Kinematics of a Particle

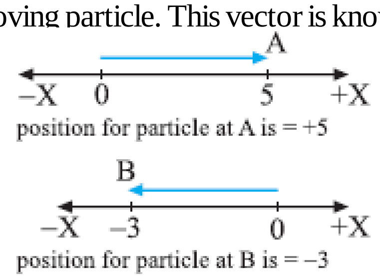
KINEMATICAL QUANTITIES — Section - 1

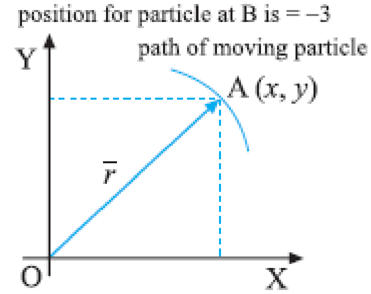
To analyse the motion of a particle, the role of four important quantities :

Position, displacement, velocity & acceleration must be clearly understood.

1. Position

The position of a particle refers to its location in the space at a certain moment of time. It is concerned with the question - " *where is the particle at a particular moment of time ?* " In general the position is measured by a vector joining a fixed point (known as **origin**) to the moving particle. This vector is known as **position vector**.





$\bar{r}$ = position vector at A

$\bar{r} = x\hat{i} + y\hat{j}$

> ***Note :*** **(i)** For a particle in straight line motion along X-axis, the position vector is always parallel to X-axis and hence has only X-component as non-zero. Therefore the position of a moving particle can be measured by the X-coordinate x (t) at a certain time instant t.
>
> **(ii)** If a particle is moving in a curve (i.e. in a plane) the position vector can have many possible directions. The position in such a case can be measured by two numbers : X-coordinate and Y-coordinate or simply the X & Y components of the position vector.

2. Displacement

It is the vector joining the initial position of the particle to its final position during an interval of time. The change in the position of a moving object is known as displacement.

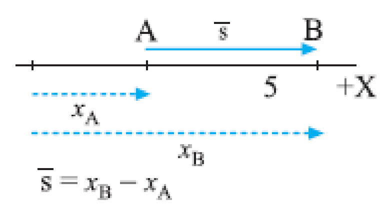
(a) For a straight line motion, if a particle goes from A to B then

s = *displacement* = AB, the displacement vector has only X-component non-zero.

Hence it is equal to the difference in the X-coordinates i.e.

$$s = x_B - x_A$$

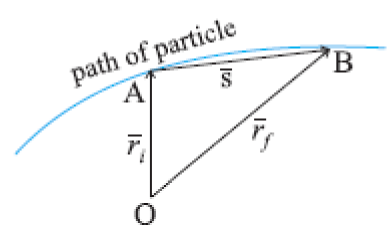
$$s = \Delta x = x_B - x_A$$



(b) If a particle goes from A to B along a curve in some time duration and if O is the origin then

$\overline{OA}$ = initial position vector = $\bar{r}_i$

$\overline{OB}$ = final position vector = $\bar{r}_f$



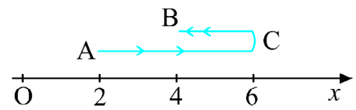
$$AB = \text{displacement vector} = \overline{OB} - \overline{OA}$$

$$\Rightarrow \qquad \bar{s} = \bar{r}_f - \bar{r}_i = \Delta \bar{r}$$

Distance :

Let a moving particle start at a point A ($x_i = 2\ m$) and later come to point B ($x_f = 4\ m$) after having turned around at C ($x = 6\ m$) as shown in the figure.

Its displacement $= \Delta x = x_f - x_i = 4 - 2 = 2\ m$.



What about the total distance travelled by the particle ?

The distance travelled i.e., *the length of the actual path* is a scalar quantity which is quite different from displacement. In the given example, distance travelled $= AC + CB = 4 + 2 = 6\ m$. In general, the distance travelled between two points may not be equal to the magnitude of the displacement between the same points. For a particle moving along a curve the distance travelled is simply the length of the curve along which it moves.

3. Velocity

The position of a moving particle is always changing with time. The rate at which the position changes with time is known as **velocity**. It is a vector quantity whose magnitude is simply the speed of the particle. The magnitude of velocity indicates how fast the particle is moving. The direction in which the particle is moving is the direction of the velocity.

(a) Average Velocity (in an interval) :

The average velocity of a moving particle over a certain time interval is defined as the displacement divided by the time duration.

$$\text{Averrage Velocity} = \frac{\text{displacement}}{\text{time interval}} = \frac{\Delta \bar{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta} j$$

For straight line motion along X-axis, we have :

$$V_{av} = \frac{\Delta x}{\Delta t}$$

The average velocity is a vector in the direction of displacement. It depends only on the net displacement and the time interval ; the details of the journey in between are of no consequence.

(b) Average Speed :

Average speed gives an idea about : "*how fast does a particle move ?*".

It is a scalar and is defined over a time interval as :

$$\text{Average Speed} = \frac{\text{distance}}{\text{time interval}}$$

In the figure discussed in the article on *distance travelled*, if the time taken from A to B is 4 sec, average velocity of the particle is

$$V_{av} = \frac{\Delta x}{\Delta t} = \frac{2}{4} = 0.5\ m/s$$

$$\text{While average speed} = \frac{\text{distance}}{\text{time interval}} = \frac{6}{4} = 1.5\ ms^{-1}$$

(c) Instantaneous Velocity (at an instant) :

The velocity at a particular moment of time is known as instantaneous velocity. The term "velocity" usually means instantaneous velocity.

4. Acceleration

It is the rate of change of velocity with time for a moving particle. A large magnitude of acceleration indicates that the velocity is changing very rapidly. We say that a body accelerates when its velocity changes in magnitude or in direction, or both.

(a) Average acceleration : (*in an interval*)

The *average acceleration* for a finite time interval is defined as :

$$\text{Average acceleration} = \frac{\text{change velocity}}{\text{time interval}}$$

Average acceleration is a ***vector quantity*** whose direction is same as that of the change in velocity.

$$a_{av} = \frac{\Delta \bar{v}}{\Delta t} = \frac{\bar{v}_f - \bar{v}_i}{\Delta t}$$

(b) Instantaneous Acceleration (*at an instant*)

The acceleration at a given instant of time is the instantaneous acceleration. We usually mean instantaneous acceleration when we say "acceleration".

- By Newton's IInd Law of Motion: $\bar{F} = m\bar{a}$
  Hence the acceleration is decided by the net force acting on a particle.
- If $\mathbf{F_{net} = 0}$, then **a = 0** and the motion is known as **uniform motion** because the velocity does not change. One can also say that the particle is moving with a constant velocity. The average velocity in an interval and the instantaneous velocity are equal in this motion.
- If the force is constant, acceleration is also constant and the motion is known as **uniformly accelerated motion.**
  Free fall (motion under gravity) is a common example of uniformly accelerated motion because the force (= mg) is constant. In this motion, average acceleration in any interval is same as the instanta neous acc.

Illustration - 1 *A bird flies north at 20 m/s for 15 s. It rests for 5 s and then flies south at 25 m/s for 10 s. For the whole trip, find*

(a) *the average speed ;* **(b)** *the average velocity ;* **(c)** *the average acceleration.*

SOLUTION :

Distance travelled towards north = AC = 20 m/s × 15 s = 300 m.

Distance travelled towards south = CB = 25 m/s × 10 s = 250 m.

$$\text{Average Speed} = \frac{\text{distance}}{\text{time}} = \frac{300+250}{15+5+10}\ m/s = 18.34\ m/s.$$

In a straight line motion, we can deal with vectors like numbers by assigning them +ve or –ve sign using a sign convention.

Let us assume north to be position direction.

$$\text{Average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{300-250}{15+5+10} = 1.67\ m/s$$

$$\text{Average Acceleration} = \overline{a_{av}} = \frac{\Delta\bar{v}}{\Delta t} = \frac{\bar{v}_f - \bar{v}_i}{\Delta t} = \frac{(-25)-(+20)}{30}\ m/s^2$$

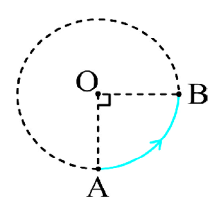
$$= -1.5\ m/s^2$$

Negative sign indicates that a_{av} is direction towards south.

Try Yourself :

At t = 0, a car is moving east at 10 m/s. Find its average acceleration between t = 0s and t = 5s if it is moving with velocity 5 m/s west at t = 5s. [Ans: 3 m/s^2 towards west)

Illustration - 2 *A particle goes along a quadrant AB of a circle of radius 5 cm with a constant speed 2.5 cm/s as shown. Find the average velocity, average speed and average acceleration over the interval AB.*



SOLUTION :

In a two-dimensional motion, we can't treat vectors as numbers. We have to analyse vector sum or difference using components or using arrow diagrams.

$$\text{Time taken} = \frac{\text{distance}}{\text{speed}} = \frac{3.14\times 5}{2\times 2.5} = 3.14\ s$$

Average velocity $= \dfrac{\text{displacement}}{\text{time}}$

$$|\bar{v}_{av}| = \frac{AB}{time} = \frac{\sqrt{5^2+5^2}}{3.14}\, cms = 2.252\ cm/s$$

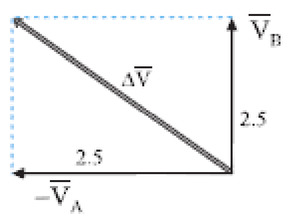
As the particle is moving with constant speed of 2.5 cm/s

$\Rightarrow$ Av. speed in any interval

= Instantaneous speed at any instant

= 2.5 cm/s

Average Acceleration $= \dfrac{\Delta\bar{v}}{\Delta t} = \dfrac{\bar{v}_B - \bar{v}_A}{\Delta t}$



$$= |\bar{a}_{av}| = \frac{\sqrt{(2.5)^2+(2.5)^2}}{3.14}\, cm/s^2 \quad = 1.126\ cm/s^2$$

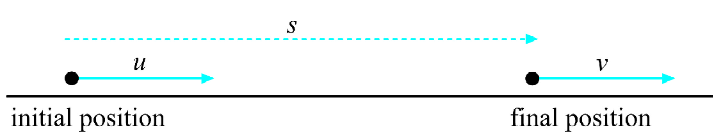
The average velocity is directed along AB and the average acceleration is directed perpendicular to AB towards O.

Try Yourself :

What is the average velocity and average acceleration in the interval AC when the particle covers a semicircle (i.e. if AC is a diameter) at the same speed ?[Ans:Av. Velocity = 1.59 cm/s², Av. Acceleration = 0.796 cm/s²]

UNIFORMLY ACCELERATED MOTION — Section - 2

For uniformly accelerated motion along a line (X-axis) during a time interval of **t** seconds, the following important results can be used.



u : initial velocity (at the beginning of interval)

a : acceleration

v : final velocity (at the end of interval)

s : displacement $(x_f - x_i)$

x_f = final coordinate (position)

x_i = initial coordinate (position)

(i) $v = u + at$

(ii) $s = ut + \frac{1}{2}at^2$; $s = ut - \frac{1}{2}at^2$; $x_f = x_i + ut + \frac{1}{2}at^2$

(Important for problem involving collisions of bodies or overtak ing of cars) See Ex-8, 9, 10

(iii) $v^2 = u^2 + 2as$

(iv) $s = \frac{1}{2}(u + v)t$

(v) *Displacement during n^{th} sec. from start* (s_n)

= Displacement at n sec. – Displacement at $(n-1)$ sec

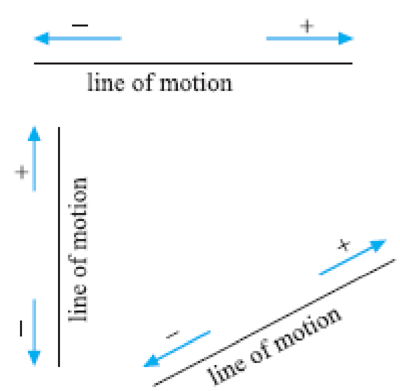
$$= un + \frac{a}{2}n^2 - [u(n-1) + \frac{a}{2}(n-1)^2] = u(n-n+1) + \frac{a}{2}[n^2 - (n-1)^2]$$

$$= u + \frac{a}{2}(2n-1)$$

Directions of Vectors in Straight Line Motion :

In straight line motion, all the vectors (position, displacement, velocity & acceleration) will have one component (along the line of motion) and there will be only two possible directions for each vector.

