M A T H E M A T I C S

RELATIONS AND FUNCTIONS

INTRODUCTION TO RELATIONS

NOTES

What you already know

- Set theory

What you will learn

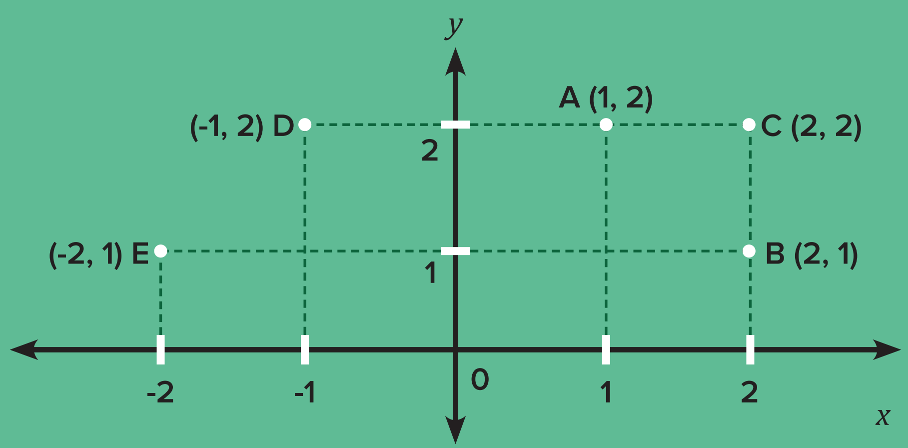
- Ordered pair
- Cartesian product
- Relation
- Inverse of a relation
- Some special relations
- Domain, range

Ordered Pair

An ordered pair consists of two objects or elements in a given fixed order.

Example: (2,3) is an ordered pair, where 2 is the first element and 3 is the second element.

Position of a point in a 2-D plane is represented by an ordered pair.



Equality of ordered pair

Two ordered pairs are equal if and only if their corresponding first and second elements are equal.

$(a_1, b_1) = (a_2, b_2)$

$\Leftrightarrow a_1 = a_2$ and $b_1 = b_2$

Note

I. Ordered pair is not termed as a set. Therefore (1, 2) is different from (2, 1).

II. In a set {2, 2} means the existence of one element but (2, 2) means two elements.

III. Order of elements is important and elements need not be distinct.

Quick Query

1. If (3a - 2, b + 3) = (2a - 1, 3), find a and b.

Cartesian Product

Let A and B be two non-empty sets. The set of all ordered pairs (a, b), where a ∈ A and b ∈ B, is known as the cartesian product of sets A and B. It is denoted by A x B.

Mathematically

A x B = {(a, b) : a ∈ A and b ∈ B}

Example

Set B

Set A	1	2	3	4
a	(a, 1)	(a, 2)	(a, 3)	(a, 4)
b	(b, 1)	(b, 2)	(b, 3)	(b, 4)
c	(c, 1)	(c, 2)	(c, 3)	(c, 4)
d	(d, 1)	(d, 2)	(d, 3)	(d, 4)

→ A x B

Example

If Set A = {1, 2, 3}, Set B = {a, b}, then

A × B = {(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)}

B × A = {(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)}

In general, A × B ≠ B × A

If A = φ or B = φ , then A × B = φ

Concept Check

1. Let A be a non-empty set such that A × A has 9 elements among which two elements are found to be (-1, 0) and (0, 1). Find set A.

(a) {-1, 0} (b) {0, 1} (c) {-1, 0, 1} (d) {-1, 1}

Number of elements in the cartesian product A × B

If A and B are finite sets, then n(A × B) = n(A) × n(B)

If either A or B is infinite, then A × B is an infinite set.

Concept Check

2. Let A have the first 10 odd natural numbers and B have the first 10 prime natural numbers. Find the number of elements common to A × B and B × A.

(a) 2^8 (b) 8^2 (c) 7^2 (d) 2^7

3. Let A be the set of all divisors of 8 and B be the set of all the divisors of 10. Find the number of elements in A × B.

4. If A = {1, 2}, B = {3, 4, 5}, C = {a, b}. Find number of elements in A × B × C.

If A and B are any two non-empty sets, then

A × B = B × A ⇔ A = B

Let A and B be two non-empty sets having m elements in common, then A × B and B × A have m^2 elements in common.

Relation

A relation from A to B is a subset of the cartesian product A × B.

Mathematically

R is a relation from A to B ⇔ R ⊆ A × B

Notations

Example: A = {1, 2, 3, 4, 5} and B = {1, 5, 7}
Define a relation R from A to B. aRb if a - b is even.
R = {(1, 1), (1, 5), (1, 7), (3, 1), (3, 5), (3, 7), (5, 1), (5, 5), (5, 7)}

R: A → B
In the above example, (1, 5) ∈ R or 1R5.
(a, b) ∈ R means a is related to b.

Quick Query

2. Let A = {1, 2, 3} and B = {4, 5, 6}. Find which of the following are relations from A to B.

- R_1 = {(1, 4), (1, 5), (1, 6)}
- R_2 = {(1, 5), (2, 4), (3, 6)}
- R_3 = {(1, 4), (1, 5), (1, 6), (3, 6), (2, 6), (3, 4)}
- R_4 = {(4, 2), (2, 6), (5, 1), (2, 4)}

Total number of relations from A to B

If n (A) = p and n (B) = q,
Total number of relations from A to B = Total number of subsets of A × B = 2^{pq}

Domain

The collection of the first elements of all the ordered pairs of a relation R is known as the domain of R.

Range

The collection of the second elements of all the ordered pairs of a relation R is known as the range of R.

Example

A = {2, 4, 5} and B = {1, 2, 3, 4, 6, 8}
R = {(x, y) : *xRy iff x divides y*}
R = {(2, 2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8)}

Domain: {2, 4}

Range: {2, 4, 6, 8}

Quick Query

3. Find the domain and range of relation R = {(1, 8), (1, 3), (2, 7), (2, 9), (5, 7), (5, 9)}.

Inverse of a Relation

Let A, B be two sets and R be a relation from A to B. The inverse of R, denoted by R^{-1}, is a relation from B to A and is defined as the following:

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$
$$\text{Thus, if } (a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$$

Example

A = {1, 2, 3, 4, 5} and B = {1, 4, 5}
Let R be a relation from A to B.
R = {$(x, y) \in$ A x B : *xRy iff* $x < y$}
R = {(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)}

Then R^{-1} = {(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4)}

R^{-1} is a relation from B to A.

Quick Query

4. Find the inverse of this relation R = {(8, 4), (13, 12), (5, 10)}.

1. Domain of R = Range of R^{-1}
2. Domain of R^{-1} = Range of R

Some special relations

Void relation

A relation R on a set A is known as a **void or an empty relation**, if no element of set A is related to any element of A.

$$\textbf{Here, } R = \emptyset \subseteq A \times A$$

For example:

R is a relation on A = {1, 2, 3} such that

R = {(a, b) : a + b = 12}

Universal relation

If each element of set A is related to every element of set A, then the relation is known as a **universal relation.**

For example:

R is a relation on A = {1, 2, 3, 4, 5} such that

R = {(a, b) : a < b or a $\geq$ b}

Identity relation

Given a set A, an identity relation on A is defined as the set of all ordered pairs (a, a) where a belongs to A.

For example:

R is a relation on A such that

A = {1, 2, 3, 4, 5} such that R = {(a, b) : a = b}

R = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)}

Summary

Key terms

1. **Ordered pair**: An ordered pair consists of two objects or elements in a given fixed order.

2. **Cartesian product**: Let A and B be two non-empty sets. The set of all ordered pairs (a, b), where $a \in A$ and $b \in B$ is known as the cartesian product of sets A and B and is denoted by A x B.

3. A **relation** from A to B is a subset of the cartesian product A x B.

4. **Domain**: The collection of the first elements of all the ordered pairs of a relation R is known as the domain of R.

5. **Range**: The collection of the second elements of all the ordered pairs of a relation R is known as the range of R.

6. **Inverse of a relation R**: $R^{-1} = \{(b, a) : (a, b) \in R\}$

7. **Void relation**: An empty relation, that is, one having no elements is a void relation.

8. **Universal relation**: If each element of set A is related to every element of set A, then the relation is known as a universal relation.

9. **Identity relation**: Given a set A, an identity relation on A is defined as the set of all ordered pairs (a, a) where a belongs to A.

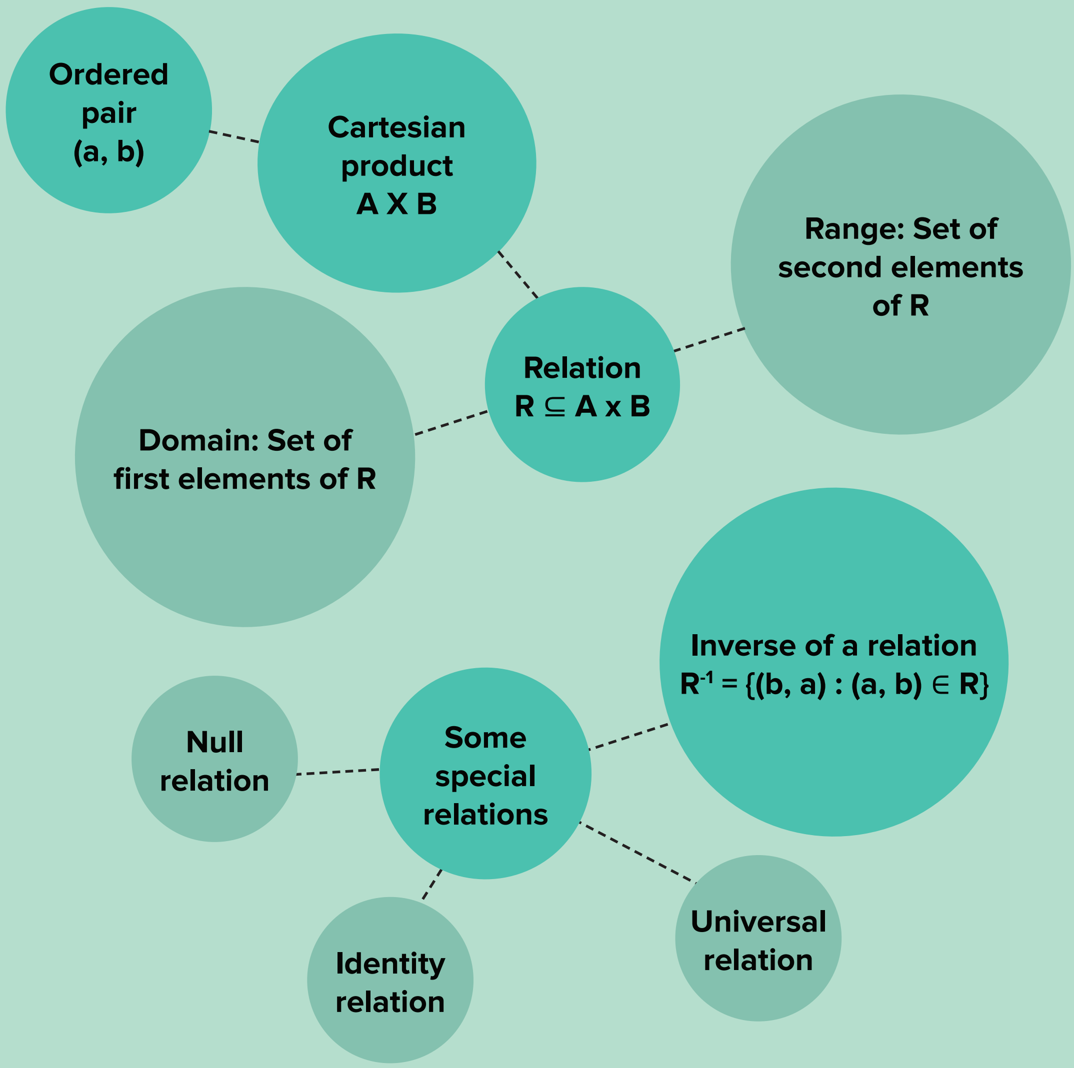
Key takeaways

1. Two ordered pairs are equal if and only if their corresponding first and second elements are equal.
2. Cartesian product: $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

