CHAPTER VI.

HARMONICAL PROGRESSION. THEOREMS CONNECTED WITH THE PROGRESSIONS.

61. Definition. Three quantities a, b, c are said to be in **Harmonical Progression** when $\frac{a}{c} = \frac{a-b}{b-c}$.

Any number of quantities are said to be in Harmonical Progression when every three consecutive terms are in Harmonical Progression.

62. *The reciprocals of quantities in Harmonical Progression are in Arithmetical Progression.*

By definition, if a, b, c are in Harmonical Progression,

$$\frac{a}{c} = \frac{a-b}{b-c};$$

$$\therefore a(b-c) = c(a-b),$$

dividing every term by abc,

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a},$$

which proves the proposition.

63. Harmonical properties are chiefly interesting because of their importance in Geometry and in the Theory of Sound: in Algebra the proposition just proved is the only one of any importance. There is no general formula for the sum of any number of quantities in Harmonical Progression. Questions in H. P. are generally solved by inverting the terms, and making use of the properties of the corresponding A. P.

64. *To find the harmonic mean between two given quantities.*

Let a, b be the two quantities, H their harmonic mean; then $\frac{1}{a}$, $\frac{1}{H}$, $\frac{1}{b}$ are in A. P.;

$$\therefore \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H},$$

$$\frac{2}{H} = \frac{1}{a} + \frac{1}{b},$$

$$H = \frac{2ab}{a+b}.$$

Example. Insert 40 harmonic means between 7 and $\frac{1}{6}$.

Here 6 is the 42^{nd} term of an A. P. whose first term is $\frac{1}{7}$; let d be the common difference; then

$$6 = \frac{1}{7} + 41d; \text{ whence } d = \frac{1}{7}.$$

Thus the arithmetic means are $\frac{2}{7}, \frac{3}{7}, \ldots\ldots\ldots \frac{41}{7}$; and therefore the harmonic means are $3\frac{1}{2}, 2\frac{1}{3}, \ldots \frac{7}{41}$.

65. If A, G, H be the arithmetic, geometric, and harmonic means between a and b, we have proved

$$A = \frac{a+b}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$G = \sqrt{ab} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$$H = \frac{2ab}{a+b} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Therefore $$AH = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = G^2;$$

that is, G is the geometric mean between A and H.

From these results we see that

$$A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$$

$$= \left(\frac{\sqrt{a} - \sqrt{b}}{\sqrt{2}}\right)^2;$$

which is positive if a and b are positive; therefore *the arithmetic mean of any two positive quantities is greater than their geometric mean.*

Also from the equation $G^2 = AH$, we see that G is intermediate in value between A and H; and it has been proved that $A > G$, therefore $G > H$; that is, *the arithmetic, geometric, and harmonic means between any two positive quantities are in descending order of magnitude.*

66. Miscellaneous questions in the Progressions afford scope for skill and ingenuity, the solution being often neatly effected by some special artifice. The student will find the following hints useful.

1. If the same quantity be added to, or subtracted from, all the terms of an A P., the resulting terms will form an A.P. with the same common difference as before. [Art. 38.]

2. If all the terms of an A.P. be multiplied or divided by the same quantity, the resulting terms will form an A.P., but with a new common difference. [Art. 38.]

3. If all the terms of a G.P. be multiplied or divided by the same quantity, the resulting terms will form a G.P. with the same common ratio as before. [Art. 51.]

4. If a, b, c, d... are in G.P., they are also in *continued proportion*, since, by definition,

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \ldots\ldots = \frac{1}{r}.$$

Conversely, a series of quantities in continued proportion may be represented by x, xr, xr^2,.......

Example 1. If a^2, b^2, c^2 are in A. P., shew that $b+c$, $c+a$, $a+b$ are in H. P.

By adding $ab+ac+bc$ to each term, we see that

$$a^2+ab+ac+bc, \quad b^2+ba+bc+ac, \quad c^2+ca+cb+ab \text{ are in A.P.};$$

that is $(a+b)(a+c)$, $(b+c)(b+a)$, $(c+a)(c+b)$ are in A. P.

∴, dividing each term by $(a+b)(b+c)(c+a)$,

$$\frac{1}{b+c}, \quad \frac{1}{c+a}, \quad \frac{1}{a+b} \text{ are in A. P.};$$

that is, $b+c$, $c+a$, $a+b$ are in H. P.