- For example, if a particle is moving in a horizontal line (X-axis), the two directions are right and left. Any vector directed towards right can be represented by a positive number and towards left can be represented by a negative number.



- For vertical or inclined motion, upward direction will be taken +ve and downward as –ve.

- For vertical motion under gravity, the only force acting on the particle is its weight (mg) i.e. the gravitational pull of the earth. Hence acceleration for this type of motion will always be $\boldsymbol{a = g}$ i.e. $\boldsymbol{a = -9.8\ m/s^2}$ **or** $\boldsymbol{-10 m/s^2}$ ***(as specified in the problem)*** *(– ve sign, because the force and acceleration are directed downwards)*.

Important Note :

(a) If acceleration (& force) is in same direction as velocity, then speed of the particle increases.

(b) If acceleration (& force) is in opposite direction to the velocity then speed decreases i.e. the particle slows down. This situation is known as **retardation**.

Illustration - 3 *A ball thrown up from the ground reaches a maximum height of 20 m. Find :*

(a) *Its initial velocity ;*
(b) *The time taken to reach the highest point ;*
(c) *Its velocity just before hitting the ground ;*
(d) *Its displacement between 0.5 s and 2.5 s ;*
(e) *The time at which it is 15 m above the ground.*

(*take* $g = 10\ m/s^2$)

SOLUTION :

(a) Using $v^2 = u^2 + 2as$ for upward motion,

$0^2 = u^2 + 2(-g)(+20) \quad \Rightarrow \quad u = 20\ m/s.$

(b) $$t = \frac{(v-u)}{a} = \frac{0-20}{-10} = 2s$$

(c) For the complete up-down trip,

$v^2 = u^2 + 2a(0)$

$\Rightarrow \quad v^2 = u^2$

$\Rightarrow \quad v = -u = -20\ m/s.$

(Negative sign is selected as we know that v is downwards)

(d) Height at $t = 0.5\ s$ is $y_1 = 20(0.5) - 5(0.5)^2 = 8.75\ m$

Height at $t = 2.5\ s$ is $y_2 = 20(2.5) - 5(2.5)^2 = 18.75\ m$

Displacement $= y_2 - y_1 = 18.75\ m - 8.75\ m = +10\ m$

$$s = ut + \frac{1}{2}at^2$$

(e) $15 = 20t - 5t^2$

$\Rightarrow \quad t = 1s, 3s.$

At $t = 1s$, ball is going up and at $t = 3s$ it is coming down. (Remember that in part (b), we get $t = 2s$ to reach the top)

Try Yourself :

Find the average velocity and average speed between $t = 0.5\ s$ and $t = 2.5\ s$ in the last example.
[Ans: Av. Velocity = +5 m/s ; Av. Speed = 6.25 m/s]

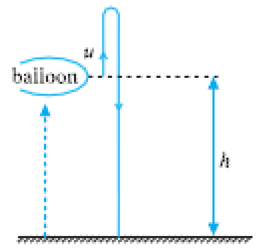
Hint : For average speed, split the interval in two parts by using the time instant ($t = 2s$) at which speed is zero.

Illustration - 4 *A balloon starting from the ground has been ascending vertically at a uniform velocity for 4 sec and a stone let fall from it reaches the ground in 6 sec. Find the velocity of the balloon and its height when the stone was let fall.*(g = 10 m/s^2)

SOLUTION :

The phrase "left fall" means that the stone is released from rest without any push.

Let u = the constant velocity with which the balloon is going up.



Initial velocity of stone (relative to ground)

= velocity of stone relative to balloon
+ velocity of balloon

$= 0 + u = u$

h = height of the balloon when the stone is dropped.

$h = 4\,u$

For the stone :

$s = u\,t + 1/2\,a\,t^2$

$-h = u\,t + 1/2\,(-g)\,t^2$

$-4\,u = 6\,u - 1/2\,g\,6^2$

$\Rightarrow\quad u = 18\ m/s.$

$\Rightarrow\quad Height = h = 4u = 72\ m.$

Exercise : Find the velocity with which the stone hits the ground. (Ans : 42 m/s downwards)

Illustration - 5 *A car starts from rest and accelerates uniformly for 10 s to a velocity of 8 m/s. It then run at a constant velocity and is finally brought to rest in 64 m with a constant retardation. The total distance covered by the car is 584 m. Find the value of acceleration, retardation and total time taken.*

SOLUTION :

The car starts from A, accelerates from A to B, run at constant velocity from B to C and retards to rest from C to D.

From A to B

$$a = \frac{v-u}{t} = \frac{8-0}{10} = 0.8\,m/s^2$$

$$s = ut + \frac{1}{2}at^2 = 0 + \frac{1}{2}(0.8)(100) = 40m$$

From B to C

$$s = BC = 584 - AB - CD$$
$$= 584 - 40 - 64 = 480\ m$$

$$s = ut + \frac{1}{2}at^2$$

$\Rightarrow \quad 480 = 8\tau + 0 \qquad \Rightarrow \qquad \tau = 60$ *seconds*

<u>From C to D</u>

$$a = \frac{v^2 - u^2}{2s} = \frac{0^2 - 8^2}{2(64)} = -0.5\ m/s^2$$

$$t = \frac{v-u}{a} = \frac{0-8}{-0.5} = 16\ seconds.$$

$\Rightarrow \quad$ total time $= t_{AB} + t_{BC} + t_{CD}$

$= 10 + 60 + 16 = 86\ sec$

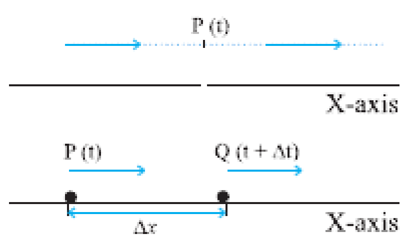
$a_{AB} = 0.8\ m/s^2$ *and* $\quad a_{CD} = -0.5\ m/s^2$

INSTANTANEOUS VELOCITY & INSTANTANEOUS ACCELERATION — Section - 3

USE OF DERIVATIVES

(a) Instantaneous Velocity :

If a car is driven along a straight road for 100 km in 2 hours, the average velocity is 50 km/hr. Can we say that the car was moving with 50 km/hr at every instant in these two hours ? Certainly not. As we can easily imagine, the car accelerated from rest and went through will all speeds like 10 km/hr, 20 km/hr etc (and it may be moving with speeds more than 50 km/hr at some instants). To deal with this, we need the concept of *instantaneous velocity*, which is the velocity at any instant of time (it is what a speedometer of a car indicates).



Consider a particle moving along X-axis. Let us aim to find the instantaneous velocity of this particle at a certain time instant t, when it passes through the point $P(t)$ as shown. To calculate the instantaneous velocity, we start with the concept of average velocity. The particle passes through the point $Q(t + \Delta t)$ Δt seconds after it passes P(t). The ratio $\Delta x/\Delta t$ is the average velocity over the interval PQ.

$$V_{av} = \frac{\Delta x}{\Delta t}(in\ PQ)$$

By taking the position $Q\,(t + \Delta t)$ more and more close to $P\,(t)$, we can make this average velocity very closely approximate the exact instantaneous velocity at P.

Taking Q closer to P means that the interval Δt shrinks and diminishes towards zero. We usually describe this situation like this : "as Q is taken closer to P, Δt approaches zero and the average velocity in PQ approaches the instantaneous velocity at P".

As $Q \to P,\ \ \Delta t \to 0$ and $V_{av}\ (in\ PQ) \to V_{inst}\ (at\ P)$

$$V_{inst.}(at\ \mathrm{P}) = \left(\frac{\Delta x}{\Delta t}\ in\ PQ\right)_{Q\to P} = \left(\frac{\Delta x}{\Delta t}\right)_{\Delta t \to 0}$$

In exact notation we write :

$$V_{inst.} = \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right) = \frac{dx}{dt}$$

In words, the instantaneous velocity at a given moment (say, t) is the limiting value of the average velocity as we let Δt approach zero. The limit as $\Delta t \to 0$ is written in calculus notation as $\frac{dx}{dt}$ and is called derivative of x with respect to t.It is important to note that we do not simply set $\Delta t = 0$, for then Δx would also be zero,and we would have an undefined number. We must consider the ratio $\frac{\Delta x}{\Delta t}$ as a whole ; and let Δt approach zero, Δx approaches zero also; but the ratio $\frac{\Delta x}{\Delta t}$ approaches some definite value, which we call the instantaneous veloctiy

Illustration - 6 *The position of a particle is given by the equation $x\,(t) = 3t^3$. Find the instantaneous velocity at instants $t = 2s$, $4s$ using the definition of instantaneous velocity.*

SOLUTION :

Let us find the instantaneous velocity $v\,(t)$ of the particle at any time instant t. Then we can substitute $t = 2s$, $4s$ for calculating particular values. Average velocity in a time interval from t to $(t + \Delta t)$ is :

$$\frac{\Delta x}{\Delta t} = \frac{3(t+\Delta t)^3 - 3t^3}{\Delta t}$$

$$= \frac{3\Delta t\left(3t^2 + \Delta t^2 + 3t.\Delta t\right)}{\Delta t} = 3\left(3t^2 + \Delta t^2 + 3t.\Delta t\right)$$

$$v\,(t) = \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right) = \lim_{\Delta t \to 0} 3\left(3t^2 + \Delta t^2 + 3t.\Delta t\right)$$

$\Rightarrow \quad v(t) = 3(3t^2 + 0 + 0) = 9t^2$

The velocity at $t = 2s$ *is* $v(t = 2) = 36$ *m/s.*

The velocity at t = 4s is $v(t = 4) = 144$ *m/s.*

Important Results :

The determination of instantaneous velocity by using the definition (*i.e. by the limiting process as in the last example*) usually involves calculations. We can find :

$v = \frac{dx}{dt}$ by using the standard results from differential calculus.

$x = u + v + \ldots\ldots \Rightarrow \frac{dx}{dt} = \frac{du}{dt} + \frac{dv}{dt} + \ldots\ldots\ldots\ldots$

$x = constant \Rightarrow \frac{dx}{dt} = 0$ (*derivative of a constant is zero*)

$x = Au \Rightarrow \frac{dx}{dt} = A\frac{du}{dt}$ (*where A is a constant*)

$x = t^n \Rightarrow \frac{dx}{dt} = n\,t^{n-1}$

$x = \sin \omega t \Rightarrow \frac{dx}{dt}\ \omega \cos \omega t$ (ω *is constant*).

$x = \cos \omega t \Rightarrow \frac{dx}{dt} = -\omega \sin \omega t$ (ω *is constant*).

$x = \log t \Rightarrow \frac{dx}{dt} = \frac{1}{t}$

$x = e^{at} \Rightarrow \frac{dx}{dt} = ae^{at}$

Illustration - 7 *A particle is moving along X-axis, its position varying with time as* $x(t) = 2t^3 - 3t^2 + 1$.

(a) *At what time instants, is its velocity zero.*

(b) *What is the velocity when it passes through origin ?*

SOLUTION :

(a) $v(t) = \frac{dx}{dt} = 2(3t^2) - 3(2t)$

$\Rightarrow \quad v(t) = 6t(t - 1)$

$\Rightarrow \quad v = 0$ for $t = 0, 1s$.

(b) It passes through origin when $x(t) = 0$.

$\Rightarrow \quad 0 = 2t^3 - 3t^2 + 1$

$\Rightarrow \quad (2t + 1)(t - 1)^2 = 0.$

$\Rightarrow \quad t = -1/2,\ 1$ sec

The particle is at origin at $t = 1$ sec and at $t = -1/2$ (*i.e.* 0.5 *sec before* $t = 0$).

