Table of Contents

Continuity Differentiability & Differentiation

- Theory 2
- Solved examples 12
- Exercise - 1 : Basic Objective Questions 33
- Exercise - 2 : Previous Year JEE Mains Questions 43
- Exercise - 3 : Advanced Objective Questions 47
- Exercise - 4 : Previous Year JEE Advanced Questions 59
- Answer Key 67

CONTINUITY

1. DEFINITION

A function $f(x)$ is said to be continuous at x = a; where a $\in$ domain of $f(x)$, if

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

i.e., LHL = RHL = value of a function at x = a

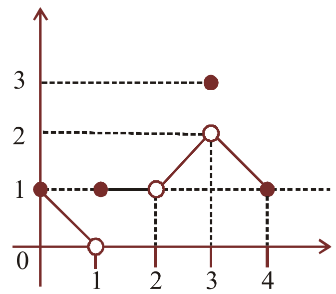
or $\lim_{x\to a} f(x) = f(a)$

12.1 Reasons of discontinuity

If $f(x)$ is not continuous at x = a, we say that $f(x)$ is discontinuous at x = a.

There are following possibilities of discontinuity :

1. $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist but they are not equal.
2. $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exists and are equal but not equal to $f(a)$.
3. $f(a)$ is not defined.
4. At least one of the limits does not exist. Geometrically, the graph of the function will exhibit a break at the point of discontinuity.



The graph as shown is discontinuous at x = 1, 2 and 3.

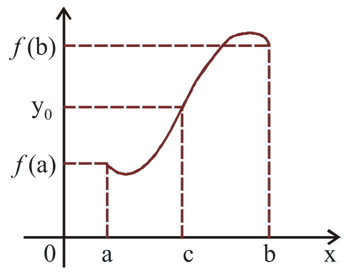
2. PROPERTIES OF CONTINUOUS FUNCTIONS

Let $f(x)$ and $g(x)$ be continuous functions at x = a. Then,

1. $c f(x)$ is continuous at x = a, where c is any constant.
2. $f(x) \pm g(x)$ is continuous at x = a.
3. $f(x) . g(x)$ is continuous at x = a.
4. $f(x)/g(x)$ is continuous at x = a, provided $g(a) \neq 0$.-
5. If $f(x)$ is continuous on [a, b] such that $f(a)$ and $f(b)$ are of opposite signs, then there exists at least one solution of equation $f(x) = 0$ in the open interval (a, b).

3. THE INTERMEDIATE VALUE THEOREM

Suppose $f(x)$ is continuous on an interval I, and a and b are any two points of I. Then if y_0 is a number between $f(a)$ and $f(b)$, their exits a number c between a and b such that $f(c) = y_0$.



The Function f, being continuous on (a,b) takes on every value between $f(a)$ and $f(b)$

Note...

That a function f which is continuous in [a, b] possesses the following properties :

(i) If $f(a)$ and $f(b)$ possess opposite signs, then there exists at least one solution of the equation $f(x) = 0$ in the open interval (a, b).

(ii) If K is any real number between $f(a)$ and $f(b)$, then there exists at least one solution of the equation $f(x) = K$ in the open interval (a, b).

4. CONTINUITY IN AN INTERVAL

(a) A function f is said to be continuous in (a, b) if f is continuous at each and every point $\in$ (a, b).

(b) A function f is said to be continuous in a closed interval [a, b] if :

(1) f is continuous in the open interval (a, b) and

(2) f is right continuous at 'a' i.e. $\text{Limit}_{x\to a^+} f(x) = f(a)$ = a finite quantity.

(3) f is left continuous at 'b'; i.e. $\text{Limit}_{x\to b^-} f(x) = f(b)$ = a finite quantity.

5. A LIST OF CONTINUOUS FUNCTIONS

	Function f (x)	Interval in which f (x) is continuous
1.	constant c	$(-\infty, \infty)$
2.	x^n, n is an integer ≥ 0	$(-\infty, \infty)$
3.	x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
4.	$\lvert x-a \rvert$	$(-\infty, \infty)$
5.	$P(x) = a_0x^n + a_1x^{n-1} + ... + a_n$	$(-\infty, \infty)$
6.	$\frac{p(x)}{q(x)}$, where p (x) and q (x) are polynomial in x	$(-\infty, \infty) - \{x ; q(x) = 0\}$
7.	sin x	$(-\infty, \infty)$
8.	cos x	$(-\infty, \infty)$
9.	tan x	$(-\infty, \infty) - \left\{(2n+1)\frac{\pi}{2} : n \in I\right\}$
10.	cot x	$(-\infty, \infty) - \{n\pi : n \in I\}$
11.	sec x	$(-\infty, \infty) - \{(2n+1)\pi/2 : n \in I\}$
12.	cosec x	$(-\infty, \infty) - \{n\pi : n \in I\}$
13.	e^x	$(-\infty, \infty)$
14.	$\log_e x$	$(0, \infty)$

6. TYPES OF DISCONTINUITIES

Type-1 : (Removable type of discontinuities)

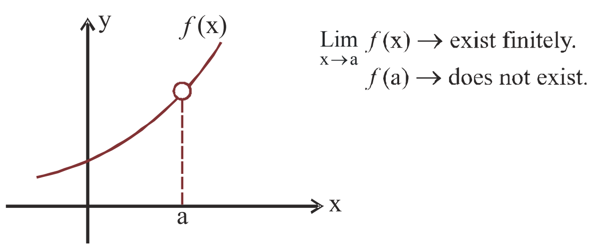
In case, $\underset{x \to c}{\text{Limit}}\, f(x)$ exists but is not equal to $f(c)$ then the function is said to have a **removable discontnuity or discontinuity of the first kind.** In this case, we can redefine the function such that $\underset{x \to c}{\text{Limit}}\, f(x) = f(c)$ and make it continuous at x = c. Removable type of discontinuity can be further classified as :

(a) Missing Point Discontinuity :

Where $\underset{x \to a}{\text{Limit}}\, f(x)$ exists finitely but $f(a)$ is not defined.

E.g. $f(x) = \frac{(1-x)(9-x^2)}{(1-x)}$ has a missing point discontinuity at x = 1, and

$f(x) = \frac{\sin x}{x}$ has a missing point discontinuity at x = 0.



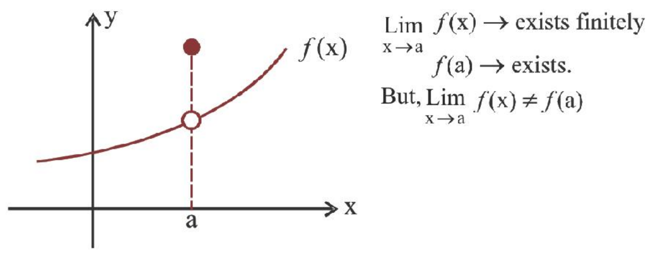
missing point discontinuity at x = a

(b) Isolated Point Discontinuity :

Where $\underset{x \to a}{\text{Limit}}\, f(x)$ exists & $f(a)$ also exists but; $\underset{x \to a}{\text{Limit}} \neq f(a)$.

E.g. $f(x) = \frac{x^2 - 16}{x - 4}$, $x \neq 4$ and $f(4) = 9$ has an isolated point discontinuity at x = 4.

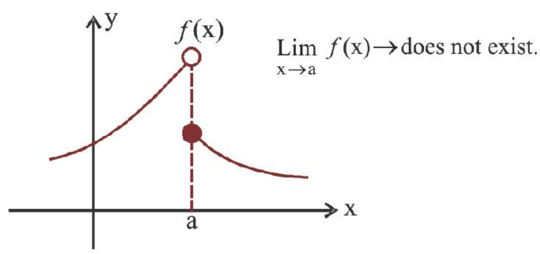
Similarly $f(x) = [x] + [-x] = \begin{bmatrix} 0 & \text{if } x \in I \\ -1 & \text{if } x \notin I \end{bmatrix}$ has an isolated point discontinuity at all $x \in I$.



Isolated point discontinuity at x = a

Type-2 : (Non-Removable type of discontinuities)

In case, $\underset{x \to a}{\text{Limit}}\, f(x)$ does not exist, then it is not possible to make the function continuous by redefining it. Such discontinuities are known as **non-removable discontinuity or discontinuity of the 2nd kind.** Non-removable type of discontinuity can be further classified as :



non-removable discontinuity at x = a

(a) Finite Discontinuity :

E.g., $f(x) = x - [x]$ at all integral x; $f(x) = \tan^{-1}\frac{1}{x}$ at $x = 0$ and

$f(x) = \frac{1}{1 + 2^{\frac{1}{x}}}$ at $x = 0$ (note that $f(0^+) = 0$; $f(0^-) = 1$)

(b) Infinite Discontiunity :

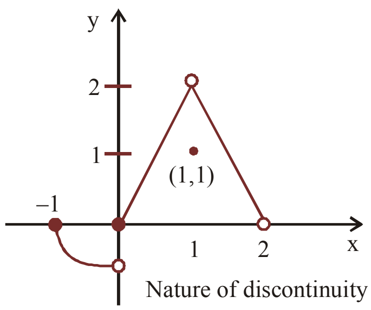
E.g., $f(x) = \frac{1}{x-4}$ or $g(x) = \frac{1}{(x-4)^2}$ at $x = 4$; $f(x) = 2^{\tan x}$

at $x = \frac{\pi}{2}$ and $f(x) = \frac{\cos x}{x}$ at $x = 0$.

(c) Oscillatory Discontinuity :

E.g., $f(x) = \sin\frac{1}{x}$ at $x = 0$.

In all these cases the value of $f(a)$ of the function at x = a (point of discontinuity) may or may not exist but $\underset{x \to a}{\text{Limit}}$ does not exist.



Nature of discontinuity

From the adjacent graph note that

– f is continuous at x = –1

– f has isolated discontinuity at x = 1

– f has missing point discontinuity at x = 2

– f has non-removable (finite type) discontinuity at the origin.

Note...

(a) In case of dis-continuity of the second kind the non-negative difference between the value of the RHL at x = a and LHL at x = a is called the **jump of discontinuity.** A function having a finite number of jumps in a given interval I is called a piece wise continuous or sectionally continuous function in this interval.

(b) All Polynomials, Trigonometrical functions, exponential and Logarithmic functions are continuous in their domains.

(c) If $f(x)$ is continuous and $g(x)$ is discontinuous at x = a then the product function $\phi(x) = f(x) . g(x)$ is not necessarily be discontinuous at x = a. e.g.

$$f(x) = x \text{ and } g(x) = \begin{cases} \sin\frac{\pi}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(d) If $f(x)$ and $g(x)$ both are discontinuous at x = a then the product function $\phi(x) = f(x) . g(x)$ is not necessarily be discontinuous at x = a . e.g.

$$f(x) = -g(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

(e) Point functions are to be treated as discontinuous eg. $f(x) = \sqrt{1-x} + \sqrt{x-1}$ is not continuous at x = 1.

(f) A continuous function whose domain is closed must have a range also in closed interval.

(g) If f is continuous at $x = a$ and g is continuous at $x = f(a)$ then the composite $g[f(x)]$ is continous at $x = a$ E.g $f(x) = \frac{x \sin x}{x^2+2}$ and $g(x) = |x|$ are continuous at x = 0, hence the composite $(gof)(x) = \left|\frac{x \sin x}{x^2+2}\right|$ will also be continuous at x = 0.

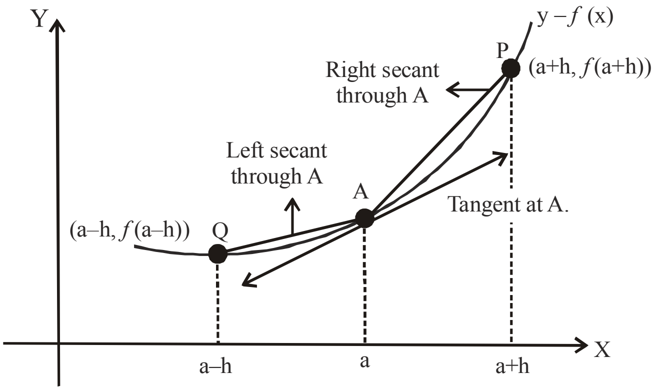
DIFFERENTIABILITY

7. DEFINITION

Let $f(x)$ be a real valued function defined on an open interval (a, b) where $c \in (a, b)$. Then $f(x)$ is said to be differentiable or derivable at x = c,

iff, $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{(x - c)}$ exists finitely.

This limit is called the derivative or differentiable coefficient of the function $f(x)$ at x = c, and is denoted by $f'(c)$ or $\dfrac{d}{dx}(f(x))_{x=c}$.



- Slope of Right hand secant = $\dfrac{f(a+h) - f(a)}{h}$ as $h \to 0$, $P \to A$ and secant (AP) $\to$ tangent at A

$\Rightarrow$ Right hand derivative $= \lim\limits_{h \to 0}\left(\dfrac{f(a+h) - f(a)}{h}\right)$

= Slope of tangent at A (when approached from right) $f'(a^+)$.

- Slope of Left hand secant = $\dfrac{f(a-h) - f(a)}{-h}$ as $h \to 0$, $Q \to A$ and secant AQ $\to$ tangent at A

$\Rightarrow$ Left hand derivative $= \lim\limits_{h \to 0}\left(\dfrac{f(a-h) - f(a)}{-h}\right)$

= Slope of tangent at A (when approached from left) $f'(a^-)$.

Thus, $f(x)$ is differentiable at x = c.

$\Leftrightarrow \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{(x - c)}$ exists finitely

$\Leftrightarrow \lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{(x - c)} = \lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{(x - c)}$

$\Leftrightarrow \lim\limits_{h \to 0} \dfrac{f(c-h) - f(c)}{-h} = \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$

Hence, $\lim\limits_{x \to c^-} \dfrac{f(x) - f(c)}{(x - c)} = \lim\limits_{h \to 0} \dfrac{f(c-h) - f(c)}{-h}$ is called the **left hand derivative** of $f(x)$ at x = c and is denoted by $f'(c^-)$ or $Lf'(c)$.

While, $\lim\limits_{x \to c^+} \dfrac{f(x) - f(c)}{x - c} = \lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ is called the **right hand derivative** of $f(x)$ at x = c and is denoted by $f'(c^+)$ or $Rf'(c)$.

If $f'(c^-) \neq f'(c^+)$, we say that $f(x)$ is not differentiable at x = c.

8. DIFFERENTIABILITY IN A SET

1. A function $f(x)$ defined on an open interval (a, b) is said to be differentiable or derivable in open interval (a, b), if it is differentiable at each point of (a, b).

2. A function $f(x)$ defined on closed interval [a, b] is said to be differentiable or derivable. "If f is derivable in the open interval (a, b) and also the end points a and b, then f is said to be derivable in the closed interval [a, b]".

i.e., $\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$ and $\lim_{x\to b^-}\frac{f(x)-f(b)}{x-b}$, both exist.

A function f is said to be a differentiable function if it is differentiable at every point of its domain.

Note..

1. If $f(x)$ and $g(x)$ are derivable at x = a then the functions $f(x)+g(x)$, $f(x)-g(x)$, $f(x).g(x)$ will also be derivable at x = a and if $g(a) \neq 0$ then the function $f(x)/g(x)$ will also be derivable at x = a.

2. If $f(x)$ is differentiable at x = a and $g(x)$ is not differentiable at x = a, then the product function $F(x) = f(x).g(x)$ can still be differentiable at x = a. E.g. $f(x) = x$ and $g(x) = |x|$.

3. If $f(x)$ and $g(x)$ both are not differentiable at x = a then the product function; $F(x) = f(x).g(x)$ can still be differentiable at x = a. E.g., $f(x) = |x|$ and $g(x) = |x|$.

4. If $f(x)$ and $g(x)$ both are not differentiable at x = a then the sum function $F(x) = f(x) + g(x)$ may be a differentiable function. E.g., $f(x) = |x|$ and $g(x) = -|x|$.

9. RELATION B/W CONTINUITY & DIFFERENTIABILITY

In the previous section we have discussed that if a function is differentiable at a point, then it should be continuous at that point and a discontinuous function cannot be differentiable. This fact is proved in the following theorem.

Theorem : If a function is differentiable at a point, it is necessarily continuous at that point. But the converse is not necessarily true,

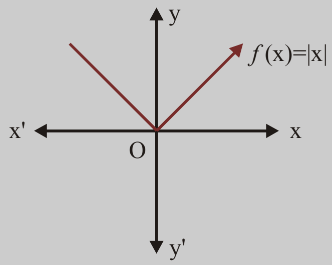
or $f(x)$ is differentiable at x = c

$\Rightarrow$ $f(x)$ is continuous at x = c.

Note..

Converse : The converse of the above theorem is not necessarily true i.e., a function may be continuous at a point but may not be differentiable at that point.

E.g., The function $f(x) = |x|$ is continuous at x = 0 but it is not differentiable at x = 0, as shown in the figure.



The figure shows that sharp edge at x = 0 hence, function is not differentiable but continuous at x = 0.

Note..

(a) Let $f'^{+}(a) = p$ & $f'^{-}(a) = q$ where p & q are finite then :

(i) p = q $\Rightarrow$ f is derivable at x = a

$\Rightarrow$ f is continuous at x = a.

(ii) p ≠ q $\Rightarrow$ f is not derivable at x = a.

It is very important to note that f may be still continuous at x = a.

In short, for a function f:

Differentiable $\Rightarrow$ Continuous;

Not Differentiable $\not\Rightarrow$ Not Continuous (i.e., function may be continuous)

But,

Not Continuous $\Rightarrow$ Not Differentiable.

(b) If a function f is not differentiable but is continuous at x = a it geometrically implies a sharp corner at x = a.

Theorem 2 : Let f and g be real functions such that fog is defined if g is continuous at x = a and f is continuous at g (a), show that fog is continuous at x = a.

DIFFERENTIATION

10. DEFINITION

(a) Let us consider a function $y = f(x)$ defined in a certain interval. It has a definite value for each value of the independent variable x in this interval.

Now, the ratio of the increment of the function to the increment in the independent variable,

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Now, as $\Delta x \to 0$, $\Delta y \to 0$ and $\frac{\Delta y}{\Delta x} \to$ finite quantity, then derivative $f(x)$ exists and is denoted by y' or $f'(x)$ or $\frac{dy}{dx}$

Thus, $f'(x) = \lim_{x\to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\frac{f(x+\Delta x) - f(x)}{\Delta x}$

(if it exits)

for the limit to exist,

$$\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f(x-h) - f(x)}{-h}$$

(Right Hand derivative) (Left Hand derivative)

(b) The derivative of a given function f at a point x = a of its domain is defined as :

$\text{Limit}_{h\to 0}\frac{f(a+h) - f(a)}{h}$, provided the limit exists & is denoted by $f'(a)$.

Note that alternatively, we can define

$f'(a) = \text{Limit}_{x\to a}\frac{f(x) - f(a)}{x - a}$, provided the limit exists.

This method is called first principle of finding the derivative of f(x).

11. DERIVATIVE OF STANDARD FUNCTION

(i) $\frac{d}{dx}(x^n) = n \, . \, x^{n-1}$; $x \in R, n \in R, x > 0$

(ii) $\frac{d}{dx}(e^x) = e^x$

(iii) $\frac{d}{dx}(a^x) = a^x \, . \ln a \ (a > 0)$

(iv) $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$

(v) $\frac{d}{dx}(\log_a |x|) = \frac{1}{x}\log_a e$

(vi) $\frac{d}{dx}(\sin x) = \cos x$

(vii) $\frac{d}{dx}(\cos x) = -\sin x$

(viii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\frac{d}{dx}(\sec x) = \sec x \, . \tan x$

(x) $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \, . \cot x$

(xi) $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$

(xii) $\frac{d}{dx}(\text{constant}) = 0$

(xiii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$, $\quad -1 < x < 1$

(xiv) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$, $\quad -1 < x < 1$

(xv) $\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{1+x^2}$, $x \in R$

(xvi) $\frac{d}{dx}\left(\cot^{-1}x\right)=\frac{-1}{1+x^2}$, $x \in R$

(xvii) $\frac{d}{dx}\left(\sec^{-1}x\right)=\frac{1}{|x|\sqrt{x^2-1}}$, $|x|>1$

(xviii) $\frac{d}{dx}\left(\operatorname{cosec}^{-1}x\right)=\frac{-1}{|x|\sqrt{x^2-1}}$, $|x|>1$

(xix) **Results :**

If the inverse functions f & g are defined by $y=f(x)$ & $x=g(y)$. Then $g(f(x))=x$.

$\Rightarrow$ $g'(f(x)).f'(x)=1$.

This result can also be written as, if $\frac{dy}{dx}$ exists & $\frac{dy}{dx}\neq 0$, then

$$\frac{dx}{dy}=1/\left(\frac{dy}{dx}\right) \text{ or } \frac{dy}{dx}.\frac{dx}{dy}=1 \text{ or } \frac{dy}{dx}=1/\left(\frac{dx}{dy}\right)\left[\frac{dx}{dy}\neq 0\right]$$

12. THEOREMS ON DERIVATIVES

If u and v are derivable functions of x, then,

(i) Term by term differentiation : $\frac{d}{dx}(u\pm v)=\frac{du}{dx}\pm\frac{dv}{dx}$

(ii) Multiplication by a constant $\frac{d}{dx}(K\,u)=K\frac{du}{dx}$, where K is any constant

(iii) **"Product Rule"** $\frac{d}{dx}(u.v)=u\frac{dv}{dx}+v\frac{du}{dx}$ known as

In general,

(a) If $u_1, u_2, u_3, u_4, \ldots, u_n$ are the functions of x, then

$$\frac{d}{dx}\left(u_1 . u_2 . u_3 . u_4 \ldots . u_n\right)$$

$$=\left(\frac{du_1}{dx}\right)\left(u_2\,u_3\,u_4\ldots u_n\right)+\left(\frac{du_2}{dx}\right)\left(u_1\,u_3\,u_4\ldots u_n\right)$$

$$+\left(\frac{du_3}{dx}\right)\left(u_1\,u_2\,u_4\ldots u_n\right)+\left(\frac{du_4}{dx}\right)\left(u_1\,u_2\,u_3\,u_5\ldots u_n\right)$$

$$+\ldots+\left(\frac{du_n}{dx}\right)\left(u_1\,u_2\,u_3\ldots u_{n-1}\right)$$

(iv) **"Quotient Rule"** $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\left(\frac{du}{dx}\right)-u\left(\frac{dv}{dx}\right)}{v^2}$ where $v\neq 0$

known as

(b) **Chain Rule :** If $y=f(u)$, $u=g(w)$, $w=h(x)$

then $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dw}.\frac{dw}{dx}$

or $\frac{dy}{dx}=f'(u).g'(w).h'(x)$

Note..

