Session 11

Inverse of a Function

Inverse of a Function

Let $f : A \to B$ be a one-one and onto function, then there exists a unique function, $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A$ and $y \in B$.

Then, g is said to be inverse of f.

Thus, $g = f^{-1} : B \to A = \{(f(x), x) | (x, f(x)) \in f\}$

Let us consider a one-one function with domain A and range B.

Where, $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$ and $f : A \to B$ is given by $f(x) = 2x$, then write f and f^{-1} as a set of ordered pairs.

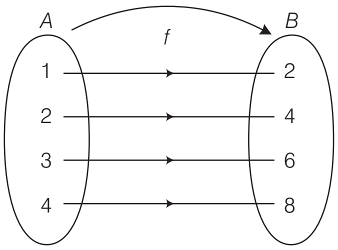

Figure 3.43

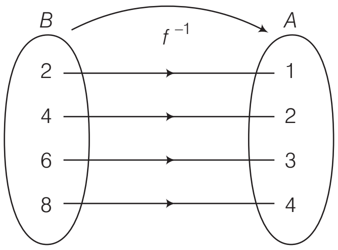

Figure 3.44

Here, member $y \in B$ arises from one and only one member $x \in A$.

So, $$f = \{(1, 2) (2, 4) (3, 6) (4, 8)\}$$

and $$f^{-1} = \{(2, 1) (4, 2) (6, 3) (8, 4)\}$$

In above function,

Domain of $f = \{1, 2, 3, 4\} =$ Range of f^{-1}.

Range of $f = \{2, 4, 6, 8\} =$ Domain of f^{-1}.

which represents for a function to have its inverse, it must be **one-one onto** or **bijective**.

Example 109 If $f(x) = 3x - 5$, find $f^{-1}(x)$.

Sol. Here, $f(x) = 3x - 5$

which is clearly bijective as it is linear in x.

Now, let $f(x) = y \Rightarrow y = 3x - 5$

$$\Rightarrow \quad x = \frac{y + 5}{3}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{y + 5}{3} \qquad [\text{as } f(x) = y \Rightarrow x = f^{-1}(y)]$$

Therefore, $$f^{-1}(x) = \frac{x + 5}{3}$$

Example 110 If $f : [1, \infty) \to [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, find $f^{-1}(x)$, (assume bijection).

[IIT JEE 2001, 2002]

Sol. Let $y = f(x)$

$$\therefore \quad y = \frac{x^2 + 1}{x} \quad \Rightarrow \quad x^2 - xy + 1 = 0$$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{y \pm \sqrt{y^2 - 4}}{2} \qquad [\text{as } f(x) = y \Rightarrow x = f^{-1}(y)]$$

$$\Rightarrow \quad f^{-1}(x) = \frac{x \pm \sqrt{x^2 - 4}}{2}$$

Since, range of inverse function is $[1, \infty)$, therefore neglecting the negative sign, we have

$$f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$$

Example 111 ***Let*** $f(x) = x^3 + 3$ ***be bijective, then find its inverse.***

Sol. Let $y = x^3 + 3$ $\qquad$ [i.e. $y = f(x)$]

$$\Rightarrow \quad x^3 = y - 3$$

$$\Rightarrow \quad x = (y - 3)^{1/3}$$

$$\Rightarrow \quad f^{-1}(y) = (y - 3)^{1/3} \quad [y = f(x) \Rightarrow f^{-1}(y) = x]$$

$$\Rightarrow \quad f^{-1}(x) = (x - 3)^{1/3}$$

Thus, $$f^{-1}(x) = (x - 3)^{1/3}$$

when $f(x) = x^3 + 3$ is bijective.

Example 112 Find the inverse of the function, (assuming onto).

$$y = \log_a (x + \sqrt{x^2 + 1}),\ (a > 1).$$

Sol. We have, $y = \log_a(x + \sqrt{x^2 + 1})$

Since, $\sqrt{x^2 + 1} > |x|$

$\therefore$ It is defined for all x.

Now, $y = \log_a(x + \sqrt{x^2 + 1})$,

which is strictly increasing when $a > 1$.