Example 2. If l the last term, d the common difference, and s the sum of n terms of an A. P. be connected by the equation $8ds=(d+2l)^2$, prove that

$$d=2a.$$

Since the given relation is true for any number of terms, put $n=1$; then

$$a=l=s.$$

Hence by substitution, $\quad 8ad=(d+2a)^2,$

or $\quad (d-2a)^2=0;$

$$\therefore\ d=2a.$$

Example 3. If the $p^{\text{th}}, q^{\text{th}}, r^{\text{th}}, s^{\text{th}}$ terms of an A.P. are in G. P., shew that $p-q,\ q-r,\ r-s$ are in G.P.

With the usual notation we have

$$\frac{a+(p-1)\,d}{a+(q-1)\,d}=\frac{a+(q-1)\,d}{a+(r-1)\,d}=\frac{a+(r-1)\,d}{a+(s-1)\,d} \qquad \text{[Art. 66. (4)];}$$

$\therefore$ each of these ratios

$$=\frac{\{a+(p-1)\,d\}-\{a+(q-1)\,d\}}{\{a+(q-1)\,d\}-\{a+(r-1)\,d\}}=\frac{\{a+(q-1)\,d\}-\{a+(r-1)\,d\}}{\{a+(r-1)\,d\}-\{a+(s-1)\,d\}}$$

$$=\frac{p-q}{q-r}=\frac{q-r}{r-s}.$$

Hence $p-q,\ q-r,\ r-s$ are in G.P.

67. The numbers 1, 2, 3,...... are often referred to as the *natural numbers*; the n^{th} term of the series is n, and the sum of the first n terms is $\frac{n}{2}(n+1)$.

68. *To find the sum of the squares of the first* n *natural numbers.*

Let the sum be denoted by S; then

$$S=1^2+2^2+3^2+\ldots\ldots+n^2.$$

We have $\quad n^3-(n-1)^3=3n^2-3n+1;$

and by changing n into $n-1$,

$$(n-1)^3-(n-2)^3=3(n-1)^2-3(n-1)+1;$$

similarly $\quad (n-2)^3-(n-3)^3=3(n-2)^2-3(n-2)+1;$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$3^3-2^3=3\,.\,3^2-3\,.\,3+1;$$

$$2^3-1^3=3\,.\,2^2-3\,.\,2+1;$$

$$1^3-0^3=3\,.\,1^2-3\,.\,1+1.$$

Hence, by addition,

$$n^3 = 3(1^2 + 2^2 + 3^2 + \dots + n^2) - 3(1 + 2 + 3 + \dots + n) + n$$

$$= 3S - \frac{3n(n+1)}{2} + n.$$

$$\therefore\ 3S = n^3 - n + \frac{3n(n+1)}{2}$$

$$= n(n+1)(n - 1 + \tfrac{3}{2});$$

$$\therefore\ S = \frac{n(n+1)(2n+1)}{6}.$$

69. *To find the sum of the cubes of the first* n *natural numbers.*

Let the sum be denoted by S; then

$$S = 1^3 + 2^3 + 3^3 + \dots\dots + n^3.$$

We have $$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1;$$

$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1;$$

$$(n-2)^4 - (n-3)^4 = 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1;$$

....

$$3^4 - 2^4 = 4 . 3^3 - 6 . 3^2 + 4 . 3 - 1;$$

$$2^4 - 1^4 = 4 . 2^3 - 6 . 2^2 + 4 . 2 - 1;$$

$$1^4 - 0^4 = 4 . 1^3 - 6 . 1^2 + 4 . 1 - 1.$$

Hence, by addition,

$$n^4 = 4S - 6(1^2 + 2^2 + \dots + n^2) + 4(1 + 2 + \dots + n) - n;$$

$$\therefore\ 4S = n^4 + n + 6(1^2 + 2^2 + \dots + n^2) - 4(1 + 2 + \dots + n)$$

$$= n^4 + n + n(n+1)(2n+1) - 2n(n+1)$$

$$= n(n+1)(n^2 - n + 1 + 2n + 1 - 2)$$

$$= n(n+1)(n^2 + n);$$

$$\therefore\ S = \frac{n^2(n+1)^2}{4} = \left\{\frac{n(n+1)}{2}\right\}^2.$$

Thus *the sum of the cubes of the first* n *natural numbers is equal to the square of the sum of these numbers.*

The formulæ of this and the two preceding articles may be applied to find the sum of the squares, and the sum of the cubes of the terms of the series

$$a,\ a + d,\ a + 2d, \dots\dots\dots$$

70. In referring to the results we have just proved it will be convenient to introduce a notation which the student will frequently meet with in Higher Mathematics. We shall denote the series

$$1 + 2 + 3 + \ldots + n \text{ by } \Sigma n;$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 \text{ by } \Sigma n^2;$$

$$1^3 + 2^3 + 3^3 + \ldots + n^3 \text{ by } \Sigma n^3;$$

where Σ placed before a term signifies the sum of all terms of which that term is the general type.

