

PERIODICITY AND EXTREME VALUES

VSAQ'S

Find the periods for the given 1-4 functions.

1. $\cos(3x + 5) + 7$

Sol. $f(x) = \cos(3x + 5) + 7$

$$\text{Period} = \frac{\text{period of } \cos x}{[\text{coefficient of } x]} = \frac{2\pi}{3}$$

2. $\tan 5x$

Sol. $f(x) = \tan 5x$

$$\text{Period} = \frac{\pi}{5}$$

3. $|\sin x|$

Sol. $f(x) = |\sin x|$

$$\text{Period} = \pi$$

$$[\because |\sin(\pi + x)| = |(-\sin x)| = |\sin x|]$$

4. $\tan(x + 4x + 9x + \dots + n^2x)$ (n any positive integer).

Sol. $f(x) = \tan(x + 4x + 9x + \dots + n^2x)$

$$= \tan(1 + 4 + 9 + \dots + n^2)x$$

$$= \tan(1^2 + 2^2 + 3^2 + \dots + n^2)x$$

$$= \tan\left[\frac{n(n+1)(2n+1)}{6}\right]x$$

$$\text{Period} = \frac{\pi}{\frac{n(n+1)(2n+1)}{6}} = \frac{6\pi}{n(n+1)(2n+1)}$$

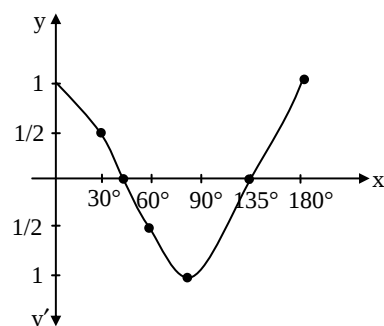
5. Find a sine function whose period is $2/3$.

Sol. $f(x) = \sin\left(\frac{2\pi}{2/3}\right)x = \sin(3\pi x)$

6. Draw $\cos 2x$ in the interval $[0, \pi]$.

Sol.

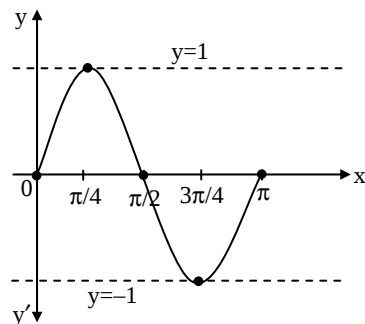
x	0	30°	45°	60°	90°	135°	180°
cos 2x	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	0	1



7. Draw $\sin 2x$ in the interval $(0, \pi)$.

Sol.

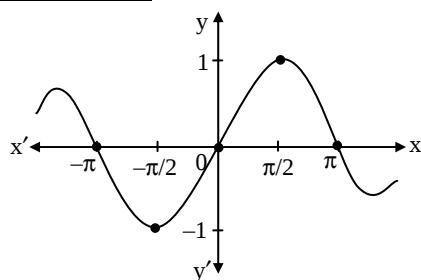
x	0	30°	60°	90°	135°	180°
$\sin 2x$	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	-1	0



8. Draw $\sin x$ in the interval $[-\pi, +\pi]$.

Sol.

x	-180° $(-\pi)$	-90° $(-\pi/2)$	0 (0)	90° $(\pi/2)$	180° (π)
$\sin x$	0	-1	0	1	0



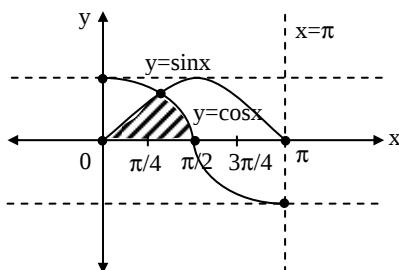
9. Sketch the region enclosed by $y = \sin x$, $y = \cos x$ and x-axis in the interval $[0, \pi]$.

Sol. $y = \sin x$

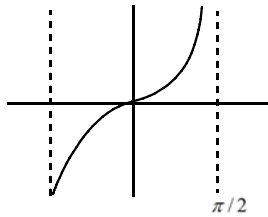
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	(π)
y	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0

$y = \cos x$

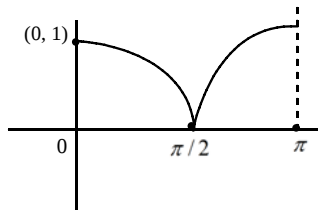
x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	(π)
y	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{2}$	-1