In general if $y=f(u)$ then $\frac{dy}{dx}=f'(u).\frac{du}{dx}$.

13. METHODS OF DIFFERENTIATION

13.1 Derivative by using Trigonometrical Substitution

Using trigonometrical transformations before differentiation shorten the work considerably. Some important results are given below :

(i) $\sin 2x = 2\sin x\cos x=\frac{2\tan x}{1+\tan^2 x}$

(ii) $\cos 2x = 2\cos^2 x-1=1-2\sin^2 x=\frac{1-\tan^2 x}{1+\tan^2 x}$

(iii) $\tan 2x=\frac{2\tan x}{1-\tan^2 x}$, $\tan^2 x=\frac{1-\cos 2x}{1+\cos 2x}$

(iv) $\sin 3x = 3\sin x-4\sin^3 x$

(v) $\cos 3x = 4\cos^3 x-3\cos x$

(vi) $\tan 3x=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}$

(vii) $\tan\left(\frac{\pi}{4}+x\right)=\frac{1+\tan x}{1-\tan x}$

(viii) $\tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}$

(ix) $\sqrt{(1 \pm \sin x)} = \left|\cos\frac{x}{2} \pm \sin\frac{x}{2}\right|$

(x) $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\left(\frac{x \pm y}{1 \mp xy}\right)$

(xi) $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right\}$

(xii) $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left\{xy \mp \sqrt{1-x^2}\,\sqrt{1-y^2}\right\}$

(xiii) $\sin^{-1}x + \cos^{-1}x = \tan^{-1}x + \cot^{-1}x = \sec^{-1}x + \text{cosec}^{-1}x = \pi/2$

(xiv) $\sin^{-1}x = \text{cosec}^{-1}(1/x)$; $\cos^{-1}x = \sec^{-1}(1/x)$; $\tan^{-1}x = \cot^{-1}(1/x)$

Note...

Some standard substitutions :

Expressions	Substitutions
$\sqrt{(a^2 - x^2)}$	$x = a \sin\theta$ or $a\cos\theta$
$\sqrt{(a^2 + x^2)}$	$x = a\tan\theta$ or $a\cot\theta$
$\sqrt{(x^2 - a^2)}$	$x = a\sec\theta$ or $a\,\text{cosec}\,\theta$
$\sqrt{\left(\frac{a+x}{a-x}\right)}$ or $\sqrt{\left(\frac{a-x}{a+x}\right)}$	$x = a\cos\theta$ or $a\cos 2\theta$
$\sqrt{(a-x)(x-b)}$ or $\sqrt{\left(\frac{a-x}{x-b}\right)}$ or $\sqrt{\left(\frac{x-b}{a-x}\right)}$	$x = a\cos^2\theta + b\sin^2\theta$
$\sqrt{(x-a)(x-b)}$ or $\sqrt{\left(\frac{x-a}{x-b}\right)}$ or $\sqrt{\left(\frac{x-b}{x-a}\right)}$	$x = a\sec^2\theta - b\tan^2\theta$
$\sqrt{(2ax - x^2)}$	$x = a(1 - \cos\theta)$

13.2 Logarithmic Differentiation

To find the derivative of :

If $y = \{f_1(x)\}^{f_2(x)}$ or $y = f_1(x) \cdot f_2(x) \cdot f_3(x) \ldots$

or $y = \frac{f_1(x) \cdot f_2(x) \cdot f_3(x) \ldots}{g_1(x) \cdot g_2(x) \cdot g_3(x) \ldots}$

then it is convenient to take the logarithm of the function first and then differentiate. This is called derivative of the logarithmic function.

Important Notes (Alternate methods)

1. If $y = \{f(x)\}^{g(x)} = e^{g(x)\ln f(x)}$ ((variable)$^{\text{variable}}$) $\{\because x = e^{\ln x}\}$

$$\therefore \frac{dy}{dx} = e^{g(x)\ln f(x)} \cdot \left\{g(x) \cdot \frac{d}{dx}\ln f(x) + \ln f(x) \cdot \frac{d}{dx} g(x)\right\}$$

$$= \{f(x)\}^{g(x)} \cdot \left\{g(x) \cdot \frac{f'(x)}{f(x)} + \ln f(x) \cdot g'(x)\right\}$$

2. If $y = \{f(x)\}^{g(x)}$

$\therefore \frac{dy}{dx} =$ Derivative of y treating $f(x)$ as constant + Derivative of y treating $g(x)$ as constant

$$= \{f(x)\}^{g(x)} \cdot \ln f(x) \cdot \frac{d}{dx} g(x) + g(x)\{f(x)\}^{g(x)-1} \cdot \frac{d}{dx} f(x)$$

$$= \{f(x)\}^{g(x)} \cdot \ln f(x) \cdot g'(x) + g(x) \cdot \{f(x)\}^{g(x)-1} \cdot f'(x)$$

13.3 Implict Differentiation : $\phi(x, y) = 0$

(i) In order to find dy/dx in the case of implicit function, we differentiate each term w.r.t. x, regarding y as a function of x & then collect terms in dy/dx together on one side to finally find dy/dx.

(ii) In answers of dy/dx in the case of implicit function, both x & y are present.

Alternate Method : If $f(x, y) = 0$

then $$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = -\frac{\text{diff. of f w.r.t. x treating y as constant}}{\text{diff. of f w.r.t. y treating x as constant}}$$

13.4 Parametric Differentiation

If $y=f(t)$ & $x=g(t)$ where t is a Parameter, then

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt} \qquad \text{...(1)}$$

Note..

1. $\frac{dy}{dx}=\frac{dy}{dt}.\frac{dt}{dx}$

2. $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}\quad\left(\because \frac{dy}{dx} \text{ in terms of t}\right)$

$$=\frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right).\frac{1}{f'(t)} \quad \{\text{From (1)}\}$$

$$=\frac{f''(t)g'(t)-g''(t)f'(t)}{\{f'(t)\}^3}$$

13.5 Derivative of a Function w.r.t. another Function

Let $y=f(x)$; $z=g(x)$ then $\frac{dy}{dz}=\frac{dy/dx}{dz/dx}=\frac{f'(x)}{g'(x)}$

13.6 Derivative of Infinite Series

If taking out one or more than one terms from an infinite series, it remains unchanged. Such that

(A) If $y=\sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+.......\infty}}}$

then $y=\sqrt{f(x)+y} \Rightarrow (y^2-y)=f(x)$

Differentiating both sides w.r.t. x, we get $(2y-1)\frac{dy}{dx}=f'(x)$

(B) If $y=\{f(x)\}^{\{f(x)\}^{\{f(x)\}^{.......\infty}}}$ then $y=\{f(x)\}^y \Rightarrow y=e^{y\ln f(x)}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{y\{f(x)\}^{y-1}.f'(x)}{1-\{f(x)\}^y.\ell n\, f(x)}=\frac{y^2 f'(x)}{f(x)\{1-y\,\ell n\, f(x)\}}$$

14. DERIVATIVE OF ORDER TWO & THREE

Let a function $y=f(x)$ be defined on an open interval (a, b). It's derivative, if it exists on (a, b), is a certain function $f'(x)$ [or (dy/dx) or y'] & is called the first derivative of y w.r.t. x. If it happens that the first derivative has a derivative on (a, b) then this derivative is called the second derivative of y w.r.t. x & is denoted by $f''(x)$ or (d^2y/dx^2) or y''.

Similarly, the 3rd order derivative of y w.r.t. x, if it exists, is defined by $\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ it is also denoted by $f'''(x)$ or y'''.

Some Standard Results :

(i) $\frac{d^n}{dx^n}(ax+b)^m=\frac{m!}{(m-n)!}.a^n.(ax+b)^{m-n}$, $m \ge n$.

(ii) $\frac{d^n}{dx^n}x^n=n!$

(iii) $\frac{d^n}{dx^n}\left(e^{mx}\right)=m^n.e^{mx}$, $m \in R$

(iv) $\frac{d^n}{dx^n}(\sin(ax+b))=a^n\sin\left(ax+b+\frac{n\pi}{2}\right)$, $n \in N$

(v) $\frac{d^n}{dx^n}(\cos(ax+b))=a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$, $n \in N$

(vi) $\frac{d^n}{dx^n}\{e^{ax}\sin(bx+c)\}=r^n.e^{ax}.\sin(bx+c+n\phi)$, $n \in N$

where $r=\sqrt{(a^2+b^2)}$, $\phi=\tan^{-1}(b/a)$.

(vii) $\frac{d^n}{dx^n}\{e^{ax}.\cos(bx+c)\}=r^n.e^{ax}.\cos(bx+c+n\phi)$, $n \in N$

where $r=\sqrt{(a^2+b^2)}$, $\phi=\tan^{-1}(b/a)$.

15. DIFFERENTIATION OF DETERMINANTS

If $F(X)=\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$,

where $f, g, h, \ell, m, n, u, v, w$ are differentiable function of x then

$$F'(x)=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ \ell(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell'(x) & m'(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}+\begin{vmatrix} f(x) & g(x) & h(x) \\ \ell(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

16. L' HOSPITAL'S RULE

If $f(x)$ & $g(x)$ are functions of x such that :

(i) $\lim\limits_{x\to a} f(x)=0=\lim\limits_{x\to a} g(x)$ or $\lim\limits_{x\to a} f(x)=\infty=\lim\limits_{x\to a} g(x)$ and

(ii) Both $f(x)$ & $g(x)$ are continuous at x = a and

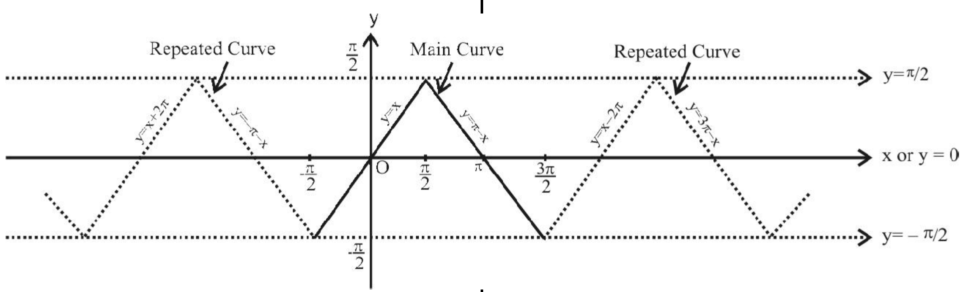
(iii) Both $f(x)$ & $g(x)$ are differentiable at x = a and

(iv) Both $f'(x)$ & $g'(x)$ are continuous at x = a, Then

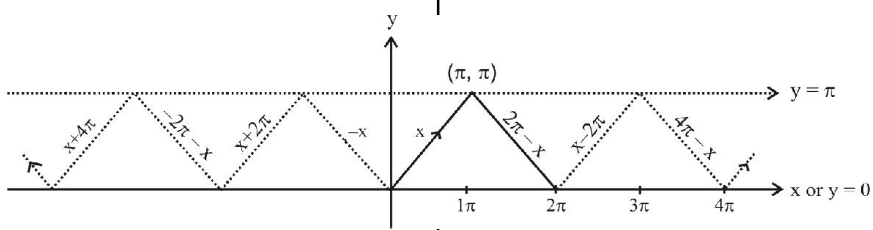
$\text{Limit}_{x\to a} \dfrac{f(x)}{g(x)}=\text{Limit}_{x\to a} \dfrac{f'(x)}{g'(x)}=\text{Limit}_{x\to a} \dfrac{f''(x)}{g''(x)}$ & so on till indeterminant form vanishes..

17. ANALYSIS & GRAPHS OF SOME USEFUL FUNCTION

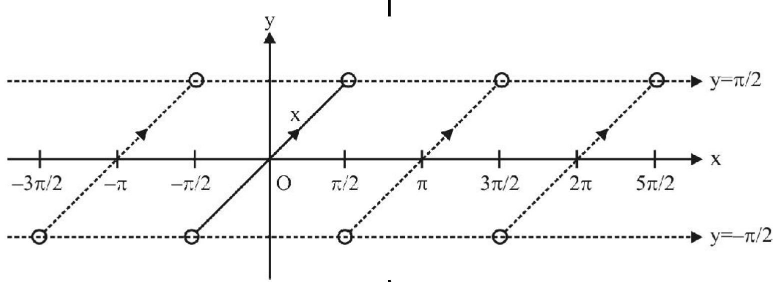
(i) $y=\sin^{-1}(\sin x)$ $\quad x\in R$; $y\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$



(ii) $y=\cos^{-1}(\cos x)$ $\quad x\in R$; $y\in[0,\pi]$



(iii) $y=\tan^{-1}(\tan x)$ $\quad x\in R-\left\{x : x=(2n+1)\dfrac{\pi}{2}, n\in Z\right\}$; $y\in\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$



SOLVED EXAMPLES

Example – 1

Show that $f(x)=\begin{cases} 5x-4, & \text{when } 0<x\le 1 \\ 4x^3-3x, & \text{when } 1<x<2 \end{cases}$

is continuous at x = 1.

Sol. We have,

$(\text{LHL at } x=1)=\lim_{x\to 1^-} f(x)=\lim_{x\to 1} 5x-4$

$[\because f(x)=5x-4, \text{ when } x\le 1]$

$=5\times 1-4=1,$

$(\text{RHL at } x=1)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} 4x^3-3x$

$[\because f(x)=4x^3-3x, x>1]$

$=4(1)^3-3(1)=1,$

and, $f(1)=5\times 1-4=1$

$[\because f(x)=5x-4, \text{ where } x\le 1]$

$\therefore \lim_{x\to 1^-} f(x)=f(1)=\lim_{x\to 1^+} f(x)$

So, f(x) is continuous at x = 1.

Example – 2

Test the continuity of the function f (x) at the origin :

$f(x)=\begin{cases} \dfrac{|x|}{x}; & x=0 \\ 1\ ; & x=0 \end{cases}$

Sol. We have,

$(\text{LHL at } x=0)=\lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0} f(-h)$

$=\lim_{h\to 0}\dfrac{|-h|}{-h}=\lim_{h\to 0}\dfrac{h}{-h}=\lim_{h\to 0} -1=-1$

and, $(\text{RHL at } x=0)=\lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0} f(h)$

$=\lim_{h\to 0}\dfrac{|h|}{h}=\lim_{h\to 0}\frac{h}{h}=\lim_{h\to 0} 1=1$

Thus, f(x) is not continuous at the origin.

Example – 3

Discuss the continuity of the function of given by $f(x)=|x-1|+|x-2|$ at $x=1$ and $x=2$.

Sol. We have,

$f(x)=|x-1|+|x-2|$

$\Rightarrow f(x)=\begin{cases} -(x-1)-(x-2), & \text{if } x<1 \\ (x-1)-(x-2), & \text{if } 1\le x<2 \\ (x-1)+(x-2), & \text{if } x\ge 2 \end{cases}$

$\Rightarrow f(x)=\begin{cases} -2x+3, & \text{if } x<1 \\ 1, & \text{if } 1\le x<2 \\ 2x-3, & x\ge 2 \end{cases}$

Continuity at x = 1:

We have,

$\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(-2x+3)=-2\times 1+3=1$

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1^+} 1=1$

and, $f(1)=1.$

$\therefore \lim_{x\to 1^-} f(x)=f(1)=\lim_{x\to 1^+} f(x)$

So, f (x) is continuous at x = 1.

Continuity at x = 2

We have,

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-} 1=1$

$\lim_{x\to 2^+} f(x)=\lim_{x\to 2}(2x-3)=2\times 2-3=1$

and, $f(2)=2\times 2-3=1.$

$\therefore \lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2)$

So, f (x) is continuous at x = 2.

Example – 4

Examine the function f(t) given by

$$f(t)=\begin{cases}\dfrac{\cos t}{\pi/2-t}; & t\neq\pi/2\\ 1 \quad ; & t=\pi/2\end{cases}$$

for continuity at t = π/2

Sol. We have,

$$(\text{LHL at } t=\pi/2)=\lim_{t\to\pi/2^-} f(t)$$

$$=\lim_{h\to0} f(\pi/2-h)=\lim_{h\to0}\frac{\cos(\pi/2-h)}{\pi/2-(\pi/2-h)}=\lim_{h\to0}\frac{\sin h}{h}=1$$

and, $(\text{RHL at } t=\pi/2)=\lim_{t\to\pi/2^+} f(t)$

$$=\lim_{h\to0} f(\pi/2+h)=\lim_{h\to0}\frac{\cos(\pi/2+h)}{\pi/2-(\pi/2+h)}$$

$$=\lim_{h\to0}\frac{-\sin h}{-h}=\lim_{h\to0}\frac{\sin h}{h}=1$$

Also, f(π/2) = 1.

$$\therefore \lim_{t\to\pi/2^-} f(t)=\lim_{t\to\pi/2^+}=f(\pi/2)$$

So, f(t) is continuous at t = π/2.

Example – 5

Prove that the greatest integer function [x] is continuous at all points except at integer points.

Sol. Let f(x) = [x] be the greatest integer function. Let k be any integer. Then,

$$f(x)=[x]=\begin{cases}k-1, & \text{if } k-1\le x<k\\ k\ , & \text{if } k\le x<k+1\end{cases} \quad \text{[By def.]}$$

Now (LHL at x = k)

$$=\lim_{x\to k^-} f(x)=\lim_{h\to0} f(k-h)=\lim_{h\to0}[k-h]$$

$$=\lim_{h\to0}(k-1)=k-1$$

$[\because k-1\le k-h<k \therefore [k-h]=k-1]$

and, (RHL at x = k)

$$=\lim_{x\to k^+} f(x)=\lim_{h\to0} f(k+h)=\lim_{h\to0}[k+h]$$

$$=\lim_{h\to0} k=k$$

$[\because k\le k+h<k+1 \therefore [k+h]=k]$

$$\therefore \lim_{x\to k^-} f(x)\neq\lim_{x\to k^+} f(x)$$

So, f(x) is not continuous at x = k.

Since k is an arbitrary integer. Therefore, f(x) is not continuous at integer points.

Let a be any real number other than an integer. Then, there exists an integer k such that k – 1 < a < k.

Now, (LHL at x = a)

$$\lim_{x\to a^-} f(x)=\lim_{h\to0} f(a-h)=\lim_{h\to0}[a-h]$$

$$=\lim_{x\to0} k-1=k-1$$

$[\because k-1<a-h<k \therefore [a-h]=k-1]$

(RHL at x = a)

$$=\lim_{x\to a^+} f(x)=\lim_{h\to0} f(a+h)$$

$$=\lim_{h\to0}[a+h]=\lim_{h\to0}(k-1)=k-1$$

$[\because k-1<a+h<k \therefore [a+h]=k-1]$

and, f(a) = k – 1

$[\because k-1<a<k \therefore [a]=k-1]$

Thus,

$$\lim_{x-a^-} f(x)=\lim_{x\to a^+} f(x)=f(a)$$

So, f(x) is continuous at x = a. Since a is an arbitrary real number, other than an integer. Therefore, f(x) is continuous at all real points except integer points.

Example – 6

Show that the function f(x) given by

$$f(x)=\begin{cases}\dfrac{e^{1/x}-1}{e^{1/x}+1}, & \text{when } x\neq0\\ 0 \quad , & \text{when } x=0\end{cases}$$ is discontinuous at x=0.

Sol. We have,

$$(\text{LHL at } x=0)=\lim_{x\to0^-} f(x)=\lim_{h\to0} f(0-h)=\lim_{h\to0} f(-h)$$

$$=\lim_{h\to0}\frac{e^{-1/h}-1}{e^{-1/h}+1}=\lim_{h\to0}\frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1}=\frac{0-1}{0+1}=-1$$

$$\left[\because \lim_{h\to 0}\frac{1}{e^{1/h}}=0\right]$$

and, (RHL at x = 0) = $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$

$$= \lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1} = \lim_{h\to 0}\frac{1-1/e^{1/h}}{1+1/e^{1/h}} = \frac{1-0}{1+0} = 1$$

So, f(x) is not continuous at x = 0 and has a discontinuity of first kind at x = 0.

Example – 7

Find the value of the constant λ so that the function given below is continuous at x = – 1.

$$f(x) = \begin{cases} \dfrac{x^2-2x-3}{x+1}, & x \neq -1 \\ \lambda, & x = -1 \end{cases}$$

Sol. Since f(x) is continuous at x = –1. Therefore,

$$\lim_{x\to -1} f(x) = f(-1)$$

$$\Rightarrow \lim_{x\to -1}\frac{x^2-2x-3}{x+1} = \lambda \qquad [\because f(-1)=\lambda]$$

$$\Rightarrow \lim_{x\to -1}\frac{(x-3)(x+1)}{x+1} = \lambda \Rightarrow \lim_{x\to -1}(x-3) = \lambda \Rightarrow -4 = \lambda$$

So, f(x) is continuous at x = –1, if λ = –4.

Example – 8

If the function f(x) defined by

$$f(x) = \begin{cases} \dfrac{\log(1+ax)-\log(1-bx)}{x}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$$

is continuous at x = 0, find k.

Sol. Since f(x) is continuous at x = 0. Therefore,

$$\lim_{x\to 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0}\frac{\log(1+ax)-\log(1-bx)}{x} = k \qquad [\because f(0)=k]$$

$$\Rightarrow \lim_{x\to 0}\left[\frac{\log(1+ax)}{x} - \frac{\log(1-bx)}{x}\right] = k$$

$$\Rightarrow \lim_{x\to 0}\frac{\log(1+ax)}{x} - \lim_{x\to 0}\frac{\log(1-bx)}{x} = k$$

$$\Rightarrow a\lim_{x\to 0}\frac{\log(1+ax)}{ax} - (-b)\lim_{x\to 0}\frac{\log(1-bx)}{(-b)x} = k$$

$$\Rightarrow a(1) - (-b)(1) = k$$

$$\left[\text{Using}: \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right]$$

$$\Rightarrow a + b = k$$

Thus, f(x) is continuous at x = 0, if k = a + b.

Example – 9

$$\text{Let } f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & \text{if } x < 0 \\ a, & \text{if } x = 0 \\ \dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}, & \text{if } x > 0 \end{cases}$$

Determine the value of a so that f (x) is continuous at x = 0.