Key results

1. If A and B are finite sets, then $n(A \times B) = n(A) \times n(B)$
2. If $n(A) = p$ and $n(B) = q$,
   Total number of relations from A to B = 2^{pq}
3. If A and B are any two non-empty sets, then
   $A \times B = B \times A \Leftrightarrow A = B$

Mind map



Self-Assessment

1. Let R be a relation on the set of natural numbers N defined by
   $R = \{(x, y) : 2x + y = 8\}$. Find its domain and range.
2. The cartesian product of two sets is given as $A \times B = \{(2, 3), (9, 3), (8, 3)\}$.
   Find A and B.
3. How many relations can be made on A?
   $A = \{x : 1 < x < 30 \text{ and } 3 \text{ divides } x\}$
4. If $A = \{1, 2, 4\}$, $B = \{2, 4, 5\}$, $C = \{2, 5\}$. Find $(A - B) \times (B - C)$.
   (a) $\{(1, 2), (1, 5), (2, 5)\}$ (b) $\{(1, 4)\}$ (c) $\{2, 4\}$ (d) $\{(1, 2), (2, 4)\}$

Answers

Quick Query

1. $(3a - 2, b + 3) = (2a - 1, 3)$
   $3a - 2 = 2a - 1$
   $a = 1$
   $b + 3 = 3$
   $b = 0$
   $a = 1, b = 0$

2. R_1, R_2, R_3 are all relations from A to B.
   All their ordered pairs have the first element belonging to A and the second element belonging to B.
   It is sufficient to say that (4, 2) is an element of R_4 but 4 does not belong to A. Hence, R_4 is not a relation from A to B.

3. Domain: {1, 2, 5}
   Range: {3, 7, 8, 9}

4. Inverse $R^{-1} = \{(4, 8), (12, 13), (10, 5)\}$

Concept Check

1. Solution : We have,
   $A \neq \emptyset$ and $n(A \times A) = 9$
   Now, $n(A \times A) = n(A) \times n(A)$
   $\Rightarrow n(A) \times n(A) = 9$
   $\Rightarrow (n(A))^2 = 9 \Rightarrow n(A) = 3$
   Also, $(-1, 0), (0, 1) \in A \times A$
   $\Rightarrow A = \{-1, 0, 1\}$

2. Solution: We have,
   $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$
   $B = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$
   $\therefore n(A \cap B) = 7$
   $\Rightarrow n((A \times B) \cap (B \times A)) = (n(A \cap B))^2 = 7^2$
   $\Rightarrow$ Number of elements common to A x B and B x A = 49

3. A is the set of all divisors of 8
   $A = \{1, 2, 4, 8\}$
   B is the set of all divisors of 10.
   $B = \{1, 2, 5, 10\}$
   $n(A \times B) = n(A) \times n(B)$
   $= 4 \times 4 = 16$

4. Number of elements in
   $A \times B \times C = n(A) \times n(B) \times n(C)$
   $\Rightarrow 2 \times 3 \times 2 = 12$

Self-Assessment

1. $R = \{(x, y) : 2x + y = 8\}$
   $R = \{(1, 6), (2, 4), (3, 2)\}$
   Domain = {1, 2, 3}
   Range = {2, 4, 6}

2. $A \times B = \{(2, 3), (9, 3), (8, 3)\}$
   $A = \{2, 9, 8\}$
   $B = \{3\}$

3. $A = \{3, 6, 9, 12, 15, 18, 21, 24, 27\}$
   Number of elements in A = 9
   Total number of relations from
   A to A $= 2^{n(A) \times n(A)}$
   $= 2^{81}$

4. $(A - B) = \{1\}$
   $(B - C) = \{4\}$
   Therefore, $(A - B) \times (B - C) = \{(1, 4)\}$

M A T H E M A T I C S

RELATIONS AND FUNCTIONS

MORE ON RELATIONS AND INTRODUCTION TO FUNCTIONS

NOTES

What you already know

- Cartesian product
- Relations

What you will learn

- Functions
- Image, Preimage
- Domain, Codomain, Range
- Some standard functions

Function

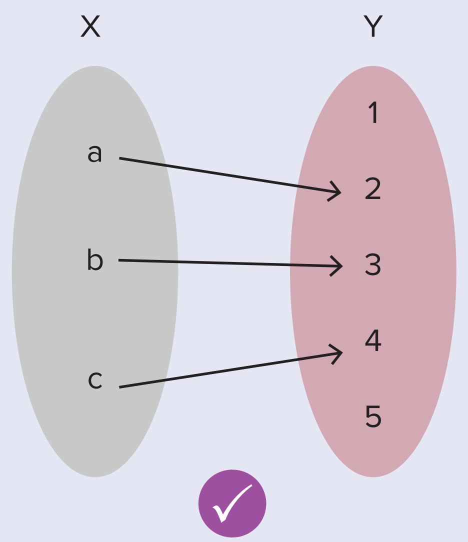
A function 'f' from a non-empty set A to a non-empty set B is a rule or a correspondence under which:

Every element of A is associated with exactly one element of B.

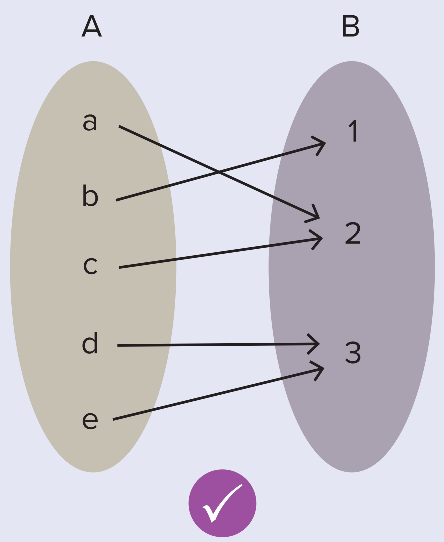
Notation: f is a function from A to B

$f : A \rightarrow B$

Examples

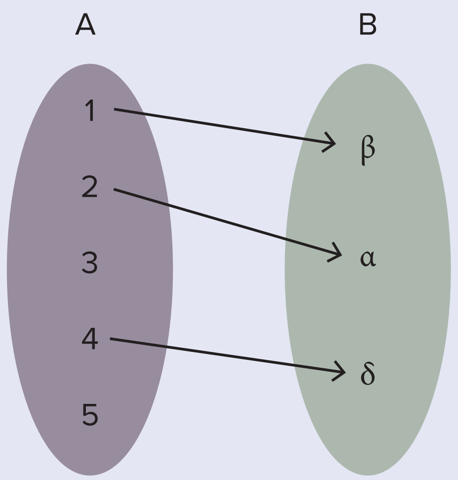


Every element of A is associated with exactly one element of B.



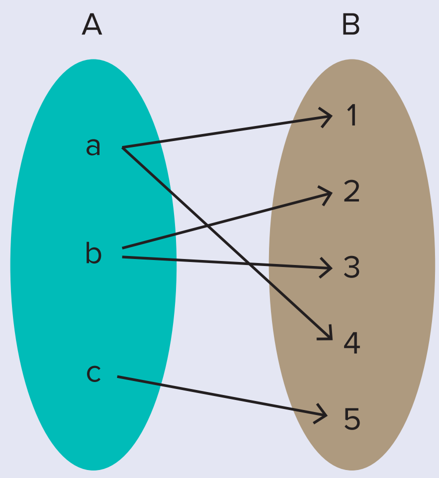
Every element of A is associated with **exactly one** element of B.

Non-Examples



This isn't a function because:

- 3 is not associated with any element of B.
- 5 is not associated with any element of B.



This isn't a function because:

- a is associated with two elements of B.
- b is associated with two elements of B.

- A function is a special relation from A to B such that :
  Every element of A is related to exactly one element of B.
- Number of functions for the mapping f : A → B will be $n(B)^{n(A)}$

Quick Query

1. Let R be a relation from A to B. Determine whether it is a function:
   A = {1, 2, 3, 4}
   B = {5, 10, 15}
   R = {(1, 5), (1, 10), (2, 10), (3, 15), (4, 15)}
2. Let R be a relation from A to B. Determine whether it is a function:
   A = {1, 2, 3, 4}
   B = {5, 10, 15}
   R = {(1, 5), (2, 10), (3, 15)}
3. Let R be a relation from A to B. Determine whether it is a function:
   A = {1, 2, 3, 4}
   B = {5, 10, 15}
   R = {(1, 5), (2, 10), (3, 10), (4, 10)}

Image and Preimage

Let f : A → B be a function. Let a ∈ A. Then, it is associated to exactly one element of B, say b. Then, we write b = f(a)

Image

b is called 'the f-image of a' or 'image of a under f' or 'the value of function f at a'.

Preimage

a is the preimage of b under the function f.

Examples

f : {1, 2, 3} → {3, 4, 5, 6, 7, 8, 9}

$f(x) = 2x + 1$

Here, $f(1) = 2(1) + 1 = 3$

3 is the image of 1 under f and 1 is the preimage of 3 under f

$f(2) = 2(2) + 1 = 5$

5 is the image of 2 under f and 2 is the preimage of 5 under f

$f(3) = 2(3) + 1 = 7$

7 is the image of 3 under f and 3 is the preimage of 7 under f

Concept Check

1. Let f be a function from A to B

   A = {1, 2, 3}

   B = {5, 10, 15}

   f (1) = 5; f (2) = 10; f (a) = 15.

   Find the preimage of 15.

Domain, Codomain and Range of a function

Domain

Let f : A → B, then the set A is known as the domain of f .

Notation: D(f) = A

Codomain

Let f : A → B, then the set B is known as the co-domain of f .

Notation: C(f) = B

Range

Let f : A → B, then the set of all the images of elements of A under f in B is known as the range of f .

Notation: R(f) = {f(x) : x ∈ A}

- Range ⊆ codomain

Examples

$f : \{1, 2, 3\} \rightarrow \{3, 4, 5, 6, 7, 8, 9\}$

$f(x) = 2x + 1$

Domain	Codomain	Range
{1, 2, 3}	{3, 4, 5, 6, 7, 8, 9}	{3, 5, 7}

- Functions having domain and co-domain both as subsets of R are known as real functions or real valued functions of a real variable.
- A real function is generally described by some general formula.
- For example, f: R → R given by $f(x) = x^2 + x + 1$.
- The domain of the real function f is the set of all those real numbers x for which the expression for f(x) assumes real values only i.e., D(f) is set of all those real numbers x for which f(x) is meaningful.

Quick Query

4. Find the domain, codomain, and range of the following function:

   $f : Z \rightarrow Z$

   $f(x) = 2x.$

Some Standard Functions

- Constant function
- Identity function
- Polynomial function
- Rational function
- Greatest integer function
- Signum function
- Fractional part function
- Exponential function
- Logarithmic function
- Absolute value function

Constant Function

The function $f : R \rightarrow R$ defined by $f(x) = c \ \forall \ x \in R$, where c is a real constant is known as a **constant function.**

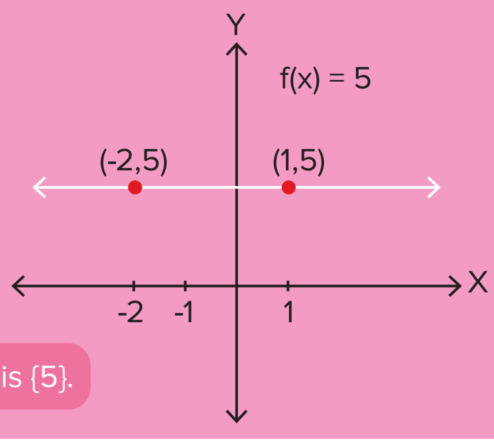
Example : f(x) = 5

Graph: f(x) = 5

The graph of f(x) = 5 is a straight line parallel to x-axis passing through (0, 5).