$v(t = 1) = 0\ m/s$

$v(t = -0.5) = 6(-0.5)(-0.5 - 1) = 4.5\ m/s.$

(b) Instantaneous Acceleration :

The instantaneous acceleration of a particle is its acceleration at a particular instant of time. It is defined as the derivative (rate of change) of velocity with respect to time :

$$\bar{a} = \frac{d\bar{v}}{dt} \quad \lim_{\Delta t \to 0} \left(\frac{\Delta \bar{v}}{\Delta t} \right)$$

Try Yourself :

Find an expression for acceleration in term of t in the last example.
What is the acceleration at $t = 0.5$ s ?[*Ans:* $a(t) = 12t - 6\ m/s^2$; $0\ m/s^2$ *at* $t = 0.5$ *s*]

GRAPHS (STRAIGHT LINE MOTION) — Section - 4

With the help of graphs we visualise the variation of *position* (x), *velocity* (v), and *acceleration* (a) of a moving particle with time. Plotting *time* (t) on X–axis and x, v, a on Y–axis we get three useful graphs:

(i) ***x–t*** graphs **(ii)** ***v–t*** graphs **(iii)** ***a–t*** graphs

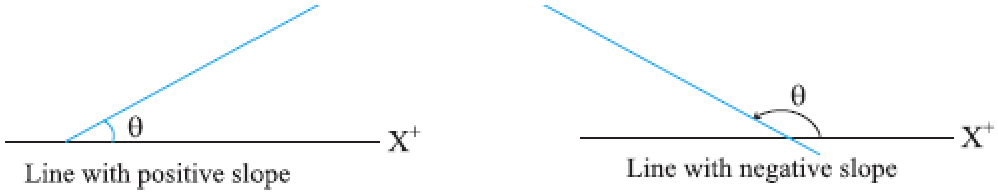
Illustration:

A particle is resting on X-axis at the point (3,0). *Draw its x-t graph.The equation is :* $x(t) = 3$.*As the particle is at rest, its X-coordinate is constant with the time and hence a horizontal line* $x = 3$ *is the x- t graph.*

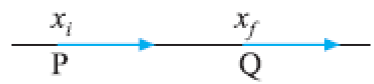
Graphical Interpretation of some Quantities

Slope of a Line

If any line makes angle θ with the positive direction of X-axis, we define its slope = *tan* θ
Note that all horizontal lines have zero slope. If θ is acute, slope is positive and if θ is obtuse, the slope is negative.

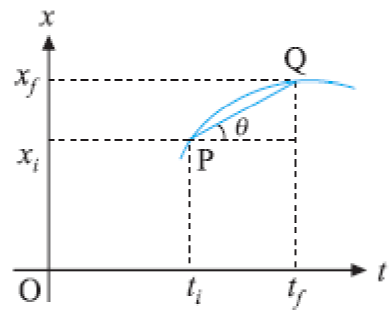


Average Velocity



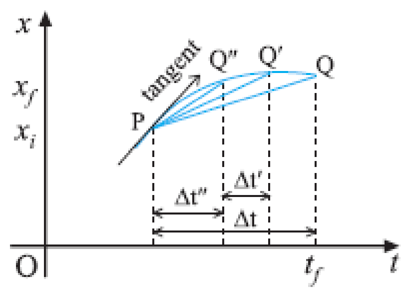
If a particle passes a point $P(x_i)$ at time $t = t_i$ and reaches $Q\,(x_f)$ at a later time instant $t = t_f$, its average velocity in the interval PQ

is $V_{av} = \dfrac{\Delta x}{\Delta t} = \dfrac{x_f - x_i}{t_f - t_i} = \tan\theta$ where θ = angle that PQ makes with horizontal.



This expression suggests that the average velocity is equal to the slope of the line (chord) joining the points corresponding to P, Q on the $\boldsymbol{x - t}$ graph.

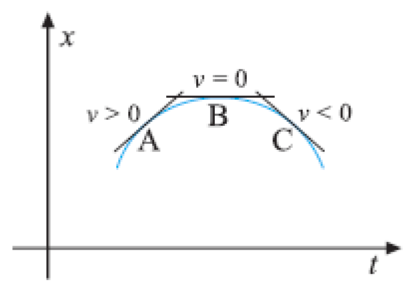
Instantaneous Velocity



Consider the motion of the particle between the two points P and Q on the $x - t$ graph shown. As the point Q is brought closer and closer to the point P, the time interval between PQ ($\Delta t, \Delta t'$, $\Delta t''$, ...) get progressively smaller. The average velocity for each time interval is the slope of the appropriate dotted line (PQ, PQ', PQ'' . . . .). As the point Q approaches P, the time interval approaches zero, but at the same time the slope of the dotted line approaches that of the tangent to the curve at the point P.

As $\Delta t \to 0$, V_{av} ($= \Delta x/\Delta t$) $\to V_{inst.}$

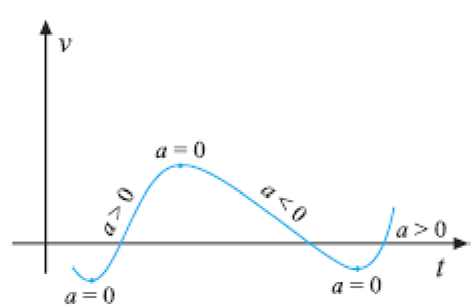
Geometrically, As $\Delta t \to 0$, chord $PQ \to$ tangent at P.



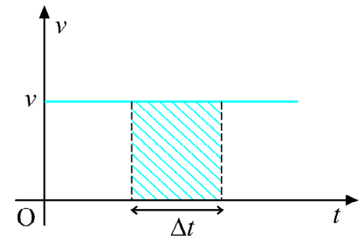
Hence the instantaneous velocity at P in the slope of the tangent at P in $x - t$ graph. When the slope of the $x - t$ graph is positive, v is negative (as at the point A in figure). At C, v is negative because the tangent has negative slope. The instantaneous velocity at point B (turning point) is zero as the slope is zero.

Instantaneous Acceleration :

The derivative of velocity with respect to time is the slope of the tangent in velocity time ($v - t$) graph.



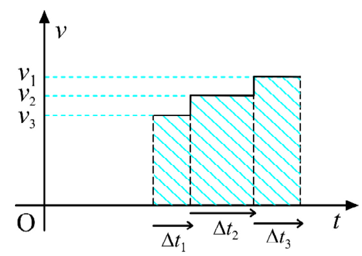
Displacement in v–t graph :



Let us now discuss the problem of determining displacement from a $v - t$ graph. For motion at constant velocity, the $v - t$ graph is a horizontal line, as shown in the figure. Since $v = \frac{\Delta x}{\Delta t}$, the displacement Δx in a time integral Δt is given by $\Delta x = v\Delta t$.

This is just the area of the shaded rectangle of height v and width Δt.
Note that the unit of this area is $(m/s)(s) = m$.

$\Delta x = u\,\Delta t$ = height of rectangle × base = area of rectangle under $v - t$ graph.



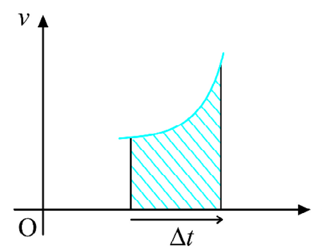
Let us now consider the case when the velocity is not constant. Let the velocity be v_1 for Δt_1 seconds, v_2 for Δt_2 seconds, v_3 for Δt_3 seconds.

Displacement = $\Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3$

$= v_1\,\Delta t_1 + v_2\,\Delta t_2 + v_3\,\Delta t_3$

= sum of areas of the three shaded rectangles.

$\Rightarrow$ Displacement = Δx = area under $v - t$ graph.



For the most general case, the $v - t$ graph can be a curve i.e. velocity may change continuously with time as shown. To calculate displacement during an interval Δt, we divide this interval into many small intervals :

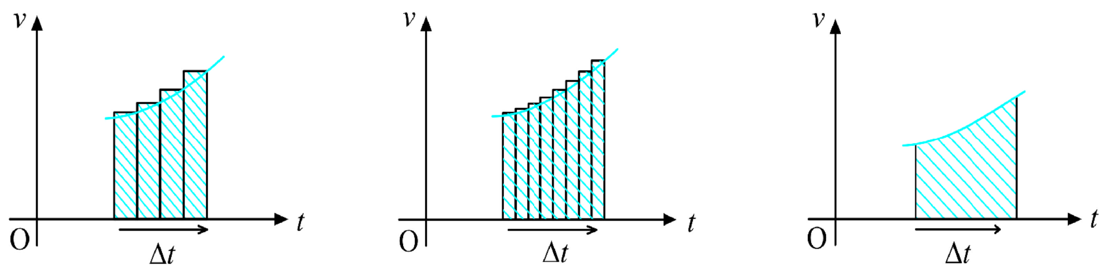
$$(\Delta t = \Delta t_1 + \Delta t_2, + \text{- - - - - - -} + \Delta t_n).$$

If the number of sub-intervals is made very large (i.e. n goes on increasing), each interval becomes very small ($\Delta t_i \to 0$) and displacement during each of these sub-intervals may be taken as the area of the shaded rectangles as shown. The approximation improves as the number of rectangles (i.e. sub-intervals) is increased.

Displacement = $(v_1\,\Delta t_1 + v_2\,\Delta t_2 + \text{- - - - -}\; v_n\,\Delta t_n)$ as $n \to \infty$, $\Delta t_i \to 0$

$\Rightarrow$ Displacement = total area under the $v - t$ curve.

Since a negative velocity causes a negative displacement, areas below the time axis are taken negative.
In a similar way, we can see that $\Delta v = a\,\Delta t$ leads to the conclusion that area under $a - t$ graph gives the change in velocity Δv during that interval.

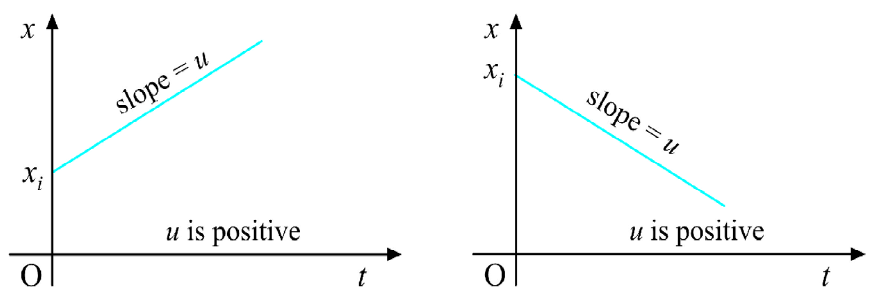


As the number of sub-intervals is increased ($n \to \infty$),the sum of the areas of the rectangles approaches the area under the curve.

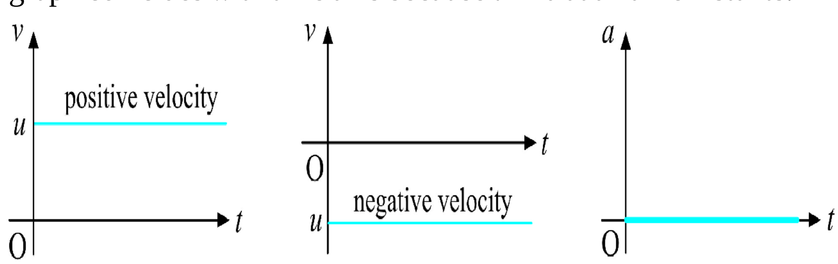
Motion with Uniform Velocity u :

Consider a particle moving along X-axis with uniform velocity u starting from the point $x = x_i$ at $t = 0$.
Equations of x , v, a are :

$x(t) = x_i + ut \quad ; \quad v(t) = u \quad ; \quad a(t) = 0$



- xt graph is a straight line of slope u through x_i.
- As velocity is constant, $v - t$ graph is a horizontal line.
- $a - t$ graph coincides with time axis because $a = 0$ at all time instants.