Sol. For f (x) to be continuous at x = 0, we must have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = a \qquad \text{... (i)}$$

Now, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0}\dfrac{1-\cos 4x}{x^2}$

$$\left[\because f(x) = \frac{1-\cos 4x}{x^2} \text{ for } x < 0\right]$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0}\frac{2\sin^2 2x}{x^2}$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = 2\lim_{x\to 0}\left(\frac{\sin 2x}{x}\right)^2$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = 2\times 4 . \lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^2 = 8(1)^2 = 8 \qquad \text{... (ii)}$$

and, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0}\dfrac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4}$

$$\left[\because f(x) = \frac{\sqrt{x}}{\sqrt{16+\sqrt{x}}-4} \text{ for } x > 0\right]$$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0} \frac{\sqrt{x}}{16+\sqrt{x}-16}.(\sqrt{16+\sqrt{x}}+4)$$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0} (\sqrt{6+\sqrt{x}}+4) = 4+4=8 \quad \text{... (iii)}$$

From (i), (ii) and (iii), we get a = 8.

Example – 10

Determine the value of the constant m so that the function

$$f(x) = \begin{cases} m(x^2-2x), & \text{if } x<0 \\ \cos x, & \text{if } x \ge 0 \end{cases} \text{ is continuous}$$

Sol. When x < 0, we have

$f(x) = m(x^2 - 2x)$, which being a polynomial is continuous at each x < 0.

When x > 0, we have

f (x) = cos x, which being a cosine function is continuous at eaxh x > 0.

So, consider the point x = 0.

We have,

(LHL at x = 0) = $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} (x^2-2x) = 0$, for all values of m

and (RHL at x = 0) = $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} \cos x = 1$

Clearly, $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$ for all values of m.

So, f (x) cannot be made continuous for any value of m.

In other words, the value of m does not exist for which f(x) can be made continuous.

Example – 11

$$\text{If } f(x) = \begin{cases} 1, & \text{if } x \le 3 \\ ax+b, & \text{if } 3<x<5 \\ 7, & \text{if } 5 \le x \end{cases}$$

Determine the values of a and b so that f(x) is continuous.

Sol. The given function is a constant function for all x < 3 and for all x > 5 so it is continuous for all x < 3 and for all x > 5. We know that a polynomial function is continuous. So, the given function is continuous for all x ∈ (3, 5). Thus, f(x) is continuous at each x ∈ R except possibly at x = 3 and x = 5.

At, x = 3, we have

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} 1 = 1$, $\lim_{x\to 3^+} f(x) = \lim_{x\to 3} ax+b = 3a+b$, and, f(3) = 1

For f(x) be continuous at x = 3, we must have

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3)$$

$\Rightarrow$ 1 = 3a + b

At x = 5, we have ... (i)

$\lim_{x\to 5^-} f(x) = \lim_{x\to 5} ax+b = 5a+b$; $\lim_{x\to 5^+} f(x) = \lim_{x\to 5}$ 7 = 7, and, f(5) = 7

For f(x) to be continuous at x = 5, we must have

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$$

$\Rightarrow$ 5a + b = 7

Solving (i) and (ii), we get a = 3, b = – 8 ... (ii)

Example – 12

Show that the function f (x) = | sin x + cos x | is continuous at x = π.

Sol. Let g(x) = sin x + cos x and h (x) = | x |. Then, f (x) = hog (x). In order to prove that f (x) is continuous at x = π. It is sufficient to prove that g (x) is continuous at x = π and h (x) is continuous at y = g (π) = sin π + cos π = –1.

Now,

$$\lim_{x\to\pi} g(x) = \lim_{x\to\pi} (\sin x + \cos x) = \sin\pi + \cos\pi = -1$$

and $g(\pi) = -1$

$$\therefore \lim_{x\to\pi} g(x) = g(\pi)$$

So, g (x) is continuous at x = π.

We have, g (π) = –1 ⇒ y = g (π) = –1.

Now, $\lim_{y\to -1} h(y) = \lim_{y\to -1} |y| = \lim_{y\to -1} -y = -(-1) = 1$

and, h(g (π)) = h (–1) = |–1 | = 1.

$$\therefore \lim_{y\to -1} h(y) = h(g(\pi))$$

$$\Rightarrow \lim_{g(x)\to -1} h(g(x)) = h(g(\pi))$$

$$\Rightarrow \lim_{g(x)\to g(\pi)} h(g(x)) = h(g(\pi))$$

$\Rightarrow$ h is continuous at g (π)

Hence, f (x) = hog (x) is continuous at x = π.

Example – 13

Show that the function $f(x) = \begin{cases} x-1, & \text{if } x<2 \\ 2x-3, & \text{if } x \geq 2 \end{cases}$ is not differentiable at $x = 2$.

Sol. We have,

$$(\text{LHD at } x=2) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2}$$

$$\Rightarrow (\text{LHD at } x=2) = \lim_{x\to 2} \frac{(x-1)-(4-3)}{x-2}$$

$[\because f(x) = x-1 \text{ for } x<2]$

$$\Rightarrow (\text{LHD at } x=2) = \lim_{x\to 2} \frac{x-2}{x-2} = \lim_{x\to 2} 1 = 1$$

and, $$(\text{RHD at } x=2) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2}$$

$$\Rightarrow (\text{RHD at } x=2) = \lim_{x\to 2} \frac{(2x-3)-(4-3)}{x-2}$$

$[\because f(x) = 2x-3 \text{ for } x\geq 2]$

$$\Rightarrow (\text{RHD at } x=2) = \lim_{x\to 2} \frac{2x-4}{x-2} = \lim_{x\to 2} 2 = 2$$

$\Rightarrow$ (LHD at $x=2$) $\neq$ (RHD at $x=2$).

So, $f(x)$ is not differentiable at $x = 2$.

Example – 14

Show that the function $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is differentiable at $x = 0$ and $f'(0) = 0$.

Sol. We have,

$$(\text{LHD at } x=0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow (\text{LHD at } x=0) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{0-h-0}$$

$$\Rightarrow (\text{LHD at } x=0) = \lim_{h\to 0} \frac{f(-h)-f(0)}{-h}$$

$$\Rightarrow (\text{LHD at } x=0) = \lim_{h\to 0} \frac{(-h)^2 \sin\left(\frac{1}{-h}\right)-0}{-h}$$

$$\Rightarrow (\text{LHD at } x=0) = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right)$$

$\Rightarrow$ (LHD at $x=0$) $= 0 \times$ (an oscillating number between -1 and 1) $= 0$

$$(\text{RHD at } x=0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow (\text{RHD at } x=0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{0+h-0}$$

$$\Rightarrow (\text{RHD at } x=0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)-0}{h}$$

$$\Rightarrow (\text{RHD at } x=0) = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right)$$

$\Rightarrow$ (RHD at $x=0$) $= 0 \times$ (an oscillating number between -1 and 1) $= 0$

$\therefore$ (LHD at $x=0$) = (RHD at $x=0$) $= 0$.

So, $f(x)$ is differentiable at $x = 0$ and $f'(0) = 0$.

Example – 15

Discuss the differentiability of $f(x) = |x-1| + |x-2|$.

Sol. We have,

$f(x) = |x-1| + |x-2|$

$$\Rightarrow f(x) = \begin{cases} -(x-1)-(x-2) \text{ for } x<1 \\ x-1-(x-2) \text{ for } 1\leq x<2 \\ (x-1)+(x-2) \text{ for } x\geq 2 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} -2x+3, & x<1 \\ 1, & 1\leq x<2 \\ 2x-3, & x\geq 2 \end{cases}$$

When $x < 1$, we have

$f(x) = -2x + 3$ which, being a polynomial function is continuous and differentiable.

When $1 \leq x < 2$, we have

$f(x) = 1$ which, being a constant function, is differentiable on $(1, 2)$.

When $x \geq 2$, we have

$f(x) = 2x - 3$ which, being a polynomial function, is differentiable for all $x > 2$. Thus, the possible points of non-differentiability of $f(x)$ are $x = 1$ and $x = 2$.

Now,

$$(\text{LHD at } x=1) = \lim_{x\to 1^-} \frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow (\text{LHD at } x=1) = \lim_{x\to 1} \frac{(-2x+3)-1}{x-1}$$

$[\because f(x) = -2x+3 \text{ for } x<1]$

$\Rightarrow$ $(\text{LHD at } x=1) = \lim_{x\to 1} \frac{-2(x-1)}{x-1} = -2$

$(\text{RHD at } x=1) = \lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1}$

$\Rightarrow$ $(\text{RHD at } x=1) = \lim_{x\to 1} \frac{1-1}{x-1} = 0$

$[\because f(x)=1 \text{ for } 1\le x<2]$

$\therefore$ $(\text{LHD at } x=1) \ne (\text{RHD at } x=1)$

So, $f(x)$ is not differentiable at $x=1$.

$(\text{LHD at } x=2) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2}$

$\Rightarrow$ $(\text{LHD at } x=2) = \lim_{x\to 2} \frac{1-(2\times 2-3)}{x-2}$

$\left[\begin{array}{l}\because f(x)=1 \text{ for } 1\le x<2 \\ \quad \text{and } f(2)=2\times 2-3\end{array}\right]$

$\Rightarrow$ $(\text{LHD at } x=2) = \lim_{x\to 2} \frac{1-1}{x-2} = 0.$

$(\text{RHD at } x=2) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2}$

$\Rightarrow$ $(\text{RHD at } x=2) = \lim_{x\to 2} \frac{(2x-3)-(2\times 2-3)}{x-2}$

$[\because f(x)=2x-3 \text{ for } x\ge 2]$

$\Rightarrow$ $(\text{RHD at } x=2) = \lim_{x\to 2} \frac{2x-4}{x-2} = \lim_{x\to 2} \frac{2(x-2)}{x-2} = 2$

$\therefore$ $(\text{LHD at } x=2) \ne (\text{RHD at } x=2)$

So, $f(x)$ not differentiable at $x=2$.

Remark It should be noted that the function f(x) given by

$f(x) = |x-a_1| + |x-a_2| + (x-a_3| + ... + |x-a_n|$

is not differentiable at $x = a_1, a_2, a_3, ..., a_n$.

Example – 16

Discuss the differentiability of

$$f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & , \quad x\ne 0 \\ 0 & , \quad x=0 \end{cases} \text{ at } x=0.$$

Sol. We have,

$$f(x) = \begin{cases} xe^{-\left(\frac{1}{x}+\frac{1}{x}\right)} = xe^{-2/x}, & x\ge 0 \\ xe^{-\left(\frac{-1}{x}+\frac{1}{x}\right)} = x \quad , & x<0 \\ 0 \quad , & x=0 \end{cases}$$

Now,

$(\text{LHD at } x=0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$

$\Rightarrow$ $(\text{LHD at } x=0) = \lim_{x\to 0} \frac{x-0}{x-0} = 1$

$[\because f(x)=x \text{ for } x<0 \text{ and } f(0)=0]$

and,

$(\text{RHD at } x=0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$

$\Rightarrow$ $(\text{RHD at } x=0) = \lim_{x\to 0} \frac{xe^{-2/x}-0}{x}$

$\left[\begin{array}{l}\because f(x)=xe^{-2/x} \text{ for } x>0 \\ \quad \text{and } f(0)=0\end{array}\right]$

$\Rightarrow$ $(\text{RHD at } x=0) = \lim_{x\to 0} e^{-2/x} = 0.$

$\Rightarrow$ $(\text{LHD at } x=0) \ne (\text{RHD at } x=0)$

So, $f(x)$ is not differentiable at $x=0$.

Example – 17

Find the values of a & b so that the function is continuous for $0\le x\le \pi$

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0\le x<\frac{\pi}{4} \\ 2x\cot x + b, & \frac{\pi}{4}\le x\le \frac{\pi}{2} \\ a\cos 2x - b\sin x, & \frac{\pi}{2}<x\le \pi \end{cases}$$

Sol. Since, $f(x)$ is continuous for $0\le x\le \pi$

$\therefore$ $\text{RHL}\left(\text{at } x=\frac{\pi}{4}\right) = \text{LHL}\left(\text{at } x=\frac{\pi}{4}\right)$

$\Rightarrow$ $\left(2.\frac{\pi}{4}\cot\frac{\pi}{4}+b\right) = \left(\frac{\pi}{4}+a\sqrt{2}.\sin\frac{\pi}{4}\right)$

$\Rightarrow$ $\frac{\pi}{2}+b = \frac{\pi}{4}+a$

$\Rightarrow$ $a-b = \frac{\pi}{4}$ ....(i)

Also, $\text{RHL}\left(\text{at } x=\frac{\pi}{2}\right) = \text{LHL}\left(\text{at } x=\frac{\pi}{2}\right)$

$\Rightarrow$ $\left(a\cos\frac{2\pi}{2} - b\sin\frac{\pi}{2}\right) = \left(2.\frac{\pi}{2}.\cot\frac{\pi}{2}+b\right)$

$\Rightarrow$ $-a-b = b$

$\Rightarrow\ a + 2b = 0$ ...(ii)

From eqs. (i) and (ii), $a = \dfrac{\pi}{6}$ and $b = \dfrac{-\pi}{12}$

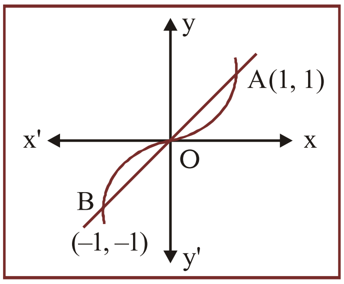
Example – 18

Let $f : R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. Show that the set of points $\{-1, 0, 1\}$, $f(x)$ is not differentiable.

Sol. $f(x) = \max\{x, x^3\}$ considering the graph separately, $y = x^3$ and $y = x$.

$$\text{Now, } \begin{cases} f(x) = x & \text{in } (-\infty, -1] \\ x^3 & \text{in } [-1, 0] \\ x & \text{in } [0, 1] \\ x^3 & \text{in } [1, \infty) \end{cases}$$

The point of consideration are



$f'(-1^-) = 1$ and $f'(-1^+) = 3$
$f'(-0^-) = 0$ and $f'(0^+) = 1$
$f'(1^-) = 1$ and $f'(1^+) = 3$
Hence, f is not differentiable at $-1, 0, 1$.

Example – 19

Show that the function $f(x)$ is continuous at $x = 0$ but its derivative does not exists at $x = 0$

$$\text{if } f(x) = \begin{cases} x \sin(\log x^2); & x \neq 0 \\ 0; & x = 0 \end{cases}$$

Sol. $\text{LHL} = \lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} f(-h) \sin \log(-h)^2$

$$= -\lim_{h\to 0} h \sin \log h^2$$

As $h \to 0$, $\log h^2 \to -\infty$.
Hence $\sin \log h^2$ oscillates between -1 and $+1$.

$$\text{L.H.L} = -\lim_{h\to 0}(h) \times \lim_{h\to 0}(\sin \log h^2)$$

$$= -0 \times (\text{number between } -1 \text{ and } +1) = 0$$

$$\text{R.H.L} = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} h \sin \log h^2 = \lim_{h\to 0} h \lim_{h\to 0} \sin \log h^2$$

$$= 0 \times (\text{oscillating between } -1 \text{ and } +1) = 0$$

$f(0) = 0$ (Given)
$\Rightarrow$ L.H.L. = R.H.L. = $f(0)$
Hence $f(x)$ is continuous at $x = 0$.
Test for differentiability :

$$Lf'(0) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{-h \sin \log(-h)^2 - 0}{-h}$$

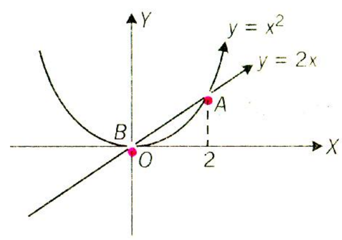
$$= \lim_{h\to 0} \sin(\log h^2)$$

As the expression oscillates between -1 and $+1$, the limit does not exists.
$\Rightarrow$ Left hand derivative is not defined.
Hence the function is not differentiable at $x = 0$

Example – 20

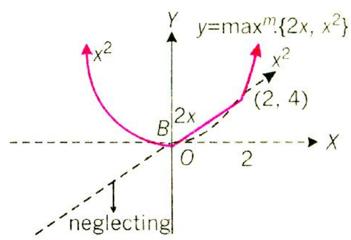
Draw graph for $y = \max.\{2x, x^2\}$ and discuss the continuity and differentiability.

Sol. Here, to draw, $y = \max.\{2x, x^2\}$



Firstly plot $y = 2x$ and $y = x^2$ on graph and put $2x = x^2 \Rightarrow x = 0, 2$ (i.e., their point of intersection).

Now, since $y = \max.\{2x, x^2\}$ we have to neglect the curve below point of intersections thus, the required graph is, as shown.



Thus, from the given graph $y = \max.\{2x, x^2\}$ we can say $y = \max.\{2x, x^2\}$ is continuous for all $x \in R$.

But $y = \max.\{2x, x^2\}$ is differentiable for all $x \in R - \{0, 2\}$

Note...

One must remember the formula we can write,

$$\max.\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} + \left|\frac{f(x)-g(x)}{2}\right|$$

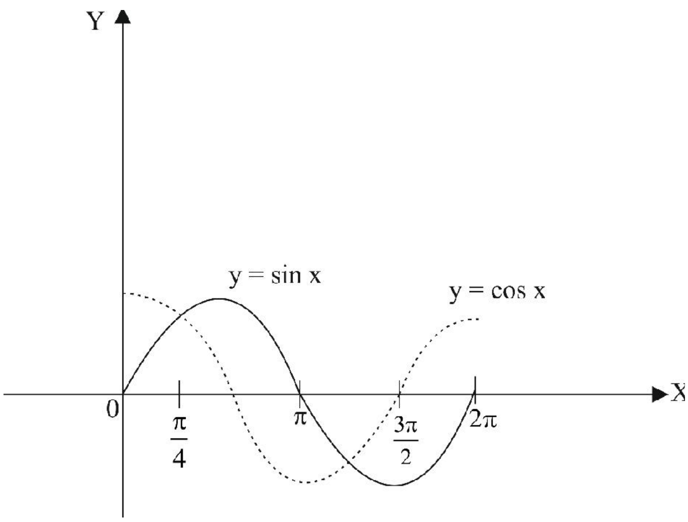
$$\min.\{f(x), g(x)\} = \frac{f(x)+g(x)}{2} - \left|\frac{f(x)-g(x)}{2}\right|$$

Example – 21

Find the number of points of non-differentiability of $f(x) = \max.\{\sin x, \cos x, 0\}$ in $(0, 2n\pi)$.

Sol. Here, we know sin x and cos x are periodic with period 2π. Thus we could sketch the curve as; (In the interval 0 to 2π)

Which shows



$y = \max.\{\sin x, \cos x, 0\}$

$$= \begin{cases} \cos x, & 0 < x < \frac{\pi}{4} \text{ or } \frac{3\pi}{2} < x < 2\pi \\ 0, & \pi < x < \frac{3\pi}{2} \\ \sin x, & \frac{\pi}{4} < x < \pi \end{cases}$$

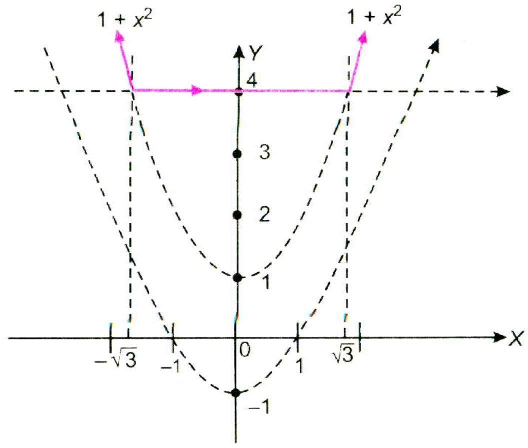
Clearly, $y = \max.\{\sin x, \cos x, 0\}$ is not differentiable at 3 points when $x = (0, 2\pi)$.

Thus, $y = \max.\{\sin x, \cos x, 0\}$ is not differentiable at 3n points.

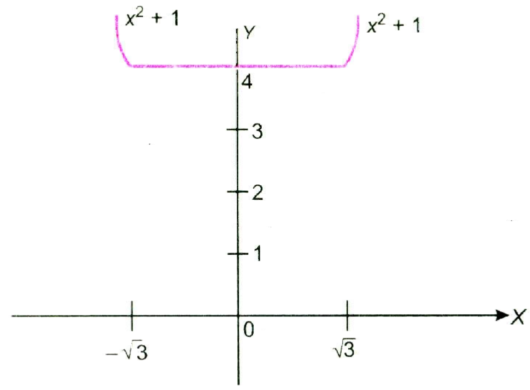
Example – 22

Let $f(x) = \text{maximum}\ \{4, 1 + x^2, x^2 - 1\}\ \forall\ x \in R$. Then find the total number of points, where $f(x)$ is not differentiable.

Sol. We have discussed in last chapter for sketching maximum $\{4, 1 + x^2, x^2 - 1\}$ as



OR



Thus, from above graph we can simply say,

$f(x)$ is not differentiable at $x = \pm\sqrt{3}$.

And it could be defined as :

$$f(x) = \begin{cases} 4, & -\sqrt{3} \le x \le \sqrt{3} \\ x^2 + 1, & x \le -\sqrt{3} \text{ or } x \ge \sqrt{3} \end{cases}$$

Example – 23

Let $f(x+y) = f(x) + f(y)$ for all $x, y \in R$. If $f(x)$ is continuous at $x = 0$, show that $f(x)$ is continuous at all x.

Sol. Since $f(x)$ is continuous at $x = 0$. Therefore,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(0+h) = f(0)$$

$$\Rightarrow \lim_{h\to 0} f(0+(-h)) = \lim_{h\to 0} f(0+h) = f(0)$$

$$\Rightarrow \lim_{h\to 0} [f(0)+f(-h)] = \lim_{h\to 0} [f(0)+f(h)] = f(0)$$

[Using : $f(x+y) = f(x) + f(y)$]

$$\Rightarrow f(0) + \lim_{h\to 0} f(-h) = f(0) + \lim_{h\to 0} f(h) = f(0)$$

$$\Rightarrow \lim_{h\to 0} f(-h) = \lim_{h\to 0} f(h) = 0 \qquad \text{... (i)}$$

Let a be any real number. Then,

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h) = \lim_{h\to 0} f(a+(-h))$$

$$\Rightarrow \lim_{x\to a^-} f(x) = \lim_{h\to 0} [f(a)+f(-h)]$$

[$\because f(x+y) = f(x) + f(y)$]

$$\Rightarrow \lim_{x\to a^-} f(x) = f(a) + \lim_{h\to 0} f(-h)$$

$$\Rightarrow \lim_{x\to a^-} f(x) = f(a) + 0 \qquad \text{[Using (i)]}$$

$$\Rightarrow \lim_{x\to a^-} f(x) = f(a). \qquad \text{and,}$$

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$$

$$\Rightarrow \lim_{x\to a^+} f(x) = \lim_{h\to 0} [f(a)+f(h)]$$

[$\because f(x+y) = f(x) + f(y)$]

$$\Rightarrow \lim_{x\to a^+} f(x) = f(a) + \lim_{h\to 0} f(h)$$

$$\Rightarrow \lim_{x\to a^+} f(x) = f(a) + 0 = f(a) \qquad \text{[Using (i)]}$$

Thus, we have

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$$

$\therefore$ $f(x)$ is continuous at $x = a$.