Domain: Domain of f(x) = 5 is R.

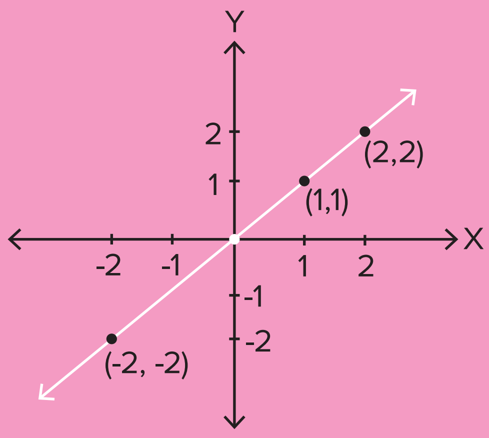
Range: Range of f(x) = 5 is {5}.



Identity Function

The function f: R → R defined by $f(x) = x \ \forall \ x \in R$ is known as **identity function**.

Graph: $f(x) = x$



The graph of $f(x) = x$ is a straight line passing through the origin inclined at an angle of 45° with the positive direction of x-axis.

Domain: Domain of $f(x) = x$ is R.

Range: Range of $f(x) = x$ is R.

Polynomial Function

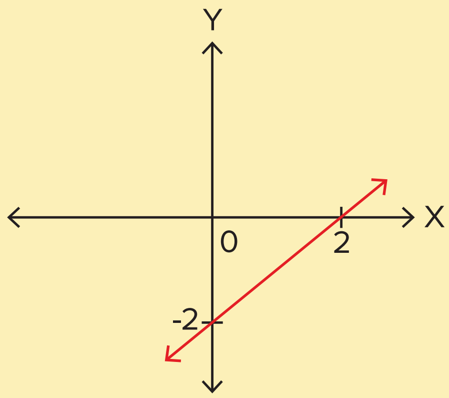
A polynomial function is a function that can be written in the form

$$f(x) = a_0 + a_1 x + a_2 x^2 \ldots . \ a_n x^n \text{ and } n \in W$$

where each a_0, a_1, etc. represents a real number, and where n is a whole number (including 0).

Example: Linear polynomial: $ax + b$; $a \neq 0$

Graph: $f(x) = x - 2$



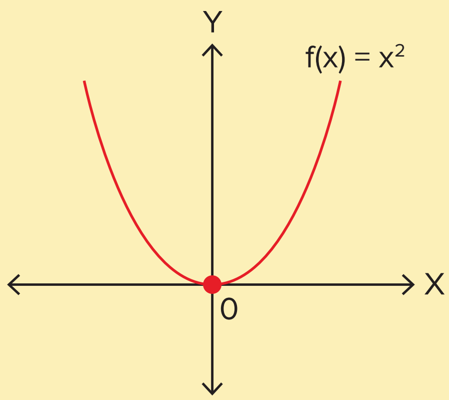
The graph of any linear polynomial function $f(x) = ax + b$; $a \neq 0$ is a straight line in the cartesian plane.

Domain: Domain of $f(x) = x - 2$ is R.

Range: Range of $f(x) = x - 2$ is R.

Quadratic function: $f(x) = ax^2 + bx + c$; $a \neq 0$

Graph: $f(x) = x^2$



The graph of any quadratic polynomial function $f(x) = ax^2 + bx + c$; $a \neq 0$ is a parabola.

Domain: Domain of $f(x) = x^2$ is R.

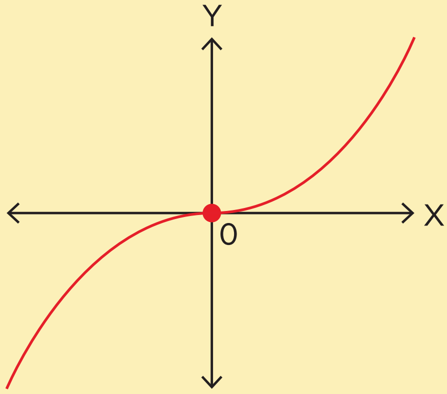
Range: Range of $f(x) = x^2$ is $[0, \infty)$.

Cubic function: $f(x) = ax^3 + bx^2 + cx + d$; $a \neq 0$

Graph: $f(x) = x^3$

Domain: Domain of $f(x) = x^3$ is R.

Range: Range of $f(x) = x^3$ is R.



Rational Function

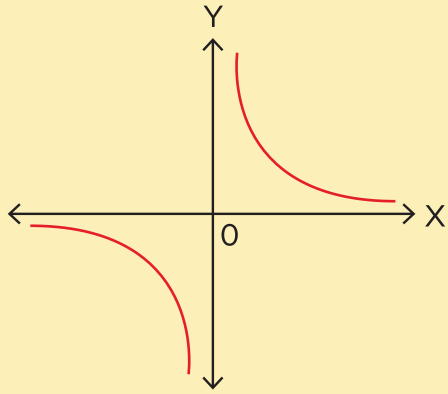
A real function defined as $f(x) = \frac{g(x)}{h(x)}$, where g(x) and h(x) are polynomial functions and $h(x) \neq 0$ is known as a rational function.

Example: 1. $f(x) = \frac{7}{x}$ 2. $f(x) = \frac{x^2 + 2}{x}$ 3. $f(x) = \frac{1}{x}$

Graph: $f(x) = \frac{1}{x}$

Domain: Domain of $f(x) = \frac{1}{x}$ is R - {0}

Range: Range of $f(x) = \frac{1}{x}$ is R - {0}



Quick Query

5. Find the domain of the rational function.

$f(x) = \frac{5}{(x - 2)(x + 3)}$

Concept Check

2. Find the domain of the rational function.

$f(x) = \frac{5x}{x\,(x + 8)}$

Modulus Function (Absolute Value Function)

The function f : R → R defined by $f(x) = |x|$ where

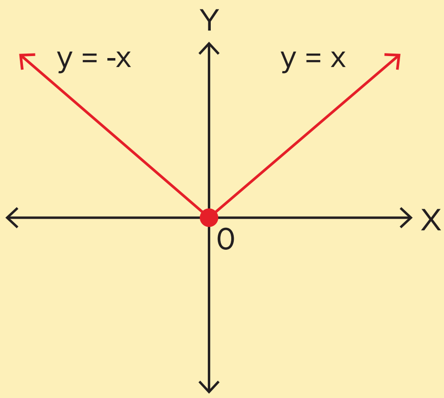
$$|x| = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$$

is known as the modulus or the absolute value function.

Graph: $f(x) = |x|$

Domain: Domain of $f(x) = |x|$ is R.

Range: Range of $f(x) = |x|$ is $[0, \infty)$.



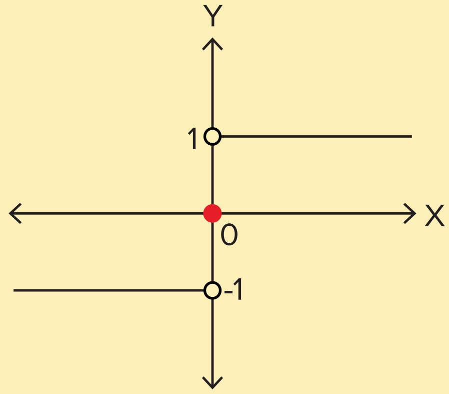
Concept Check

3. Find the domain and range of $f(x) = |x - 1|$.

Signum function

$$f(x) = \text{sgn}(x) = \begin{cases} 1; & x > 0 \\ 0; & x = 0 \\ -1; & x < 0 \end{cases} \quad \text{or} \quad f(x) = \text{sgn}(x) = \begin{cases} \frac{|x|}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$$

Graph: f(x) = sgn(x)



Domain: The domain of the signum function is R.

Range: The range of the signum function is {-1, 0, 1}.

Concept Check

4. Draw graph of $f(x) = \text{sgn}(x^2 - 1)$.

5. The equation $x^2 - 5x\,\text{sgn}(x^2 - 4) + 6 = 0$ has ____________ number of solutions.

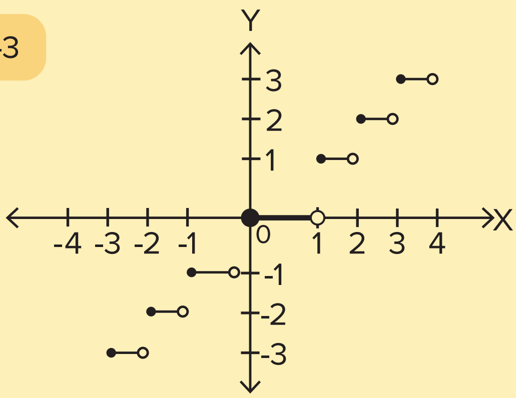
Greatest Integer Function

The real function $f: R \to R$ defined by $f(x) = [x], x \in R$ assumes the value of the greatest integer less than or equal to x and is known as the greatest integer function.

$[x] = -1$ for $-1 \leq x < 0$
$[x] = 0$ for $0 \leq x < 1$
$[x] = 1$ for $1 \leq x < 2$
$[x] = 2$ for $2 \leq x < 3$, and so on

Example: $[2.45] = 2$, $[3.6] = 3$, $[-0.67] = -1$, $[-2.38] = -3$

Graph: f(x) = [x]



Domain: The domain of greatest integer function is R.

Range: The range of greatest integer function is Z.

Properties

1. $[x] \le x < [x] + 1$
2. $x - 1 < [x] \le x$
3. $[x + m] = [x] + m;\ m \in Z$
4. $[-x] = \begin{cases} -[x] & x \in Z \\ -[x] - 1 & x \notin Z \end{cases}$ that is $[x] + [-x] = \begin{cases} 0 & x \in Z \\ -1 & x \notin Z \end{cases}$
5. If $[x] \ge n$ then $x \ge n,\ n \in Z$
6. If $[x] > n$ then $x \ge n + 1,\ n \in Z$
7. If $[x] \le n$ then $x < n + 1,\ n \in Z$
8. If $[x] < n$ then $x < n,\ n \in Z$
9. If $[x] + [x + \frac{1}{n}] + [x + \frac{2}{n}] + [x + \frac{3}{n}] + \ldots + [x + \frac{n-1}{n}] = [nx],\ n \in Z$
10. If $[x] = [\frac{x}{2}] + [\frac{x+1}{2}]$

$[\frac{1}{4}] + [\frac{1}{4} + \frac{1}{200}] + [\frac{1}{4} + \frac{2}{200}] + [\frac{1}{4} + \frac{3}{200}] + [\frac{1}{4} + \frac{4}{200}] + \ldots\ldots + [\frac{1}{4} + \frac{199}{200}] = \ldots\ldots$

Solution:

By Property 9; $n = 200$, $x = \frac{1}{4}$ then $[nx] = [200 . \frac{1}{4}] = 50$

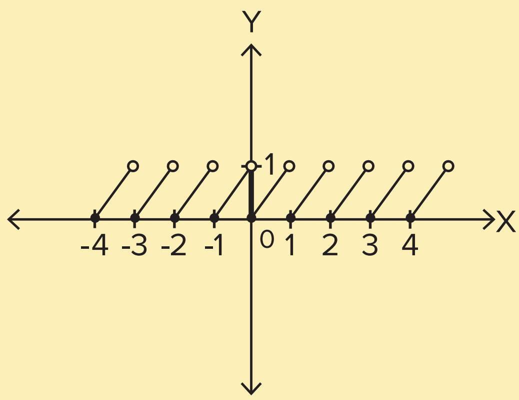
Fractional Part Function

It is denoted by $\{x\}$. It is defined as $\{x\} = x - [x]$.

Example:

$\{0.23\} = 0.23 - [0.23] = 0.23 - 0 = 0.23$
$\{2.78\} = 2.78 - [2.78] = 2.78 - 2 = 0.78$
$\{-1.67\} = -1.67 - [-1.67] = -1.67 + 2 = 0.33$
$\{-3.35\} = -3.35 - [-3.35] = -3.35 + 4 = 0.65$

Graph: $f(x) = \{x\}$



Domain: The domain of fractional part function is R.