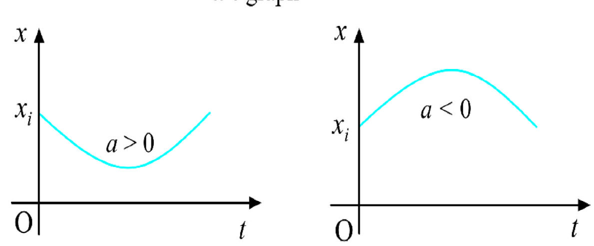
Uniformly Accelerated Motion ($a \neq 0$)

$$x(t) = x_i + ut + \frac{1}{2}at^2$$

$$v(t) = u + at$$

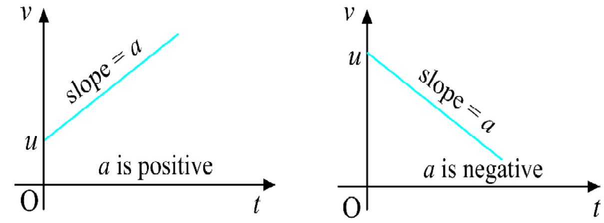
$$a(t) = a$$

x-t graph



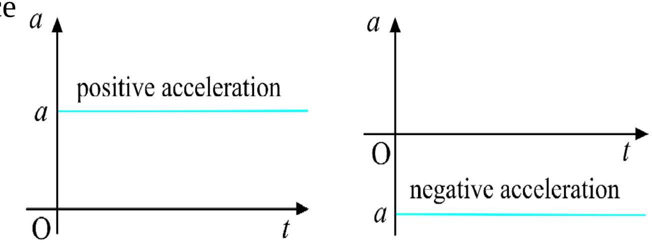
- $x(t)$ is a quadratic polynomial in terms of t. Hence x-t graph is a parabola. For a > 0, parabola opens upwards and for a < 0, parabola opens downwards

v-t graph



- $v(t)$ is a linear polynomial in terms of t. Hence v-t graph is a straight line of slope a.
- a–t graph is a horizontal line because a is constant.

a-t graph



IMPORTANT POINTS TO REMEMBER :

- For uniformly accelerated motion ($a \neq 0$), x-t graph is a parabola (opening upwards if $a > 0$ and opening downwards if $a < 0$). The slope of tangent at any point of the parabola gives the velocity at that instant.
- For uniformly accelerated motion ($a \neq 0$), v-t graph is straight line whose slope gives the acceleration of the particle.
- **In general, the slope of tangent in *x-t* graph is velocity and the slope of tangent in *v-t* graph is the acceleration.**
- The area under a-t graph gives the change in velocity.
- The area between the v-t graph and the time-axis gives the distance travelled by the particle, if we take all areas as positive ; shaded area = distance covered in the time interval $t = t_1$ to $t = t_2$. (*see example* 9)
- Area under v-t graph gives displacement if areas below the t-axis are taken negative. (*see example*

Illustration - 8 *A car accelerates from rest at the rate of* 1 m/s^2 *for* 5 *seconds and then retards at the same rate till it comes to rest. Draw the x-t, v-t and a-t graphs.*

SOLUTION :

(i) Velocity acquired after 5 *sec*.

$$= u + at = 0 + 1\,(5) = 5\ m/s.$$

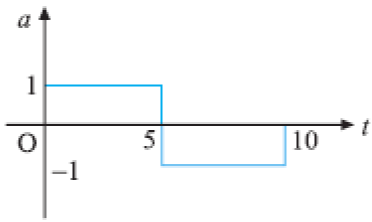
(ii) Time taken to come to rest

$$= \frac{0-5}{-1} = 5\ sec$$

Car starts at $t = 0$ and accelerates till $t = 5s$.

(iii) The car starts slowing down at $t = 5s$ and comes to rest at $t = 10$ s.

a-t Graph



(a) During $t = 0$ to $t = 5s$, $a = +1\ m/s^2$

(b) During $t = 5s$ to $t = 10s$ $a = -1\ m/s^2$

v-t graph

As acceleration is uniform, *v-t* graph is a straight line.

(a) $t = 0s$ to $t = 5s$

Velocity increases at a constant rate from 0 *m/s* to 5 *m/s*

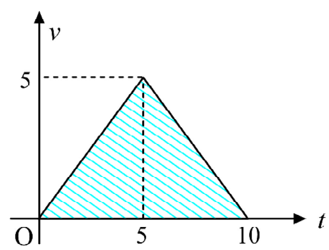
(b) $t = 5s$ to $t = 10s$

Velocity decreases from 5 *m/s* to 0 *m/s* in 5 sec.

In the v-t graph the area of shaded triangle

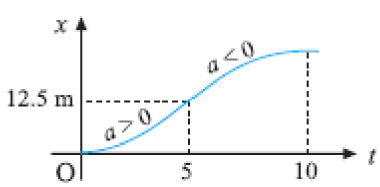
= *distance covered*

= 1/2 (5)(10) = 25 *m*.



x-t graph

As acceleration is uniform the *x-t* graph will be a parabola.



(a) $t = 0s$ to $t = 5s$

As acceleration is positive, the parabola opens upwards and slope (velocity) varies from 0 *m/s* to 5 *m/s*. *x* varies from 0 *m* to *x* (5)

= area under v-t graph

$$= \frac{5 \times 5}{2} = 12.5\ m$$

(b) $t = 5s$ to $t = 10s$

As acceleration is negative, parabola opens downwards and slope (velocity) varies from 5 m/s to 0 m/s. x varies from x (5) to x (10)

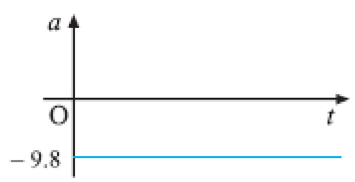
$$\frac{5\times5}{2} = \frac{25}{2}m$$

Illustration - 9 *A ball is thrown vertically upwards with a speed of* 9.8 *m/s from the ground. Draw the x-t, v-t and a-t graph for its motion.*

SOLUTION :

a-t Graph

As the acceleration of the ball remains $a = 9.8\ m/s^2$ throughout the motion, a-t graph is a horizontal line.

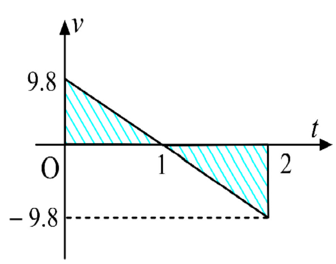


Time taken by the ball to reach top

$$t = \frac{0-9.8}{-9.8} = 1\ sec$$ and the height attained is

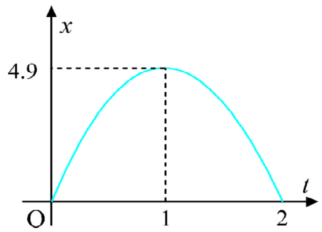
$$x = \frac{0^2 - 9.8^2}{2(-9.8)} = 4.9\ m$$

v-t Graph



As acceleration is constant, v-t graph is a straight line of slope = $a = -9.8 m/s^2$. Velocity varies from +9.8 m/s at t = 0s to zero at t = 1s and finally to – 9.8 m/s at t = 2s.

x-t Graph



As $a < 0$, x–t graph is a parabola opening downwards. The x-coordinate of the ball is zero initially, increases to 4.9m at t = 1s and finally become zero again when it reaches back after 2 *seconds*.

***Note* :**

(i) Velocity at the top = 0 As the slope of the tangent to the x–t graph at t = 1 is zero.

(ii) In v-t graph :

Distance covered = shaded area (*taking area above or below t-axis as positive*)

$= 2\,(1/2 \times 1 \times 9.8) = 9.8\ m.$

displacement = area (*taking area below t-axis as negative*)

$= 4.9 + (-4.9) = 0\ m$

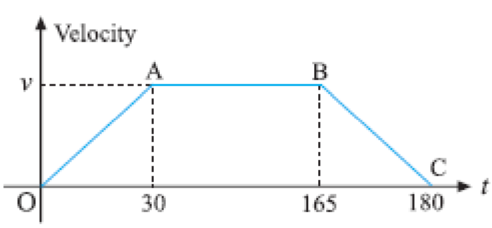
(iii) Slope of line in v–t graph = acceleration

$= -9.8\ m/s^2$

(iv) Between t = 0 and t = 1, velocity (slope in x-t graph) is +*ve* as the ball is going up. Between t = 1 and t = 2, velocity (slope in x-t graph) is –*ve* as the ball is coming down.

Illustration - 10 *A train travels in* 3 *minutes a distance of* 3.15 *Km from rest at one station to rest at another station. It is uniformly accelerated for the Ist* 30 *seconds and uniformly retarded for the last* 15 *seconds, the speed being constant for the remaining time. Find the maximum velocity, acceleration and retardation. Use v-t graph to solve the problem.*

SOLUTION:



Let v = max. speed attained by the train.

From t = 0 to t = 30 *sec*, acceleration is positive.

From t = 30 to t = 165 *sec*, acceleration is zero.

From t = 165 to t = 180 *sec*, acceleration is negative.

Area under $OABC$ = distance covered

$$\frac{1}{2}(OC + AB)\, v = 3150$$

$$\frac{1}{2}(180 + 135)\, v = 3150$$

$\Rightarrow$ $\quad$ **$v = 20\ m/s$.**

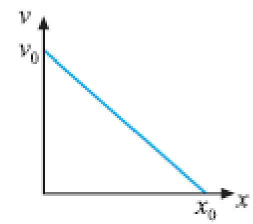
Slope of OA = acceleration in 1st part

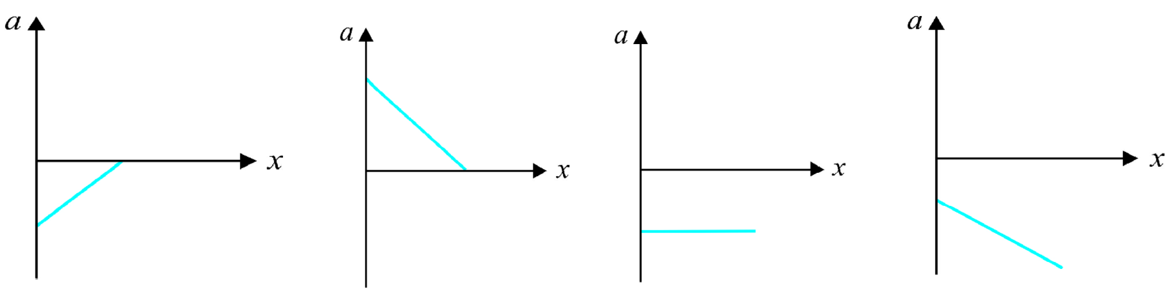
$$= \frac{v}{30} = \frac{2}{3}\mathbf{0.667\ m/s^2}.$$

Slope of BC = acceleration in 3rd part

$$= -\frac{v}{15} = -\frac{4}{3} = -1.334\ m/s^2.$$

Illustration - 11 *The given graph shows the variation of velocity with displacement. Which one of the graph given below represents the variation of acceleration with displacement ?*





(A) (B) (C) (D)

SOLUTION :

From the v-x graph, we can do an analysis for acceleration using

$$a = v\frac{dv}{dx} = v\,(\text{slope of tangent in } v\text{-}x \text{ graph})$$

From the graph, we have $v(x) = \left(\dfrac{-v_0}{x_0}\right)x + v_0$

Slope of tangent $= \dfrac{dy}{dx} = \dfrac{-v_0}{x_0}$

This slope is constant as v-x graph is a straight line.

$$\Rightarrow \quad a = v\left(\frac{-v_0}{x_0}\right)$$

$$a(x) = \left(\frac{-v_0 x}{x_0} + v_0\right)\left(\frac{-v_0}{x_0}\right) = \frac{v_0^2}{x_0^2}x - \frac{v_0^2}{x_0}$$

$\Rightarrow$ a-x graph is a straight line with slope $\frac{v_0^2}{x_0^2}$

and intercept on Y-axis equal to $\frac{-v_0^2}{x_0}$

$\Rightarrow$ (A) is correct.

Alternative Approach :

(i) At x = 0, $a = v\frac{dv}{dx} = v_0\left(\frac{-v_0}{x_0}\right) = -ve$

(ii) At x = x_0, $a = v\frac{dv}{dx} = 0\left(\frac{-v_0}{x_0}\right) = 0$

Hence (A) is the only possible answer.

