Check your Intelligence — Chapter 43

CHECK YOUR INTELLIGENCE IN MATHEMATICS

1. Here I have proved the wrong statement that

$$\lim_{x \to 0} \cos\frac{1}{x} = 0$$

find the mistake in the following proof.

Proof : Let $f(x) = \begin{cases} x^2 \sin 1/x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

Now let us apply Lagrange's theorem on this function in the interval $[0, x]$

Clearly is differentiable for any x by Lagrange's theorem

$x^2 \sin\frac{1}{x} = 2\xi \sin\frac{1}{\xi} - \cos\frac{1}{\xi}$

Hence $\cos\frac{1}{\xi} = 2\xi \sin\frac{1}{\xi} - x^2 \sin\frac{1}{x}$ where

As x tends to zero, ξ $(\because \xi \in (0, x))$ will also tend to zero therefore passing to the limit, we obtain $\lim_{\xi \to 0} \cos\frac{1}{\xi} = 0$.

2. Here I have proved the wrong statement that $\pi = 2\sqrt{2}$. Find the mistake in the following proof.

Proof : Consider the integral $I = \int_0^\pi xf(\sin x)\,dx$, where $f(\sin x)$ is any function of sin x. In accordance with a standard treatment, make the substitution $x = \pi - x'$, and then drop dashes.

Thus $\quad I = \int_\pi^0 (\pi - x) f\{\sin(\pi - x)\}\,d(-x) = \int_0^\pi (\pi - x) f(\sin x)\,dx.$

Hence $\quad 2\int_0^\pi xf(\sin x)\,dx = \pi \int_0^\pi f(\sin x)\,dx.$

Take, in particular,

$$f(u) = u \sin^{-1}(u), \text{ so that } f(\sin x) = \sin x.x = x \sin x.$$

Then the relation is

$$2\int_0^\pi x^2 \sin x dx = \pi \int_0^\pi x \sin x dx.$$

But $\quad \int_0^\pi x^2 \sin x dx = \pi^2 - 4, \int_0^\pi x \sin x dx = \pi.$

Hence $\quad 2(\pi^2 - 4) = \pi^2, \text{or } \pi^2 = 8.$

so that $\quad \pi = 2\sqrt{2}.$

3. Here I have proved the wrong statement that every angle is multiple of two right angles. Find the mistake in following proof.

Proof : Let θ be an angle (complex) satisfying the relation $\tan\theta = i$.

Then, if A is any angle,

$$\tan(A+\theta)=\frac{\tan A+\tan\theta}{1-\tan A\tan\theta}=\frac{\tan A+i}{1-i\tan A}=i=\tan\theta.$$

Thus, $\tan(A+\theta)=\tan\theta,$ so that $A+\theta=n\pi+\theta,$ or $A=n\pi$ for any angle A.

4. Here I have proved the wrong statement that $0=1$. Find the mistake in the following proof.

Proof : Consider the integral $I=\int\frac{dx}{x}$. Integrate by parts :

$$I=\int 1.(1/x)dx=x(1/x)-\int x(-1/x^2)dx=1+\int\frac{dx}{x}=1+I \qquad \text{Hence } 0=1.$$

5. Here I have proved the wrong statement that $2=1$. Find the mistake in the following proof.

Proof : Let $f(x)$ be any given function.

Then $\int_1^2 f(x)dx=\int_0^2 f(x)dx-\int_0^1 f(x)dx.$

If we write $x=2y$ in the first integral on the right, then

$$\int_0^2 f(x)dx=2\int_0^1 f(2y)dy=2\int_0^1 f(2x)dx,$$

on renaming the variable. Suppose, in particular, that the function $f(x)$ is such that

$$f(2x)=\frac{1}{2}f(x)$$

for all values of x. Then

$$\int_1^2 f(x)dx=2\int_0^1\frac{1}{2}f(x)dx-\int_0^1 f(x)dx=0.$$

Now the relation $f(2x)=\frac{1}{2}f(x)$ is satisfied by the function $f(x)=\frac{1}{x}$

Hence $\int_1^2\frac{dx}{x}=0,$ so that $\log 2=0$ or $2=1.$

6. Here I have proved the wrong statement that, if $f(\theta)$ is any function of θ, then $\int_0^\pi f(\theta)\cos\theta d\theta=0.$

Find the mistake in the following proof.

Proof : Substitute $\sin\theta=t$ so that $\cos\theta d\theta=dt,$ and write $f\{\sin^{-1}t\}=g(t)$

The limits of integration are 0, 0 since $\sin 0=0$ and $\sin\pi=0.$ Hence the integral is $\int_0^0 g(t)dt=0.$

Corollary : The special case when $f(\theta)=\cos\theta$ is of interest. Then the integral is

$$\int_0^\pi\cos^2\theta d\theta=\frac{1}{2}\int_0^\pi(1+\cos 2\theta)d\theta=\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^\pi=\frac{1}{2}\pi, \text{ hence } \frac{1}{2}\pi=0.$$

7. Here I have proved the wrong statement that, $1=0.$ Find the mistake in the following proof.

Proof : $S=1-1+1-1+1-1+\ldots$

Then, grouping in pairs,

$$S=(1-1)+(1-1)+(1-1)+\ldots=0+0+0+\ldots=0.$$

Also, grouping alternatively in pairs,

$$S=1-(1-1)-(1-1)-(1-1)-\ldots=1-0-0-0\ldots=1.$$

Hence $1 = 0.$

8. Here I have proved the wrong statement that, -1 is positive. Find the mistake in the following proof.

Proof : Let $S = 1 + 2 + 4 + 8 + 16 + 32 + ...$

Then S is positive. Also, multiplying each side by 2,

$$2S = 2 + 4 + 8 + 16 + 32 + ... = S - 1$$

Hence $S = -1,$ so that -1 is positive.

9. Here I have proved the wrong statement that, 0 is positive (*i.e.*, greater than zero).
Find the mistake in the following proof.

Proof : Write $u = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + ...$ and $v = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + ...$

Then $2v = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... = u + v$ so that $u - v = 0.$

But, on subtracting corresponding terms,

$$u - v = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + ...$$

where each bracketed terms is greater than zero.

Thus $u - v$ is greater that zero or 0 is greater that zero.

10. Here I have proved the wrong statement that $-2\pi = 0.$ Find the mistake in the following proof.

Proof : As we know that $e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$ it follows that for any x

$$e^{ix} = e^{ix}.e^{2xi} = e^{i(x+2x)} \Rightarrow \left(e^{ix}\right)^{i} = \left(e^{i(x+2\pi)}\right)^{i} \Rightarrow e^{-x} = e^{-(x+2x)}$$

$$\Rightarrow e^{-x} = e^{-x}.e^{-2x} \Rightarrow e^{-2x} = 1 \Rightarrow -2\pi = 0$$