Range: The range of fractional part function is [0, 1).

Properties

1. $0 \le \{x\} < 1$
2. $\{x + m\} = \{x\};\ m \in Z$
3. $\{-x\} = \begin{cases} 1 - \{x\}, & x \notin Z \\ 0, & x \in Z \end{cases}$

Quick Query

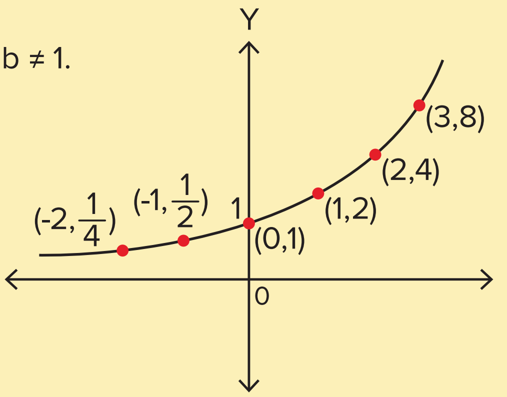
6. Solve [3x + 2] = 5.

Concept Check

6. 4{x} = x + [x]. Find x.

Exponential Function

Exponential functions are the functions of the form $f(x) = b^x$, where b is the base and x is the exponent. Here, $b > 0$ and $b \neq 1$.



Graph: $f(x) = 2^x$

Domain: The domain of $f(x) = 2^x$ is R.

Range: The range of $f(x) = 2^x$ is $(0, \infty)$

Logarithmic Function

Given an exponential function $y = b^x$, x is the input and y is the output. If we reverse the input and the output, we get its inverse, that is, the logarithmic function. Log function is written as $f(x) = \log_b x$, where b is the base of the logarithm, $b > 0$, $b \neq 1$ and $x > 0$

Example:

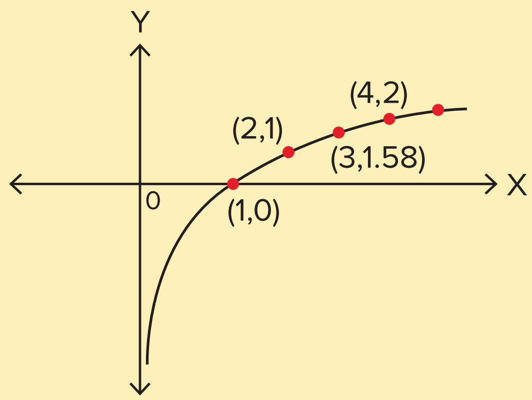
If $y = \log_2 x$ then $x = 2^y$

$\log_2 2 = 1$ as $2^1 = 2$

$\log_2 4 = 2$ as $2^2 = 4$

$\log_2 8 = 3$ as $2^3 = 8$

$\log_2 16 = 4$ as $2^4 = 16$



Graph: $f(x) = \log_2 x$

Domain: The domain of $f(x) = \log_2 x$ is $(0, \infty)$.

Range: The range of $f(x) = \log_2 x$ is R.

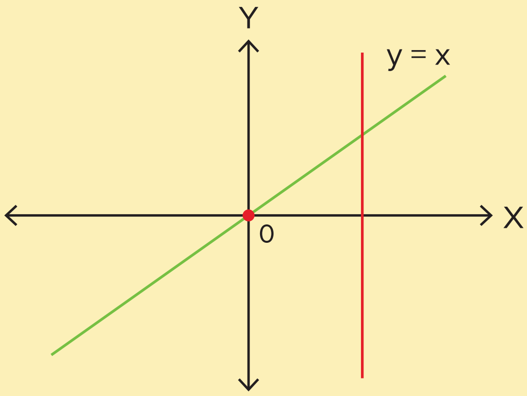
Quick Query

7. Find the value of $\log_a a^3$. ($a > 0$; $a \neq 1$).

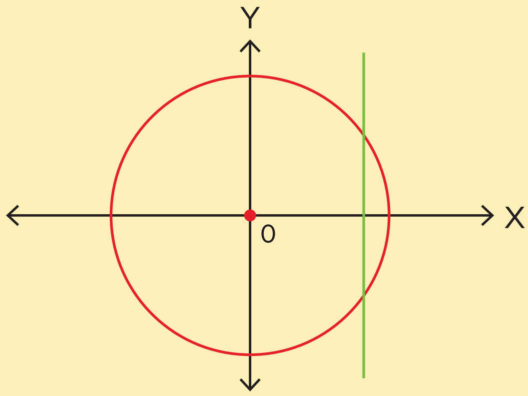
Line Test

Given the graph of a curve, how do we tell whether it represents a function?
If we draw a line parallel to the y-axis and it intersects the given curve/diagram at only one point then, the given curve/diagram represents a function.

Example:



This is the graph of a function. Draw a vertical line anywhere. It intersects the graph at one point only.



This is not a function. The vertical line is intersecting the graph at two points.

Summary Sheet

Key Takeaways

1. A function 'f' from a non-empty set A to a non-empty set B is a rule or a correspondence under which every element of A is associated with exactly one element of B.
2. A function is a special relation from A to B such that every element of A is related to exactly one element of B.
3. The function $f: R \rightarrow R$ defined by $f(x) = c \ \forall \ x \in R$, where c is a real constant is known as a **constant function**.
4. The function $f: R \rightarrow R$ defined by $f(x) = x \ \forall \ x \in R$ is called **identity function**.
5. A function defined as f(x) = polynomial is known as a **polynomial function**.
6. A real function defined as $f(x) = \frac{g(x)}{h(x)}$, where g(x) and h(x) are polynomial functions and $h(x) \neq 0$ is known as a **rational function**.
7. The function $f: R \rightarrow R$ defined by $f(x) = |x|$ where
$|x| = \begin{cases} x; & x \geq 0 \\ -x; & x < 0 \end{cases}$ is known as the **modulus function**.
8. The **signum function** gives the sign of a real number. It is defined as:

$$f(x) = \text{sgn}(x) = \begin{cases} 1; & x > 0 \\ 0; & x = 0 \\ -1; & x < 0 \end{cases} \quad \text{or} \quad f(x) = \text{sgn}(x) = \begin{cases} \frac{|x|}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$$

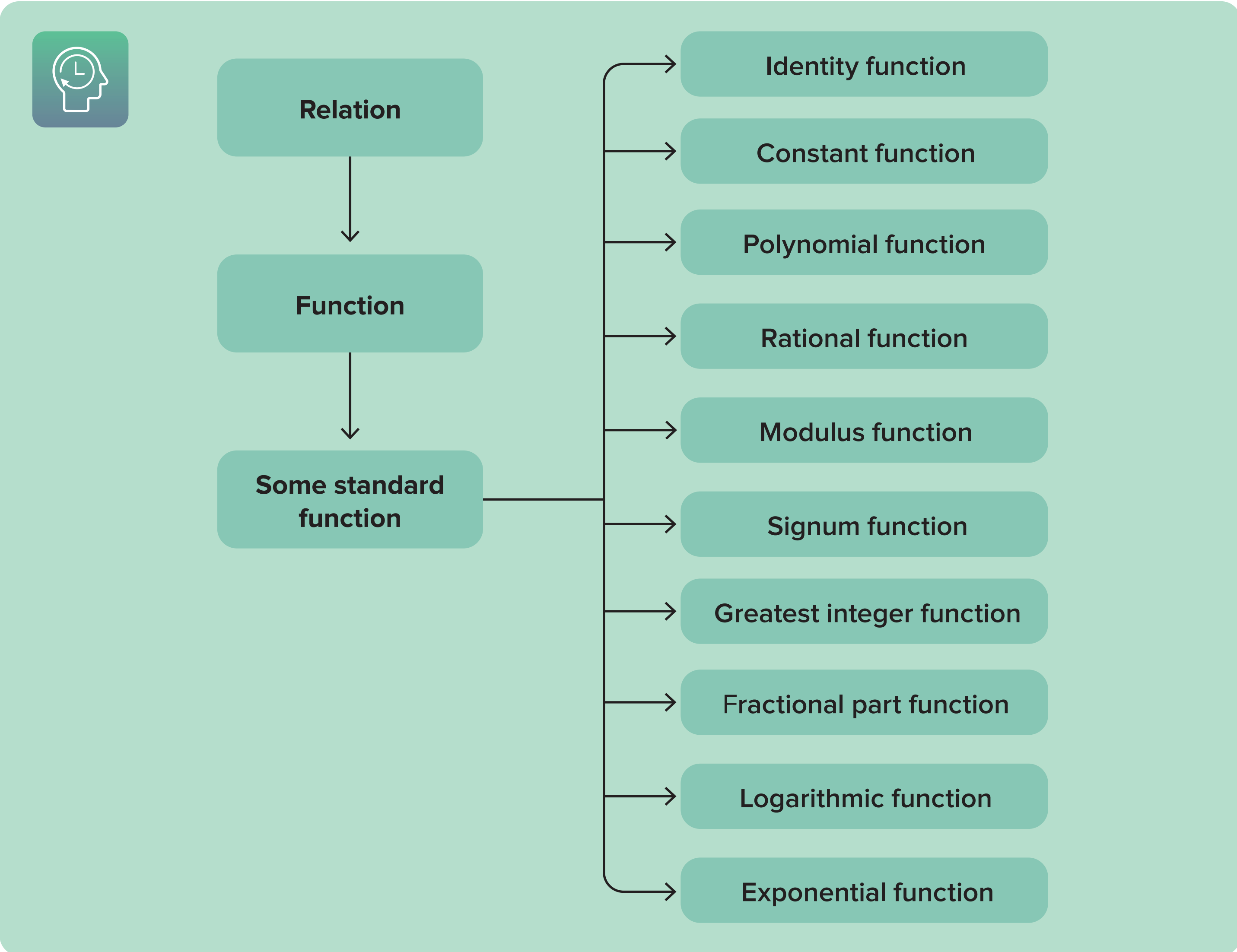
9. The real function $f: R \rightarrow R$ defined by $f(x) = [x]$, $x \in R$ assumes the value of the greatest integer less than or equal to x and is known as the **greatest integer function**.
10. **Fractional Part function** is denoted by $\{x\}$. It is defined as $\{x\} = x - [x]$.
11. **Exponential functions** are the functions of the form:
    $f(x) = b^x$ where b is the base and x is the exponent. Here, $b > 0$ and $b \neq 1$
12. **Logarithmic function** is the inverse of exponential function.
13. **Line test:** Given the graph of a curve, how do we tell whether it represents a function?
    If we draw a line parallel to the y-axis and it intersects the given curve/diagram at only one point then, the given curve/diagram represents a function.

Key Terms

1. $f: A \rightarrow B$
   $b = f(a)$
   b is known as '**the f-image of** a' or '**image of a under** f' or '**the value of function** f **at** a'
   a is the **preimage** of b under the function f
2. Let $f: A \rightarrow B$, then the set A is known as the **domain** of f .
3. Let $f: A \rightarrow B$, then the set B is known as the **co-domain** of f .
4. Let $f: A \rightarrow B$, then the set of all the images of elements of A in B under f is known as the **range** of f .

Key Results

1. Range $\subseteq$ codomain.
2. $[x] \leq x < [x] + 1$
3. $x - 1 < [x] \leq x$
4. $[x + m] = [x] + m;\ m \in Z$.
5. $[-x] = \begin{cases} -[x] & x \in Z \\ -[x] - 1 & x \notin Z \end{cases}$ that is $[x] + [-x] = \begin{cases} 0 & x \in Z \\ -1 & x \notin Z \end{cases}$
6. $[x] + [x + \frac{1}{n}] + [x + \frac{2}{n}] + [x + \frac{3}{n}] + \ldots + [x + \frac{n-1}{n}] = [nx],\ n \in Z$
7. $[x] = [\frac{x}{2}] + [\frac{x+1}{2}]$
8. $0 \leq \{x\} < 1$
9. $\{x + m\} = \{x\}\ ;\ m \in Z$
10. $\{-x\} = \begin{cases} 1 - \{x\}, & x \notin Z \\ 0\ , & x \in Z \end{cases}$
11. Any real value can be expressed as a sum of its greatest integer function and fractional part function.
    $x = [x] + \{x\}$
12. If $y = \log_a x$, then $a^y = x$.