MOTION WITH NON-UNIFORM ACCELERATION — Section - 5

Use of definite integrals

In the previous section we have seen that the displacement for a variable velocity in an interval $\Delta t = (t_2 - t_1)$ can be calculated by dividing this interval into n sub – intervals.

$$t = \Delta t_1 + \Delta t_2 + \Delta t_3 + \text{- - - - -} + \Delta t_n$$

$$\Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3 + \text{- - - - -} + \Delta x_n$$

As the number of intervals is increased endlessly (i.e. n $\rightarrow \infty$), each interval becomes infinitely small

$(\Delta t_i \rightarrow 0, i = 1, 2, \text{- - - - - -} n)$ and we can approximately treat velocity as constant during one such interval.

$$\Delta x = (v_1 \Delta t_1 + v_2 \Delta t_2 + \text{- - - - -} + v_n \Delta t_n.) \text{ as } n \rightarrow \infty \text{ and } \Delta t_i \rightarrow 0.$$

$$\Delta x = \lim_{\substack{\Delta t_i \rightarrow 0 \\ n \rightarrow \infty}} \left[\sum_{i=1}^{n} v_i \, \Delta t_i\right] = \int_{t_1}^{t_2} v(t)\,dt$$

The expression in the right hand side is called the definite integral of v (t) between $t = t_1$ and $t = t_2$. A similar discussion leads us to calculate that *change in velocity* $= \Delta v = v_f - v_i = \int_{t_1}^{t_2} a(t)\,dt$

Short note on calculation of integrals :

We have two types of integrals : *Indefinite integrals and Definite integrals.*

Indefinite integrals are basically anti-derivatives i.e., they are inverse of derivatives. For ex ample, we know that derivative of $t2 = 2t$ and this means that the indefinite integral of $2t$ is $t2$. Similarly, the derivative of tn is $n\ t\ n$-1 and hence the indefinite integral of $n\ t\ n$-1 is tn.

Proceeding in a similar way, we get :

$$\int t^n dt = \frac{t^{n+1}}{n+1} \qquad \ldots\ldots\text{(I)}$$

Note that the derivative of $\frac{t^{n+1}}{n+1}$ is t^n.

(*We can add a constant C to the right hand side of equation* (*I*) *because the derivative of a con stant is zero.*)

Definite integrals are calculated over some intervals i.e., between an upper limit and a lower limit. To calculate a definite integral, first find its indefinite integral (anti-derivative) and then substitute upper and lower limits and subtract.

Illustrations :

1. $\int t' dt = \frac{t^{1+1}}{1+1}$ from (I)

$\Rightarrow \quad \int_1^2 t\,dt = \left.\frac{t^2}{2}\right]_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$

2. $\int_0^4 \sqrt{t}\,dt = \left.\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^4 = \left[\frac{2}{3}t\sqrt{t}\right]_0^4 = \frac{16}{3}$

3. $\int_2^5 \frac{1}{t^2}dt = \int_2^5 t^{-2}dt$

$= \left.\frac{t^{-2+1}}{-2+1}\right]_2^5$

$= \left[\frac{-1}{t}\right]_2^5$

$= \left(\frac{-1}{5}\right) - \left(\frac{-1}{2}\right) = \frac{3}{10}$

Some quantities defined as derivatives and integrals.

$v(t) = \frac{dx}{dt}$	v = *slope of x - t graph*
$a(t) = \frac{dv}{dt}$	a = *slope of v - t graph*
$F(t) = \frac{dp}{dt}$	F = *slope of p – t graph* (*p* = *linear momentum*)

$\Delta x = \int dx = \int_{t_1}^{t_2} v(t)\,dt$	D x = *area under v - t graph*
$\Delta v = \int dv = \int_{t_1}^{t_2} a(t)\,dt$	Δv = *area under* $a - t$ *graph*
$\Delta p = \int dp = \int_{t_1}^{t_2} F(t)\,dt$	Δp = *area under* $F - t$ *graph*
$W = \int dW = \int_{x_1}^{x_2} F(x)\,dx$	W = *area under* $F - x$ *graph*

Important result for integration :

1. **(a)** $\int t^n dt = \frac{t^{n+1}}{n+1} (n \neq -1)$ **(b)** $\int (at+b)^n dt = \frac{1}{a}\frac{(at+b)^{n+1}}{(n+1)}$

2. **(a)** $\int \frac{dt}{t} = \log t \Rightarrow \int_{t_1}^{t_2} \frac{dt}{t} = \log \frac{t_2}{t_1}$ **(b)** $\int \frac{dt}{at+b} = \frac{1}{a} \ell og(at+b)$

3. **(a)** $\int e^t\, dt = e^t$ **(b)** $\int e^{at}\, dt = \frac{e^{at}}{a}$

4. **(a)** $\int \sin t = -\cos t$ **(b)** $\int \sin at \frac{-\cos at}{a}$

5. **(a)** $\int \cos t = \sin\, t$ **(b)** $\int \cos at\, dt = \frac{\sin at}{a}$

Illustrations :

1. $\int_2^3 \frac{1}{v} dv = \log v \Big]_2^3$

$= \log 3 - \log 2$

$= \log (3/2)$

2. $\int_0^2 \left(e^{2t} - t\right) dt = \int_0^2 e^{2t} dt - \int_0^2 t\, dt$

$= \left[\frac{e^{2t}}{2}\right]_0^2 - \left[\frac{t^2}{2}\right]_0^2$

$= \frac{e^4 - 5}{2}$

Exercise : Evaluate

(a) $\int_1^4 \frac{dt}{t}$ (b) $\int_0^{\pi/4} \sin 2x\, dx$ (c) $\int_0^1 \left(t^2 - e^{-t}\right) dt$

Ans : (a) $\log 4$, (b) 1/2, (c) $\frac{1}{e} - \frac{2}{3}$

Solving Problems involving Non-Uniform Acceleration :

(a) Acceleration depends on time *t*

(i) Steps to find *v* (*t*) from *a* (*t*)

By definition of acceleration, we have :

$$a = \frac{dv}{dt}$$

After substituting the expression for acceleration in left hand side, we separate the variables.

If $a(t)$ is in terms of t, $dv = a(t)\, dt$

Integrating on both sides,

$$\int_{v(0)}^{v(t)} dv = \int_0^t a(t)\, dt$$

Where $v(0)$ = initial velocity at time $t = 0$

We get an expression for $v(t)$

(ii) Steps to find *x* (*t*) from *v* (*t*)

To get $x(t)$, we put $v(t) = \frac{dx}{dt}$

$\Rightarrow \quad dx = v(t)\, dt$

Integrating on both sides,

$$\int_{x(0)}^{x(t)} dx = \int_0^t v(t)\, dt$$

Where $x(0)$ = position at $t = 0$.

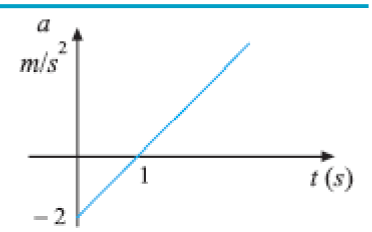
We get an expression for $x(t)$.

Illustration - 12 *The acceleration of a particle varies with time as shown.*

(i) *Find an expression for velocity in terms of t and draw v-t graph*

(ii) *Derive x (t) and hence calculate the displacement of the particle in the interval from t = 2 sec. to t = 4 sec.*

Assume that x = 0 and v = 0 at t = 0.



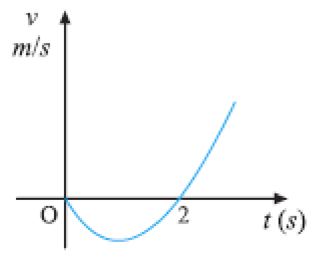
SOLUTION :

(i) *The a – t graph leads to the following expression for a(t).*

$a(t) = 2t - 2$

As acceleration is in terms of time t,

Put it as $\frac{dv}{dt} = 2t - 2$



Take all terms involving t and dt to the RHS.

$dv = (2t - 2)\,dt$

$\int_0^{v(t)} dv = \int_0^t (2t-2)\,dt$

$\Rightarrow v(t) = \int_0^t (2t-2)\,dt$

$= \int_0^t 2t\,dt - \int_0^t 2dt = 2\int_0^t t\,dt - 2\int_0^t dt$

$= 2\left[\frac{t^2}{2}\right]_0^t - 2[t]_0^t$

$\Rightarrow v(t) = t^2 - 2t$

$\Rightarrow$ *The v – t graph is a parabola.*

(ii) $v(t) = \frac{dx}{dt} = t^2 - 2t$

$dx = \left(t^2 - 2t\right) dt$

Integrating on both sides,

$\int_0^{x(t)} dx = \int_0^t \left(t^2 - 2t\right) dt$

$x(t) = \frac{t^3}{3} - t^2$

$\Delta x = x(4) - x(2)$

$= \left(\frac{4^3}{3} - 4^2\right) - \left(\frac{2^3}{3} - 2^2\right) = \frac{20}{3}\ m$

(b) Acceleration depends on position *x*

(i) Steps to find *v* (*x*) from *a* (*x*)

$a = \frac{dv}{dt} \Rightarrow a = \frac{dx}{dt}\frac{dv}{dx}$

$\Rightarrow a = v\frac{dv}{dx}$ by substituting $\frac{dx}{dt}$ as *v*.

This is another important expression for acceleration.

If *a* (*x*) is in terms of *x* $\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a(x)\,dx$

Hence we get an expression for *v* (*x*).

(ii) Steps to find *x* (*t*) from *v* (*x*)

Using $v(x) = \frac{dx}{dt}$, we get :

$\frac{dx}{v(x)} = dt$

Integrate on both sides.

$\int_{x(0)}^{x(t)} \frac{dx}{v(x)} = \int_0^t dt$

Hence we get an expression for *x* (*t*).

Illustration - 13 *A particle is travelling along X-axis with an acceleration which varies as :*

$$a(x) = -4x$$

(i) *Derive the expression for v (x). Assume that the particle starts from rest at x = 1m.*

(ii) *Hence find the maximum possible speed of the particle.*

SOLUTION :

(i) $$a(x) = v\frac{dv}{dx}$$

$$v\frac{dv}{dx} = -4x$$

$$\int_0^v v\,dv = -4\int_1^x x\,dx$$

$$\frac{v^2}{2} - \frac{0^2}{2} = -4\left(\frac{x^2}{2} - \frac{1}{2}\right)$$

$$v^2 = 4\left(1 - x^2\right)$$

$$v(x) = \pm 2\sqrt{1 - x^2}$$

(ii) Speed is maximum for $x = 0$

$\Rightarrow$ maximum speed = 2 *m*/*s*.

(c) Acceleration depends on velocity v :

(i) Steps to find *v* (*t*) from *a* (*v*)

$$a(v) = \frac{dv}{dt}$$

Separate the variables on two sides of the equation.

$$\frac{dv}{a(v)} = dt$$

Integrating on both sides,

$$\int_{v(0)}^{v(t)} \frac{dv}{a(v)} = \int_0^t dt$$

We get an expression for *v* (*t*).

(ii) Steps to find *v* (*x*) from *a* (*v*)

$$a(v) = v\frac{dv}{dx}$$

Separate the variables on two sides of the equation.

$$v\frac{dv}{a(v)} = dx$$

Integrating on both sides,

$$\int_{v_0}^{v} \frac{v\,dv}{a(v)} = \int_{x_0}^{x} dx$$

On integrating, we get a relation between *x* and *v*.

Using $\int_{x_0}^{x} \frac{dx}{v(x)} = \int_0^t dt$, we can relate *x* and *t*.