10. Draw the graph of $y = \tan x$ in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



11. Draw the graph of $y = \cos^2 x$ in $(0, \pi)^{-\pi/2}$



12. Find the period of the function $f(x) = 2\sin\frac{\pi x}{4} + 3\cos\frac{\pi x}{3}$

$$\text{Period of } \sin\frac{\pi x}{4} \text{ is } \frac{2\pi}{\pi/4} = 8$$

$$\text{Period of } \cos\frac{\pi x}{3} \text{ is } \frac{2\pi}{\pi/3} = 6$$

Period of given function is L.C.M of 8, 6

$\therefore$ Period is 24

13. If $a \leq \cos \theta + 3\sqrt{2} \sin\left(\theta + \frac{\pi}{4}\right) + 6 \leq b$, then find the largest value of a and smallest value of b.

$$\text{Sol. } f = \cos \theta + 3\sqrt{2} \sin\left(\theta + \frac{\pi}{4}\right) + 6$$

$$= \cos \theta + 3\sqrt{2} \left[\sin \theta \cos \frac{\pi}{4} + \cos \theta \sin \frac{\pi}{4} \right] + 6$$

$$= \cos \theta + 3\sqrt{2} \left[\frac{\sin \theta}{\sqrt{2}} + \frac{\cos \theta}{\sqrt{2}} \right] + 6$$

$$= \cos \theta + \frac{3\sqrt{2}}{\sqrt{2}} [\sin \theta + \cos \theta] + 6$$

$$= \cos \theta + 3\sin \theta + 3\cos \theta + 6$$

$$= 3\sin \theta + 4\cos \theta + 6$$

$$\text{Minimum value} = C - \sqrt{a^2 + b^2}$$

$$= 6 - \sqrt{3^2 + 4^2} = 6 - \sqrt{25} = 6 - 5 = 1$$

$$\text{Maximum value} = C + \sqrt{a^2 + b^2}$$

$$= 6 + 5 = 11$$

$\therefore a = 1, b = 11.$

14. Find the periods for the following function $\cos^4 x$.

Sol. Let $f(x) = \cos^4 x = (\cos^2 x)^2$

$$\begin{aligned} &= \left[\frac{1 + \cos 2x}{2} \right]^2 \\ &= \frac{1 + 2\cos 2x + \cos^2 2x}{4} \\ &= \frac{1}{4} \left[1 + 2\cos 2x + \frac{1 + \cos 4x}{2} \right] \\ &= \frac{1}{8} [1 + 4\cos 2x + 1 + \cos 4x] \\ &= \frac{1}{8} [3 + 4\cos 2x + \cos 4x] \end{aligned}$$

Period at $\cos 2x = \frac{2\pi}{2} = \pi$

Period at $\cos 4x = \frac{2\pi}{4} = \frac{\pi}{2}$

L.C.M. of $\left(\pi, \frac{\pi}{2} \right) = \pi$

$\therefore$ Period of $f(x) = \pi$.

15. $\frac{5\sin x + 3\cos x}{4\sin 2x + 5\cos x}$

Sol. Let $f(x) = \frac{5\sin x + 3\cos x}{4\sin 2x + 5\cos x}$

Period of $\sin x = 2\pi$

Period of $\cos x = 2\pi$

Period of $\sin 2x = \frac{2\pi}{2} = \pi$

Period of $\cos 2x = \frac{2\pi}{2} = \pi$

Period of $f(x) =$

L.C.M. of $\{2\pi, 2\pi, \pi, \pi\} = 2\pi$

16. . Find the maximum and minimum values of $4\sin^2 x + 5\cos^2 x$

Solution:

$$\begin{aligned} 4\sin^2 x + 5\cos^2 x &= 4(1 - \cos^2 x) + 5\cos^2 x \\ &= \cos^2 x + 4 \end{aligned}$$

Maximum value of $\cos^2 x = 1$

$\therefore$ Maximum value of $\cos^2 x + 4$ is $1 + 4 = 5$

Minimum value of $\cos^2 x = 0$

$\therefore$ Minimum value of $\cos^2 x + 4$ is $0 + 4 = 4$

17. Find the minimum and maximum value of $3 \cos x + 4 \sin x$.

Sol. Let $f(x) = 3 \cos x + 4 \sin x$

$$\begin{aligned}\text{Maximum value of } f &= C + \sqrt{a^2 + b^2} \\ &= 0 + \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5\end{aligned}$$

$$\begin{aligned}\text{Minimum value of } f &= C - \sqrt{a^2 + b^2} \\ &= -\sqrt{4^2 + 3^2} = -\sqrt{16 + 9} = -\sqrt{25} = -5\end{aligned}$$

18. Find the range of $7 \cos x - 24 \sin x + 5$.

Sol. Minimum value of $f = C - \sqrt{a^2 + b^2}$

$$\begin{aligned}&= 5 - \sqrt{(-24)^2 + 7^2} \\ &= 5 - \sqrt{576 + 49} \\ &= 5 - \sqrt{625} = 5 - 25 = -20\end{aligned}$$

$$\begin{aligned}\text{Maximum value of } f &= C + \sqrt{a^2 + b^2} \\ &= 5 + \sqrt{625} = 5 + 25 = 30\end{aligned}$$