Thus, one-one. Also, given that $f(x)$ is onto. [where $y = f(x)$]

Hence, the given function is invertible.

Now, $y = \log_a (x + \sqrt{x^2 + 1})$

$\Rightarrow \quad a^y = x + \sqrt{x^2 + 1}$ and $a^{-y} = \sqrt{x^2 + 1} - x$

$\Rightarrow \quad x = \frac{1}{2}(a^y - a^{-y})$

Hence, the inverse in the form $y = f^{-1}(x)$ is,

$$y = \frac{1}{2}(a^x - a^{-x})$$

Graphical Representation of Invertible Functions

Let (h, k) be a point on the graph of the function f. Then, (k, h) is the corresponding point on the graph of inverse of f, i.e. g.

The line segment joining the points (h, k) and (k, h) is bisected at right angle by the line $y = x$.

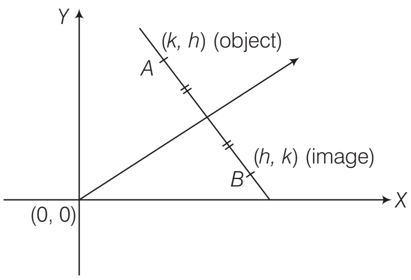

Figure 3.45

So, that the two points play object-image role in the line $y = x$ as plane mirror.

It follows that the graph of $y = f(x)$ and its inverse written in form $y = g(x)$ are symmetrical about the line $y = x$.

The graphs $y = f(x)$ and $y = f^{-1}(x)$, if they intersect then they meet on the line $y = x$ only. Hence, the solutions of $f(x) = f^{-1}(x)$ are also the solutions of $f(x) = x$.

Example 113 Let $f : R \to R$ be defined by $f(x) = \frac{e^x - e^{-x}}{2}$. Is $f(x)$ invertible? If so, find its inverse.

Sol. Let us check for invertibility of $f(x)$.

(a) **One-one** Here, $f'(x) = \frac{e^x + e^{-x}}{2}$

$\Rightarrow \quad f'(x) = \frac{e^{2x} + 1}{2e^x}$, which is strictly increasing as $e^{2x} > 0$ for all x. Thus, it is one-one.

(b) **Onto** Let $y = f(x)$

$\Rightarrow \quad y = \frac{e^x - e^{-x}}{2}$, where y is strictly monotonic.

Hence, range of $f(x) = (f(-\infty), f(\infty))$

$\Rightarrow$ Range of $f(x) = (-\infty, \infty)$

So, range of $f(x) =$ codomain

Hence, $f(x)$ is one-one and onto.

(c) **To find f^{-1}** $\quad y = \frac{e^{2x} - 1}{2e^x}$

$\Rightarrow \quad e^{2x} - 2e^x y - 1 = 0 \Rightarrow e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$

$\Rightarrow \quad x = \log (y \pm \sqrt{y^2 + 1})$

$\Rightarrow \quad f^{-1}(y) = \log (y \pm \sqrt{y^2 + 1})$

[as $f(x) = y \Rightarrow x = f^{-1}(y)$]

Since, $e^{f^{-1}(x)}$ is always positive.

So, neglecting the negative sign.

Hence, $f^{-1}(x) = \log (x + \sqrt{x^2 + 1})$

Example 114 Let $f : [1/2, \infty) \to [3/4, \infty)$, where $f(x) = x^2 - x + 1$. Find the inverse of $f(x)$. Hence, solve the equation $x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$.

Sol. (a) $f(x) = x^2 - x + 1$

$\Rightarrow f(x) = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$, which is clearly one-one and onto in given domain and codomain.

Thus, its inverse can be obtained.