Illustration - 14 *A stone is thrown up with an initial speed v_0. There is a resisting acceleration $k\sqrt{v}$, due to air, where v is instantaneous velocity and k is some positive constant. Find the time taken to reach the highest point and the maximum height attained by the particle*

SOLUTION :

(a) To find the stopping time, we put $a = \frac{dv}{dt}$

As the force is opposite to motion, retardation is taking place. So $\frac{dv}{dt}$ must be negative.

$$\frac{dv}{dt} = -k\sqrt{v}$$

$$\int_{v_0}^{0} \frac{dv}{\sqrt{v}} = -k\int_0^t dt$$

$$2\sqrt{v_0} = kt$$

$$t = \frac{2\sqrt{v_0}}{k}$$

(b) To find the stopping distance, we put $a = v\frac{dv}{dx}$

$$v\frac{dv}{dx} = -k\sqrt{v}$$

$$\int_{v_0}^{0} \sqrt{v}\, dt = -k\int_0^x dx$$

$$\frac{2}{3}v_0^{3/2} = kx$$

$$x = \frac{2v_0^{3/2}}{3k}$$

Illustration - 15 *A particle of mass m is projected in a resisting medium whose resistive force is F = kv and the initial velocity is v_0.*

(a) *Find the expression for position and velocity in terms of time.*

(b) *Find the time after which the velocity becomes $v_0/2$.*

SOLUTION :

<u>Derivation of v (t) from a (v)</u>

(a) As retardation is taking place, $\frac{dv}{dt}$ must be negative.

Acceleration $= \frac{dv}{dt} = \frac{-kv}{m}$

$$\int_{v_0}^{v} \frac{dv}{v} = \int_0^t -k\frac{dt}{m}$$

$$\Rightarrow \quad \log\frac{v}{v_0} = \frac{-kt}{m}$$

<u>Derivation of $x(t)$ from $v(t)$</u>

$$\Rightarrow \quad \frac{dx}{dt} = v_0 e^{\frac{-kt}{m}}$$

$$\Rightarrow \quad dx = v_0 e^{\frac{-kt}{m}} dt$$

$$\Rightarrow \quad \int_0^x dx = \int_0^x v_0 e^{\frac{-kt}{m}} dt$$

$$\Rightarrow \quad x = v_0\left[\frac{e^{-kt/m}}{-k/m}\right]_0^t$$

$$\left[using \int e^{at}\,dt = \frac{e^{at}}{a}\right]$$

$$\Rightarrow \quad x(t) = \frac{mv_0}{k}\left(1 - e^{-kt/m}\right) \quad \text{..... (ii)}$$

(b) Substituting. $v = v = \frac{v_0}{2}$ in (i), we get :

$$t = \frac{m}{k} \log 2.$$

SUBJECTIVE SOLVED EXAMPLES

Example - 1 *A helicopter takes off along the vertical with an acceleration a = 3 m/s² and zero initial velocity. After a certain time t_1, a bullet is fired from the helicopter. At the point of take off on ground, the sound of the shot is heard at a time t_2 = 30 s after the take - off of helicopter. Find the velocity of the helicopter at the moment when the bullet is fired assuming that the velocity of sound is c = 320 m/s.*

SOLUTION :

The height of the helicopter at the instant $t = t_1$

when the bullet is fired is $h = (0)t_1 + \frac{1}{2}at_1^2$.

The time t_2 = 30s is the sum of the time taken by the helicopter to reach height h and the time taken by the sound to reach the ground level.

$$t_2 = t_1 + \frac{h}{c} \quad \Rightarrow \quad t_2 = t_1 + \frac{\alpha t_1^2}{2c}$$

Solving as a quadratic in t_1, we get :

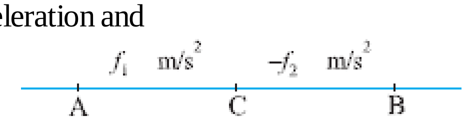
$$t_1 = \frac{\sqrt{c^2 + 2act_2} - c}{a}$$

$$\Rightarrow \quad v = at_1 = \sqrt{c^2 + 2act_2} - c = 80 \text{ m/s}.$$

Example - 2 *A train travels from rest at one station to rest at another in the same straight line distant ℓ. It moves over the first part of the distance with an acceleration of f_1 m/s² and for the remainder with retardation of f_2 m/s². Find time taken to complete the journey.*

SOLUTION :

Let AC be the distance covered with acceleration and



CB be the distance covered with retardation. Let v be

the velocity at C.

$$\Rightarrow \quad \ell = AC + BC$$

$$\Rightarrow \quad \ell = \frac{v^2 - 0^2}{2f_1} + \frac{0^2 - v^2}{-2f_2}$$

$$\Rightarrow \quad v^2 = \frac{2f_1 f_2}{f_1 + f_2}\ell$$

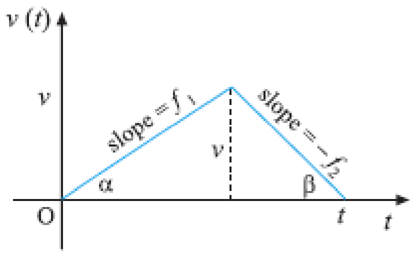
Total time $= t = t_{AC} + t_{CB} = \dfrac{v-0}{f_1} + \dfrac{0-v}{-f_2}$

$\Rightarrow \quad t = v\left(\dfrac{1}{f_1} + \dfrac{1}{f_2}\right)$

$= \sqrt{v^2\dfrac{(f_1+f_2)^2}{f_1^2 f_2^2}} = \sqrt{\dfrac{2(f_1+f_2)\ell}{f_1 \; f_2}}$

Graphical Approach :

The $v-t$ graph has two straight segments of slopes f_1 *and* $-f_2$ *as shown.*



Area under graph $= \ell$

$\Rightarrow \qquad \dfrac{1}{2} v\, t = \ell$

$\Rightarrow \qquad v = \dfrac{2\ell}{t}$

$t = \dfrac{v}{\tan\alpha} + \dfrac{v}{\tan\beta}$

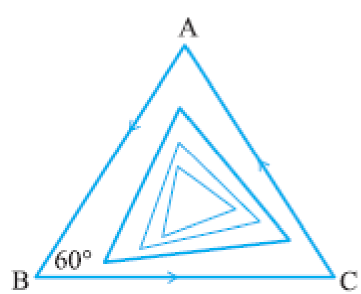
$= \dfrac{v}{f_1} + \dfrac{v}{f_2} = \dfrac{2\ell}{t}\left(\dfrac{1}{f_1} + \dfrac{1}{f_2}\right)$

$\Rightarrow \qquad t = \sqrt{\dfrac{2\ell(f_1+f_2)}{f_1 \; f_2}}$

Example - 3 *Three particles are located at the vertices of an equilateral triangle of side a. They all start moving simultaneously with a constant speed v but move in such a way that the first particle is continually headed for the second, the second for the third and the third for the first. Where and when will the particles converge ?*

SOLUTION :

As the particles are moving with equal speeds, symmetry leads us to conclude that the triangle joining the particles *A, B, C* always remains equilateral. The size of the triangle diminishes and it rotates as shown.



Whenever two moving bodies approach each other (come closer), the rate at which their separation decreases with time is known as velocity of approach and is equal to the component of relative velocity along the line joining them.

$V_{app.}$ = component of V_{AB} along AB

It can easily be seen from the figure that the velocity at which *A* approaches *B* (the component of the velocity of *A* relative to *B* along *AB*) is always 3*v*/2.

$$\overline{v_r} = \overline{v_A} - \overline{v_B}$$

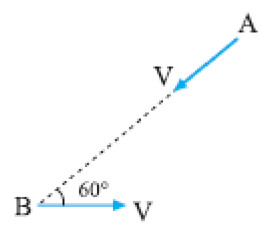
To get velocity of approach, subtract components of V_A and V_B along AB.

$\Rightarrow \quad V_{app} = v + v \cos 60° = 3v/2.$

This velocity of approach is the rate which the separation between *A* and *B* decreases with time. As this rate is constant, the time after which the separation decreases from *a* to zero is simply:

$$t = \frac{\text{decrease in separation}}{\text{rate of decrease }(V_{app})}$$

$$t = \frac{a}{V_{app}} \Rightarrow t = \frac{2a}{3v}$$



Example - 4 *Two trains are approaching each other on a long straight track with constant speed of v km/hr each. When the trains are* ℓ *km apart, a bird just in front of one train flies at a speed* ω *km/hr* ($\omega > v$) *towards the other train. When it arrives just in front of that train, it turns and flies back towards the first train. In this way, it flies back and forth between the two trains until the final moment when it is sandwiched between the trains*

(a) *Find the total distance travelled by the bird.*

(b) *Taking* ℓ = 20 km, *v* = 50 *km/hr*, ω = 70 *km/hr, draw the v-t and x-t graphs for the problem.*

SOLUTION :

The bird makes infinite trips of decreasing intervals.

(a) Total time interval for bird's motion = time taken by the trains to collide

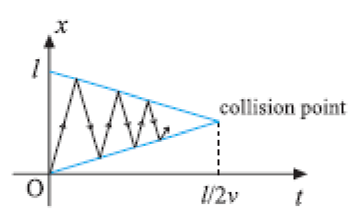
$$\Rightarrow \text{Total time} \frac{\text{inital seperation}}{\text{velocity of approach}}$$

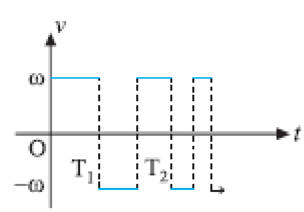
$$= \frac{\ell}{2v}$$

$\Rightarrow$ Total distance covered by the bird

$$= \omega\left(\frac{\ell}{2v}\right)$$

(b) As the bird moves with constant speed and re verses its direction at the end of each trip, its *v-t* graph is a series of horizontal segment of decreas ing lengths. In the *x-t* graph, the trains have straight lines of slopes +50 and –50 *km/hr*. The bird's *x-t* graph is a series of lines with slopes +70 and –70 *km/hr.*





Example - 5 *A rubber ball is released from a height of 4.90 m above the floor. It bounces repeatedly, always rising to 81/100 of the height through which it falls.*

(a) *Ignoring the practical fact that the ball has a finite size (in other words, treating the ball as point mass that bounces an infinite number of times), show that its total distance of travel is 46.7 m.*

(b) *Determine the time required for the infinite number of bounces.*

(c) *Determine the average speed*

SOLUTION :

Let $h = 4.9\ m$

(a) Distance travelled

$$= h + 2\left[\frac{81}{100}h + \left(\frac{81}{100}\right)^2 h + \ldots\ldots\ldots\ldots\right]$$

$$= h + 2h\left(\frac{0.81}{1-0.81}\right)$$

$$= 4.9 + \frac{9.8 \times 0.81}{0.19} = \mathbf{46.7\,m}$$

(b) Time required to fall through height h

$$= \sqrt{\frac{2h}{g}}$$

Total time $= \sqrt{\frac{2h}{g}} + 2\left(\sqrt{\frac{2nh}{g}} + \sqrt{\frac{2n^2h}{g}} + \ldots\ldots\ldots\ldots\right)$

where $n = \frac{81}{100}$

Total time $= \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2h}{g}}\left(\frac{\sqrt{n}}{1-\sqrt{n}}\right)$

$$= \sqrt{\frac{2h}{g}}(1+18) = \mathbf{19\ sec}$$

(c) Average Speed $= \frac{\text{distance travelled}}{\text{time taken}}$

$$= \frac{46.7}{19} = \mathbf{2.46\ m/s}$$

Example - 6 *A stone is dropped from the top of a tower. When it crosses a point 5 m below the top, another stone is let fall from a point 25 m below the top. Both stones reach the bottom of the tower simultaneously. Find the height of the tower.*

SOLUTION :

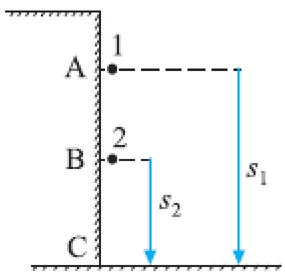
Let us take downward direction as the positive direction. At the moment when the first stone crosses A, its velocity

$$= \sqrt{0 + 2(-9.8)(-5)}$$

$$= +7\sqrt{2}\ m/s$$

Let t = time taken by each stone to reach the ground after the second stone is dropped.