Example 1. Sum the series

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots \text{to } n \text{ terms.}$$

The n^{th} term $= n(n+1) = n^2 + n$; and by writing down each term in a similar form we shall have two columns, one consisting of the first n natural numbers, and the other of their squares.

$$\therefore \text{ the sum} = \Sigma n^2 + \Sigma n$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{2n+1}{3} + 1\right\}$$

$$= \frac{n(n+1)(n+2)}{3}.$$

Example 2. Sum to n terms the series whose n^{th} term is $2^{n-1} + 8n^3 - 6n^2$.

Let the sum be denoted by S; then

$$S = \Sigma\, 2^{n-1} + 8\Sigma n^3 - 6\Sigma n^2$$

$$= \frac{2^n - 1}{2-1} + \frac{8n^2(n+1)^2}{4} - \frac{6n(n+1)(2n+1)}{6}$$

$$= 2^n - 1 + n(n+1)\{2n(n+1) - (2n+1)\}$$

$$= 2^n - 1 + n(n+1)(2n^2 - 1).$$

EXAMPLES. VI. a.

1. Find the fourth term in each of the following series:

   (1) $2,\ 2\frac{1}{2},\ 3\frac{1}{3}, \ldots$

   (2) $2,\ 2\frac{1}{2},\ 3, \ldots$

   (3) $2,\ 2\frac{1}{2},\ 3\frac{1}{8}, \ldots$

2. Insert two harmonic means between 5 and 11.

3. Insert four harmonic means between $\frac{2}{3}$ and $\frac{2}{13}$.

4. If 12 and $9\frac{3}{5}$ are the geometric and harmonic means, respectively, between two numbers, find them.

5. If the harmonic mean between two quantities is to their geometric means as 12 to 13, prove that the quantities are in the ratio of 4 to 9.

6. If a, b, c be in H. P., shew that

$$a : a-b = a+c : a-c.$$

7. If the m^{th} term of a H. P. be equal to n, and the n^{th} term be equal to m, prove that the $(m+n)^{\text{th}}$ term is equal to $\frac{mn}{m+n}$.

8. If the p^{th}, q^{th}, r^{th} terms of a H. P. be a, b, c respectively, prove that

$$(q-r)\,bc+(r-p)\,ca+(p-q)\,ab=0.$$

9. If b is the harmonic mean between a and c, prove that

$$\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}.$$

Find the sum of n terms of the series whose n^{th} term is

10. $3n^2-n$. 11. $n^3+\frac{3}{2}n$. 12. $n\,(n+2)$.

13. $n^2\,(2n+3)$. 14. 3^n-2^n. 15. $3\,(4^n+2n^2)-4n^3$.

16. If the $(m+1)^{\text{th}}$, $(n+1)^{\text{th}}$, and $(r+1)^{\text{th}}$ terms of an A. P. are in G. P., and m, n, r are in H. P., shew that the ratio of the common difference to the first term in the A. P. is $-\frac{2}{n}$.

17. If l, m, n are three numbers in G. P., prove that the first term of an A. P. whose l^{th}, m^{th}, and n^{th} terms are in H. P. is to the common difference as $m+1$ to 1.

18. If the sum of n terms of a series be $a+bn+cn^2$, find the n^{th} term and the nature of the series.

19. Find the sum of n terms of the series whose n^{th} term is

$$4n\,(n^2+1)-(6n^2+1).$$

20. If between any two quantities there be inserted two arithmetic means A_1, A_2; two geometric means G_1, G_2; and two harmonic means H_1, H_2; shew that $G_1G_2 : H_1H_2 = A_1+A_2 : H_1+H_2$.

21. If p be the first of n arithmetic means between two numbers, and q the first of n harmonic means between the same two numbers, prove that the value of q cannot lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$.

22. Find the sum of the cubes of the terms of an A. P., and shew that it is exactly divisible by the sum of the terms.

Piles of Shot and Shells.

71. *To find the number of shot arranged in a complete pyramid on a square base.*

Suppose that each side of the base contains n shot; then the number of shot in the lowest layer is n^2; in the next it is $(n-1)^2$; in the next $(n-2)^2$; and so on, up to a single shot at the top.

$$\therefore\ S = n^2 + (n-1)^2 + (n-2)^2 + \ldots + 1$$

$$= \frac{n(n+1)(2n+1)}{6} \qquad \text{[Art. 68.]}$$

72. *To find the number of shot arranged in a complete pyramid the base of which is an equilateral triangle.*

Suppose that each side of the base contains n shot; then the number of shot in the lowest layer is

$$n + (n-1) + (n-2) + \ldots\ldots + 1;$$

that is,

$$\frac{n(n+1)}{2} \text{ or } \frac{1}{2}(n^2+n).$$

In this result write $n-1$, $n-2$,...... for n, and we thus obtain the number of shot in the 2nd, 3rd,...... layers.

$$\therefore\ S = \tfrac{1}{2}(\Sigma n^2 + \Sigma n)$$

$$= \frac{n(n+1)(n+2)}{6} \qquad \text{[Art. 70.]}$$

73. *To find the number of shot arranged in a complete pyramid the base of which is a rectangle.*

Let m and n be the number of shot in the long and short side respectively of the base.