Let $\quad y = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$

$\Rightarrow \quad x - \frac{1}{2} = \pm \sqrt{y - \frac{3}{4}} \Rightarrow x = \frac{1}{2} \pm \sqrt{y - \frac{3}{4}}$

$\Rightarrow \quad f^{-1}(y) = \frac{1}{2} + \sqrt{y - \frac{3}{4}} \quad [f(x) = y \Rightarrow x = f^{-1}(y)]$

[neglecting – ve sign as x is always +ve]

$$\Rightarrow \quad f^{-1}(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$$

(b) To solve $x^2 - x + 1 = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$, since solution of $f(x) = f^{-1}(x)$ are solutions of $f(x) = x$.

i.e. $\quad f(x) = x \Rightarrow x^2 - x + 1 = x$

$\Rightarrow \quad x^2 - 2x + 1 = 0 \;\Rightarrow\; (x-1)^2 = 0$

$\therefore\; x = 1$ is the required solution.

Properties of Inverse Functions

(i) *The inverse of a bijection is unique.*

Proof Let $f : A \to B$ be a bijection. If possible let $g : B \to A$ and $h : B \to A$ be two inverse functions of f. Also, let $a_1, a_2 \in A$ and $b \in B$ such that $g(b) = a_1$ and $h(b) = a_2$, then $g(b) = a_1 \Rightarrow f(a_1) = b$

$$h(b) = a_2 \Rightarrow f(a_2) = b$$

But, since f is one-one, so $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

$\Rightarrow \quad g(b) = h(b), \forall\, b \in B$

(ii) *If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are identity functions on the sets A and B, respectively.*

Remarks

(a) The graphs of f and g are the mirror images of each other in the line $y = x$. As shown, in the figure given below a point (x', y') corresponding to $y = x^2 (x \geq 0)$ changes to (y', x') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

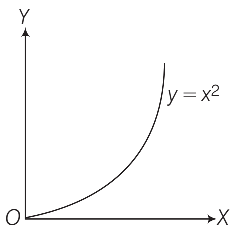

Figure 3.46

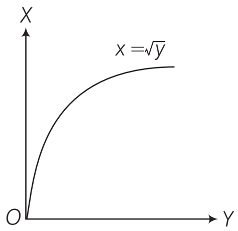

Figure 3.47

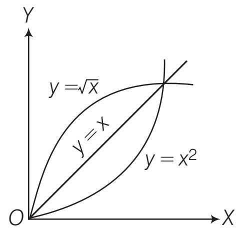

Figure 3.48

(b) If $f(x)$ has its own inverse as in $f(x) = \frac{1}{x}$, then $f(x) = f^{-1}(x)$ will have infinite solutions but $f(x) = f^{-1}(x) = x$ will have only one solution.

(iii) *The inverse of a bijection is also a bijection.*

Proof Let $f : A \to B$ be a bijection and $g : B \to A$ be its inverse. We have to show that g is one-one and onto.

One-one Let $g(b_1) = a_1$ and

$$g(b_2) = a_2 : a_1, a_2 \in A \text{ and } b_1, b_2 \in B$$

Then, $\quad g(b_1) = g(b_2)$

$\Rightarrow \quad a_1 = a_2$

$\Rightarrow \quad f(a_1) = f(a_2)$

$\Rightarrow \quad b_1 = b_2 \quad$ [$\because f$ is a bijection]

[$\because g(b_1) = a_1 \Rightarrow b_1 = f(a_1)$, $g(b_2) = a_2 \Rightarrow b_2 = f(a_2)$]

which proves that g is one-one.

Onto Again, if $a \in A$, then

$a \in A \Rightarrow \exists\, b \in B$ such that $f(a) = b$

[by definition of f]

$\Rightarrow \exists\, b \in B$ such that $g(b) = a$ [$\because f(a) = b \Rightarrow a = g(b)$]

which proves that g is onto.

Hence, g is also a bijection.

(iv) If f and g are two bijections $f : A \to B$, $g : B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1} o g^{-1}$.

Proof Since, $f : A \to B$ and $g : B \to C$ are two bijections.

$\therefore\; gof : A \to C$ is also a bijection.

[by theorem, the composite of two bijections is a bijection]

As such gof has an inverse function $(gof)^{-1} : C \to A$.

We have to show that $(gof)^{-1} = f^{-1} o g^{-1}$.