$s_1 - s_2 = 20$



$$\Rightarrow \left(+7\sqrt{2}\,t + \frac{1}{2}gt^2\right) - \left(+\frac{1}{2}gt^2\right) = +20$$

$$\Rightarrow t = \frac{20}{7\sqrt{2}} \text{ sec.}$$

$$\Rightarrow BC = \left(0 \times t + \frac{1}{2}gt^2\right)$$

$$\Rightarrow BC = \frac{1}{2} \times 9.8 \times \frac{400}{49 \times 2} = 20m$$

$$\Rightarrow OC = OB + BC = 25 + 20 = \mathbf{45\ m}.$$

Example - 7 *In a car race, car A takes time t less to finish than car B and passes the finishing point with a velocity V more than car B. Assuming the cars starts from rest and travel with constant accelerations a_1 and a_2 respectively. Show that $V^2 = a_1 a_2 t^2$.*

SOLUTION :

Let s = distance to be travelled by each car. Let V_A, V_B be the final velocities of the cars and t_A, t_B be the time taken by them to cover distance s.

We have, $V_A - V_B = V$

and $t_B - t_A = t$

$$\Rightarrow \sqrt{2a_1 s} - \sqrt{2a_2 s} = V$$

(*Using* $v^2 = u^2 + 2as$)

and $$\sqrt{\frac{2s}{a_2}} - \sqrt{\frac{2s}{a_1}} = t$$

(*Using* $s = \frac{1}{2}at^2$)

Dividing the above two equations, we get :

$$\Rightarrow \frac{\sqrt{a_1} - \sqrt{a_2}}{\frac{1}{\sqrt{a_2}} - \frac{1}{\sqrt{a_1}}} = \frac{V}{t}$$

$$\Rightarrow \sqrt{a_1 a_2} = \frac{V}{t}$$

$$\Rightarrow \mathbf{V^2 = a_1 a_2 t^2}$$

Example - 8 *The driver of a car moving at 30 m/s suddenly sees a truck that is moving in the same direction at 10 m/s and is 60 m ahead. The maximum deceleration of the car is 5 m/s^2.*

(a) *Will the collision occur if the driver's reaction time is zero ? If so, when ?*

(b) *If the car driver's reaction time of 0.5 s is included, what is the minimum magnitude of deceleration required to avoid the collision ?*
to avoid the collision ?

SOLUTION :

(a) Let us assume that they collide after t seconds.

Position of car after t seconds :

$$x_C(t) = 30t + 1/2(-5)(t)^2$$

Position of truck after t seconds :

$$x_T(t) = 60 + 10t$$

Now, $x_C(t) = x_T(t)$

$\Rightarrow$ $30t + 1/2(-5)(t)^2 = 60 + 10t$

$\Rightarrow$ $5t^2 - 40t + 120 = 0$

$\Rightarrow$ t does not have any real value

Hence collision does not occur.

(b) During 0.5 seconds, separation reduces to

$60 - 30(0.5) + 10(0.5) = 50\ m.$

Let a be the **magnitude** of acceleration

To avoid collision, the truck and the car should not be at the same position at any time instant.

$\Rightarrow$ $x_C(t) = x_T(t)$ must not give any real value of t.

i.e. $30t + \frac{1}{2}(-a)\,t^2 = 50 + 10t$ must have non-real roots.

$-(1/2)\,at^2 + 20t - 50 = 0$

$\Rightarrow$ $D = 400 - 4 \times \frac{a}{2} \times 50 < 0$

$\Rightarrow$ $a > 4$

Thus, minimum magnitude of deceleration is 4 m/s^2.

Example - 9 *A particle is dropped from the top of a tower of height h and at the same moment, another particle is projected upward from the bottom. They meet when the upper one has descended one third of the height of the tower. Find the ratio of their velocities when they meet and the initial velocity of the lower.*

SOLUTION :

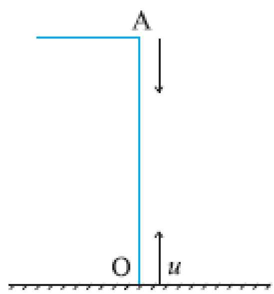
Let t = time taken before colliding.

Taking the line of motion as X-axis and ground as origin. The final positions of the particles after time t are :

For the upper :

$$x_1 = x_i + ut + \frac{1}{2}at^2$$

$$\Rightarrow \quad x_1 = h + 0 - \frac{1}{2}gt^2$$



For the lower :

$$x_2 = x_i + ut + \frac{1}{2}at^2$$

$$\Rightarrow \quad x_2 = 0 + ut - \frac{1}{2}gt^2$$

At time t, $x_1 = x_2$

$$\Rightarrow \quad h - \frac{1}{2}gt^2 = ut - \frac{1}{2}gt^2$$

$$\Rightarrow \quad h = ut \qquad \ldots\ldots\text{(i)}$$

Also we have for the upper particle,

$$\Rightarrow \quad s_1 = -\frac{1}{3}h - \frac{1}{2}gt^2 = \frac{-h}{3}$$

$$\Rightarrow \quad h = \frac{3}{2}gt^2 \qquad \ldots\ldots\text{(ii)}$$

Combining (i) and (ii), we get : $h = \frac{3}{2}g\frac{h^2}{u^2}$

$$\Rightarrow \quad u = \sqrt{\frac{3gh}{2}}$$

Final velocities are : $v_1 = 0 - gt$ and $v_2 = u - gt$

$$\Rightarrow \quad \frac{v_1}{v_2} = \frac{-gt}{u - gt}$$

Using $\quad u = \frac{h}{t} = \frac{3}{2}gt$

$$\frac{v_1}{v_2} = \frac{-gt}{\frac{3}{2}gt - gt} = -2$$

The ratio is negative because v_1 and v_2 are opposite

Hence the ratio of speed is 2 : 1.

Example - 10 *An elevator whose floor to the ceiling distance is 2.50 m, starts ascending with a constant acceleration of 1.25 m/s^2. One second after the start, a bolt begins falling from the ceiling of the elevator. Calculate :*

(a) *free fall time of the bolt.*

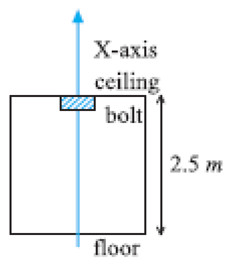
(b) *the displacement and distance covered by the bolt during the free fall in the reference frame fixed to the shaft of the elevator.*

SOLUTION :

We will take the line of motion of elevator and bolt as X-axis and the floor's initial position (*when bolt starts falling*) as origin.

(a) At the moment when the bolt starts falling, the speed of the elevator and the bolt

$$= 0 + at = 0 + (1.25)\,1 = 1.25\ m/s$$



t = time after which bolt strikes the floor.

The X-coordinates of the bolt and floor at time t are:

$$x_b = x_i + u\,t + 1/2\,a\,t^2$$

$$= 2.5 + 1.25\,t - 1/2\,g\,t$$

Note : *As the bolt is freely falling, its acceleration is negative = – g*

$$x_f = x_i + u\,t + 1/2\,a\,t^2 = 0 + 1.25\,t + 1/2\,(1.25)\,t^2$$

Note : *The acceleration of floor remains same at* 1.25 m/s^2. As the bolt strikes the floor at time t,

$$x_b = x_f$$

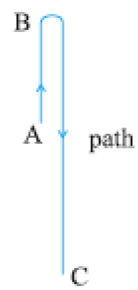
$$\Rightarrow\quad 2.5 + 1.25t - 1/2\,gt^2 = 0 + 1.25t + 1/2\,(1.25)t^2$$

$$\Rightarrow\quad t = 2/3\ sec.$$

(b) $$S_b = ut + \frac{1}{2}at^2 = 1.25\left(\frac{2}{3}\right) + \frac{1}{2}(-10)\frac{4}{9}$$

$$= -1.39\ m$$

The bolt goes up from A to B and then comes down from B to C.



$\Rightarrow$ Distance covered

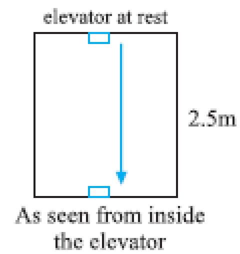
$$= 2\,(AB) + (AC)$$

$$2\left|\frac{0^2 - u^2}{-2g}\right| + \left|-1.39\right|$$

$$= \frac{2\,(1.25)^2}{20} + 1.39 = \mathbf{1.546\ m}$$

Alternative Approach for part (a)

Let us analyse the motion of the bolt as seen in the reference frame of the elevator



$$S_{be} = -2.5m,\quad u_{be} = 0\ m/s$$

$$a_{be} = a_b - a_e = (-10) - (+1.25) = -11.25\ m/s^2$$

$$S_{be} = u_{be}t + \frac{1}{2}a_{be}t^2$$

$$-2.5 = 0t - \frac{1}{2}(11.25)\,t^2 \quad\Rightarrow\quad t = \frac{2}{3}s$$

Example - 11 *A particle is projected vertically upwards from earth's surface with a velocity just sufficient to carry it to infinity. Find the time it takes in reaching a height h taking the radius of earth as R and the acceleration due to gravity at the surface as g.*

SOLUTION :

The retarding force of earth at a distance x from centre is $\frac{GMm}{x^2}$

$$\Rightarrow \quad \text{Magnitude of acceleration} = \frac{GM}{x^2}$$

$$\Rightarrow \quad a(x) = \frac{-GM}{x^2} = \frac{-gR^2}{x^2}.$$

As acceleration is in terms of x, we put it as v $\frac{dv}{dx}$.

$$\Rightarrow \quad v\frac{dv}{dx} = \frac{-gR^2}{x^2}$$

$$\Rightarrow \quad \int_{v_0}^{v} v\,dv = gR^2 \int_R^x \frac{dx}{x^2}$$

$$\Rightarrow \quad \frac{v^2 - v_0^2}{2} = -gR^2\left(\frac{1}{R} - \frac{1}{x}\right) \qquad \text{......(i)}$$

For $x = \infty$, *v (as given in the problem)* $= 0$

$$\Rightarrow \quad v_0^2 = 2gR$$

$$\Rightarrow \quad v^2 = \frac{+2gR^2}{x} \qquad \text{[by substituting } v_o^2 = 2gR \text{ in (i)]}$$

$$\Rightarrow \quad v = \frac{dx}{dt} = \sqrt{\frac{2gR^2}{x}}$$

$$\Rightarrow \quad \int_R^{R+h} \sqrt{x}\,dx = \int_0^t \sqrt{2gR^2}\,dt$$

$$\Rightarrow \quad \frac{2}{3}\left[(R+h)^{\frac{3}{2}} - R^{\frac{3}{2}}\right] = \sqrt{2gR^2}\,t$$

$$\Rightarrow \quad t = \frac{1}{3}\sqrt{\frac{2R}{g}}\left[\left(1+\frac{h}{R}\right)^{\frac{3}{2}} - 1\right]$$

Example - 12 *A particle moves along a horizontal path, such that its velocity is given by* $v = \left(3t^2 - 6t\right)\ m/s$, *where t is the time in seconds. If it is initially located at the origin O,determine the distance travelled by the particle in time interval from t = 0 to t = 3.5s and the particle's average velocity and average speed during the same time interval.*

SOLUTION :

$$v(t) = 3t^2 - 6t$$

$$\frac{dx}{dt} = 3t^2 - 6t$$

$$\int_0^x dx = \int_0^t \left(3t^2 - 6t\right) dt$$

$$x(t) = t^3 - 3t^2 = t^2(t-3)$$

(a) Average velocity $= \frac{\Delta x}{\Delta t} = \frac{x(3.5) - x(0)}{3.5}$

$$= \frac{\left[(3.5)^3 - 3(3.5)^2\right] - 0}{3.5} = 1.75\, m/s$$

(b) To calculate distance travelled, we have to locate the turning points (where $v = 0$) i.e. the instants when the particle changes direction of motion.

$$v(t) = 3t(t-2) = 0$$

$\Rightarrow$ $t = 0s,\ 2s$

Distance travelled $= \Delta x_1| + |\Delta x_2|$

$$= |x(2) - x(0)| + |x(3.5) - x(2)|$$

Using $x(t) = t^2(t-3)$ we get :

$$x(2) = 2^2(2-3) = -4m$$

$$x(3.5) = (3.5)^2\ (3.5-3)$$

$$= 6.125\ m$$

$\Rightarrow$ Distance travelled

$$= -4 - 0| + |6.125 - (-4)| = 4 + 10.128 = 14.125\ m$$

(c) Average speed $= \frac{\text{Distance covered}}{\text{time}} = \frac{14.125}{3.5} = 4.036\ m/s$

Example - 13 *Two particles A and B start moving simultaneously along the line joining them in the same direction with accelerations of 1 m/s² and 2 m/s² and speeds 3 m/s and 1 m/s respectively. Initially A is 10 m behind B. What is the minimum distance between them ?*

SOLUTION :

Let the initial position of A be the origin and AB be the X-axis

We use $x(t) = x(0) + ut + \frac{1}{2}at^2$ in such problems.