Since a is an arbitrary real number. So, $f(x)$ is continuous at all $x \in R$.

Example – 24

If $f : R \to R$ is continuous and satisfies the relation

$f(x+y) + f(x-y) = 2\,f(x) + 2\,f(y)$

and $f(1) = 1$, then f(3) is equal to____.

Sol. In the given relation, taking $x = y = 0$, we have $f(0) = 0$. Also $x = 0$ implies

$f(y) + f(-y) = 0 + 2\,f(y)$

$\Rightarrow f(-y) = f(y)$

Again if we put $y = x$ in the given relation we get

$f(2x) = 4\,f(x) = 2^2 f(x)$

Now replacing y with 2x in the given relation we obtain

$f(3x) + f(-x) = 2f(x) + 2\,f(2x)$

Therefore [$\because f(-x) = f(x)$]

$f(3x) = f(x) + 2f(2x)$

$= f(x) + 2.2^2 f(x)$

$= 3^2 f(x)$

Therefore by induction, we have $f(n\,x) = n^2 f(x)$ for all positive integers n. Replacing n with –n and observing that $f(-x) = f(x)\ \forall\ x$, we have

$f(-nx) = f(nx) = n^2 f(x) = (-n)^2 f(x)$

Therefore $f(nx) = n^2 f(x)$ for all integers x. Also

$f(n) = n^2 \quad (\because f(1) = 1)$

If $x = p/q$ is rational, then

$q^2 f(x) = f(qx) = f(p) = p^2 f(1) = p^2 (\because f(1) = 1)$

Therefore

$$f(x) = \frac{p^2}{q^2} = x^2 \text{ for all rational}$$

If x is irrational, then let $[x_n]$ be a sequence of rational numbers such that $x_n \to x$ as $n \to \infty$. Since f is continuous, by Theorem we have

$f(x_n) \to f(x)$ as $n \to \infty$

But

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (x_n^2) = x^2$$

Therefore $f(x) = x^2$ when x is irrational. Also $f(x) = x^2$ for all real x. Hence

$f(3) = 3^2 = 9$

Example – 25

If f is a real-valued function defined for all x ≠ 0, 1 and satisfying the relation

$$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2}{x}-\frac{2}{1-x}$$

Then $\lim_{x\to 2} f(x)$ is______.

Sol. Given relation is

$$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2}{x}-\frac{2}{1-x} \quad \text{...(1)}$$

Replacing x with $\frac{1}{1-x}$ in above equation Eq. we have

$$f\left(\frac{1}{1-x}\right)+f\left(\frac{x-1}{x}\right)=2(1-x)+\frac{2(1-x)}{x} \quad \text{...(2)}$$

Again replacing x with $\frac{1}{1-x}$ in Eq. (1), we get

$$f\left(\frac{x-1}{x}\right)+f(x)=-2x-\frac{2x}{1-x} \quad \text{...(3)}$$

Now adding Eqs. (1) and (3) and subtracting Eq. (2) gives

$$2f(x)=\left(\frac{2}{x}-\frac{2}{1-x}-2x-\frac{2x}{1-x}\right)-2(1-x)-\frac{2(1-x)}{x}$$

$$=\left(\frac{2}{x}-\frac{2(1-x)}{x}\right)+\left(\frac{-2}{1-x}-\frac{2x}{1-x}\right)-2x-2(1-x)$$

$$=\frac{2x}{x}-\frac{2(1+x)}{(1-x)}-2$$

$$=2+\frac{2(x+1)}{x-1}-2$$

$$=\frac{2(x+1)}{x-1}$$

Therefore

$$f(x)=\frac{x+1}{x-1}$$

Taking limit we get

$$\lim_{x\to 2}\left(\frac{x+1}{x-1}\right)=\frac{2+1}{2-1}=3$$

Example – 26

If f is a real-valued function satisfying the relation

$$f(x)+2f\left(\frac{1}{x}\right)=3x$$

for all real x ≠ 0, then $\lim_{x\to 0}(\sin x)\,f(x)$ is equal to

(a) 1 (b) 2

(c) 0 (d) ∞

Ans. (b)

Sol. We have $f(x)+2f\left(\frac{1}{x}\right)=3x$ ...(1)

Replacing x with 1/x, we have

$$2f(x)+f\left(\frac{1}{x}\right)=\frac{3}{x} \quad \text{...(2)}$$

From Eqs. (1) and (2) we get

$$f(x)=\frac{2}{x}-x$$

Therefore

$$\lim_{x\to 0}(\sin x)\,f(x)=\lim_{x\to 0}\left(\frac{2\sin x}{x}-x\sin x\right)$$

$$=2(1)-0=2.$$

Example – 27

P (x) is a polynomial such that P(x) + P (2x) = $5x^2 - 18$.

Then $\lim_{x\to 3}\left(\frac{P(x)}{x-3}\right)=$

(a) 6 (b) 9

(c) 18 (d) 0

Ans. (a)

Sol. Since $5x^2 - 18$ is a quadratic polynomial and P(x) + P (2x) = $5x^2 - 18$ it follows that P(x) must be a quadratic polynomial. Suppose

$$P(x) = ax^2 + bx + c$$

By hypothesis

$(ax^2 + bx + c) + (4ax^2 + 2bx + c) = 5x^2 - 18$

or $5ax^2 + 3bx + 2c = 5x^2 - 18$

This gives a = 1, b = 0, c = –9

So $P(x) = x^2 - 9$

Therefore $\lim_{x\to 3}\frac{P(x)}{x-3}=\lim_{x\to 3}(x+3)=6$

Example – 28

Differentiate the following function w.r.t.x :
$\log \sin x^2$

Sol. We have,

Let $y = \log \sin x^2$.

Putting $v = x^2$ and $u = \sin x^2 = \sin v$, we get

$y = \log u$, $u = \sin v$ and $v = x^2$

$$\therefore \frac{dy}{du} = \frac{1}{u}, \frac{du}{dv} = \cos v \text{ and } \frac{dv}{dx} = 2x.$$

Now,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{u} \times \cos v \times 2x = \frac{1}{\sin v} \cos v \times 2x$$

$[\because 4 = \sin v]$

$$\Rightarrow \frac{dy}{dx} = \cot v.2x = 2x \cot x^2$$

$[\because v = x^2]$

Hence, $\frac{d}{dx}\left(\log \sin x^2\right) = 2x \cot x^2$

Example – 29

Differentiate the following functions with respect to x :
(i) $\log(\sec x + \tan x)$ (ii) $e^{x \sin x}$

Sol. (i) Let $y = \log(\sec x + \tan x)$.

Putting $u = \sec x + \tan x$, we get

$y = \log u$ and $u = \sec x + \tan x$

$$\therefore \frac{dy}{du} = \frac{1}{u} \text{ and } \frac{du}{dx} = \sec x \tan x + \sec^2 x.$$

Now, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{u}.(\sec x \tan x + \sec^2 x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec x + \tan x} \sec x(\tan x + \sec x) = \sec x.$$

(ii) Let $y = e^{x \sin x}$.

Putting $u = x \sin x$, we get

$y = e^u$ and $u = x \sin x$

$$\therefore \frac{dy}{du} = e^u \text{ and } \frac{du}{dx} = x \cos x + \sin x.$$

Now, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\Rightarrow \frac{dy}{dx} = e^u.(x \cos x + \sin x) = e^{x \sin x}(x \cos x + \sin x)$$

Example – 30

If $y = \frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \log\sqrt{1-x^2}$, then prove that $\frac{dy}{dx} = \frac{\sin^{-1} x}{\left(1-x^2\right)^{3/2}}$.

Sol. We have, $y = x \sin^{-1} x\left(1-x^2\right)^{-1/2} + \frac{1}{2}\log\left(1-x^2\right)$.

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left\{x.\sin^{-1} x.\left(1-x^2\right)^{1-/2}\right\} + \frac{1}{2}\frac{d}{dx}\left\{\log\left(1-x^2\right)\right\}$$

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} x.\left(1-x^2\right)^{-1/2}\frac{d}{dx}(x) + x.\frac{d}{dx}\left(\sin^{-1} x\right).\left(1-x^2\right)^{-1/2}$$

$$+ x \sin^{-1} x.\frac{d}{dx}\left(1-x^2\right)^{-1/2} + \frac{1}{2}.\frac{1}{1-x^2}\frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^{-1} x}{\sqrt{1-x^2}}.1 + x \times \frac{1}{\sqrt{1-x^2}} \times \frac{1}{\sqrt{1-x^2}}$$

$$+ x \sin^{-1} x.\left(-\frac{1}{2}\right)\left(1-x^2\right)^{-3/2}\frac{d}{dx}\left(1-x^2\right) + \frac{1}{2\left(1-x^2\right)}(0-2x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^{-1} x}{\sqrt{1-x^2}} + \frac{x}{1-x^2} - \frac{x}{2}\frac{\left(\sin^{-1} x\right)}{\left(1-x^2\right)^{3/2}}(0-2x) - \frac{x}{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^{-1} x}{\sqrt{1-x^2}} + \frac{x^2 \sin^{-1} x}{\left(1-x^2\right)^{3/2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^{-1} x}{\sqrt{1-x^2}}\left\{1 + \frac{x^2}{1-x^2}\right\} = \frac{\sin^{-1} x}{\sqrt{1-x^2}} \times \frac{1}{1-x^2} = \frac{\sin^{-1} x}{\left(1-x^2\right)^{3/2}}$$

Example – 31

If $y=\dfrac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}$, show that

$\dfrac{dy}{dx}=\dfrac{-2a^2}{x^3}\left\{1+\dfrac{a^2}{\sqrt{a^4-x^4}}\right\}.$

Sol. We have,

$$y=\frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}=\frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}-\sqrt{a^2-x^2}}\cdot\frac{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}{\sqrt{a^2+x^2}+\sqrt{a^2-x^2}}$$

$$\Rightarrow\quad y=\frac{\left[\sqrt{a^2+x^2}+\sqrt{a^2-x^2}\right]^2}{\left(a^2+x^2\right)-\left(a^2-x^2\right)}=\frac{a^2+x^2+a^2-x^2+2\sqrt{a^2+x^2}\sqrt{a^2-x^2}}{2x^2}$$

$$\Rightarrow\quad y=\frac{2a^2+2\sqrt{a^4-x^4}}{2x^2}$$

$$\Rightarrow\quad y=\frac{a^2}{x^2}+\frac{\sqrt{a^4-x^4}}{x^2}$$

$$\Rightarrow\quad y=a^2x^{-2}+\sqrt{a^4-x^4}\,x^{-2}$$

$$\Rightarrow\quad \frac{dy}{dx}=a^2\frac{d}{dx}\left(x^{-2}\right)+\frac{d}{dx}\left\{\sqrt{a^4-x^4}\,x^{-2}\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=-2a^2x^{-3}+(-2)x^{-3}\sqrt{a^4-x^4}+\left(x^{-2}\right)\frac{1}{2}\left(a^4-x^4\right)^{-1/2}\frac{d}{dx}\left(a^4-x^4\right)$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{-2a^2}{x^3}-\frac{2}{x^4}\sqrt{a^4-x^4}+\frac{1}{2x^2\sqrt{a^4-x^4}}\left(-4x^3\right)$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{-2a^2}{x^3}-\frac{2}{x^3}\sqrt{a^4-x^4}-\frac{2x}{\sqrt{a^4-x^4}}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{-2a^2}{x^3}-2\left\{\frac{\sqrt{a^4-x^4}}{x^3}+\frac{x}{\sqrt{a^4-x^4}}\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{-2a^2}{x^3}-2\left\{\frac{a^4-x^4+x^4}{x^3\sqrt{a^4-x^4}}\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{-2a^2}{x^3}-\frac{2a^4}{x^3\sqrt{a^4-x^4}}=\frac{-2a^2}{x^3}\left\{1+\frac{a^2}{\sqrt{a^4-x^4}}\right\}$$

Example – 32

If $x\sqrt{1+y}+y\sqrt{1+x}=0$, prove that $\dfrac{dy}{dx}=-\dfrac{1}{(x+1)^2}$

Sol. We have,

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

$\Rightarrow\quad x\sqrt{1+y}=-y\sqrt{1+x}$

$\Rightarrow\quad x^2(1+y)=y^2(1+x)$ [On squaring both sides]

$\Rightarrow\quad x^2-y^2=y^2x-x^2y$

$\Rightarrow\quad (x+y)(x-y)=-xy(x-y)$

$\Rightarrow\quad x+y=-xy$ [$\because x-y\neq 0$. as $y=x$ does not satisfy the given equation]

$\Rightarrow\quad x=-y-xy$

$\Rightarrow\quad y(1+x)=-x$

$$\Rightarrow\quad y=-\frac{x}{1+x}$$

$$\Rightarrow\quad \frac{dy}{dx}=-\left\{\frac{(1+x).1-x(0+1)}{(1+x)^2}\right\}$$

$$\Rightarrow\quad \frac{dy}{dx}=-\frac{1}{(1+x)^2}$$

Example – 33

If $\sin y=x\sin(a+y)$, prove that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}$

Sol. Differentiating both sides of the given relation with respect to x, we get

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left[x\sin(a+y)\right]$$

$$\Rightarrow\quad \cos y\frac{dy}{dx}=1.\sin(a+y)+x\cos(a+y)\cdot\frac{d}{dx}(a+y)$$

$$\Rightarrow\quad \cos y\frac{dy}{dx}=\sin(a+y)+x\cos(a+y).\frac{dy}{dx}$$

$$\Rightarrow\quad \cos y\frac{dy}{dx}-x\cos(a+y)\frac{dy}{dx}=\sin(a+y)$$

$$\Rightarrow\quad \left\{\cos y-x\cos(a+y)\right\}\frac{dy}{dx}=\sin(a+y)$$

$$\Rightarrow \left\{\cos y - \frac{\sin y}{\sin(a+y)} . \cos(a+y)\right\}\frac{dy}{dx} = \sin(a+y)$$

$$\left[\begin{array}{l}\because \sin y = x\sin(a+y) \\ \therefore x = \dfrac{\sin y}{\sin(a+y)}\end{array}\right]$$

$$\Rightarrow \left\{\frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin(a+y)}\right\}\frac{dy}{dx} = \sin(a+y)$$

$$\Rightarrow \frac{\sin(a+y-y)}{\sin(a+y)} \times \frac{dy}{dx} = \sin(a+y) \Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Example – 34

If $\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$, prove that

$\dfrac{dy}{dx} = \dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$, where $-1 < x < 1$ and $-1 < y < 1$.

Sol. Putting $x^3 = \sin A$ and $y^3 = \sin B$ in the given relation, we get

$$\Rightarrow \sqrt{1-\sin^2 A} + \sqrt{1-\sin^2 B} = a(\sin A - \sin B)$$

$$\Rightarrow 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) = 2a\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$\Rightarrow \cot\left(\frac{A-B}{2}\right) = a \Rightarrow \frac{A-B}{2} = \cot^{-1}(a)$$

$\Rightarrow$ $A - B = 2\cot^{-1}(a)$

$\Rightarrow$ $\sin^{-1} x^3 - \sin^{-1} y^3 = 2\cot^{-1}(a)$.

Differentiating both sides with respect to x, we get

$$\frac{1}{\sqrt{1-x^6}} \times \frac{d}{dx}(x^3) - \frac{1}{\sqrt{1-y^6}} \times \frac{d}{dx}(y^3) = 0.$$

$$\Rightarrow \frac{1}{\sqrt{1-x^6}} \times 3x^2 - \frac{1}{\sqrt{1-y^6}} \times 3y^2\frac{dy}{dx} = 0.$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2}\sqrt{\frac{1-y^6}{1-x^6}}$$

Example – 35

Differentiate the following functions with respect to x :

(i) x^{x^x} (ii) $(x^x)^x$

Sol. (i) Let $y = x^{x^x}$. Then,

$y = e^{x^x} . \log x$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{x^x} . \log x \frac{d}{dx}(x^x . \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^{x^x} \frac{d}{dx}(e^{x\log x} . \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^{x^x}\left\{\log x . \frac{d}{dx}(e^{x\log x}) + e^{x\log x} . \frac{d}{dx}(\log x)\right\}$$

$$\Rightarrow \frac{dy}{dx} = x^{x^x}\left\{\log x . e^x \log x \frac{d}{dx}(x\log x) + e^{x\log x} . \frac{1}{x}\right\}$$

$$\Rightarrow \frac{dy}{dx} = x^{x^x}\left\{\log x . x^x\left(x . \frac{1}{x} + \log x\right) + x^x . \frac{1}{x}\right\}$$

$$\Rightarrow \frac{dy}{dt} = x^{x^x}\left\{x^x(1+\log x) . \log x + \frac{x^x}{x}\right\}$$

$$\Rightarrow \frac{dy}{dt} = x^{x^x} . x^x\left\{(1+\log x) . \log x + \frac{1}{x}\right\}$$

(ii) Let $y = (x^x)^x$. Then,

$y = x^{x . x} = x^{x^2}$

$\Rightarrow$ $y = e^{x^2} . \log x$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{x^2} . \log x \frac{d}{dx}(x^2 . \log x)$$

$$\Rightarrow \frac{dy}{dx} = e^{x^2} . \log x\left(\log x . \frac{d}{dx}(x^2) + x^2 . \frac{d}{dx}(\log x)\right)$$

$$\Rightarrow \frac{dy}{dx} = x^{x^2}\left(\log x . 2x + x^2 . \frac{1}{x}\right)$$

$$\left[\because e^{x^2} . \log x = x^{x^2}\right]$$

$$\Rightarrow \frac{dy}{dx} = x^{x^2}(2x . \log x + x)$$

$$\Rightarrow \frac{dy}{dx} = x . x^{x^2}(2\log x + 1).$$

Example – 36

Differentiate : $(\log x)^x + x^{\log x}$ with respect to x.

Sol. Let $y = (\log x)^x + x \log x$. Then,

$$y = e^{\log(\log x)^x} + e^{\log(x^{\log x})}$$

$$\Rightarrow \quad y = e^{x \log(\log x)} + e^{\log x \,.\, \log x}$$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = e^{x\log(\log x)} \cdot \frac{d}{dx}\{x \log(\log x)\} + e^{(\log x)^2} \frac{d}{dx}(\log x)^2$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^x \left\{\log(\log x).\frac{d}{dx}(x) + x.\frac{d}{dx}(\log(\log x))\right\} + x^{\log x}$$

$$\left\{2(\log x.\frac{d}{dx}(\log x))\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^x \left\{\log(\log x) + x.\frac{1}{\log x}.\frac{1}{x}\right\} + x^{\log x}\left\{2\log x.\frac{1}{x}\right\}$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^x \left\{\log(\log x) + \frac{1}{\log x}\right\} + x^{\log x}\left\{\frac{2\log x}{x}\right\}.$$

Example – 37

Differentiate the following functions with respect to x :

$$x^{\cot x} + \frac{2x^2 - 3}{x^2 + x + 2}$$

Sol. Let $y = x^{\cot x} + \frac{2x^2-3}{x^2+x+2}$. Then,

$$y = e^{\cot x.\log x} + \frac{2x^2-3}{x^2+x+2}$$

$$\left[\because x^{\cot x} = e^{\log x^{\cot x}} = e^{\cot x.\log x}\right]$$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\cot x\,.\,\log x}\right) + \frac{d}{dx}\left(\frac{2x^2-3}{x^2+x+2}\right)$$

$$\frac{dy}{dx} = e^{\cot x.\log x}\frac{d}{dx}(\cot x.\ \log x)$$

$$+\frac{(x^2+x+2)\frac{d}{dx}(2x^2-3) - (2x^2-3)\frac{d}{dx}(x^2+x+2)}{(x^2+x+2)^2}$$

$$\frac{dy}{dx} = x^{\cot x}\left\{\log x\ .\ \frac{d}{dx}(\cot x) + \cot x\ .\ \frac{d}{dx}(\log x)\right\}$$

$$+\frac{(x^2+x+2)(4x) - (2x^2-3)(2x+1)}{(x^2+x+2)^2}$$

$$\frac{dy}{dx} = x^{\cot x}\left\{-\operatorname{cosec}^2 x.\log x + \frac{\cot x}{x}\right\} + \frac{2x^2+14x+3}{(x^2+x+2)^2}$$

Example – 38

If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, find $\frac{dy}{dx}$.

Sol. Let $z = \frac{2x-1}{x^2+1}$. Then,

$$y = f(z)$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\{f(z)\} = \frac{d}{dz}\{f(z)\}\cdot\frac{dz}{dx}$$

$$\Rightarrow \frac{dy}{dx} = f'(z)\frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$$

$$\Rightarrow \frac{dy}{dx} = f'(z)\left\{\frac{2(x^2+1)-(2x-1)2x}{(x^2+1)^2}\right\}$$

$$\Rightarrow \frac{dy}{dx} = \sin z^2\ \frac{2(x^2+1)-(4x^2-2x)}{(x^2+1)^2}$$

$$\left[\begin{array}{l}\because f'(x) = \sin x^2 \\ \therefore f'(z) = \sin z^2\end{array}\right]$$

$$\Rightarrow \frac{dy}{dx} = 2\sin\left(\frac{2x-1}{x^2+1}\right)\left\{\frac{1+x-x^2}{\left(x^2+1\right)^2}\right\}$$

Example – 39

Given that $\cos\frac{x}{2}.\ \cos\frac{x}{4}.\ \cos\frac{x}{8}... = \frac{\sin x}{x}$, prove that

$$\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{4} + ... = \operatorname{cosec}^2 x - \frac{1}{x^2}$$

Sol. We have,

$$\cos\frac{x}{2}.\ \cos\frac{x}{4}.\ \cos\frac{x}{8}... = \frac{\sin x}{x}$$

Taking log on both sides, we get

$$\log\cos\frac{x}{2}+\log\cos\frac{x}{4}+\log\cos\frac{x}{8}...=\log\sin x-\log x$$

Differentiating both sides with respect to x, we get

$$-\frac{1}{2}\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}-\frac{1}{4}\frac{\sin\frac{x}{4}}{\cos\frac{x}{4}}-\frac{1}{8}\frac{\sin\frac{x}{8}}{\cos\frac{x}{8}}...=\frac{\cos x}{\sin x}-\frac{1}{x}$$

$$\Rightarrow -\frac{1}{2}\tan\frac{x}{2}-\frac{1}{4}\tan\frac{x}{4}-\frac{1}{8}\tan\frac{x}{8}...=\cot x-\frac{1}{x}$$

Differentiating both sides with respect to x, we get

$$-\frac{1}{2^2}\sec^2\frac{x}{2}-\frac{1}{4^2}\sec^2\frac{x}{4}-\frac{1}{8^2}\sec^2\frac{x}{8}...=-\operatorname{cosec}^2x+\frac{1}{x^2}$$

$$-\frac{1}{2^2}\sec^2\frac{x}{2}+\frac{1}{4^2}\sec^2\frac{x}{4}+\frac{1}{8^2}\sec^2\frac{x}{8}...=\operatorname{cosec}^2x-\frac{1}{x^2}$$

Example – 40

If $y=a^{x^{a^{x\ldots\infty}}}$, prove that $\frac{dy}{dx}=\frac{y^2\log y}{x(1-y\log x\,.\,\log y)}$.