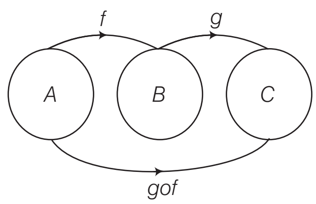

Figure 3.49

Now, let $a \in A$, $b \in B$, $c \in C$ such that

$$f(a) = b \text{ and } g(b) = c$$

So, $\quad (gof)(a) = g[f(a)] = g(b) = c$

Now, $\quad f(a) = b \Rightarrow a = f^{-1}(b) \quad$ ...(i)

$g(b) = c \Rightarrow b = g^{-1}(c) \quad$ ...(ii)

$(gof)(a) = c \Rightarrow a = (gof)^{-1}(c) \quad$ ...(iii)

Also, $(f^{-1} o g^{-1})(c) = f^{-1}[g^{-1}(c)] \quad$ [by definition]

$$= f^{-1}(b) \quad \text{[by Eq. (ii)]}$$

$$= a \quad \text{[by Eq. (i)]}$$

$$= (gof)^{-1}(c) \quad \text{[by Eq. (iii)]}$$

$\therefore \quad (gof)^{-1} = f^{-1} o g^{-1}$,

which proves the theorem.

Example 115 Let $g(x)$ be the inverse of $f(x)$ and $f'(x) = \dfrac{1}{1+x^3}$. Find $g'(x)$ in terms of $g(x)$.

Sol. We know, if $g(x)$ is inverse of $f(x)$.

$\Rightarrow \quad g\{f(x)\} = x \Rightarrow g'\{f(x)\} \cdot f'(x) = 1$

$\Rightarrow \quad g'\{f(x)\} = \dfrac{1}{f'(x)} = 1 + x^3 \Rightarrow g'\{f(g(x))\} = 1 + (g(x))^3$

$\Rightarrow \quad g'(x) = 1 + (g(x))^3 \quad [\because f(g(x)) = x]$

Example 116 ***If*** $f : R \to R$ ***is defined by*** $f(x) = x^2 + 1$, ***find the value of*** $f^{-1}(17)$ ***and*** $f^{-1}(-3)$.

Sol. $f(x) = x^2 + 1$; $f^{-1}(17) \Rightarrow f(x) = 17 \Rightarrow x^2 + 1 = 17$

$\Rightarrow \quad x = \pm 4$ and $f^{-1}(-3)$

$\Rightarrow \quad f(x) = -3 \Rightarrow x^2 + 1 = -3$

$\Rightarrow \quad x^2 = -4$ [which is not possible]

Hence, $\quad f^{-1}(17) = \pm 4$ and $f^{-1}(-3) = \phi$

Example 117 If the function f and g are defined as $f(x) = e^x$ and $g(x) = 3x - 2$, where $f : R \to R$ and $g : R \to R$, find the function fog and gof. A***lso,*** find the domain of $(fog)^{-1}$ and $(gof)^{-1}$.

Sol. $(fog)(x) = f\{g(x)\}$

$\Rightarrow \quad f\{g(x)\} = f(3x - 2)$

$\Rightarrow \quad f\{g(x)\} = e^{3x-2} \quad$ ...(i)

and $\quad (gof)(x) = g\{f(x)\}$

$\Rightarrow \quad g\{f(x)\} = g\{e^x\}$

$\Rightarrow \quad g\{f(x)\} = 3e^x - 2 \quad$ ...(ii)

For finding $(fog)^{-1}$ and $(gof)^{-1}$.

Let $\quad (fog)(x) = y = e^{3x-2}$

$\Rightarrow \quad 3x - 2 = \log y \quad \Rightarrow \quad x = \dfrac{\log y + 2}{3}$

$\Rightarrow \quad (fog)^{-1} y = \dfrac{\log y + 2}{3}$ and $(fog)^{-1} x = \dfrac{\log x + 2}{3}$

and domain of $(fog)^{-1}$ is $x > 0$, i.e. $x \in (0, \infty)$.