The top layer consists of a single row of $m-(n-1)$, or $m-n+1$ shot;

in the next layer the number is $2(m-n+2)$;

in the next layer the number is $3(m-n+3)$;

and so on;

in the lowest layer the number is $n(m-n+n)$.

$$\therefore\ S = (m-n+1) + 2(m-n+2) + 3(m-n+3) + \ldots + n(m-n+n)$$
$$= (m-n)(1+2+3+\ldots+n) + (1^2+2^2+3^2+\ldots+n^2)$$
$$= \frac{(m-n)\,n\,(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{n(n+1)}{6}\{3(m-n)+2n+1\}$$
$$= \frac{n(n+1)(3m-n+1)}{6}.$$

74. *To find the number of shot arranged in an incomplete pyramid the base of which is a rectangle.*

Let a and b denote the number of shot in the two sides of the top layer, n the number of layers.

In the top layer the number of shot is ab;

in the next layer the number is $(a+1)(b+1)$;

in the next layer the number is $(a+2)(b+2)$;

and so on;

in the lowest layer the number is $(a+\overline{n-1})(b+\overline{n-1})$

or $$ab+(a+b)(n-1)+(n-1)^2.$$

$$\therefore\ S = abn + (a+b)\Sigma(n-1) + \Sigma(n-1)^2$$
$$= abn + \frac{(n-1)\,n\,(a+b)}{2} + \frac{(n-1)\,n\,(2\,.\,\overline{n-1}+1)}{6}$$
$$= \frac{n}{6}\{6ab + 3(a+b)(n-1) + (n-1)(2n-1)\}.$$

75. In numerical examples it is generally easier to use the following method.

Example. Find the number of shot in an incomplete square pile of 16 courses, having 12 shot in each side of the top.

If we place on the given pile a square pile having 11 shot in each side of the base, we obtain a complete square pile of 27 courses;

and number of shot in the complete pile $= \frac{27 \times 28 \times 55}{6} = 6930$; [Art. 71.]

also number of shot in the added pile $= \frac{11 \times 12 \times 23}{6} = 506$;

$\therefore$ number of shot in the incomplete pile $= 6424$.

EXAMPLES. VI. b.

Find the number of shot in

1. A square pile, having 15 shot in each side of the base.

2. A triangular pile, having 18 shot in each side of the base.

3. A rectangular pile, the length and the breadth of the base containing 50 and 28 shot respectively.

4. An incomplete triangular pile, a side of the base having 25 shot, and a side of the top 14.

5. An incomplete square pile of 27 courses, having 40 shot in each side of the base.

6. The number of shot in a complete rectangular pile is 24395; if there are 34 shot in the breadth of the base, how many are there in its length?

7. The number of shot in the top layer of a square pile is 169, and in the lowest layer is 1089; how many shot does the pile contain?

8. Find the number of shot in a complete rectangular pile of 15 courses, having 20 shot in the longer side of its base.

9. Find the number of shot in an incomplete rectangular pile, the number of shot in the sides of its upper course being 11 and 18, and the number in the shorter side of its lowest course being 30.

10. What is the number of shot required to complete a rectangular pile having 15 and 6 shot in the longer and shorter side, respectively, of its upper course?

11. The number of shot in a triangular pile is greater by 150 than half the number of shot in a square pile, the number of layers in each being the same; find the number of shot in the lowest layer of the triangular pile.

12. Find the number of shot in an incomplete square pile of 16 courses when the number of shot in the upper course is 1005 less than in the lowest course.

13. Shew that the number of shot in a square pile is one-fourth the number of shot in a triangular pile of double the number of courses.

14. If the number of shot in a triangular pile is to the number of shot in a square pile of double the number of courses as 13 to 175; find the number of shot in each pile.

15. The value of a triangular pile of 16 lb. shot is £51; if the value of iron be 10*s.* 6*d.* per cwt., find the number of shot in the lowest layer.

16. If from a complete square pile of n courses a triangular pile of the same number of courses be formed; shew that the remaining shot will be just sufficient to form another triangular pile, and find the number of shot in its side.