After time t,

$$x_B(t) = 10 + 1t + \frac{1}{2}(2)t^2$$

$$x_A(t) = 0 + 3t + \frac{1}{2}(1)\,t^2$$

The separation S, at any instant t is equal to the difference of their coordinates.

$$S = x_B - x_A$$

$$S(t) = 10 - 2t + \frac{t^2}{2} \equiv at^2 + bt + c$$

As $S(t)$ is a quadratic polynomial in t, its minimum value occurs at $t = \frac{-b}{2a}$

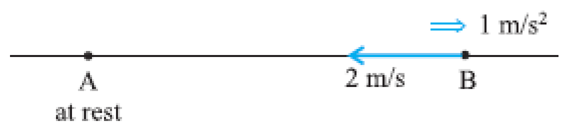
$$\Rightarrow \quad \text{S is minimum at } t = \frac{-(-2)}{1} = 2s$$

$$S_{min} = S(2) = 10 - 2(2) + \frac{4}{2} = 8\,m$$

Alternative Approach :

We can also analyse the problem relative to the particle A.

The following figure shows the situation as it is seen by A.



$$u_{BA} = u_B - u_A = 1 - 3 = -2m/s$$

$$a_{BA} = a_B - a_A$$

$$2 - 1 = 1m/s^2$$

As B has an acceleration opposite to its velocity, it slows down to rest and then reverses its direction of motion. Finally it goes away from A. The minimum separation occurs at the instant when B has zero speed.

$$v_{BA} = -2 + It$$

$$0 = -2 + t \quad \Rightarrow \quad t = 2s$$

$$BP = \frac{0^2 - (-2)^2}{2(1)} = 2m$$

Minimum separation = AP = 10 – 2 = 8 m.

Example - 14 *An elevator without a ceiling is ascending with a constant speed of 6 m/s. A boy on the elevator throws a ball directly upward, from a height of 2.0 m above the elevator floor. At this time the elevator floor is 30 m above the ground. The initial speed of the ball with respect to the elevator is 9 m/s.* (Take g = 10 m/s^2)

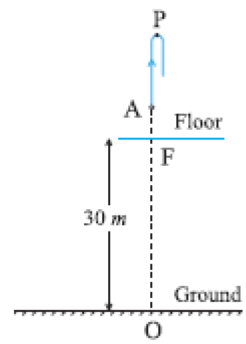
(a) *What maximum height above the ground does the ball reach ?*

(b) *How long does the ball take to return to the elevator floor ?*

SOLUTION :

(a) As the height is to be calculated from ground, we analyse the problem relative to ground.

The balls is thrown up from *A* and goes up to *P* before coming down again.



During AP,

$$u_{ball} = u_{be} + u_e$$

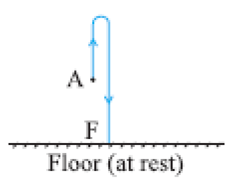
$$= 9 + 6 = 15\ m/s$$

$$s = \frac{v^2 - u^2}{2a} = \frac{0^2 - 15^2}{2(10)} = 11.25\,m$$

Maximum height from ground = OP = OF + FA + AP

= 30 + 2 + 11.25 = 43.25 *m*

(b) To find the time taken to hit the floor again, it is easier to do the analysis relative to the elevator.



$$u_{be} = 9m/s$$

$$a_{be} = a_b - a_e$$

$$= (-g) - 0$$

$$= -g = -10\,m/s^2$$

$$S_{be} = -AF = -2m$$

$$S_{be} = u_{be}t + \frac{1}{2}a_{be}t^2$$

$$-2 = 9t - 5t^2$$

$\Rightarrow$ $t = 2s.$

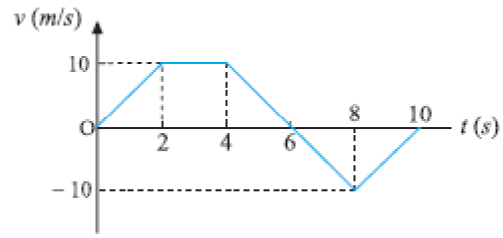
Example - 15 *Velocity-time graph of a particle moving in a straight line is shown in figure. At time $t = 0$, position of the particle from origin is 10 m. Find :*

(a) *acceleration of particle at $t = 5s$, $3s$ and $9s$.*

(b) *Find the position of particle at $t = 10$ sec.*

(c) *write down x-t equation for time interval :*

(i) $0 \le t \le 2s$, **(ii)** $4s \le t \le 8s$



SOLUTION :

(a) a = slope in v-t graph

At $t = 5s$

$$a = \text{slope of BC} = -\frac{10}{2} = -5\,m/s^2$$

At $t = 3s$

a = slope of AB = 0 m/s^2

At $t = 9s$

$$a = \text{slope of DE} = \frac{10}{2} = 5\,m/s^2$$

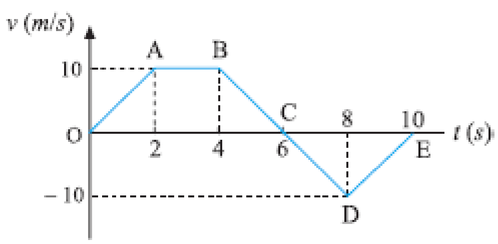
(b) $x(10)$ = $x(0)$ + displacement

= $x(0)$ + Area under v-t graph

= 10 + ar(OABC) – ar(CDE)

= 10 + 40 – 20

= 30 m



(c) (i) $$x(t) = x(0) + ut + \frac{1}{2}at^2$$

$$= 10 + 0t + \frac{1}{2}\text{slope of OA)}\, t^2$$

$$= 10 + \frac{5}{2}t^2$$

(ii) During this interval,

a = slope of BD = – 5 m/s^2

$x(t) = x(4)$ + displacement during $(t – 4)\,s$

$$= x(4) + u(t-4) + \frac{1}{2}a(t-4)^2$$

where u = velocity at $t = 4s$

= $x(0)$ + Area under

$$OAB + 10(t-4) - \frac{5}{2}(t-4)^2$$

$= 10 + 30 + 10t – 40 – 2.5\,(t – 4)^2$

$$= -40 + 30t - 2.5t^2$$

Note : In general,

$$x(t) = x(t_0) + V_0(t - t_0) + \frac{1}{2}a(t - t_0)^2$$

where V_0 = velocity at $t = t_0$.

THINGS TO REMEMBER

1. **Displacement** is the vector joining the initial position of the particle to its final position during an interval of time. The change in the position of a moving object is known as displacement.

2. The **distance travelled** i.e., *the length of the actual path* is a scalar quantity which is quite different from displacement.

3. The **average velocity** of a moving particle over a certain time interval is defined as the displacement divided by the time duration.

$$\text{Average Velocity} = \frac{\text{displacement}}{\text{time interval}} = \frac{\Delta \bar{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j}$$

For straight line motion along X-axis, we have :

$$V_{av} = \frac{\Delta x}{\Delta t}$$

4. **Average speed** is a scalar and is defined over a time interval as :

$$\text{Average Speed} = \frac{\text{distance}}{\text{time interval}}$$

5. The **average acceleration** for a finite time interval is defined as :

$$\text{Average acceleration} = \frac{\text{change in velocity}}{\text{time interval}}$$

Average acceleration is a *vector quantity* whose direction is same as that of the change in velocity.

$$a_{av} = \frac{\Delta \bar{v}}{\Delta t} = \frac{\bar{v}_f - \bar{v}_i}{\Delta t}$$

6. The **instantaneous velocity** at a given moment (say, *t*) is the limiting value of the average velocity as we let Δt approach zero.

$$V_{inst.} = \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right) = \frac{dx}{dt}$$

7. The **instantaneous acceleration** of a particle is its acceleration at a particular instant of time. It is defined as the derivative (rate of change) of velocity with respect to time :

$$\bar{a} = \frac{d\bar{v}}{dt} \quad \lim_{\Delta t \to 0}\left(\frac{\Delta \bar{v}}{\Delta t}\right)$$

8. **Important Ideas for Straight Line Motion :**

(a) If $\mathbf{F_{net} = 0}$, then $\mathbf{a = 0}$ and the motion is known as **uniform motion** because the velocity does not change. One can also say that the particle is moving with a constant velocity. The average velocity in an interval and the instantaneous velocity are equal in this motion.

(b) If the force is constant, acceleration is also constant and the motion is known as **uniformly accelerated motion**. *Free fall* (motion under gravity) is a common example of uniformly accelerated motion because the force (= mg) is constant. In this motion, average acceleration in any interval is same as the instantaneous acc.

(c) If acceleration (& force) is in same direction as velocity, then speed of the particle increases.

(d) If acceleration (& force) is in opposite direction to the velocity then speed decreases i.e. the particle slows down. This situation is known as **retardation**.

9. Free Fall

(a) Acceleration = – g

(b) Time taken to fall through a height h from rest = $\sqrt{\frac{2h}{g}}$

(c) If thrown up with speed u :

(i) $h_{max} = \frac{u^2}{2g}$ maximum height attained)

(ii) Time taken to go up = time taken to come down = u/g

(iii) Total time to come back = $2u/g$

(d) Initial velocity required to go up to a height h :

$$u = \sqrt{2gh}$$

(e) If dropped from a height h, the particle hits the ground with speed :

$$v = \sqrt{2gh}$$

10. Important Results for Derivatives :

(i) $\frac{d}{dt} t^n = n\, t^{n-1}$

(ii) $\frac{d}{dt} \sin\omega t = \omega \cos \omega t$ (ω *is constant*).

(iii) $\frac{d}{dt} \cos\omega t = -\omega \sin \omega t$ (ω *is constant*).

(iv) $\frac{d}{dt} \log t = \frac{1}{t}$

(v) $\frac{d}{dt} e^{at} = ae^{at}$

11. Important Points to Remember about Graphs :

(a) For uniformly accelerated motion ($a \neq 0$), x-t graph is a parabola (opening upwards if $a > 0$ and opening downwards if $a < 0$). The slope of tangent at any point of the parabola gives the velocity at that instant.

(b) For uniformly accelerated motion ($a \neq 0$), v-t graph is straight line whose slope gives the acceleration of the particle.

(c) In general, the slope of tangent in x-t graph is velocity and the slope of tangent in v-t graph is the acceleration.

(d) The area under a-t graph gives the change in velocity.

(e) The area between the v-t graph and the time-axis gives the distance travelled by the particle, if we take all areas as positive ; shaded area = distance covered in the time interval $t = t_1$ to $t = t_2$.

(g) From the v-x graph, we can do an analysis for acceleration using $a = v\dfrac{dv}{dx} = v$ (slope of tangent in v-x graph)

12. Important Results for Integration :

(i) (a) $\int t^n dt = \dfrac{t^{n+1}}{n+1} \quad (n \neq -1)$ **(b)** $\int (at + b)^n dt = \dfrac{1}{a}\dfrac{(at+b)^{n+1}}{(n+1)}$

(ii) (a) $\int \dfrac{dt}{t} = \log t \Rightarrow \int_{t_1}^{t_2} \dfrac{dt}{t} = \log \dfrac{t_2}{t_1}$ **(b)** $\int \dfrac{dt}{at+b} = \dfrac{1}{a}\ell og\,(at+b)$

(iii) (a) $\int e^t\, dt = e^t$ **(b)** $\int e^{at}\, dt = \dfrac{e^{at}}{a}$

(iv) (a) $\int \sin t = -\cos t$ **(b)** $\int \sin at = \dfrac{-\cos at}{a}$

(v) (a) $\int \cos t = \sin t$ **(b)** $\int \cos at\, dt = \dfrac{\sin at}{a}$