Self-Assessment

1. If $y = 2[x] + 3$, $y = 3[x - 2] + 5$ then find $[x + y]$.
2. Find the domain of $g(x) = \sqrt{1 - x}$.
3. What is the domain of $f(x) = \log_2 \{x\}$?
4. Draw the graph of $f(x) = |x + 2|$, $g(x) = |x| + 2$.

Answer

Quick Query

1. It is not a function because 1 is related to 5 and 10 both.
2. It is not a function because 4 is not related to any element of B.
3. Yes, it is a function.
4. Domain: Z
   Codomain: Z
   Range: All even integers.
5. The domain will contain those values where the function is defined. So, we will remove those values from R that make the denominator 0.
   The domain of this function is R - {2, -3}

6. $5 \le 3x + 2 < 6$

$3 \le 3x < 4$

$1 \le x < \frac{4}{3}$

$x \in [1, \frac{4}{3})$

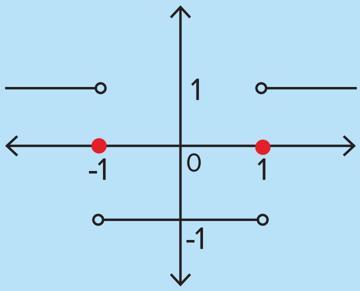
7. $y = \log_a a^3$

$a^y = a^3$

$y = 3$

Concept Check

1. Since f is a function a has to be 3, the preimage of 15.
2. The domain of this function is R - {0, - 8}.
   A word of caution: Do not cancel the x from numerator and denominator before finding the domain.
3. Any real value can be put as input. So, Domain : R.
   The modulus function will give only non-negative values. So Range $\subseteq [0, \infty)$. Will it take all of these values? Let us check. Let $y \in [0, \infty)$.
   $y = |x - 1|$ then $y + 1$ is a real number that when we input, we get y as the output.
   $f(y + 1) = |y + 1 - 1| = |y| = y$ as $y \ge 0$. So, Range : $[0, \infty)$
4. $$f(x) = \begin{cases} -1 & x^2 - 1 < 0 \\ 0 & x^2 - 1 = 0 \\ 1 & x^2 - 1 > 0 \end{cases}$$



$$f(x) = \begin{cases} -1 & , -1 < x < 1 \\ 0 & , x = 1, -1 \\ 1 & x > 1, x < -1 \end{cases}$$

5. Case I: $x^2 - 4 < 0$

$-2 < x < 2$

$x^2 - 5 \times (-1) + 6 = 0$

$x = -2, -3 \Rightarrow$ No solution

Case II: $x^2 - 4 = 0$

$x = -2, 2$

$x^2 - 5x\ (0) + 6 = 0 \Rightarrow x^2 + 6 = 0 \Rightarrow$ No solution

Case III: $x^2 - 4 > 0$

$x \in (-\infty, -2) \cup (2, \infty)$

$x^2 - 5x\ (1) + 6 = 0$

$x = 2, 3 \Rightarrow 3$ is a solution.

So, there is just 1 solution that is 3.

6. $4\{x\} = [x] + \{x\} + [x]$

$\Rightarrow 3\{x\} = 2[x]$

But $0 \le \{x\} < 1$

$\Rightarrow 0 \le 3\{x\} < 3$

$0 \le 2[x] < 3$

$\Rightarrow 0 \le [x] < \frac{3}{2}$

$\Rightarrow [x] = 0, 1$

$3\{x\} = 2[x]$

$\Rightarrow 3(x - [x]) = 2[x]$

$\Rightarrow x = \frac{5}{3}[x]$

$x = 0, \frac{5}{3}$

Self-Assessment

1. $2[x] + 3 = 3[x - 2] + 5$

$2[x] + 3 = 3[x] - 6 + 5$

$4 = [x]$

$y = 2(4) + 3 = 11$

$[x + y] = [11 + x] = 11 + [x] = 11 + 4 = 15$

2. $1 - x \ge 0$

$x \le 1$

$x \in (-\infty, 1]$

So, domain is $(-\infty, 1]$.

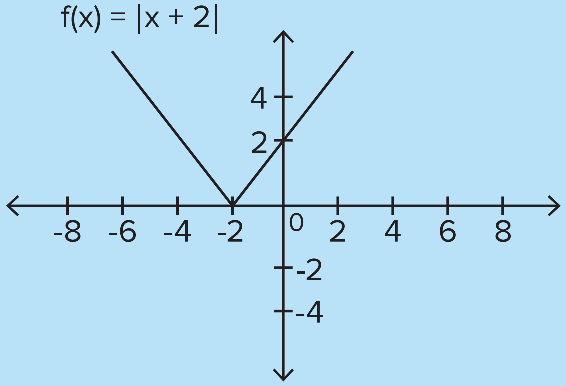
3. The domain of $\log_2$ is $(0, \infty)$

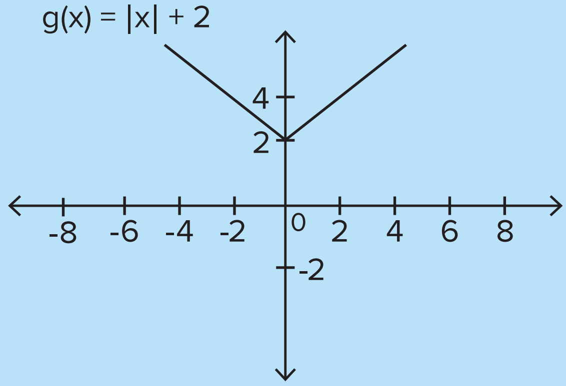
The range of $\{x\}$ is $[0, 1)$

So, x can take all real values except those for which $\{x\}$ is 0.

That is, R - Z is the required domain.

4.





M A T H E M A T I C S

RELATIONS AND FUNCTIONS

DOMAIN OF FUNCTIONS

NOTES

What you already know

- What is a function?
- Domain and Range of a function
- Some standard functions

What you will learn

- Domain of a function (in detail)

Domain of a function

The domain of a real function *f* is the set of all those real numbers *x* for which *f*(*x*) assumes real values, i.e., *D*(*f*) is the set of all those reals *x* for which *f*(*x*) is meaningful.

For two functions *f* and *g*

1. $D(f + g) = D(f) \cap D(g)$
2. $D(f - g) = D(f) \cap D(g)$
3. $D(f \times g) = D(f) \cap D(g)$
4. $D(\frac{f}{g}) = D(f) \cap D(g) - \{x \in D(g);\ g(x) = 0\}$

Some of the standard formats to find the domain

(a) $\sqrt{f(x)} \Rightarrow f(x) \geq 0$

(b) $\frac{1}{\sqrt{f(x)}} \Rightarrow f(x) > 0$

(c) $\frac{k}{f(x)} \Rightarrow f(x) \neq 0$, K is a constant

(d) $\log_b a$, $a > 0$, $b > 0$, $b \neq 1$

(e) $\frac{1}{\log_b a}$, $a > 0$, $b > 0$, $a \neq 1$, $b \neq 1$

Quick Query

Q.1. Find the domain of $y = \frac{1}{x}$

Q.2. Find the domain of $y = \sqrt{x}$

Q.3. Find the domain of $y = \sqrt{2x - x^2}$

Concept Check

Q.1. Find the domain of

$y = \sqrt{x - 1} + \sqrt{6 - x}$

Q.2. Find the domain of

$y = \frac{\sqrt{x}}{\sqrt{x}}$

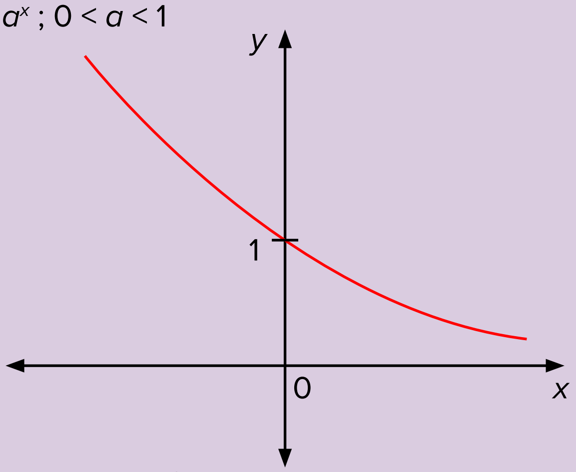
$y = \frac{\sqrt{x}}{\sqrt{x}}$ is not the same as $y = 1$.

Exponential function

$f(x) = a^x$; $a > 0$ and $a \neq 1$

Case 1

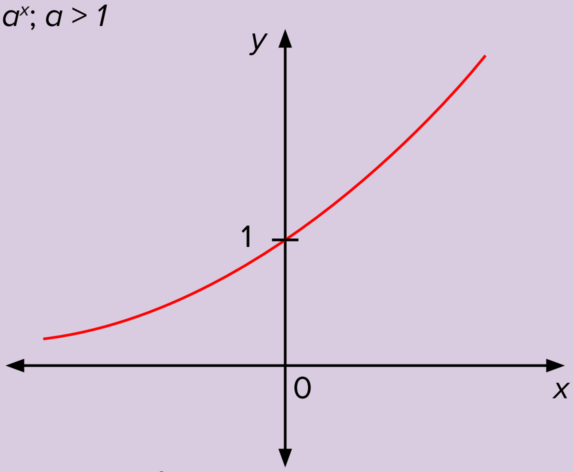
a^x ; $0 < a < 1$



Decreasing function
Domain: R
Range : (0, ∞)

Case 2

a^x; $a > 1$



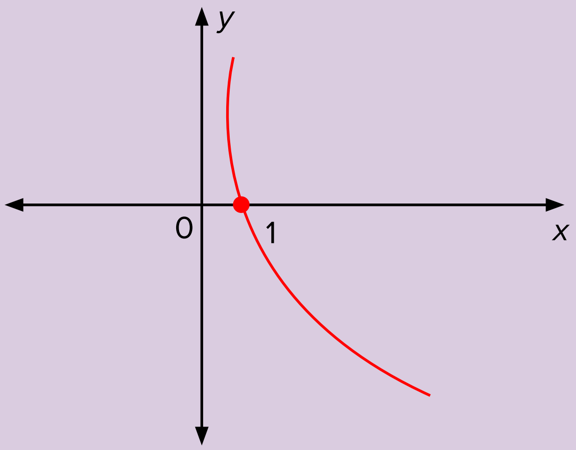
Increasing function
Domain: R
Range : (0, ∞)

Logarithmic function

$f(x) = \log_a x$; $a > 0$ and $a \neq 1$, $x > 0$

Case 1

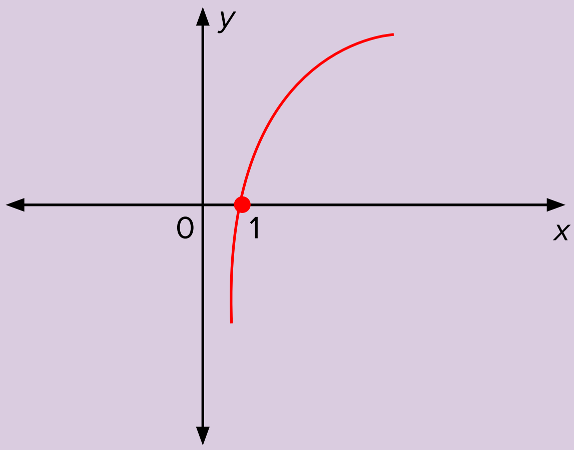
$\log_a x$; $0 < a < 1$



Decreasing function
Domain: (0, ∞)
Range : R

Case 2

$\log_a x$; $a > 1$



Increasing function
Domain: (0, ∞)
Range : R

Quick Query

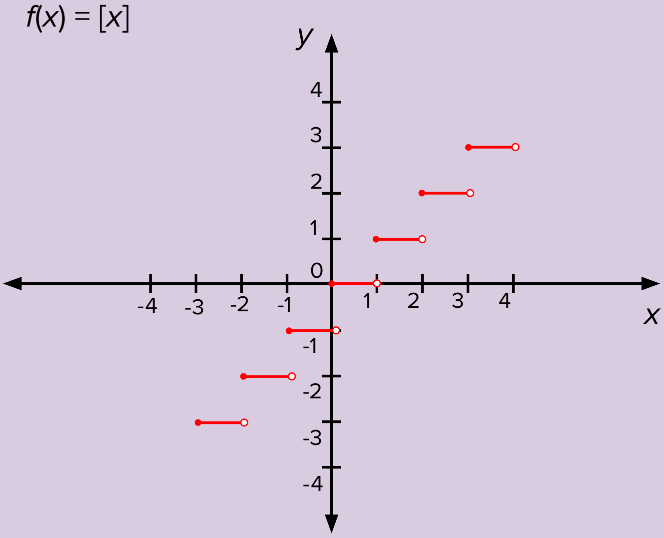
Q.4. Find the domain of $y = \log_e x$.