Sol. The given series may be written as

$$y=a^{(x^y)}$$

$\Rightarrow$ $\log y = x^y\log a$ [Taking log of both sides]

$\Rightarrow$ $\log(\log y)=y\log x+\log(\log a)$

$$\Rightarrow \frac{1}{\log y}\frac{d}{dx}(\log y)=\frac{dy}{dx}\,.\,\log x+y.\frac{d}{dx}(\log x)+0$$

[Differentiating both sides with respect to x]

$$\Rightarrow \frac{1}{\log y}.\frac{1}{y}\frac{dy}{dx}=\frac{dy}{dx}.\log x+y.\frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{1}{y\log y}-\log x\right\}=\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}\left\{\frac{1-y\log y.\log x}{y\log y}\right\}=\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y^2\log y}{x\{1-y\log y.\log x\}}.$$

Example – 41

If $y=e^{x+e^{x+e^{x+\ldots\text{to }\infty}}}$, show that $\frac{dy}{dx}=\frac{y}{1-y}$

Sol. The given function may be written as

$y=e^{x+y}$

$\Rightarrow$ $\log y=(x+y)\,.\log e$ [Taking log of both sides]

$\Rightarrow$ $\log y=x+y\,[\because \log e=1]$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx}=1+\frac{dy}{dx} \qquad \text{[Differentiating with respect to x]}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y}-1\right)=1\Rightarrow\frac{dy}{dx}=\frac{y}{1-y}$$

Example – 42

Find $\frac{dy}{dx}$ in the following cases :

(i) $x=a\left\{\cos t+\frac{1}{2}\log\tan^2\frac{t}{2}\right\}$ and $y = a\sin t$

(ii) $x=a(\theta-\sin\theta)$ and $y=a(1-\cos\theta)$

Sol. We have,

(i) $x=a\left\{\cos t+\frac{1}{2}\log\tan^2\frac{t}{2}\right\}$ and $y=a\sin t$

$\Rightarrow$ $x=a\left\{\cos t+\frac{1}{2}2\log\tan\frac{t}{2}\right\}$ and $y=a\sin t$

$\Rightarrow$ $x=a\left\{\cos t+\log\tan\frac{t}{2}\right\}$ and $y=a\sin t$.

Differentiating with respect to t, we get

$$\frac{dx}{dt}=a\left\{-\sin t+\frac{1}{\tan t/2}\sec^2\frac{t}{2}\,.\,\frac{1}{2}\right\} \text{ and } \frac{dy}{dt}=a\cos t\ t$$

$$\Rightarrow \frac{dx}{dt}=a\left\{-\sin t+\frac{1}{2\sin(t/2)\cos(t/2)}\right\} \text{ and } \frac{dy}{dt}=a\cos t$$

$$\Rightarrow \frac{dx}{dt}=a\left\{-\sin t+\frac{1}{\sin t}\right\} \text{ and } \frac{dy}{dt}=a\cos t$$

$$\Rightarrow \frac{dx}{dt}=\frac{a\cos^2 t}{\sin t} \text{ and } \frac{dy}{dt}=a\cos t$$

$$\Rightarrow \frac{dx}{dt} = a\left\{\frac{-\sin^2 t + 1}{\sin t}\right\} \text{ and } \frac{dy}{dx} = a \cos t$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}} = \tan t$$

(ii) We have,

$x = a(\theta - \sin\theta)$ and $y = a(1 - \cos\theta)$

Differentiating with respect to θ, we get

$$\frac{dx}{d\theta} = a(1-\cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot\frac{\theta}{2}.$$

Example – 43

If $x = \sqrt{a^{\sin^{-1} t}}$, $y = \sqrt{a^{\cos^{-1} t}}$, $a > 0$ and $-1 < t < 1$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Sol. We have,

$$x = \sqrt{a^{\sin^{-1} t}} \text{ and } y = \sqrt{a^{\cos^{-1} t}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{2}\left(a^{\sin^{-1} t}\right)^{-1/2}\frac{d}{dt}\left(a^{\sin^{-1} t}\right) \text{ and } \frac{dy}{dx} = \frac{1}{2}\left(a^{\cos^{-1} t}\right)^{-1/2}\frac{d}{dt}\left(a^{\cos^{-1} t}\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{2}\left(a^{\sin^{-1} t}\right)^{-1/2}\left(a^{\sin^{-1} t}\log_e a\right).\frac{d}{dt}\left(\sin^{-1} t\right) \text{ and ,}$$

$$\frac{dy}{dt} = \frac{1}{2}\left(a^{\cos^{-1} t}\right)^{-1/2}\left(a^{\cos^{-1} t}\log_e a\right).\frac{d}{dt}\left(\cos^{-1} t\right)$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{2}\left(a^{\sin^{-1} t}\right)^{1/2}(\log_e a)\times\frac{1}{\sqrt{1-t^2}} = \frac{x\log_e a}{2\sqrt{1-t^2}} \text{ and ,}$$

$$\frac{dy}{dt} = \frac{1}{2}\left(a^{\cos^{-1} t}\right)^{1/2}(\log_e a)\times\frac{-1}{\sqrt{1-t^2}} = \frac{-y\log_e a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y\log_e a}{2\sqrt{1-t^2}} \times \frac{2\sqrt{1-t^2}}{x\log_e a} = \frac{-y}{x}$$

Example – 44

Differentiate $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ with respect to $\cos^{-1} x^2$

Sol. Let $u = \tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}$ and $v = \cos^{-1} x^2$.

Putting $x^2 = \cos\theta$, we get

$$u = \tan^{-1}\left\{\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right\}$$

$$\Rightarrow u = \tan^{-1}\left\{\frac{\sqrt{2\cos^2\theta/2}-\sqrt{2\sin^2\theta/2}}{\sqrt{2\cos^2\theta/2}+\sqrt{2\sin^2\theta/2}}\right\}$$

$$\Rightarrow u = \tan^{-1}\left\{\frac{\cos\theta/2-\sin\theta/2}{\cos\theta/2+\sin\theta/2}\right\}$$

$$\Rightarrow u = \tan^{-1}\left\{\frac{1-\tan\theta/2}{1+\tan\theta/2}\right\}$$

[Dividing numerator and denominator by $\cos\theta/2$]

$$\Rightarrow u = \tan^{-1}\left\{\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

$$\Rightarrow u = \frac{\pi}{4}-\frac{1}{2}\theta = \frac{\pi}{4}-\frac{1}{2}\cos^{-1} x^2 \quad [\because x^2 = \cos\theta \therefore \theta \cos^{-1} x^2]$$

$$\therefore \frac{du}{dx} = -\frac{1}{2}\times\frac{-2x}{\sqrt{1-x^4}} = \frac{x}{\sqrt{1-x^4}}$$

and , $v = \cos^{-1} x^2 \Rightarrow \frac{dv}{dx} = \frac{-2x}{\sqrt{1-x^4}}$

So, $\frac{du}{dv} = \frac{du/dx}{dv/dx} = -\frac{1}{2}$

Example – 45

Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, if

(i) $x \in (-1, 1)$ (ii) $x \in (1, \infty)$

(iii) $x \in (-\infty, -1)$

Sol. Let $u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Putting $x = \tan\theta$, we have

$$u = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) \text{ and } v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$\Rightarrow$ $u = \tan^{-1}(\tan 2\theta)$ and $v = \sin^{-1}(\sin 2\theta)$

(i) When $x \in (-1, 1)$.

We have,

$x \in (-1, 1)$ and $x = \tan\theta$

$$\Rightarrow -1 < \tan\theta < 1 \Rightarrow -\frac{\pi}{4} < \theta < \frac{\pi}{4} \Rightarrow -\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$$

$\Rightarrow$ $\tan^{-1}(\tan 2\theta) = 2\theta$ and $\sin^{-1}(\sin\theta) = 2\theta$

$\therefore$ $u = 2\theta$ and $v = 2\theta$

$\Rightarrow$ $u = 2\tan^{-1}x$ and $v = 2\tan^{-1}x$

$[\because x = \tan\theta \Rightarrow \theta \tan^{-1}x]$

$$\Rightarrow \frac{du}{dx} = \frac{2}{1+x^2} \text{ and } \frac{dv}{dx} = \frac{2}{1+x^2}$$

$$\Rightarrow \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}} = 1$$

(ii) When $x \in (1, \infty)$

We have,

$x \in (1, \infty)$ and $x = \tan\theta$

$$\Rightarrow 1 < \tan\theta < \infty \Rightarrow \frac{\pi}{4} < \theta < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < 2\theta < \pi$$

$\therefore$ $u = \tan^{-1}(\tan 2\theta) = \tan^{-1}\{-\tan(\pi - 2\theta)\}$
$= \tan^{-1}\{\tan(2\theta - \pi)\} = 2\theta - \pi$

$\Rightarrow$ $u = 2\tan^{-1}x - \pi$ $[\because \theta = \tan^{-1}x]$

$$\Rightarrow \frac{du}{dx} = \frac{2}{1+x^2} - 0 = \frac{2}{1+x^2}$$

and, $v = \sin^{-1}(\sin 2\theta) = \sin^{-1}(\sin(\pi - 2\theta)) = \pi - 2\theta$
$= \pi - 2\tan^{-1}x$

$$\Rightarrow \frac{dv}{dx} = 0 - \frac{2}{1+x^2} = \frac{-2}{1+x^2}$$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+x^2}}{\frac{-2}{1+x^2}} = -1$$

(iii) When $x \in (-\infty, -1)$.

We have,

$x = \tan\theta$ and $x \in (-\infty, -1)$

$$\Rightarrow -\infty < \tan\theta < -1 \Rightarrow -\frac{\pi}{2} < \theta < -\frac{\pi}{4} \Rightarrow -\pi < 2\theta < -\frac{\pi}{2}$$

$$\therefore u = \tan^{-1}(\tan 2\theta) = \tan^{-1}\{\tan(\pi + 2\theta)\} = \pi + 2\theta = \pi + 2\tan^{-1}x$$

$$\Rightarrow \frac{du}{dx} = 0 + \frac{2}{1+x^2} = \frac{2}{1+x^2}$$

and, $v = \sin^{-1}(\sin 2\theta) = \sin^{-1}\{-\sin(\pi + 2\theta)\}$

$\Rightarrow$ $v = \sin^{-1}(\sin(-\pi - 2\theta)) = -\pi - 2\theta = -\pi - 2\tan^{-1}x$

$$\Rightarrow \frac{dv}{dx} = -\frac{2}{1+x^2}$$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\frac{2}{1+x^2}}{-\frac{2}{1+x^2}} = -1$$

Example – 46

If $y = A\cos(\log x) + B\sin(\log x)$, prove that

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

Sol. We have,

$y = A\cos(\log x) + B\sin(\log x)$.

On differentiating with respect to x, we get

$$\frac{dy}{dx} = -\frac{1}{x}A\sin(\log x) + \frac{B}{x}\cos(\log x)$$

$\Rightarrow \quad x\frac{dy}{dx} = -A\sin(\log x) + B\cos(\log x).$

On differentiating again with respect to x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -A\frac{\cos(\log x)}{x} - B\frac{\sin(\log x)}{x}$$

$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\{A\cos(\log x) + B\sin(\log x)\}$

$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -y$

$\Rightarrow \quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Example – 47

If $y = x\log\left(\frac{x}{a+bx}\right)$, prove that $x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$

Sol. We have,

$$y = x\log\left(\frac{x}{a+bx}\right)$$

$\Rightarrow \quad y = x[\log x - \log(a+bx)]$

$\Rightarrow \quad \frac{y}{x} = \log x - \log(a+bx)$

On differentiating with respect to x, we get

$$\frac{x\frac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{1}{a+bx}\frac{d}{dx}(a+bx)$$

$\Rightarrow \quad x\frac{dy}{dx} - y = x^2\left\{\frac{1}{x} - \frac{b}{a+bx}\right\}$

$\Rightarrow \quad x\frac{dy}{dx} - y = \frac{ax}{a+bx}$

Differentiating both sides of (i) with respect to x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx).a - ax(0+b)}{(a+bx)^2}$$

$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$

$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2}$ [Multiplying both sides by x^2]

$\Rightarrow \quad x^3\frac{d^2y}{dx^2} = \left(\frac{ax}{a+bx}\right)^2$

From (i) and (ii), we have

$$x^3\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)^2$$

Example – 48

If $y = \sin^{-1}\left\{x\sqrt{(1-x)} - \sqrt{x}\sqrt{1-x^2}\right\}$ then find $\frac{dy}{dx}$.

Sol. We have $y = \sin^{-1}\left\{x\sqrt{(1-x)} - \sqrt{x}\sqrt{1-x^2}\right\}$

Putting $x = \sin\theta$ and $\sqrt{x} = \sin\phi$

then $y = \sin^{-1}\{\sin\theta\cos\phi - \cos\theta\sin\phi\} = \sin^{-1}\sin(\theta - \phi) = \theta - \phi$

$= \sin^{-1}x - \sin^{-1}\sqrt{x}$

$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}\sin^{-1}x - \frac{d}{dx}\sin^{-1}\sqrt{x}$

$$= \frac{1}{\sqrt{(1-x^2)}} - \frac{1}{\sqrt{1-x}}\frac{d}{dx}\sqrt{x} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}$$

Example – 49

If $u = f(x^2)$, $v = g(x^3)$, $f'(x) = \sin x$ and $g'(x) = \cos x$ then find $\frac{du}{dv}$

Sol. Differentiating $u = f(x^2)$ and $v = g(x^3)$ w.r.t.x we get

$$\frac{du}{dx} = f'(x^2).2x = \sin(x^2).2x$$

$$\{\because f'(x) = \sin x \Rightarrow f'(x^2) = \sin(x^2)\}$$

$$\frac{dv}{dx} = g'(x^3).3x^2 = \cos(x^3).3x^2$$

$$\{\because g'(x) = \cos x \Rightarrow g'(x^3) = \cos(x^3)\}$$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{\sin(x^2).2x}{\cos(x^3).3x^2} = \frac{2}{3x}\cdot\frac{\sin x^2}{\cos x^3}$$

Example – 50

If $\sqrt{(1-x^{2n})}+\sqrt{(1-y^{2n})}=a(x^n-y^n)$ then prove that $\frac{dy}{dx}=\frac{x^{n-1}}{y^{n-1}}\sqrt{\left(\frac{1-y^{2n}}{1-x^{2n}}\right)}$.

Sol. We have $\sqrt{(1-x^{2n})}+\sqrt{(1-y^{2n})}=a(x^n-y^n)$ ...(1)

$$\left.\begin{aligned}\text{Putting } x^n=\sin\theta &\Rightarrow \theta=\sin^{-1}x^n\\ \text{and } y^n=\sin\phi &\Rightarrow \phi=\sin^{-1}y^n\end{aligned}\right\} \quad ...(2)$$

then (1), becomes $\cos\theta+\cos\phi=a(\sin\theta-\sin\phi)$

$$\Rightarrow\quad 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)=a.2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

$$\Rightarrow\quad \cot\left(\frac{\theta+\phi}{2}\right)=a \Rightarrow \theta-\phi=2\cot^{-1}a$$

$$\Rightarrow\quad \sin^{-1}x^n-\sin^{-1}y^n=2\cot^{-1}a \qquad \{\text{from (2)}\}$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{\sqrt{(1-x^{2n})}}.nx^{n-1}-\frac{1}{\sqrt{(1-y^{2n})}}.ny^{n-1}\frac{dy}{dx}=0$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{x^{n-1}}{y^{n-1}}\sqrt{\left(\frac{1-y^{2n}}{1-x^{2n}}\right)} \qquad \text{(Remember)}$$

Corollary : (i) For n = 1

$$\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=a(x-y) \text{ then } \frac{dy}{dx}=\sqrt{\left(\frac{1-y^2}{1-x^2}\right)}$$

(ii) For n = 2

$$\sqrt{(1-x^4)}+\sqrt{(1-y^4)}=a(x^2-y^2)$$

$$\text{then } \frac{dy}{dx}=\frac{x}{y}\sqrt{\left(\frac{1-y^4}{1-x^4}\right)}$$

(iii) For n = 3

$$\sqrt{(1-x^6)}+\sqrt{(1-y^6)}=a(x^3-y^3)$$

$$\text{then } \frac{dy}{dx}=\frac{x^2}{y^2}\sqrt{\left(\frac{1-y^6}{1-x^6}\right)}$$

Example – 51

Find the derivative of f (tan x) with respect to g (sec x) at x = π/4, if $f'(1)=2,\ g'(\sqrt{2})=4$.

Sol. Let $u=f(\tan x)$

$$\therefore\quad \frac{du}{dx}=f'(\tan x).\sec^2 x \qquad ...(1)$$

and let $v=g(\sec x)$

$$\frac{dv}{dx}=g'(\sec x).\sec x\tan x \quad ...(2)$$

From (1) and (2)

$$\frac{du}{dv}=\frac{\left(\frac{du}{dx}\right)}{\left(\frac{dv}{dx}\right)}=\frac{f'(\tan x).\sec^2 x}{g'(\sec x).\sec x\tan x}=\frac{f'(\tan x)}{g'(\sec x)}.\frac{1}{\sin x}$$

$$\therefore\quad \left.\frac{du}{dv}\right|_{x=\pi/4}=\frac{f'(1)}{g'(\sqrt{2})}.\frac{1}{(1/\sqrt{2})}=\frac{2.\sqrt{2}}{4}=\frac{1}{\sqrt{2}}$$

Note...

In general if $y=f(u)$ then $\frac{dy}{dx}=f'(u).\frac{du}{dx}$

Example – 52

If f, g, h are differentiable functions of x and

$$\Delta(x)=\begin{vmatrix} f & g & h\\ (xf)' & (xg)' & (xh)'\\ (x^2f)'' & (x^2g)'' & (x^2h)''\end{vmatrix}, \text{ prove that}$$

$$\frac{d\Delta(x)}{dx}=\begin{vmatrix} f & g & h\\ f' & g' & h'\\ (x^3f'')' & (x^3g'')' & (x^3h'')'\end{vmatrix}$$

Sol. $$\Delta(x)=\begin{vmatrix} f & g & h\\ f+xf' & g+xg' & h+xh'\\ 2f+4xf'+x^2f'' & 2g+4xg'+x^2g'' & 2h+4xh'+x^2h''\end{vmatrix}$$

$$\{\because (x^2f)'=2xf+x^2f'\ (x^2f)''=2(f+xf')+2xf'+x^2f''\}$$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ 2xf'+x^2f'' & 2xg'+x^2g'' & 2xh'+x^2h'' \end{vmatrix},$$

$R_3 \to R_3 - 2R_1, R_2 \to R_2 - R_1$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' & x^2g'' & x^2h'' \end{vmatrix}, R_3 \to R_3 - 2R_2$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix},$$

taking the common factor x from R_2 to R_3

$$\therefore \frac{d\Delta(x)}{dx} = \begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$+ \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3f'' & x^3g'' & x^3h'' \end{vmatrix}$$

$$+ \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

$$= 0 + x^3 \begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ f'' & g'' & h'' \end{vmatrix}$$

$$+ \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

$$= 0 + x^3 \times 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

Example – 53

Find the sum of

$\sin x + 3\sin 3x + 5\sin 5x + ... + (2k-1)\sin(2k-1)x.$

Sol. Let $S = \cos x + \cos 3x + \cos 5x + ... + \cos(2k-1)x$.

Here the angles are in AP whose first term = x, common diff. = 2x.

$$\therefore \quad S = \frac{\sin\frac{k.2x}{2}}{\sin\frac{2x}{2}} \cos\left\{\frac{x+(2k-1)x}{2}\right\}$$

$$= \frac{\sin kx}{\sin x}\cos kx = \frac{\sin 2kx}{2\sin x}$$

$$\therefore \quad \cos x + \cos 3x + \cos 5x + ... + \cos(2k-1)x = \frac{\sin 2kx}{2\sin x}$$

Differentiating w.r.t.x,

$-\{\sin x + 3\sin 3x + 5\sin 5x + ... + (2k-1)\sin(2k-1)x\}$

$$= \frac{1}{2}\cdot\frac{2k\cos 2kx.\sin x - \sin 2kx.\cos x}{\sin^2 x}$$

$\therefore \quad \sin x + 3\sin 3x + 5\sin 5x + ... + (2k-1)\sin(2k-1)x$

$$= \frac{-1}{2\sin^2 x}[k\{\sin(2k+1)x - \sin(2k-1)x\} - \frac{1}{2}\{\sin(2k+1)x + \sin(2k-1)x\}]$$

$$= \frac{1}{4\sin^2 x}[(2k+1)\sin(2k-1)x - (2k-1)\sin(2k+1)x]$$

Example – 54

Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p is constant then

find $\frac{d^3}{dx^3}\{f(x)\}$ at $x = 0$.

Sol. We have

$$f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\frac{d^3}{dx^3}\{f(x)\}=\begin{vmatrix}\frac{d^3}{dx^3}x^3 & \frac{d^3}{dx^3}\sin x & \frac{d^3}{dx^3}\cos x\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}$$

($\because$ All elements in second row and third row are constants)

$$=\begin{vmatrix}3! & \sin\left(x+\frac{3\pi}{2}\right) & \cos\left(x+\frac{3\pi}{2}\right)\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}$$

$$\left.\frac{d^3}{dx^3}f(x)\right|_{x=0}=\begin{vmatrix}6 & \sin\frac{3\pi}{2} & \cos\frac{3\pi}{2}\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}=\begin{vmatrix}6 & -1 & 0\\ 6 & -1 & 0\\ p & p^2 & p^3\end{vmatrix}=0$$

($\because$ first and second rows are identical).