Again, let $(gof)x = y = 3e^x - 2 \Rightarrow e^x = \dfrac{y+2}{3}$

$\Rightarrow \quad x = \log\left(\dfrac{y+2}{3}\right)$

$\Rightarrow \quad (gof)^{-1} y = \log\left(\dfrac{y+2}{3}\right)$

$\Rightarrow \quad (gof)^{-1} x = \log\left(\dfrac{x+2}{3}\right)$

and domain of $(gof)^{-1}$ is $\dfrac{x+2}{3} > 0$.

Hence, domain of $(gof)^{-1}$ is $x > -2$, i.e. $x \in (-2, \infty)$.

Example 118 If $f(x) = ax + b$ and the equation $f(x) = f^{-1}(x)$ be satisfied by every real value of x, then

(a) $a = 2, b = -1$ (b) $a = -1, b \in R$

(c) $a = 1, b \in R$ (d) $a = 1, b = -1$

Sol. If $f(x) = ax + b$

$\Rightarrow \quad f^{-1}(x) = \dfrac{x}{a} - \dfrac{b}{a}$

Since, $f(x) = f^{-1}(x), \forall x \in R$

$\Rightarrow \quad \dfrac{1}{a} = a$ and $b = -\dfrac{b}{a} \Rightarrow a = -1$ and $b \in R$

Hence, (b) is the correct answer.

Example 119 If $g(x)$ is the inverse of $f(x)$ and $f'(x) = \sin x$, then $g'(x)$ is equal to

(a) sin (g(x)) (b) cosec (g(x))

(c) tan (g(x)) (d) None of these

Sol. Given, $g(x) = f^{-1}(x)$

So, $\quad x = f(g(x))$

On differentiating w.r.t. 'x', we get $1 = f'(g(x)) \cdot g'(x)$

Therefore, $g'(x) = \dfrac{1}{f'(g(x))} = \dfrac{1}{\sin(g(x))}$

$\therefore \quad g'(x) = \text{cosec}(g(x))$

Hence, (b) is the correct answer.

Example 120 If A and B are the points of intersection of $y = f(x)$ and $y = f^{-1}(x)$, then

(a) A and B necessarity lie on the line $y = x$

(b) A and B must be coincident

(c) slope of line AB may be -1

(d) None of the above

Sol. If solution of $f(x) = f^{-1}(x)$ doesn't lie on $y = x$, then they must be of the form (α, β) and (β, α).

$\therefore$ Slope of line AB may be -1. Hence, (c) is the correct answer.

General Results

If x, y are independent variables, then

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$.

(ii) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n, n \in R$.

(iii) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$.

(iv) $f(x + y) = f(x) = f(y) \in f(x) = k$, where k is constant.

(v) $f(x)$ takes rational values for all $x \Rightarrow f(x)$ is a constant function.

Exercise for Session 11

1. Find the inverse of the following functions

(i) $f(x) = \sin^{-1}\left(\dfrac{x}{3}\right), x \in [-3, 3]$

(ii) $f(x) = \log_e (x^2 + 3x + 1), x \in [1, 3]$

(iii) $f(x) = 5^{\log_e x}, x > 0$

(iv) $f(x) = \log_e (x + \sqrt{x^2 + 1})$

(v) $f(x) = \begin{cases} x, & x < 1 \\ x^2, & 1 \le x \le 4 \\ 8\sqrt{x}, & x > 4 \end{cases}$

2. If the function $f : [1, \infty) \to [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then find $f^{-1}(x)$.

Answers

Exercise for Session 11

1. (i) $f^{-1}(x) = 3 \sin x$

(ii) $f^{-1}(x) = \dfrac{-3 + \sqrt{5 + 4e^x}}{2}$

(iii) $f^{-1}(x) = x^{\log_5 e}, x > 0$

(iv) $f^{-1}(x) = \dfrac{1}{2}(e^x - e^{-x})$

(v) $f^{-1}(x) = \begin{cases} x, & x < 1 \\ \sqrt{x}, & 1 \le x \le 16 \\ \dfrac{x^2}{64}, & x > 16 \end{cases}$

2. $f^{-1}(x) = \dfrac{1 + \sqrt{1 + 4 \log_2 x}}{2}, x > 0$