Concept Check

Q.3. Find the domain of $y = \frac{1}{\ln x}$

Greatest Integer function:

$f(x) = [x]$

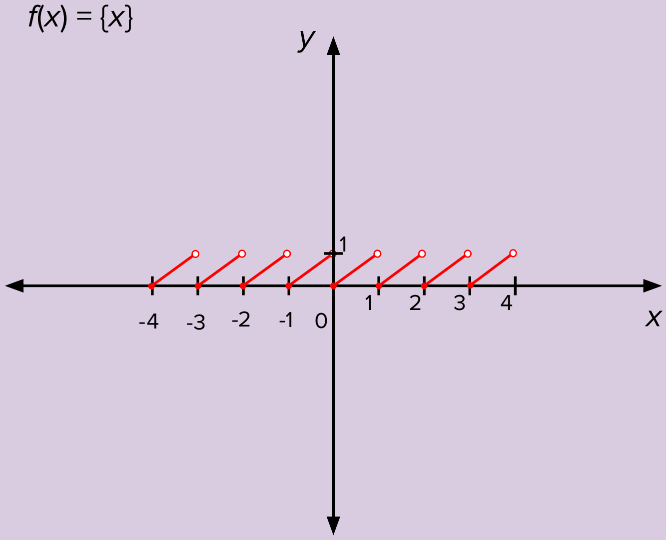


Domain: R

Range : Z

Fractional Part function:

$f(x) = \{x\}$



Domain: R

Range : [0,1)

Concept Check

Q.4. Find the domain of $y = \frac{1}{[x]}$

Concept Check

Q.5. Find the domain of $y = \frac{1}{\{x\}}$

Example

1 . Find the domain of the function $f(x) = \frac{1}{\log_{2x}(x+1)}$

Solution

For

$\log_a x$ to be defined, $a > 0$ and $a \neq 1$

and $x > 0$. So, $2x > 0$ and $2x \neq 1$

$x > 0$ and $x \neq \frac{1}{2}$ ________A

$x + 1 > 0$

$x > -1$ ______________B

and

The denominator must not be 0.

$\log_{2x}(x+1) \neq 0$

that is $x + 1 \neq 1$

$x \neq 0$ ______________C

Combining A, B and C

Domain : $(0, \infty) - \left\{\frac{1}{2}\right\}$

Example

2 . Find the domain of the function $f(x) = \log_{2\{x\} - 3}(x^2 - 5x + 13)$

Solution

$2\{x\} - 3 > 0$ and $2\{x\} - 3 \neq 1$

$\{x\} > \frac{3}{2}$ and $\{x\} \neq 2$

$0 \leq \{x\} < 1 \; \forall x \in R$

Since $\{x\}$ can't be greater than $\frac{3}{2}$ for any x, there are no such x.

Domain : Φ

Example

3. Find the domain of the function $f(x) = \dfrac{\sqrt{x + 5}}{\log_{10}(9 - x)}$

Solution

Domain(Numerator)

$x + 5 \geq 0$

$x \geq -5$

$x \in [-5, \infty)$

Domain (Denominator)

$9 - x > 0$

$x < 9$

$x \in (-\infty, 9)$

Where denominator is 0:

$9 - x = 1$

$x = 8$

Domain(f) = Domain(Numerator) ∩ Domain(Denominator) - {x where Denominator = 0}

$= [-5, \infty) \cap (-\infty, 9) - \{8\}$

$= [-5, 9) - \{8\}$

Example

4 . Find the domain of the function $f(x) = \sqrt{x^2 - x - 20} + \dfrac{1}{\sqrt{x^2 - 5x - 14}}$

Solution

Domain of $\sqrt{x^2 - x - 20}$

$x^2 - x - 20 \geq 0$

$(x - 5)(x + 4) \geq 0$

$x \in (-\infty, -4] \cup [5, \infty)$

Domain of $\dfrac{1}{\sqrt{x^2 - 5x - 14}}$

= Domain(Numerator) ∩ Domain(Denominator) - {x where Denominator = 0}

Domain (Denominator)

$x^2 - 5x - 14 \geq 0$

$(x - 7)(x + 2) \geq 0$

$x \in (-\infty, -2] \cup [7, \infty)$

Where denominator is 0

$(x-7)(x+2) = 0$

$x = -2, 7$

Domain $= (-\infty, -2] \cup [7, \infty) - \{-2, 7\}$

$= (-\infty, -2) \cup (7, \infty)$

Domain(f) = Domain of $\sqrt{x^2 - x - 20} \cap$ Domain of $\dfrac{1}{\sqrt{x^2 - 5x - 14}}$

$= (-\infty, -4] \cup (7, \infty)$

Summary

Key Terms

The domain of a real function f is the set of all those real numbers x for which $f(x)$ assumes real values i.e., $D(f)$ is the set of all those reals x for which $f(x)$ is meaningful.

Key takeaways

1. Exponential function: $f(x) = a^x$; $a > 0$ and $a \neq 1$.

Case 1:

a^x ; $0 < a < 1$

Decreasing function

Domain: R

Range : $(0, \infty)$

Case 2:

a^x ; $a > 1$

Increasing function

Domain: R

Range : $(0, \infty)$

2. Logarithmic function: $f(x) = \log_a x$; $a > 0$ and $a \neq 1$.

Case 1:

$\log_a x$; $0 < a < 1$

Decreasing function

Domain: $(0, \infty)$

Range : R

Case 2:

$\log_a x$; $a > 1$

Increasing function

Domain: $(0, \infty)$

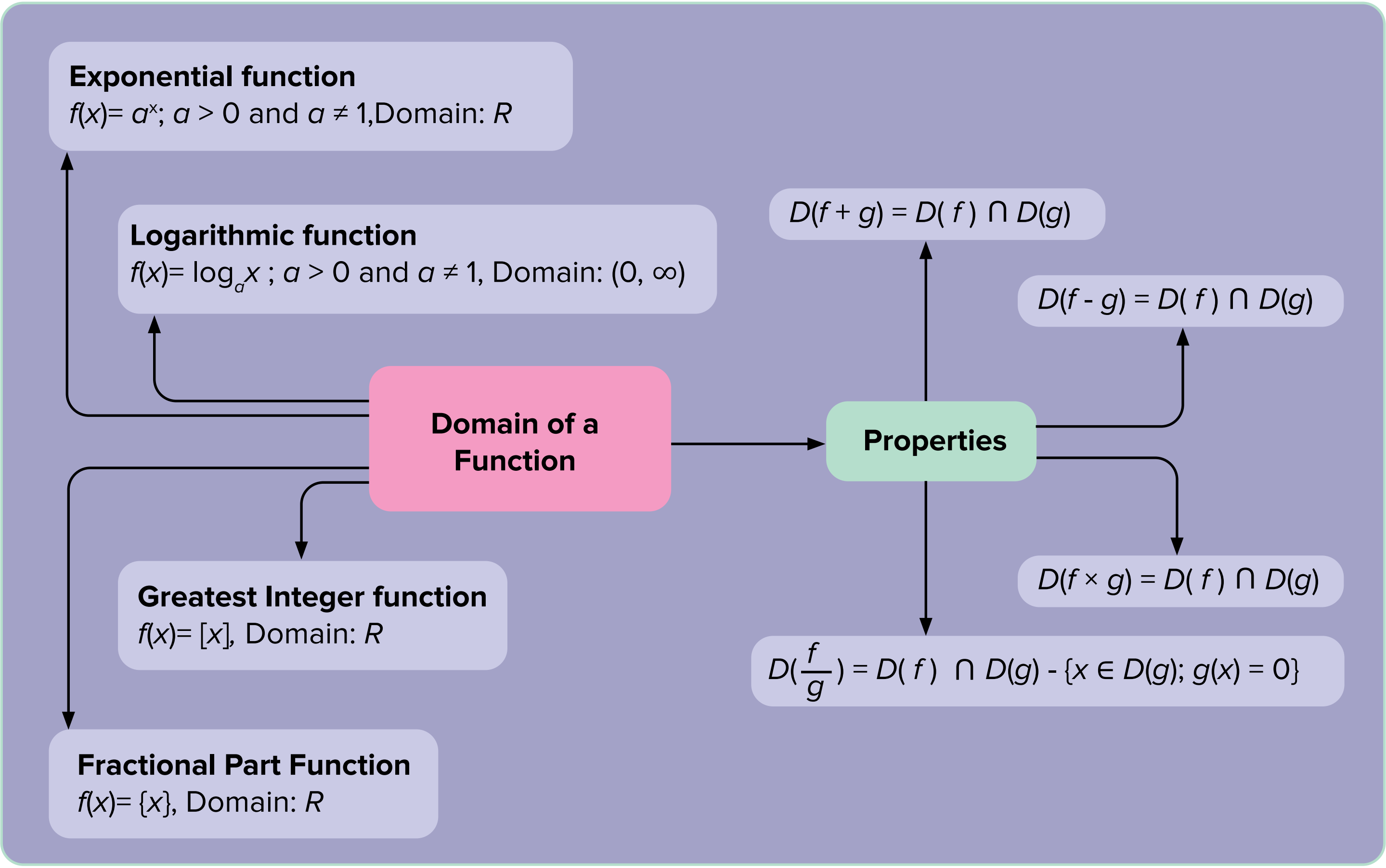
Range : R

3. Greatest Integer function: $f(x) = [x]$
Domain: R
Range : Z

4. Fractional Part Function: $f(x) = \{x\}$
Domain: R
Range : $[0, 1)$

Key results

1. $D(f+g) = D(f) \cap D(g)$
2. $D(f-g) = D(f) \cap D(g)$
3. $D(f \times g) = D(f) \cap D(g)$
4. $D(\frac{f}{g}) = D(f) \cap D(g) - \{x \in D(g); g(x) = 0\}$



Self-Assessment

1. Find the domain of $f(x) = \dfrac{20}{\sqrt{x - |x|}}$

2. Find the domain of $f(x) = \dfrac{1}{\sqrt{[x]^2 - [x] - 6}}$

3. Find the domain of $f(x) = \dfrac{2}{\sqrt{9 - x^2}} \log(x^3 - x)$

4. Find the domain of $y = f(x)$ satisfying $4^x + 4^y = 4$

Answers

Quick Query

1. Domain: R - {0}

2. Domain: $[0, \infty)$

3. $2x - x^2 \geq 0$
   $x(x - 2) \leq 0$
   $x \in [0, 2]$
   Domain: [0, 2]

4. $\log_e x = \ln x$
   $e \sim 2.718$
   Domain: $(0, \infty)$

Concept Check

1. $f(x) = \sqrt{x - 1}$
   $D(f) = [1, \infty)$
   $g(x) = \sqrt{6 - x}$
   $D(g) = (-\infty, 6]$
   $D(f + g) = D(f) \cap D(g)$
   Domain: [1, 6]