Example – 55

If $y=\tan^{-1}\frac{1}{(x^2+x+1)}+\tan^{-1}\frac{1}{(x^2+3x+3)}$ $+\tan^{-1}\frac{1}{(x^2+5x+7)}+\tan^{-1}\frac{1}{(x^2+7x+13)}+\ldots$ to n terms. Find $\frac{dy}{dx}$.

Sol. Since

$$y=\tan^{-1}\frac{1}{(x^2+x+1)}+\tan^{-1}\frac{1}{(x^2+3x+3)}$$
$$+\tan^{-1}\frac{1}{(x^2+5x+7)}+\tan^{-1}\frac{1}{(x^2+7x+13)}+\ldots \text{ to n terms.}$$

$$=\tan^{-1}\left(\frac{(x+1)-x}{1+x(x+1)}\right)+\tan^{-1}\left(\frac{(x+2)-(x+1)}{1+(x+1)(x+2)}\right)$$
$$+\tan^{-1}\left(\frac{(x+3)-(x+2)}{1+(x+2)(x+3)}\right)+\tan^{-1}\left(\frac{(x+4)-(x+3)}{1+(x+3)(x+4)}\right)$$
$$+\ldots\ldots\ldots+\tan^{-1}\left(\frac{(x+n)-(x+n-1)}{1+(x+n-1)(x+n)}\right)$$
$$=\tan^{-1}(x+1)-\tan^{-1}x+\tan^{-1}(x+2)-\tan^{-1}(x+1)$$
$$+\tan^{-1}(x+3)-\tan^{-1}(x+2)+\tan^{-1}(x+4)-\tan^{-1}(x+3)$$
$$+\ldots\ldots+\tan^{-1}(x+n)-\tan^{-1}(x+n-1)=\tan^{-1}(x+n)-\tan^{-1}x$$

$$\therefore\quad \frac{dy}{dx}=\frac{1}{1+(x+n)^2}-\frac{1}{(1+x^2)}.$$

Example – 56

If $(a+bx)e^{y/x}=x$, show that $x^3y''=(xy'-y)^2$.

Sol. We have $(a+bx)e^{y/x}=x$

$\Rightarrow\quad e^{y/x}=\frac{x}{(a+bx)}$ ...(1)

Taking logarithm of both sides, we have

$$\frac{y}{x}=\ln x-\ln(a+bx)$$

Differentiating both sides w.r.t. x, we get

$$\frac{xy'-y}{x^2}=\frac{1}{x}-\frac{b}{(a+bx)}$$

$\Rightarrow\quad xy'-y=\frac{ax}{(a+bx)}=ae^{y/x}$ {from (1)}

Again taking logarithm of both sides, we have

$$\ln(xy'-y)=\ln a+\frac{y}{x}$$

Again differentiating both sides w.r.t. x, we get

$\Rightarrow\quad \frac{(xy''+y'-y')}{(xy'-y)}=0+\frac{(xy'-y.1)}{x^2}$

Hence $x^3y''=(xy'-y)^2$

EXERCISE - 1 : BASIC OBJECTIVE QUESTIONS

Checking continuity at a point

1. Let $f(x) = \frac{1-\tan x}{4x-\pi}$, $x \neq \frac{\pi}{4}$, $x \in \left[0, \frac{\pi}{2}\right]$. $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is

(a) – 1/2 (b) 1/2

(c) 1 (d) –1

2. If f(x) be a continuoes function and g(x) be discontinuous function, then f(x) + g(x) is a

(a) continuous function (b) discontinuous function

(c) can't say anything (d) none of these

3. The point of discontinuity of the function

$f(x) = \frac{1+\cos 5x}{1+\cos 4x}$, is

(a) $x = 2$ (b) $x = \frac{\pi}{6}$

(c) $x = \pi$ (d) $x = \frac{\pi}{4}$

4. The function $f(x) = \begin{cases} \frac{1}{4^x - 1}; & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous

(a) everywhere except at x = 0 and x = 1

(b) nowhere

(c) everywhere

(d) everywhere except at x = 0

5. The function $f(x) = (1+x)^{\cot x}$ is not defined at x = 0. The value of f (0) so that f (x) becomes continuous at x = 0 is

(a) 1 (b) 0

(c) e (d) none of these

6. The function $f(x) = \frac{1}{u^2 + u - 2}$, where $u = \frac{1}{x-1}$, is discontinuous at the points

(a) $x = -2, 1, \frac{1}{2}$ (b) $x = \frac{1}{2}, 1, 2$

(c) $x = 1, 0$ (d) none of these

7. If $f : R \to R$ is defined by

$$f(x) = \begin{cases} \frac{x-2}{x^2 - 3x + 2}, & \text{if } x \in R - \{1, 2\} \\ 2, & \text{if } x = 1 \\ 1, & \text{if } x = 2 \end{cases},$$

then $\lim_{x \to 2} \frac{f(x) - f(2)}{x-2} =$

(a) 0 (b) –1

(c) 1 (d) –1/2

8. If $f(x) = \begin{cases} \frac{1-\sin^2 x}{3\cos^2 x}, & x < \frac{\pi}{2} \\ a, & x = \frac{\pi}{2} \\ \frac{b(1-\sin x)}{(\pi - 2x)^2}, & x > \frac{\pi}{2} \end{cases}$. Then $f(x)$ is continuous at $x = \frac{\pi}{2}$, if

(a) $a = \frac{1}{3}, b = 2$ (b) $a = \frac{1}{3}, b = \frac{8}{3}$

(c) $a = \frac{2}{3}, b = \frac{8}{3}$ (d) None of these

Finding unknown when function is continuous

9. Let $f(x)=\begin{cases}\dfrac{(x^3+x^2-16x+20)}{(x-2)^2} & \text{if } x\neq 2\\ k & \text{if } x=2\end{cases}$

If f(x) is continuous for all x, then k =

(a) 7 (b) 2

(c) 0 (d) –1

10. If the function $f(x)=\begin{cases}\dfrac{x^2-(A+2)x+A}{x-2}, & \text{for } x\neq 2\\ 2\quad , & \text{for } x=2\end{cases}$ is continuous at x = 2, then

(a) A = 0 (b) A = 1

(c) A = –1 (d) None of these

11. The function $f(x)=\dfrac{\ln(1+ax)-\ln(1-bx)}{x}$ is not defined at x = 0. The value which should be assigned to f at x = 0 so that it is continuous at x = 0, is

(a) a – b (b) a + b

(c) *l*n a + *l*n b (d) None of these

12. Let $f(x)=\begin{cases}\dfrac{\tan x-\cot x}{x-\frac{\pi}{4}}, & x\neq\dfrac{\pi}{4}\\ a, & x=\dfrac{\pi}{4}\end{cases}$

Then the value of a so that f(x) is continuous at $x=\dfrac{\pi}{4}$ is

(a) 2 (b) 4

(c) 3 (d) 1

13. $F(x)=\begin{cases}ax^2+b, & 0\le x<1\\ 4, & x=1\\ x+3, & 1<x\le 2\end{cases}$ then the value of (a, b) for which f(x) cannot be continuous at x = 1.

(a) (2, 2) (b) (3, 1)

(c) (4, 0) (d) (5, 12)

14. If $f(x)=\begin{cases}\dfrac{A+3\cos x}{x^2}, & x<0\\ B\tan\dfrac{\pi}{[x+3]}, & x\ge 0\end{cases}$

where [.] represents greatest integer function is continuous at x = 0. Then,

(a) $A=-3, B=-\sqrt{3}$ (b) $A=3, B=-\dfrac{\sqrt{3}}{2}$

(c) $A=-3, B=-\dfrac{\sqrt{3}}{2}$ (d) $A=-\dfrac{\sqrt{3}}{2}, B=-3$

15. Let $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2}, & x<0\\ a, & x=0\\ \dfrac{\sqrt{x}}{(16+\sqrt{x})^{1/2}-4}, & x>0\end{cases}$

The value of ‘a’ for which f (x) becomes continuous at 0 must be

(a) 2 (b) 4

(c) 6 (d) 8

16. The value of f(0) so that the function $f(x)=\dfrac{2x-\sin^{-1}x}{2x+\tan^{-1}x}$ is continous at each point on its domain is

(a) 2 (b) $\dfrac{1}{3}$

(c) $\dfrac{2}{3}$ (d) $-\dfrac{1}{3}$

17. $F(x)=(x-1)^{\frac{1}{2-x}}$ is not defined at x = 2. If f(x) is continuous, then F(2) is equal to

(a) e (b) e^{-1}

(c) e^{-2} (d) 1

18. If the function $f(x)=\begin{cases} x+a\sqrt{2}\sin x, & 0\le x<\frac{\pi}{4} \\ 2x\cot x+b, & \frac{\pi}{4}\le x\le\frac{\pi}{2} \\ a\cos 2x-b\sin x, & \frac{\pi}{2}<x\le\pi \end{cases}$

is continuous in the interval [0, π], then

(a) $a=\frac{\pi}{6}, b=\frac{\pi}{12}$ (b) $a=-\frac{\pi}{6}, b=\frac{\pi}{12}$

(c) $a=-\frac{\pi}{6}, b=-\frac{\pi}{12}$ (d) $a=\frac{\pi}{6}, b=-\frac{\pi}{12}$

19. Let $f(x)=\begin{cases} \frac{x-4}{|x-4|}+a, & x<4 \\ a+b, & x=4 \\ \frac{|x-4|}{x-4}+b, & x>4 \end{cases}$ then f (x) is continuous

at $x=4$, when

(a) $a=b=0$ (b) $a=b=1$

(c) $a=-1, b=1$ (d) $a=1, b=-1$

20. If the function $f(x)=\begin{cases} Ax-B, & x\le 1 \\ 3x, & 1<x<2 \\ Bx^2-A, & x\ge 2 \end{cases}$ be

continuous at x = 1 and discontinuous at x = 2, then

(a) $A=3+B, B\ne 3$ (b) $A=3+B, B=3$

(c) $A=3+B$ (d) none of these

21. If $f(x)=\frac{2-(256-7x)^{1/8}}{(5x+32)^{1/5}-2}$ $(x\ne 0)$; then for f to be continous everywhere, f(0) is equal to

(a) –1 (b) 1

(c) 2^6 (d) none of these

Mixed Problems of Continuous Differentiability

22. The function $f(x)=\sin^{-1}(\cos x)$ is

(a) discontinuous at x = 0

(b) continuous at x = 0

(c) differentiable at x = 0

(d) None of these

23. The function $f(x)=1+|\sin x|$ is

(a) continuous no where

(b) continuous every where and no differentiable at x = 0

(c) differentiable no where

(d) differentiable at x = 0

24. For the function $f(x)=(\pi-x)\frac{\cos x}{|\sin x|}$; $x\ne\pi$, $f(\pi)=1$, which of the following statements is true ?

(a) $f(\pi-0)=-1$ at $f(\pi+0)=1$

(b) f (x) is continuous at $x=\pi$

(c) f (x) is differentiable at $x=\pi$

(d) None of these

25. The function $f(x)=\begin{cases} |2x-3|.[x], & x\ge 1 \\ \sin\left(\frac{\pi x}{2}\right), & x<1 \end{cases}$

(where [x] denotes greatest integer ≤ x)

(a) continuous at x = 2

(b) differentiable at x = 1

(c) continuous but not differentiable at x = 1

(d) None of these

26. If f (x) = $\begin{cases} \frac{[x]-1}{x-1}, & x\ne 1 \\ 0, & x=1 \end{cases}$, where [x] denotes greatest integer ≤ x. then f(x) is

(a) continuous as well as differentiable at x = 1

(b) differentiable but not continuous at x = 1

(c) continuous but not differentiable at x = 1

(d) neither continuous nor differentiable at x = 1

27. If $f(x)=\begin{cases} 3^x, & -1\le x\le 1 \\ 4-x, & 1<x\le 4 \end{cases}$, then f (x) is

(a) continuous as well as differentiable at x = 1

(b) continuous but not differentiable at x = 1

(c) differentiable but not continuous at x = 1

(d) none of the above

28. The set of points where the function f (x) = |x–1| e^x is differentiable is

(a) R (b) R–{1}

(c) R – {–1} (d) R – {0}

29. If $f(x) = x + \frac{x}{1+x} + \frac{x}{(1+x)^2} + \ldots$ to ∞, then at $x = 0$, $f(x)$

(a) $\lim_{x\to 0} f(x)$ does not exist

(b) is discontinuous

(c) is continuous but not differentiable

(d) is differentiable

30. Let $f(x+y) = f(x)\,f(y)$ for all $x, y \in R$.

Suppose that $f(3) = 3$ then, $f'(3)$ is equal to

(a) 22 (b) 44

(c) 28 (d) none of these

31. If $f(x) = x^3 \operatorname{sgn} x$, then

(a) f is derivable at $x = 0$

(b) f is continuous but not derivable at $x = 0$

(c) LHD at $x = 0$ is 1

(d) RHD at $x = 0$ is 1

32. The set of points where the function

$f(x) = [x] + |1 - x|, -1 \le x \le 3$

where [.] denotes the greatest integer function, is not differentiable, is

(a) $\{-1, 0, 1, 2, 3\}$ (b) $\{-1, 0, 2\}$

(c) $\{0, 1, 2, 3\}$ (d) $\{-1, 0, 1, 2\}$

33. Let $f(x) = \lim_{n\to\infty} (\sin x)^{2n}$. Then, which one of the following is incorrect ?

(a) continuous at $x = \pi/2$

(b) discontinuous at $x = \pi/2$

(c) discontinuous at $x = -\pi/2$

(d) discontinuous at infinite number of points.

34. If $f(x) = \begin{cases} \frac{|x+2|}{\tan^{-1}(x+2)}; & x \ne -2 \\ = 2; & x = -2 \end{cases}$ then $f(x)$ is

(a) continuous at $x = -2$

(b) not continuous at $x = -2$

(c) differentiable at $x = -2$

(d) continuous but not diff. at $x = -2$

35. For the function $f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$ which of the following is incorrect ?

(a) continuous at $x = 1$ (b) continuous at $x = 3$

(c) derivable at $x = 1$ (d) derivable at $x = 3$

36. The number of points at which the function $f(x) = |x - 0.5| + |x-1| + \tan x$ does not have a derivative in interval $(0, 2)$ is

(a) 1 (b) 2

(c) 3 (d) 4

37. $F(x) = \begin{cases} [\cos \pi x], & x < 1 \\ |x-2|, & 2 > x \ge 1 \end{cases}$, then f(x) is where [.] denotes greatest integer fraction.

(a) discontinous and non-diff. at $x = -1$ and $x = 1$

(b) continuous and differentiable at $x = 0$

(c) discontinuous at $x = \frac{1}{2}$

(d) cont. but & t not diff. at $x = 2$

38. A function is defined as follows :

$f(x) = \begin{cases} x^3; & x^2 < 1 \\ x; & x^2 \ge 1 \end{cases}$ The function is

(a) dis continuous at $x = 1$

(b) differentiable at $x = 1$

(c) continuous but not differentiable at $x = 1$

(d) none of these

39. If $f(x) = \begin{cases} \frac{x(3e^{1/x}+4)}{2-e^{1/x}}; & x \ne 0 \\ 0; & x = 0 \end{cases}$ then

(a) $\lim_{x\to 0} f(x) = 1$

(b) $f(x)$ is continuous at $x = 0$

(c) $f(x)$ is differentiable at $x = 0$

(d) None of these

40. If $f(x)=\sqrt{1-\sqrt{1-x^2}}$ then f(x) is

(a) continuous on [–1, 1] and differentiable on (–1, 1)

(b) continuous on [–1, 1] and differentiable on $(-1,0)\cup(0,1)$

(c) continuous and differentiable on [–1, 1]

(d) None of these

Questions Based on Basic Differentiation

(Product Rule, Quotient Rule)

41. Derivative of x^6+6^x with respect to x is

(a) 12x (b) x + 4

(c) $6x^5+6^x \log 6$ (d) $6x^5+x6^{x-1}$

42. If $y=\frac{a+bx^{3/2}}{x^{5/4}}$ and y′ = 0 at x = 5, then the ratio a : b is equal to

(a) $\sqrt{5}:1$ (b) 5 : 2

(c) 3 : 5 (d) 1 : 2

43. If $y=\log_a x+\log_x a+\log_x x+\log_a a$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{x}+x\log a$ (b) $\frac{\log a}{x}+\frac{x}{\log a}$

(c) $\frac{1}{x\log a}+x\log a$ (d) $\frac{1}{x\log a}-\frac{\log a}{x(\log x)^2}$

44. If $y=\tan^{-1}\left[\frac{\sin x+\cos x}{\cos x-\sin x}\right]$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{\pi}{4}$

(c) 0 (d) 1

45. If $y=(1+x^2)\tan^{-1}x-x$, then $\frac{dy}{dx}$ is equal to

(a) $\tan^{-1}x$ (b) $2x\tan^{-1}x$

(c) $2x\tan^{-1}x-1$ (d) $\frac{2x}{\tan^{-1}x}$

46. If $y=2^x\,.\,3^{2x-1}$, then $\frac{dy}{dx}$ is equal to

(a) (log 2) (log 3) (b) (log 18)

(c) $(\log 18^2)\,y^2$ (d) (log 18) y

Questions Based on Chain Rule

47. If *f* be a polynomial then, the second derivative of $f(e^x)$ is

(a) $f'(e^x)$ (b) $f''(e^x)\,e^x+f'(e^x)$

(c) $f''(e^x)\,e^{2x}+f''(e^x)$ (d) $f''(e^x)\,e^{2x}+f'(e^x)\,e^x$

48. If $f(x)=\sqrt{1+\cos^2(x^2)}$, then the falue of $f'\left(\frac{\sqrt{\pi}}{2}\right)$ is

(a) $\frac{\sqrt{\pi}}{6}$ (b) $-\sqrt{\frac{\pi}{6}}$

(c) $\frac{1}{\sqrt{6}}$ (d) $\frac{\pi}{\sqrt{6}}$

49. If $y=e^{\tan x}$, then $\cos^2 x\,\frac{d^2y}{dx^2}=$

(a) $(1-\sin 2x)\frac{dy}{dx}$ (b) $-(1+\sin 2x)\frac{dy}{dx}$

(c) $(1+\sin 2x)\frac{dy}{dx}$ (d) None of these

50. If *g* is the inverse of *f* and $f'(x)=\frac{1}{1+x^3}$ then g' (x) is equal to

(a) $1+[g(x)]^3$ (b) $\frac{1}{1+[g(x)]^3}$

(c) $[g(x)]^3$ (d) None of these

51. If $y=\left(x+\sqrt{1+x^2}\right)^n$, then $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}$ is

(a) n^2y (b) $-n^2y$

(c) – y (d) $2x^2y$

52. If $f(x) = \log_x (\log_e x)$, then $f'(x)$ at $x = e$ is

(a) e (b) $\frac{1}{e}$

(c) $\frac{2}{e}$ (d) 0

53. If $f(x) = e^x g(x)$, $g(0) = 2$, $g'(0) = 1$, then $f'(0)$ is

(a) 1 (b) 3

(c) 2 (d) 0

Questions Based on Impticit Function

54. If $2^x + 2^y = 2^{x+y}$, then the value of $\frac{dy}{dx}$ at $x = y = 1$, is

(a) 0 (b) –1

(c) 1 (d) 2

55. If $x^y = e^{x-y}$, then $\frac{dy}{dx}$ is equal to

(a) $(1 + \log x)^{-1}$ (b) $(1 + \log x)^{-2}$

(c) $\log x \,.\, (1 + \log x)^{-2}$ (d) None of these

56. If $e^y + xy = e$, then the value of $\frac{d^2y}{dx^2}$ for $x = 0$, is

(a) $1/e$ (b) $1/e^2$

(c) $1/e^3$ (d) e

57. If $2x^2 - 3xy + y^2 + x + 2y - 8 = 0$, then $\frac{dy}{dx} =$

(a) $\frac{3y-4x-1}{2y-3x+2}$ (b) $\frac{3y+4x+1}{2y+3x+2}$

(c) $\frac{3y-4x+1}{2y-3x-2}$ (d) $\frac{3y-4x+1}{2y+3x+2}$

58. If $x^2 + y^2 = t + \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y}{x}$ (b) $-\frac{y}{x}$

(c) $\frac{x}{y}$ (d) $-\frac{x}{y}$

Questions Based on Parametric Functions

59. If $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$, then $\frac{dy}{dx}$ is equal to

(a) $-\frac{y}{x}$ (b) $\frac{y}{x}$

(c) $-\frac{x}{y}$ (d) $\frac{x}{y}$

60. If $x = a\cos^4\theta$, $y = a\sin^4\theta$, then $\frac{dy}{dx}$ at $\theta = \frac{3\pi}{4}$ is

(a) –1 (b) 1

(c) $-a^2$ (d) a^2

61. If $t \in (0, ½)$ and $x = \sin^{-1}(3t - 4t^3)$ and $y = \cos^{-1}\left(\sqrt{1-t^2}\right)$, then $\frac{dy}{dx}$ is equal to

(a) 1/2 (b) 2/5

(c) 3/2 (d) 1/3

62. If $y = A\cos nx + B\sin nx$, then $\frac{d^2y}{dx^2} =$

(a) $-n^2y$ (b) $-y$

(c) n^2y (d) none of these

63. If $x = a\sin\theta$ and $y = b\cos\theta$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $\frac{a}{b^2}\sec^2\theta$ (b) $-\frac{b}{a}\sec^2\theta$

(c) $\frac{b}{a^2}\sec^3\theta$ (d) $-\frac{b}{a^2}\sec^3\theta$

64. If $x = \cos\theta$, $y = \sin 5\theta$, then $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} =$

(a) $-5y$ (b) $5y$

(c) $25y$ (d) $-25y$

Questions based on Differentiation of a function w.r.t. another function.

