log x + \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} \left[\frac{1}{y} - \log x \right] = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{1}{y} - \log x} = \frac{y^2}{x - y \log x}$$

- If the variables x and y are expressed in the form of $x = f(t)$ and $y = g(t)$, then they are said to be in parametric form. In this case, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{g'(t)}{f'(t)}$, provided $f'(t) \neq 0$

- If $y = f(x)$, then $\frac{dy}{dx} = f'(x)$ and $\frac{d^2y}{dx^2}$ or $f''(x) = \frac{d}{dx} \left(\frac{dy}{dx} \right)$

Here, $f''(x)$ or $\frac{d^2y}{dx^2}$ is called the second order derivative of y with respect to x .

- Rolle's Theorem:**

If $f: [a, b] \rightarrow \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$, where a and b are some real numbers, then there exists some $c \in (a, b)$ such that $f'(c) = 0$

Example: Verify Rolle's Theorem for the function:

$f(x) = 2x^2 - 17x + 30$ in the interval $\left[\frac{5}{2}, 6\right]$.

Solution:

$$f(x) = 2x^2 - 17x + 30$$

$$\therefore f'(x) = 4x - 17$$

The function $f(x)$ being a polynomial, is continuous on $\left[\frac{5}{2}, 6\right]$ and is differentiable on $\left(\frac{5}{2}, 6\right)$.

$$\text{Also, } f\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right)^2 - 17\left(\frac{5}{2}\right) + 30 = 0$$

$$\text{And, } f(6) = 2(6)^2 - 17 \times 6 + 30 = 0$$

$$\therefore f\left(\frac{5}{2}\right) = f(6)$$

Therefore, we can apply Rolle's Theorem for $f(x)$.

According to this theorem, there exists $c \in \left(\frac{5}{2}, 6\right)$ such that $f'(c) = 0$

$$\text{We have } f'(x) = 4x - 17$$

$$\therefore f'(c) = 0$$

$$\Rightarrow 4c - 17 = 0$$

$$\Rightarrow c = \frac{17}{4} \in \left(\frac{5}{2}, 6\right)$$

Therefore, Rolle's Theorem is verified.

- **Mean value theorem:**

If $f: [a, b] \rightarrow \mathbf{R}$ is continuous on $[a, b]$ and differentiable on (a, b) , then there exists some $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

Example: Verify Mean Value Theorem for the function:

$$f(x) = 2x^2 - 17x + 30 \text{ in the interval } \left[\frac{5}{2}, 6\right].$$

Solution:

$$f(x) = 2x^2 - 17x + 30$$

$$\therefore f'(x) = 4x - 17$$

The function $f(x)$ being a polynomial, is continuous on $\left[\frac{5}{2}, 6\right]$ and is differentiable on $\left(\frac{5}{2}, 6\right)$.

$$\text{Also, } f\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right)^2 - 17\left(\frac{5}{2}\right) + 30 = 0$$

$$\text{And, } f(6) = 2(6)^2 - 17 \times 6 + 30 = 0$$

$$\therefore f\left(\frac{5}{2}\right) = f(6)$$

$$\text{Now, } \frac{f(6) - f\left(\frac{5}{2}\right)}{6 - \frac{5}{2}} = 0$$

According to Mean Value Theorem (MVT), there exists $c \in \left(\frac{5}{2}, 6\right)$ such that $f'(c) = 0$

$$\therefore 4c - 17 = 0$$

$$\Rightarrow c = \frac{17}{4} \in \left(\frac{5}{2}, 6\right)$$

Therefore, M.V.T is verified.