2. $y = \frac{\sqrt{x}}{\sqrt{x}}$
   $f(x) = \sqrt{x}$
   $D(f) = [0, \infty)$
   $g(x) = \sqrt{x}$
   $D(g) = [0, \infty)$
   $D(\frac{f}{g}) = D(f) \cap D(g) - \{x \in D(g); g(x) = 0\}$
   Domain: $[0, \infty) - \{0\} = (0, \infty)$

3. $y = \frac{1}{\ln x}$
   $f(x) = 1$
   $D(f) = R$
   $g(x) = \ln x$
   $D(g) = (0, \infty)$
   $D(\frac{f}{g}) = D(f) \cap D(g) - \{x \in D(g); g(x) = 0\}$
   Domain: $(0,\infty) - \{1\} = (0, 1) \cup (1, \infty)$

4. Domain: R - [0, 1)

5. Domain: $R - Z$

Self-Assessment

1. $D(f) = D(20) \cap D(\sqrt{x - |x|}) - \{x : \sqrt{x - |x|} = 0\}$
   $D(20) = R$
   $D(\sqrt{x - |x|})$
   $x - |x| \geq 0$
   $|x| \leq x$
   $x \in [0, \infty)$

   When denominator becomes 0
   $x - |x| = 0$
   $|x| = x$
   $x \in [0, \infty)$
   $D(f) = R \cap [0, \infty) - [0, \infty)$
   $D(f) = \varphi$

2. Domain(f) = Domain(Numerator) ∩ Domain(Denominator) - {x where Denominator = 0}

Domain of Numerator: R

Domain of Denominator

$[x]^2 - [x] - 6 \geq 0$

$([x] - 3)([x] + 2) \geq 0$

$[x] \in (-\infty, -2] \cup [3, \infty)$

$[x] \in \{...,-5,-4,-3, -2, 3, 4 ,5 ,6 ,....\}$

$x \in (-\infty, -1) \cup [3, \infty)$

Where denominator becomes 0

$[x] \in \{-2, 3\}$

$x \in [-2, -1) \cup [3, 4)$

Domain(f)= $[R \cap (-\infty, -1) \cup [3, \infty)] - [-2, -1) \cup [3, 4)$

$= (-\infty, -2) \cup [4, \infty)$

3. Domain (f) = Domain(Numerator) ∩ Domain(Denominator) - {x where Denominator = 0}

Domain of Numerator $2\log(x^3 - x)$:

$x^3 - x > 0$

$x(x^2 - 1) > 0$

$x(x - 1)(x + 1) > 0$

$x \in (-1, 0) \cup (1, \infty)$

Domain of Denominator $\sqrt{9 - x^2}$:

$9 - x^2 \geq 0$

$(x - 3)(x + 3) \leq 0$

$x \in [-3, 3]$

Where denominator becomes 0

$9 - x^2 = 0$

$x = \pm 3$

Domain(f) = $(-1, 0) \cup (1, \infty) \cap [-3, 3] - \{-3, 3\}$

$= (-1, 0) \cup (1, 3)$

4. $4^y = 4 - 4^x$

$y = \log_4(4 - 4^x)$

$4 - 4^x > 0$

$4^x < 4$

$x \in (- \infty, 1)$

RELATIONS AND FUNCTIONS

RANGE OF FUNCTIONS

What you already know

- Some standard functions
- Domain of a function

What you will learn

- Range of functions (in detail)

Range of a Function

Let f be a function from A to B.

$f : A \to B$

The set of all the images of elements of A under f in B is known as the range of f.

$R(f) = \{ f(x) : x \in A \}$

Let f: {1, 2, 3} → R ; f(x) = x + 3.

$f(1) = 4$
$f(2) = 5$
$f(3) = 6$
Range: {4, 5, 6}.

Let W → R ; f(x) = 2x + 1

$f(0) = 2(0) + 1 = 1$
$f(1) = 2(1) + 1 = 3$
$f(2) = 2(2) + 1 = 5$
$f(3) = 2(3) + 1 = 7$
Range: Set of all odd natural numbers.

If f: N → R ; f(x) = x^2, find its range.

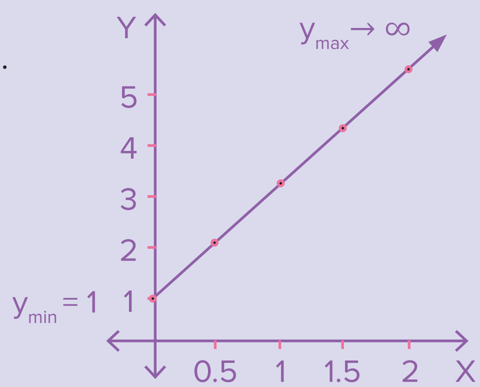
$f(1) = 1$ $f(2) = 4$ $f(3) = 9$
Range: All natural numbers that are perfect squares.

Quick Query

1. If f: Z → R ; $f(x) = x^2$, then find its range.

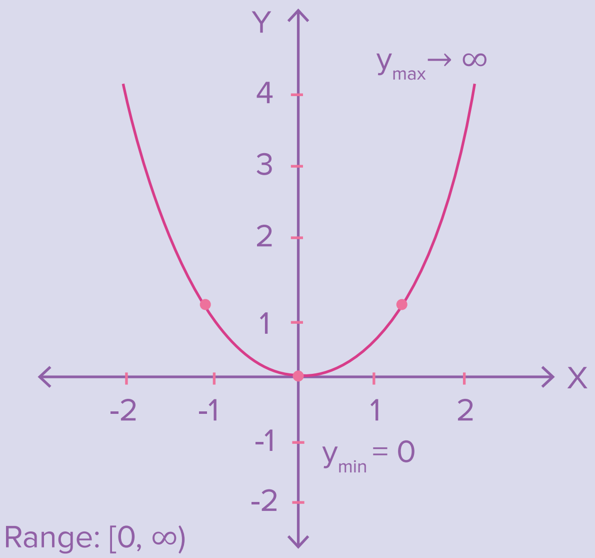
Finding Range Using Graphs

(1) If $f : [0, \infty) \to R ; f(x) = 2x + 1$, find its range.

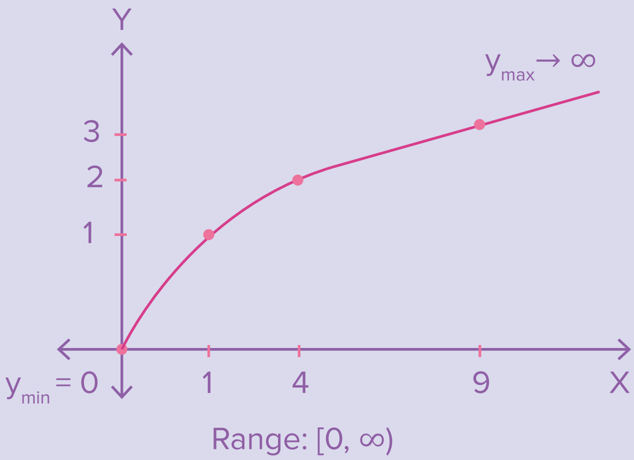


Range: $[1, \infty)$

(2) If $f: R \to R; f(x) = x^2$, find its range.



(3) If $f: [0, \infty) \to R; f(x) = \sqrt{x}$, then find its range.



Concept Check

(1) If $f: R \to R; f(x) = x^2 + 1$, find its range.
(2) If $f: R \to R; f(x) = x^2 + 2$, find its range.

Quick Query

Q.2. If $f: [0, \infty) \to R; f(x) = x^2$, find its range.

Methods to Find Range of a Function

I. By converting function of x into function of y

Convert $y = f(x)$ into $x = g(y)$

↓

Find domain of g

↓

Domain of g will be same as range of f

f: R - {-2} → R ; $f(x) = \frac{x-1}{x+2}$, then find its range.

Solution:

$y = \frac{x - 1}{x + 2}$

$xy + 2y = x - 1$

$x(1 - y) = 1 + 2y$

$x = \frac{1 + 2y}{1 - y}$

$x = g(y)$

Domain(g) = R - {1}

So, Range(f) = R - {1}

II. Finding range of quadratic functions

For $f(x) = ax^2 + bx + c$; $a \neq 0$

$= a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$

$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$

$= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$

For $a > 0$;

Range: $\left[\frac{4ac - b^2}{4a}, \infty\right)$

For $a < 0$;

Range: $\left(-\infty, \frac{4ac - b^2}{4a}\right]$

If $f: R \to R$; $f(x) = x^2 - 7x + 5$, then find its range.

$f(x) = x^2 - 7x + 5$

$= x^2 - 7x + \left(\frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 5$

$= \left(x + \frac{-7}{2}\right)^2 - \frac{49}{4} + 5$

$= \left(x - \frac{7}{2}\right)^2 + \frac{20 - 49}{4}$

$= \left(x - \frac{7}{2}\right)^2 - \frac{29}{4}$

Range: $\left[\frac{-29}{4}, \infty\right)$

III. Finding range of the functions of the form:

$\frac{\text{Linear}}{\text{Quadratic}}$ or $\frac{\text{Quadratic}}{\text{Quadratic}}$

Step 1: Let $y = \frac{\text{Linear}}{\text{Quadratic}}$ or $\frac{\text{Quadratic}}{\text{Quadratic}}$, then cross multiply and make quadratic in x.

Step 2. D ≥ 0

Note: $\frac{\text{Linear}}{\text{Quadratic}}$ or $\frac{\text{Quadratic}}{\text{Quadratic}}$ cancellation of common factor is not allowed

If f: $R \to R$; $f(x) = x^2 - 7x + 5$, then find its range.

Solution:

$y = x^2 - 7x + 5$

$x^2 - 7x + 5 - y = 0$

$x \in R \Rightarrow D \geq 0$

$\Rightarrow 49 - 4(1)(5 - y) \geq 0$

$\Rightarrow 4(5 - y) \leq 49$

$\Rightarrow 20 - 4y \leq 49$

$\Rightarrow 4y \geq -29$

$\Rightarrow y \geq \frac{-29}{4}$

Range: $\left[\frac{-29}{4}, \infty\right)$

If f: $R \to R$; $f(x) = \frac{x^2}{x^2 + 1}$, then find its range.

Solution:

$y = \frac{x^2}{x^2 + 1}$

$yx^2 + y = x^2$

$x^2(1 - y) = y$

$x^2 = \frac{y}{(1 - y)}$

$\frac{y}{1 - y} \geq 0$ ($\because x^2 \geq 0$)

$y \in [0, 1)$

Range: [0, 1)

Concept Check 3

3. If $f : R - \{2\} \rightarrow R$; $f(x) = \dfrac{x^2 - 4}{x - 2}$, then find $R(f)$.

IV. Finding range of modulus function

To find the range of modulus, divide the function into regions, and plot the function region wise.

Example:

Find the range of $f(x) = |x - 3| - |x - 9|$.