65. The derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'\left(\sqrt{2}\right) = 4$, is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$

(c) 1 (d) None of these

66. The derivative of e^{x^3} with respect to $\log x$ is

(a) e^{x^3} (b) $3x^2\, 2e^{x^3}$

(c) $3x^3 e^{x^3}$ (d) $3x^2 e^{x^3} + 3x^2$

67. The derivative of $\log_{10} x$ with respect to x^2 is

(a) $\frac{1}{2x^2}\log_e 10$ (b) $\frac{2}{x^2}\log_{10} e$

(c) $\frac{1}{2x^2}\log_{10} e$ (d) None of these

68. If $y = e^{\sin^{-1} x}$ and $u = \log x$, then $\frac{dy}{du}$ is

(a) $\frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}}$ (b) $x e^{\sin^{-1} x}$

(c) $\frac{x e^{\sin^{-1} x}}{\sqrt{1-x^2}}$ (d) $\frac{e^{\sin^{-1} x}}{x}$

69. The derivative of $\sin^2 x$ with respect to $\cos^2 x$ is

(a) $\tan^2 x$ (b) $\tan x$

(c) $-\tan x$ (d) None of these

70. Let $f(x) = \frac{(x+1)^2(x-1)}{(x-2)^3}$, then $f'(0)$ is

(a) $-\frac{9}{8}$ (b) $-\frac{11}{8}$

(c) $-\frac{13}{8}$ (d) None of these

Logarithmic Differentiation

71. $\frac{d}{dx}\left[\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right]$ is equal to

(a) 1 (b) $\frac{x^2+1}{x^2-4}$

(c) $\frac{x^2-1}{x^2-4}$ (d) $e^x \cdot \frac{x^2-1}{x^2-4}$

72. The derivative of $y = x^{\ln x}$ is

(a) $x^{\ln x} \ln x$ (b) $x^{\ln x - 1} \ln x$

(c) $2x^{\ln x - 1} \ln x$ (d) $x^{\ln x - 2}$

73. If $y = \{f(x)\}^{\phi(x)}$, then $\frac{dy}{dx}$ is

(a) $e^{\phi(x)\log f(x)}\left\{\frac{\phi(x)\, d f(x)}{f(x)\, dx} + \log f(x) . \frac{d\phi(x)}{dx}\right\}$

(b) $\frac{\phi(x)}{f(x)}\left(\frac{df(x)}{dx}\right) + \frac{d\phi(x)}{dx}\log f(x)$

(c) $e^{\phi(x)\log f(x)}\left\{\phi(x)\frac{f'(x)}{f(x)} + \phi'(x)\log f'(x)\right\}$

(d) None of these

74. If $y = \log_{\cos x} \sin x$, then $\frac{dy}{dx}$ is equal to

(a) $(\cot x \log \cos x + \tan x \log \sin x)/ (\log \cos x)^2$

(b) $(\tan x \log \cos x + \cot x \log \sin x)/ (\log \cos x)^2$

(c) $(\cot x \log \cos x + \tan x \log \sin x)/ (\log \sin x)^2$

(d) None of these

75. Let $f(x) = \left(x^x\right)^x$ and $g(x) = x^{\left(x^x\right)}$ then

(a) $f'(1) = 1$ and $g'(1) = 2$

(b) $g'(1) = 2$ and $f'(1) = 2$

(c) $f'(1) = 1$ and $g'(1) = 0$

(d) $f'(1) = 1$ and $g'(1) = 1$

Differentiation of Infinite Sereis

76. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+...\infty}}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{\cos x}{2y-1}$ (b) $\frac{-\cos x}{2y-1}$

(c) $\frac{\sin x}{1-2y}$ (d) $\frac{-\sin x}{1-2y}$

77. If $y=\sqrt{x+\sqrt{y+\sqrt{x+\sqrt{y+...\infty}}}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y+x}{y^2-2x}$ (b) $\frac{y^3-x}{2y^2-2xy-1}$

(c) $\frac{y^3+x}{2y^2-x}$ (d) None of these

78. If $y=x^{x^{x^{x^{...\infty}}}}$, then $x(1-y\log x)\frac{dy}{dx}$

(a) x^2 (b) y^2

(c) xy^2 (d) xy

79. If $y=x+\cfrac{1}{x+\cfrac{1}{x+\cfrac{1}{x+.....\infty}}}$, then find $\frac{dy}{dx}$

(a) $\frac{y}{2y-x}$ (b) $\frac{x}{2x-y}$

(c) $\frac{y}{y-x}$ (d) $\frac{x}{x-y}$

80. If $y=\cfrac{\sin x}{1+\cfrac{\cos x}{1+\cfrac{\sin x}{1+\cfrac{\cos x}{1+\sin x}.....\infty}}}$ then $y'(0)$ is

(a) 1 (b) 0

(c) $\frac{1}{2}$ (d) 2

81. For $|x|<1$, let $y=1+x+x^2+...$ to ∞, then $\frac{dy}{dx}$ equal to

(a) $\frac{x}{y}$ (b) $\frac{x^2}{y^2}$

(c) $\frac{x}{y^2}$ (d) xy^2+y

Differentiation Based on Trigonometric Substitution

82. If $x \in (0, 1)$The derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is

(a) –1 (b) 1

(c) 2 (d) 4

83. Let $f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, find $f'(1/2)$

(a) $\frac{5}{8}$ (b) $\frac{6}{7}$

(c) $\frac{8}{5}$ (d) $\frac{7}{6}$

84. Find the derivative of $y=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ at $x=0$

(a) 0 (b) $\frac{1}{4}$

(b) $\frac{1}{2}$ (d) None

85. Let $f(x)=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$, then $f'(0)$ is

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) 2

86. Let $f(x) = 2\tan^{-1}x + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$. Then.

(a) $f'(2) = f'(3)$ (b) $f'(2) = 0$

(c) $f'\left(\dfrac{1}{2}\right) = \dfrac{16}{5}$ (d) All the above

87. If $f(x) = 2\tan^{-1}x + \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, then

(a) $f'(-2) = \dfrac{4}{5}$

(b) $f'(-1) = -1$

(c) $f'(x) = 0$ for all $x < 0$

(d) None of these

88. If $y = \cos^{-1}\left(\dfrac{2\cos x - 3\sin x}{\sqrt{13}}\right)$; $0 < x < \dfrac{\pi}{2}$, then $\dfrac{dy}{dx}$ is

(a) zero (b) constant = 1

(c) constant ≠ 1 (d) none of these

Problems Based on Higher Order Derivatives

89. If $y^2 = ax^2 + bx + c$ where a, b, c are constants, then $y^3 \dfrac{d^2y}{dx^2}$, is equal to

(a) a constant

(b) a function of x

(c) a function of y

(d) a function of x and y both

90. If $y = x + e^x$, then $\dfrac{d^2x}{dy^2}$ is

(a) e^x (b) $-\dfrac{e^x}{(1+e^x)^3}$

(c) $-\dfrac{e^x}{(1+e^x)^2}$ (d) $\dfrac{1}{(1+e^x)^2}$

91. If $x^2 + y^2 = 1$, then

(a) $yy'' - 2(y')^2 + 1 = 0$ (b) $yy'' + (y')^2 + 1 = 0$

(c) $yy'' + (y')^2 - 1 = 0$ (d) $yy'' + 2(y')^2 + 1 = 0$

92. If $\sqrt{x+y} + \sqrt{y-x} = c$ then $\dfrac{d^2y}{dx^2}$ is

(a) $\dfrac{2x}{c^2}$ (b) $\dfrac{2}{c^3}$

(c) $-\dfrac{2}{c^2}$ (d) $\dfrac{2}{c^2}$

93. Let x = sin ($l$nt) and y = cos ($l$nt) then $\dfrac{d^2y}{dx^2}$ is

(a) $-\dfrac{1}{y^2}$ (b) $-\dfrac{1}{y^3}$

(c) $\dfrac{1}{y^2}$ (d) $\dfrac{1}{y^3}$

94. If $y = a\cos(\log x) + b\sin(\log x)$ where a, b are parameters, the $x^2y'' + xy'$ is equal to

(a) y (b) –y

(c) 2y (d) –2y

Problems Based on Existence of Differentiation

95. If $f(x) = \sqrt{x^2 - 10x + 25}$, then the derivative of $f(x)$ on the interval [0, 7] is

(a) 1 (b) –1

(c) 0 (d) Does not exist

96. If $f(x) = \sqrt{x^2 + 6x + 9}$, then $f'(x)$ is equal to

(a) 1 for $x < -3$ (b) –1 for $x < -3$

(c) 1 for all $x \in R$ (d) None of these

97. If $f(x) = |(x-4)(x-5)|$, then $f'(x)$ is equal to

(a) $-2x + 9$, for all $x \in R$ (b) $2x - 9$ if $4 < x < 5$

(c) $-2x + 9$ if $4 < x < 5$ (d) None of these

98. If $y = |\cos x| + |\sin x|$ then $\dfrac{dy}{dx}$ at $x = \dfrac{2\pi}{3}$ is

(a) $\dfrac{1-\sqrt{3}}{2}$ (b) 0

(c) $\dfrac{1}{2}(\sqrt{3}-1)$ (d) none of these

99. If $f(x) = \log|x|, x \neq 0$ then $f'(x)$ equals

(a) $\dfrac{1}{|x|}$ (b) $\dfrac{1}{x}$

(c) $-\dfrac{1}{x}$ (d) None of these

100. If $f(x) = \sin\left\{\frac{\pi}{3}[x] - x^2\right\}$ where [x] denotes the greatest integer less than or equal to x, then $f'\left(\sqrt{\pi/3}\right)$ is equal to

(a) $\sqrt{\pi/3}$ (b) $-\sqrt{\pi/3}$

(c) $-\sqrt{\pi}$ (d) None of these

Misc. Problems

101. If $f(x) = \left(\frac{x^a}{x^b}\right)^{a+b}\left(\frac{x^b}{x^c}\right)^{b+c}\left(\frac{x^c}{x^a}\right)^{c+a}$ then $f'(x)$ is equal to

(a) 1 (b) 0

(c) x^{a+b+c} (d) None of these

102. If $y = \frac{1}{1+x^{\beta-\alpha}+x^{\gamma-\alpha}} + \frac{1}{1+x^{\alpha-\beta}+x^{\gamma-\beta}} + \frac{1}{1+x^{\alpha-\gamma}+x^{\beta-\gamma}}$ then $\frac{dy}{dx}$.

(a) 0 (b) 1

(c) $(\alpha+\beta+\gamma)\, x^{\alpha+\beta+\gamma-1}$ (d) None of these

103. If $f(x) = \left(\frac{\sin^m x}{\sin^n x}\right)^{m+n}\left(\frac{\sin^n x}{\sin^p x}\right)^{n+p}\left(\frac{\sin^p x}{\sin^m x}\right)^{p+m}$, then $f'(x)$ is equal to

(a) 0 (b) 1

(c) $\cos^{m+n+p} x$ (d) None of these

104. If $y = (1+x)(1+x^2)(1+x^4)\ldots\left(1+x^{2^n}\right)$, then $\frac{dy}{dx}$ at $x = 0$ is

(a) –1 (b) 1

(c) 0 (d) None of these

105. If $y = (1+x^{1/4})(1+x^{1/2})(1-x^{1/4})$, then $\frac{dy}{dx} =$

(a) 1 (b) –1

(c) x (d) $\sqrt{x}$

106. If $f'(x) = \sqrt{2x^2-1}$ and $y = f(x^2)$ then $\frac{dy}{dx}$ at $x = 1$ is

(a) 2 (b) 1

(c) –2 (d) none of these

107. A triangle has two of its vertices at $P(a, 0)$, $Q(0, b)$ and the third vertex $R(x, y)$ is moving along the straight line $y = x$. If $ab < (a+b)x$ and A be the area of the triangle, then $\frac{dA}{dx} =$

(a) $\frac{a-b}{2}$ (b) $\frac{a-b}{4}$

(c) $\frac{a+b}{2}$ (d) $\frac{a+b}{4}$

108. If $f(x) = \cos x \,.\, \cos 2x \,.\, \cos 4x \,.\, \cos 8x \,.\, \cos 16x$, then the value of $f'\left(\frac{\pi}{4}\right)$ is

(a) 1 (b) $\sqrt{2}$

(c) $\frac{1}{\sqrt{2}}$ (d) 0

109. If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x) = \sin x^2$, then $\frac{dy}{dx}$ is equal to

(a) $\sin\left(\frac{2x-1}{x^2+1}\right)^2\left\{\frac{x^2+2x+2}{(x^2+1)^2}\right\}$

(b) $\sin\left(\frac{2x-1}{x^2+1}\right)^2\left\{\frac{2+2x-2x^2}{(x^2+1)^2}\right\}$

(c) $\sin\left(\frac{2x-1}{x^2+1}\right)^2\left\{\frac{2+2x-x^2}{(x^2+1)}\right\}$

(d) None of these

110. If $y = \left[\tan^{-1}\frac{1}{1+x+x^2} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \ldots \text{upto n terms}\right]$ then $y'(0)$ equals

(a) $\frac{-1}{n^2+1}$ (b) $\frac{-n^2}{n^2+1}$

(c) $\frac{n^2}{n^2+1}$ (d) None of these

EXERCISE - 2 : PREVIOUS YEAR JEE MAINS QUESTIONS

1. If $f(1)=1, f'(1)=2$, then $\lim_{x\to 1}\frac{\sqrt{f(x)}-1}{\sqrt{x}-1}$ is **(2002)**

(a) 2 (b) 4

(c) 1 (d) 1/2

2. If $y=(x+\sqrt{1+x^2})^n$, then $(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}$ is **(2002)**

(a) n^2y (b) $-n^2y$

(c) $-y$ (d) $2x^2y$

3. If sin y = x sin (a + y), then $\frac{dy}{dx}$ is **(2002)**

(a) $\frac{\sin a}{\sin^2(a+y)}$ (b) $\frac{\sin^2(a+y)}{\sin a}$

(c) $\sin a \sin^2(a+y)$ (d) $\frac{\sin^2(a-y)}{\sin a}$

4. If $x^y = e^{x-y}$, then $\frac{dy}{dx}$ is **(2002)**

(a) $\frac{1+x}{1+\log x}$ (b) $\frac{1-\log x}{1+\log x}$

(c) not defined (d) $\frac{\log x}{(1+\log x)^2}$

5. If $f(x+y)=f(x).f(y)\ \forall$ x, y and $f(5)=2$, $f'(0)=3$, then $f'(5)$ is **(2002)**

(a) 0 (b) 1

(c) 6 (d) 2

6. If $f(x)=x^n$, then the value of

$f(1)-\frac{f'(1)}{1!}+\frac{f''(1)}{2!}-\frac{f'''(1)}{3!}+...+\frac{(-1)^n f^n(1)}{n!}$ is **(2003)**

(a) 2^{n-1} (b) 0

(c) 1 (d) 2^n

7. Let $f(x)$ be a polynomial function of second degree. If $f(1)=f(-1)$ and a, b, c are in AP, then $f'(a), f'(b)$ and $f'(c)$ are in **(2003)**

(a) AP

(b) GP

(c) HP

(d) Arithmetico-Geometric Progression

8. Let $f(a)=g(a)=k$ and their nth derivatives $f^n(a), g^n(a)$ exist and are not equal for some n. Further if

$$\lim_{x\to a}\frac{f(a)g(x)-f(a)-g(a)f(x)+g(a)}{g(x)-f(x)}=4,$$

then the value of k is equal to **(2003)**

(a) 4 (b) 2

(c) 1 (d) 0

9. If $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x\neq 0 \\ 0, & x=0\end{cases}$, then f(x) is **(2003)**

(a) continuous as well as differentiable for all x

(b) continuous for all x but not differentiable at x = 0

(c) neither differentiable nor continuous at x = 0

(d) discontinuous everywhere

10. If $x=e^{y+e^{y+...\infty}}$, x > 0, then $\frac{dy}{dx}$ is **(2004)**

(a) $\frac{x}{1+x}$ (b) $\frac{1}{x}$

(c) $\frac{1-x}{x}$ (d) $\frac{1+x}{x}$

11. Let $f(x)=\frac{1-\tan x}{4x-\pi}, x\neq\frac{\pi}{4}, x\in\left[0,\frac{\pi}{2}\right]$. If f (x) is continuous in $\left[0,\frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is **(2004)**

(a) 1 (b) 1/2

(c) –1/2 (d) –1

12. Suppose $f(x)$ is differentiable at $x = 1$ and $\lim_{h\to 0}\frac{1}{h}f(1+h)=5$, then $f'(1)$ equals **(2005)**

(a) 6 (b) 5

(c) 4 (d) 3

13. If f is a real-valued differentiable function satisfying $|f(x)-f(y)| \le (x-y)^2$, $x, y \in R$ and $f(0)=0$, then $f(1)$ equals **(2005)**

(a) 1 (b) 2

(c) 0 (d) –1

14. If $x^m y^n = (x+y)^{m+n}$, then $\frac{dy}{dx}$ is **(2006)**

(a) $\frac{x+y}{xy}$ (b) xy

(c) $\frac{x}{y}$ (d) $\frac{y}{x}$

15. The set of points, where $f(x) = \frac{x}{1+|x|}$ is differentiable, is **(2006)**

(a) $(-\infty,-1)\cup(-1,\infty)$ (b) $(-\infty,\infty)$

(c) $(0,\infty)$ (d) $(-\infty,0)\cup(0,\infty)$

16. Let $f : R \to R$ be a function defined by $f(x)=\min\{x+1, |x|+1\}$. Then, which of the following is true ? **(2007)**

(a) $f(x) \ge 1$ for all $x \in R$

(b) $f(x)$ is not differentiable at $x = 1$

(c) $f(x)$ is differentiable everywhere

(d) $f(x)$ is not differentiable at $x = 0$

17. The function $f : R/\{0\} \to R$ given by

$$f(x) = \frac{1}{x} - \frac{2}{e^{2x}-1}$$

can be made continuous at $x = 0$ by defining $f(0)$ as **(2007)**

(a) 2 (b) –1

(c) 0 (d) 1

18. Let $f(x) = \begin{cases} (x-1)\sin\frac{1}{x-1}, & \text{if } x \ne 1 \\ 0, & \text{if } x = 1 \end{cases}$

Then which one of the following is true ? **(2008)**

(a) f is differentiable at $x = 1$ but not at $x = 0$

(b) f is neither differentiable at $x = 0$ nor at $x = 1$

(c) f is differentiable at $x = 0$ and at $x = 1$

(d) f is differentiable at $x = 0$ but not at $x = 1$

19. Let $f(x) = x|x|$ and $g(x) = \sin x$

Statement I gof is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement II gof is twice differentiable at $x = 0$. **(2009)**

(a) Statement I is false, Statement II is true.

(b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.

(c) Statement I is true, Statement II is true, Statement II is not a correct explanation for Statement I.

(d) Statement I is true, Statement II is false

20. Let y be an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$. Then, $y'(1)$ equals **(2009)**

(a) –1 (b) 1

(c) log 2 (d) –log 2

21. If $f : (-1, 1) \to R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x)+2)]^2$. Then $g'(0)$ is equal to **(2010)**

(a) 4 (b) –4

(c) 0 (d) –2

22. $\frac{d^2x}{dy^2}$ equals **(2011)**

(a) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (b) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$

(c) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (d) $\left(\frac{d^2y}{dx^2}\right)^{-1}$

23. The values of p and q for which the function

$$f(x)=\begin{cases}\dfrac{\sin (p+1)\,x+\sin x}{x}, & x<0\\ q\ , & x=0\\ \dfrac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}}, & x>0\end{cases}$$

is continuous for all x in R, are **(2011)**

(a) $p=\frac{5}{2}, q=\frac{1}{2}$ (b) $p=-\frac{3}{2}, q=\frac{1}{2}$

(c) $p=\frac{1}{2}, q=\frac{3}{2}$ (d) $p=\frac{1}{2}, q=-\frac{3}{2}$

24. Define F (x) as the product of two real functions $f_1(x)=x, x\in IR$,

and $f_2(x)=\begin{cases}\sin\frac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ as follows

$$F(x)=\begin{cases}f_1(x).f_2(x), & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$$

Statement I F (x) is continuous on IR.

Statement II $f_1(x)$ and $f_2(x)$ are continuous on IR. **(2011)**

(a) Statement I is false, Statement II is true.

(b) Statement I is true, Statement II is true;

Statement II is a correct explanation for Statement I.

(c) Statement I is true, Statement II is true,

Statement II is not a correct explanation for Statement I.

(d) Statement I is true, Statement II is false

25. If $f : R\to R$ is a function defined by $f(x)=[x]\cos\left(\frac{2x-1}{2}\right)\pi$, where [x] denotes the greatest integer function, then f is **(2012)**

(a) continuous for every real x

(b) discontinuous only at x = 0

(c) discontinuous only at non-zero integral values of x

(d) continuous only at x = 0

26. Consider the function, $f(x)=|x-2|+|x-5|, x\in R$.

Statement 1 $f'(4)=0$

Statement 2 f is continuous in [2, 5], differentiable in (2, 5) and $f(2)=f(5)$. **(2012)**

(a) Statement I is false, Statement II is true.

(b) Statement I is true, Statement II is true;

Statement II is a correct explanation for Statement I.

(c) Statement I is true, Statement II is true,

Statement II is not a correct explanation for Statement I.

(d) Statement I is true, Statement II is false

27. If $y=e^{nx}$ then $\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2x}{dy^2}\right)$ is equal to:

(2014/Online Set–1)

(a) ne^{nx} (b) ne^{-nx}

(c) 1 (d) $-ne^{-nx}$

28. Let $f(x)=x|x|, g(x)=\sin x$ and $h(x)=(gof)(x)$. Then

(2014/Online Set–2)

(a) h(x) is not differentiable at x = 0.

(b) h(x) is differentiate at x = 0, but $h'(x)$ is not continuous at x = 0.

(c) $h'(x)$ is continuous at x = 0 but it is not differentiable at x = 0.

(d) $h'(x)$ is differentiable at x = 0

29. Let $f, g : R\to R$ be two functions defined by

$f(x)=\begin{cases}x\sin\left(\frac{1}{x}\right), & x\neq 0\\ 0, & x-0\end{cases}$ and $g(x)=xf(x)$

Statement I : f is a continuous function at x = 0.

Statement II : g is a differentiable function at x = 0.

(2014/Online Set–3)

(a) Both statements I and II are false.

(b) Both statements I and Ii are true.

(c) Statement I is true, statement II is false.

(d) Statement I is false, statement II is true.

30. Let f and g be two differentiable functions on R such that $f'(x)>0$ and $f'(x)<0$, for all $x\in R$. Then for all x:

(2014/Online Set–3)

(a) $f(g(x))>f(g(x-1))$ (b) $f(g(x))>f(g(x+1))$

(c) $g(f(x))>g(f(x-1))$ (d) $g(f(x))>g(f(x+1))$

31. If the function $f(x)=\begin{cases}\dfrac{\sqrt{2+\cos x}-1}{(\pi-x)^2}, & x\neq\pi\\ k, & x=\pi\end{cases}$ is continuous at $x=\pi$, then k equals: **(2014/Online Set–4)**

(a) 0 (b) $\frac{1}{2}$

(c) 2 (d) $\frac{1}{4}$

32. Let $f: R \to R$ be a function such tht $|f(x)| \le x^2$, for all $x \in R$. Then at $x = 0$ is: **(2014/Online Set–4)**

(a) continuous but not differentiate

(b) continuous as well as differentiate

(c) neigher continuous not differentiate

(d) differentiable but not continuous.