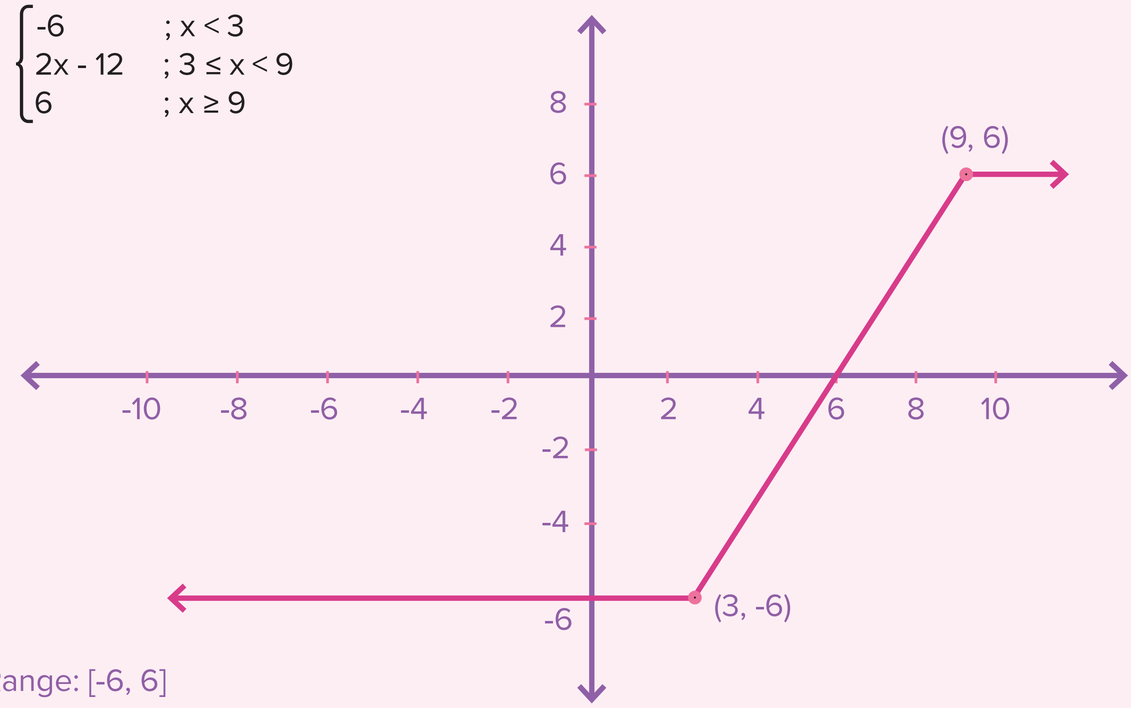
$|x - 3| = \begin{cases} x - 3; & x \geq 3 \\ -(x - 3); & x < 3 \end{cases}$ $|x - 9| = \begin{cases} x - 9; & x \geq 9 \\ -(x - 9); & x < 9 \end{cases}$

I. $x < 3$
$|x - 3| = 3 - x$
$|x - 9| = 9 - x$
$f(x) = -6$

II. $3 \leq x < 9$
$|x - 3| = x - 3$
$|x - 9| = 9 - x$
$f(x) = 2x - 12$

III. $9 \leq x$
$|x - 3| = x - 3$
$|x - 9| = x - 9$
$f(x) = 6$

$$f(x) = \begin{cases} -6 & ; x < 3 \\ 2x - 12 & ; 3 \leq x < 9 \\ 6 & ; x \geq 9 \end{cases}$$



Range: [-6, 6]

Summary Sheet

Key Takeaways

1. For finding range, converting function of x into function of y

Convert $y = f(x)$ into $x = g(y)$

↓

Find domain of g

↓

Domain of g will be same as range of f

2. Finding range of quadratic functions

$f(x) = ax^2 + bx + c; a \neq 0$

For a > 0;

Range: $\left[\frac{4ac - b^2}{4a}, \infty\right)$

For a < 0;

Range: $\left(-\infty, \frac{4ac - b^2}{4a}\right]$

3. Finding range of the functions of the form

$\frac{\text{Linear}}{\text{Quadratic}}$ or $\frac{\text{Quadratic}}{\text{Quadratic}}$

Cross-multiply and make quadratic in x

$D \geq 0$

5. Finding range of modulus function

To find the range of modulus, divide the function into regions, and plot the function region wise.

Key Terms

Let f be a function from A to B.

$f : A \rightarrow B$

The set of all the images of elements of A under f is known as the **range of** f.

Mind Map

Range of a function

- Plotting graphs
- Using formula for quadratic functions
- By converting function of x into function of y
- Modulus: Dividing function into regions
- Using D ≥ 0 for Linear/Quadratic or Quadratic/Quadratic form

Self-Assessment

1. Find the range of $f(x) = \frac{x}{x^2 - 16}$.

2. If $f: R \to R$; $f(x) = \frac{e^x - 1}{1 + e^x}$, then find $R(f)$.

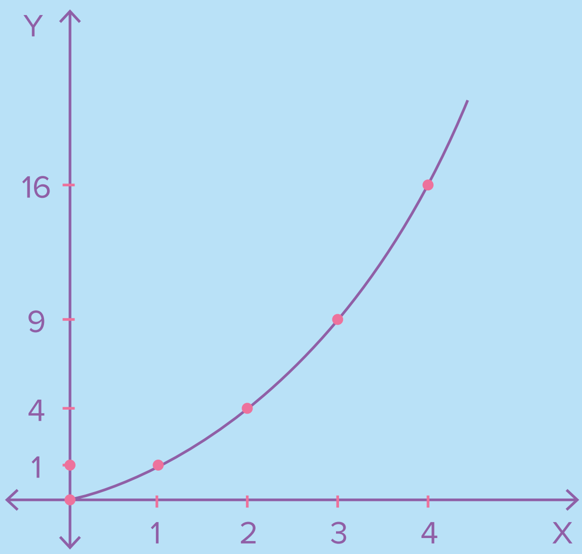
3. Find the range of $|x - 1| + |x + 2| + |x - 3|$

Answers

Quick Query

1. f(-2) = 4
   f(-1) = 1
   f(0) = 0
   f(1) = 1
   f(2) = 4
   Range: All whole numbers that are perfect squares

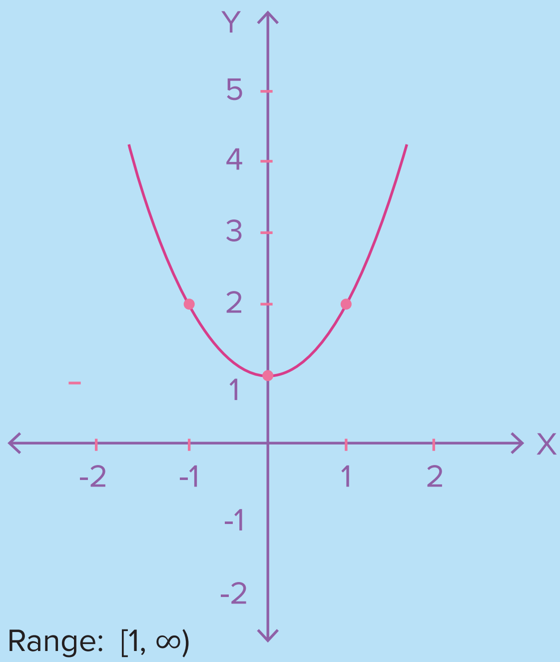
2. f: [0, ∞) → R, $f(x) = x^2$



Range: [0, ∞)

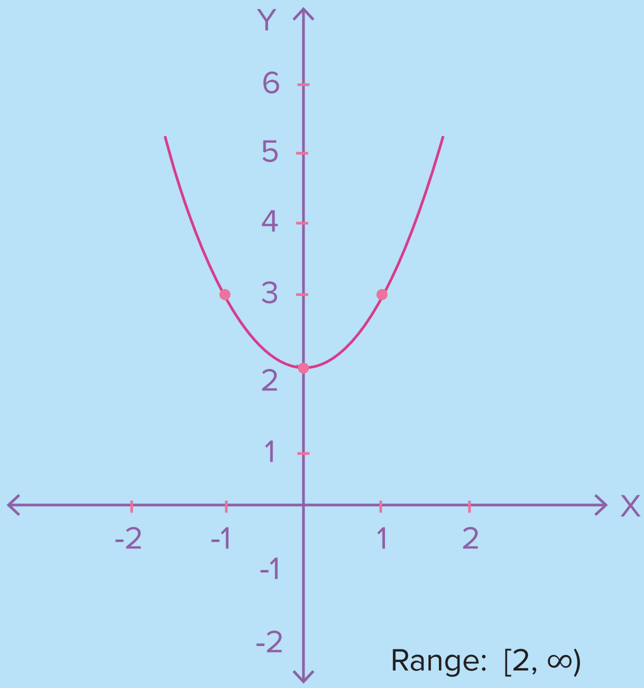
Concept Check

1. $f(x) = x^2 + 1$



Range: $[1, \infty)$

2. $f(x) = x^2 + 2$



Range: $[2, \infty)$

3. $y = \dfrac{x^2 - 4}{x - 2}$; $x \in R - \{2\}$

$y = \dfrac{(x - 2)(x + 2)}{x - 2}$

$y = x + 2$

Because $x \neq 2$, y can take any value except 4.

$R(f) = R - \{4\}$

Self-Assessment

1. Let $y = \dfrac{x}{x^2 - 16}$

$\Rightarrow y(x^2 - 16) = x$

$yx^2 - x - 16y = 0$

This is a quadratic equation in x, and in order to have solutions, the discriminant $D \geq 0$

$a = y,\ b = -1,\ c = -16y$

$\Delta = b^2 - 4ac = (-1)^2 - 4(y)(-16y) = 1 + 64y^2 \geq 0\ \forall\ y \in R$

$\Rightarrow \Delta \geq 0$ Therefore, the range is $f(x) \in R$

2. $y = \dfrac{e^x - 1}{1 + e^x}$

$y + y\,e^x = e^x - 1$

$e^x(1 - y) = 1 + y$

$e^x = \dfrac{1 + y}{1 - y}$

$x = \log_e\left(\dfrac{1 + y}{1 - y}\right) = g(y)$

$g(y) \in R$ iff $\dfrac{1 + y}{1 - y} > 0$

$R(f) = (-1, 1)$

3. Let $f(x) = |x - 1| + |x + 2| + |x - 3|$

Divide the domain of $f(x)$ in 4 parts,

$x < -2$

$-2 \le x < 1$

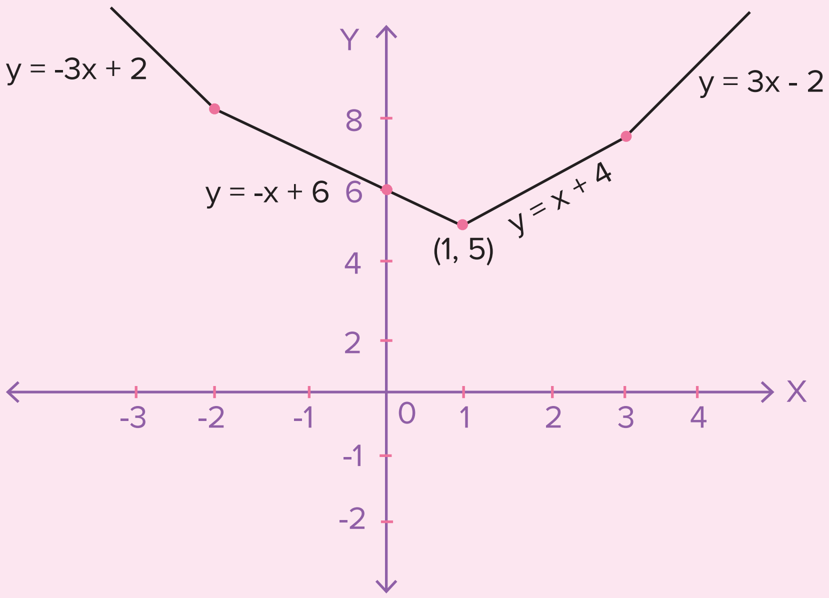
$1 \le x < 3$

$x \ge 3$

Now, for each parts of the domain $f(x)$ will be

$$f(x) = \begin{cases} -(x-1)-(x+2)-(x-3), & x < -2 \\ -(x-1)+(x+2)-(x-3), & -2 \le x < 1 \\ (x-1)+(x+2)-(x-3), & 1 \le x < 3 \\ (x-1)+(x+2)+(x-3), & x \ge 3 \end{cases}$$

$$f(x) = \begin{cases} -3x+2, & x < -2 \\ -x+6, & -2 \le x < 1 \\ x+4, & 1 \le x < 3 \\ 3x-2, & x \ge 3 \end{cases}$$



From the graph it is evident that the minimum value of $f(x)$ is 5 and maximum tends to ∞.
Therefore, the range of $f(x)$ is $[5, \infty)$.