33. If the function.

$g(x)=\begin{cases}k\sqrt{x+1}, & 0\le x\le 3\\ mx+2, & 3<x\le 5\end{cases}$ is differentiable, then the value of $k + m$ is: **(2015)**

(a) $\frac{10}{3}$ (b) 4

(c) 2 (d) $\frac{16}{5}$

34. The distance, from the origin, of the normal to the curve,

$x = 2\cos t + 2t\sin t$, $y = 2\sin t - 2t\cos t$ at $t = \frac{\pi}{4}$, is :

(2015/Online Set–1)

(a) 4 (b) 3

(c) 2 (d) $2\sqrt{2}$

35. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then : **(2016)**

(a) $g'(0) = \cos(\log 2)$

(b) $g'(0) = -\cos(\log 2)$

(c) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

(d) g is not differentiable at $x = 0$

36. If $y=\left[x+\sqrt{x^2-1}\right]^{15}+\left[x-\sqrt{x^2-1}\right]^{15}$, then

$(x^2-1)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}$ is equal to : **(2017/Online Set–1)**

(a) 125 y (b) $224\ y^2$

(c) $225y^2$ (d) 225 y

37. If $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}}$ and

$(x^2-1)\dfrac{d^2y}{dx^2}+\lambda x\dfrac{dy}{dx}+ky=0$, then $\lambda + k$ is equal to :

(2017/Online Set–2)

(a) –23 (b) –24

(c) 26 (d) –26

38. Let f be a polynomial function such that

$f(3x) = f'(x)\cdot f''(x)$, for all $x \in R$. Then :

(2017/Online Set–2)

(a) $f(2)+f'(2)=28$ (b) $f''(2)-f'(2)=0$

(c) $f''(2)-f(2)=4$ (d) $f(2)-f'(2)+f''(2)=10$

39. The value of k for which the function

$f(x)=\begin{cases}\left(\dfrac{4}{5}\right)^{\frac{\tan 4x}{\tan 5x}}, & 0<x<\dfrac{\pi}{2}\\ k+\dfrac{2}{5}, & x=\dfrac{\pi}{2}\end{cases}$

is continuous at $x = \frac{\pi}{2}$, is : **(2017/Online Set–2)**

(a) $\frac{17}{20}$ (b) $\frac{2}{5}$

(c) $\frac{3}{5}$ (d) $-\frac{2}{5}$

40. Let $S = \{t \in R : f(x) = |x-\pi|.(e^{|x|}-1)\sin|x|$ is not differentiable at t}. Then the set S is equal to: **(2018)**

(a) $\{0, \pi\}$ (b) ϕ (an empty set)

(c) $\{0\}$ (d) $\{\pi\}$

41. Let $f(x)=\begin{cases}(x-1)^{\frac{1}{2-x}}, & x>1, x\neq 2\\ k, & x=2\end{cases}$

The value of k for which f is continuous at $x = 2$ is:

(2018/Online Set–2)

(a) 1 (b) e

(c) e^{-1} (d) e^{-2}

EXERCISE - 3 : ADVANCED OBJECTIVE QUESTIONS

Single Type Questions

1. The function $f(x) = x - |x - x^2|, -1 \le x \le 1$ is continuous on the interval

(a) $[-1, 1]$ (b) $(-1, 1)$

(c) $[-1, 1] - \{0\}$ (d) $(-1, 1) - \{0\}$

2. If $f(x) = \begin{cases} 4, -3 < x < -1 \\ 5 + x, -1 \le x < 0 \\ 5 - x, 0 \le x < 2 \\ x^2 + x - 3, 2 \le x < 3 \end{cases}$, then $f\,|x|$ is

(a) differentiable but not continuous in $(-3, 3)$

(b) continuous but not differentiable in $(-3, 3)$

(c) continuous as well as differentiable in $(-3, 3)$

(d) neither continuous nor differentiable in $(-3, 3)$

3. Let $f(x) = a[x] + b\,e^{|x|} + c\,|x|^2$, where a, b and c are real constants. where [x] denotes greatest integer $\le$ x. If $f(x)$ is differentiable at x = 0, then

(a) $b = 0, c = 0, a \in R$ (b) $a = 0, c = 0, b \in R$

(c) $a = 0, b = 0, c \in R$ (d) None of these

4. If $f(x) = \lim\limits_{n\to\infty} \dfrac{(1+\cos\pi x)^n + 1}{(1+\cos\pi x)^n - 1}$, then

(a) $f(1 + 0) = 1$

(b) $f(1 - 0) = 2$

(c) $f(x)$ is continuous at x = 1

(d) $f(x)$ is not continuous at x = 1

5. If $f(x) = \begin{cases} \dfrac{[x]-1}{x-1}, & x \ne 1 \\ 0, & x = 1 \end{cases}$, where [x] denotes greatest integer $\le$ x. then $f(x)$ is

(a) continuous as well as differentiable at x = 1

(b) differentiable but not continuous at x = 1

(c) continuous but not differentiable at x = 1

(d) neither continuous nor differentiable at x = 1

6. Let f be a function defined and continuous on [2, 5]. If $f(x)$ takes rational values for all x and $f(4) = 8$ then the value of $f(3.7)$ is

(a) 0 (b) 8

(c) –1 (d) None of these

7. If $f(x) = |3 - x| + (3 + x)$ where (x) denotes the least integer greater than or equal to x, then

(a) $f(x)$ is continuous as well as differentiable at x = 3

(b) $f(x)$ is continuous but not differentiable at x = 3

(c) $f(x)$ is differentiable but not continuous at x = 3

(d) $f(x)$ is neither differentiable nor continuous at x = 3

8. If $f(x) = \begin{cases} x, \text{ when } x \text{ is rational} \\ 1 - x, \text{ when } x \text{ is irrational} \end{cases}$, then

(a) $f(x)$ is continuous for all real x

(b) $f(x)$ is discontinuous for all real x

(c) $f(x)$ is continuous only at x = 1/2

(d) $f(x)$ is discontinuous only at x = 1/2

9. If $f(x) = \begin{cases} \dfrac{x\left(3e^{1/x}+4\right)}{2-e^{1/x}}, & x \ne 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is

(a) continuous as well as differentiable at x = 0

(b) continuous but not differentiable at x = 0

(c) differentiable but not cotinuous at x = 0

(d) None of these

10. The function $f(x) = (\sin 2x)^{\tan^2 2x}$ is not defined at $x = \pi/4$. The value of $f(\pi/4)$ so that f is continuous at $x = \pi/4$ is

(a) $\sqrt{e}$ (b) $1/\sqrt{e}$

(c) 2 (d) None of these

11. If f is a periodic function, then

(a) f' and f'' are also periodic

(b) f' is periodic but f'' is not periodic

(c) f'' is periodic but f' is not periodic

(d) None of these

12. Let $f(x) = [n + p \sin x]$, $x \in (0, \pi)$, $n \in I$, p is a prime number and [x] denotes the greatest integer less than or equal to x. The number of points at which $f(x)$ is not differentiable is

(a) p – 1 (b) p
(c) 2 p + 1 (d) 2 p – 1

13. If $f(x) = (-1)^{[x^3]}$, where [.] denotes the greatest integer function, then

(a) $f(x)$ is discontinuous for $x = n^{1/3}$, where $n \in I$
(b) $f(3/2) = 1$
(c) $f'(x) = 0$ for $-1 < x < 1$
(d) None of these

14. Let $f : R \to R$ be a function such that

$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$, $f(0) = 0$ and $f'(0) = 3$. Then

(a) $f(x)$ is a quadratic function
(b) $f(x)$ is continuous but not differentiable
(c) $f(x)$ is differentiable in R
(d) $f(x)$ is bounded in R

15. If f is an even function such that $\lim_{h \to 0^+} \frac{f(h) - f(0)}{h}$ has some finite non-zero value, then

(a) f is continuous and derivable at $x = 0$
(b) f is continuous but not derivable at $x = 0$
(c) f may be discontinuous at $x = 0$
(d) None of these

16. If a function $f : R \to R$ be such that $f(x + y) = f(x) . f(y)$ for all $x, y \in R$ where $f(x) = 1 + x\,\phi(x)$ and $\lim_{x \to 0} \phi(x) = 1$, then

(a) $f'(x)$ does not exist (b) $f'(x) = 2f(x)$ for all x
(c) $f'(x) = f(x)$ for all x (d) None of these

17. Let $f(x) = a + b|x| + c|x|^4$, where a, b and c are real constants. Then $f(x)$ is differentiable at $x = 0$ if

(a) a = 0 (b) b = 0
(c) c = 0 (d) None of these

18. Let $f(x + y) = f(x). f(y)$ and $f(x) = 1 + x\, g(x)\, G(x)$ where $\lim_{x \to 0} g(x) = a$ and $\lim_{x \to 0} G(x) = b$. Then $f'(x) = k f(x)$, where k is equal to

(a) a/b (b) 1 + ab
(c) ab (d) None of these

19. Let $f(x) = Sgn(x)$ and $g(x) = x(x^2 - 5x + 6)$. The function $f(g(x))$ is discontinuous at

(a) infinitely many points (b) exactly one point
(c) exactly three points (d) no point

20. The function $f(x) = \left[x^2\left[\frac{1}{x^2}\right]\right]$, $x \neq 0$, is ([x] represents the greatest integer $\leq$ x)

(a) continuous at x = 1
(b) continuous at x = –1
(c) discontinuous at infinitely many points
(d) continuous everywhere

21. The function $f(x)$ = maximum $\left\{\sqrt{x(2-x)}, 2-x\right\}$ is non-differentiable at x equal to :

(a) 1 (b) 0, 2
(c) 0,1 (d) 1, 2

22. Let $f(x) = [n + p \sin x]$, $x \in (0, \pi)$, $n \in Z$, p is a prime number and [x] is greatest integer less than or equal to x. The number of points at which $f(x)$ is not differentiable is

(a) p (b) p –1
(c) 2p + 1 (d) 2p – 1

23. The derivative of $f(\tan x)$ w.r.t. $g(\sec x)$ at $x = \frac{\pi}{4}$, where $f'(1) = 2$ and $g'\left(\sqrt{2}\right) = 4$, is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$
(c) 1 (d) None of these

24. If $y = \tan^{-1}\sqrt{\left(\frac{1+\sin x}{1-\sin x}\right)}$, $\frac{\pi}{2} < x < \pi$, then $\frac{dy}{dx}$ equals

(a) –1/2 (b) – 1
(c) 1/2 (d) 1

25. The differential coefficient of $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ w.r.t. $\sec^{-1}\frac{1}{2x^2-1}$ at $x=\frac{1}{2}$ is equal to

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$

(c) -1 (d) None of these

26. If $y=e^{\tan x}$, then $\cos^2 x\frac{d^2y}{dx^2}=$

(a) $(1-\sin 2x)\frac{dy}{dx}$ (b) $-(1+\sin 2x)\frac{dy}{dx}$

(c) $(1+\sin 2x)\frac{dy}{dx}$ (d) None of these

27. Let $f(x)$ be a polynomial function of second degree. If $f(1)=f(-1)$ and a, b, c are in AP, then $f'(a), f'(b)$ and $f'(c)$ are in

(a) AP

(b) GP

(c) HP

(d) Arithmetico-Geometric progression

28. If $f(x)=\frac{x^2-x}{x^2+2x}$ with codomain $=R-\{1\}$, then $\frac{df^{-1}(x)}{dx}$ is equal to

(a) $-\frac{3}{(1-x)^2}$ (b) $\frac{3}{(1-x)^2}$

(c) $\frac{1}{(1-x)^2}$ (d) None of these

29. If $y=f(x)$ is an odd differentiable function defined on $(-\infty,\infty)$ such that $f'(3)=-2$, then $f'(-3)$ equals

(a) 4 (b) 2

(c) –2 (d) 0

30. If $f(x)=\log|2x|, x\neq 0$, then $f'(x)$ is equal to

(a) $\frac{1}{x}$ (b) $-\frac{1}{x}$

(c) $\frac{1}{|x|}$ (d) None of these

31. Let $y=x^3-8x+7$ and $x=f(t)$. If $\frac{dy}{dt}=2$ and $x=3$ at $t=0$, then $\frac{dx}{dt}$ at $t=0$ is given by

(a) 1 (b) $\frac{19}{2}$

(c) $\frac{2}{19}$ (d) None of these

32. If $f(x)=|x-3|$ and $\phi(x)=(fof)(x)$, then for $x>10$, $\phi'(x)$ is equal to

(a) 1 (b) 0

(c) –1 (d) None of these

33. Let $f(x)=\sin x$, $g(x)=2x$ and $h(x)=\cos x$.

If $\phi(x)=[go(fh)](x)$, then $\phi''\left(\frac{\pi}{4}\right)$ is equal to

(a) 4 (b) 0

(c) –4 (d) None of these

34. If $f(x)=\sin\left(\frac{\pi}{2}[x]-x^5\right), 1<x<2$ and $[x]$ denotes the greatest integer less than or equal to x, then $f'\left(\sqrt[5]{\frac{\pi}{2}}\right)$ is equal to

(a) $5\left(\frac{\pi}{2}\right)^{4/5}$ (b) $-5\left(\frac{\pi}{2}\right)^{4/5}$

(c) 0 (d) None of these

35. If $f(x)=|x-1|$ and $g(x)=f[f\{f(x)\}]$, then for $x>2$, $g'(x)$ is equal to

(a) -1 if $2<x<3$ (b) 1 if $2\leq x<3$

(c) 1 for all $x>2$ (d) None of these

36. Let $f(x)$ be a polynomial of degree 3 such that $f(3)=1$, $f'(3)=-1$, $f''(3)=0$ and $f'''(3)=12$. Then the value of $f'(1)$ is

(a) 12 (b) 23

(c) –13 (d) None of these

37. Let f & g be differentiable functions satisfying $g'(a) = 2, g(a) = b$ & $fog = I$ (Identity function). Then $f'(b)$ is equal to

(a) 2 (b) $\frac{2}{3}$

(c) $\frac{1}{2}$ (d) None

38. Let $f(x) = \alpha(x)\,\beta(x)\,\gamma(x)$ for all real x, where $\alpha(x), \beta(x)$ and $\gamma(x)$ are differentiable functions of x. If $f'(2) = 18 f(2)$, $\alpha'(2) = 3\alpha(2)$, $\beta'(2) = -4\beta(2)$ and $\gamma'(2) = k\gamma(2)$, then the value of k is

(a) 14 (b) 16

(c) 19 (d) None of these

39. If $y = \frac{ax+b}{x^2+c}$, where a, b, c are constants then $(2xy'+y)\,y'''$ is equal to

(a) $3(xy''+y')\,y''$ (b) $3(xy'+y'')\,y''$

(c) $3(xy''+y')\,y'$ (d) None of these

40. If $f(x) = \cos^{-1}\left[\frac{1-(\log x)^2}{1+(\log x)^2}\right]$ then $f'(e) =$

(a) does not exit (b) $\frac{2}{e}$

(c) $\frac{1}{e}$ (d) 1

41. If $y = \left(\frac{ax+b}{cx+d}\right)$, then $2\frac{dy}{dx}.\frac{d^3y}{dx^3}$ is equal to

(a) $\left(\frac{d^2y}{dx^2}\right)^2$ (b) $3\frac{d^2y}{dx^2}$

(c) $3\left(\frac{d^2y}{dx^2}\right)^2$ (d) $3\frac{d^2x}{dy^2}$

42. If $\sin y = x\sin(a+y)$, then $\frac{dy}{dx}$ is

(a) $\frac{\sin a}{\sin^2(a+y)}$ (b) $\frac{\sin^2(a+y)}{\sin a}$

(c) $\sin a \sin^2(a+y)$ (d) $\frac{\sin^2(a-y)}{\sin a}$

43. If $ax^2 + 2hxy + by^2 = 1$, then $\frac{d^2y}{dx^2}$ is equal to

(a) $\frac{ab-h^2}{(hx+by)^3}$ (b) $\frac{h^2-ab}{(hx+by)^3}$

(c) $\frac{h^2+ab}{(hx+by)^3}$ (d) None of these

44. If $x^2 + y^2 = t + \frac{1}{t}$ and $x^4 + y^4 = t^2 + \frac{1}{t^2}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y}{x}$ (b) $-\frac{y}{x}$

(c) $\frac{x}{y}$ (d) $-\frac{x}{y}$

45. If $f(x) = \sqrt{x^2 - 10x + 25}$, then the derivative of $f(x)$ on the interval [0, 7] is

(a) 1 (b) –1

(c) 0 (d) none of these

46. If $y^2 = P(x)$, a polynomial of degree $n \geq 3$, then $2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$

(a) $-P(x).P'''(x)$ (b) $P(x).P'''(x)$

(c) $P(x).P''(x)$ (d) None of these

47. If $f_r(x), g_r(x), h_r(x), r = 1, 2, 3$ are polynomials in x such that $f_r(a) = g_r(a) = h_r(a), r = 1, 2, 3$ and

$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$, then $F'(a)$ is equal to

(a) a (b) –a

(c) 0 (d) None of these.

48. If y = ksinpx, then the value of the determinant $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$ is equal to

(a) 1 (b) 0

(c) –1 (d) None of these.

where y_n denotes nth derivative of y w.r.t. x.

49. If $f(x) = \begin{vmatrix} x^n & n! & 2 \\ \cos x & \cos\frac{n\pi}{2} & 4 \\ \sin x & \sin\frac{n\pi}{2} & 8 \end{vmatrix}$ then the value of $\frac{d^n}{dx^n}\left[f(x)\right]_{x=0}$ is

(a) 0 (b) 1

(c) –1 (d) None of these

50. If $y = \begin{vmatrix} \sin x & \cos x & \sin x \\ \cos x & -\sin x & \cos x \\ x & 1 & 1 \end{vmatrix}$, then $\frac{dy}{dx}$ is equal to

(a) 1 (b) –1

(c) 0 (d) None of these

51. If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are given, then

(a) $\Delta_1 = 3(\Delta_2)^2$ (b) $\frac{d}{dx}\Delta_1 = 3\Delta_2$

(c) $\frac{d}{dx}\Delta_1 = 3(\Delta_2)^2$ (d) $\Delta_1 = 3(\Delta_2)^{3/2}$

52. Let U(x) and V(x) are differentiable functions such that $\frac{U(x)}{V(x)} = 7$. If $\frac{U'(x)}{V'(x)} = p$ and $\left(\frac{V(x)}{U(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to

(a) 1 (b) 0

(c) 7 (d) –7

53. Suppose $f(x) = e^{ax} + e^{bx}$, where $a \neq b$, and that $f''(x) - 2f'(x) - 15f(x) = 0$ for all x. Then the product ab is

(a) 25 (b) –15

(c) 9 (d) –9

54. Let $f(x) = e^x - e^{-x} - 2\sin x - \frac{2}{3}x^3$, then the least value of n for which $\left|\frac{d^n}{dx^n}f(x)\right| x = 0$ is non-zero

(a) 4 (b) 5

(c) 7 (d) 3

55. Let $f(x) = x[x]$, $x \notin I$ where [.] denotes the greatest integer function, then $f'(x)$ is equal to

(a) 2x (b) [x]

(c) 2[x] (d) None of these

56. Let $f(x) = (2x-\pi)^3 + 2x - \cos x$. The value of $\left|\frac{d}{dx}f^{-1}(x)\right|_{x=\pi}$ is

(a) $3\pi^2 + 2$ (b) – 2

(c) $\frac{1}{3\pi^2+2}$ (d) $\frac{1}{3}$

57. Let $f(x) = x^n$, $n \in W$. The number of values of n for which $f'(p+q) = f'(p) + f'(q)$ is valid for all +ve p & q is

(a) 0 (b) 1

(c) 2 (d) None of these

58. $f(x)$, g(x), h(x) are functions having non-zero derivatives. The derivative of $f(x)$ w.r.t g(x) is α(x) and derivative of g(x) w.r.t h(x) is β(x). Then derivative of h(x) w.r.t $f(x)$ =

(a) α (x) . β (x) (b) $\frac{\alpha(x)}{\beta(x)}$

(c) $\frac{1}{\alpha(x)\beta(x)}$ (d) $\frac{\beta(x)}{\alpha(x)}$

59. Let $f(x) = e^x - e^{-x} - 2\sin x - \frac{2}{3}x^3$, then the least value of n for which $\left|\frac{d^n}{dx^n} f(x)\right|_{x=0}$ is non-zero is

(a) 4 (b) 5
(c) 7 (d) 3

60. Let f (x) = 2/(x+1) and g(x) = 3x. It is given that $(fog)(x_0) = (gof)(x_0)$. Then $(gof)'(x_0)$ equals

(a) -32 (b) $\frac{32}{3}$
(c) $\frac{-32}{9}$ (d) $\frac{-32}{3}$

61. If $(\sin y)^{\sin(\pi x/2)} + \frac{\sqrt{3}}{2}\operatorname{Sec}^{-1}(2x) + 2^x \tan(\log(x+2)) = 0$ then dy/dx at x = –1 is

(a) $\frac{3}{\sqrt{\pi^2 - 3}}$ (b) $\frac{1}{\pi\sqrt{\pi^2 - 3}}$
(c) $\frac{3}{\pi\sqrt{\pi^2 - 3}}$ (d) $\frac{3\pi}{\sqrt{\pi^2 - 3}}$

62. If $y = \frac{ax^2}{(x-a)(x-b)(x-c)} + \frac{bx}{(x-b)(x-c)} + \frac{c}{x-c} + 1$ then dy/dx is equal to

(a) $\frac{-y}{x}\left(\frac{a}{x-a} + \frac{b}{x-b} + \frac{c}{x-c}\right)$

(b) $\frac{1}{x}\left(\frac{a}{x-a} + \frac{b}{x-b} + \frac{c}{x-c}\right)$

(c) $\frac{y}{x}\left(\frac{a}{x-a} + \frac{b}{x-b} + \frac{c}{x-c}\right)$

(d) $-\frac{1}{x}\left(\frac{a}{x-a} + \frac{b}{x-b} + \frac{c}{x-c}\right)$

63. Function f : R → R satisfies the functional equation

$$f(x-y) = \frac{f(x)}{f(y)}$$

If $f'(0) = p$ and $f'(a) = q$, then $f'(-a)$ is

(a) $\frac{p^2}{q}$ (b) $\frac{q}{p}$
(c) $\frac{p}{q}$ (d) q

64. Let $f(x) = 2^{x(x-1)}$ for all $x \ge 1$. Then $(f^{-1})'(4)$ is $(1/k)\log_2 e$ where the value of k is

(a) 4 (b) 8
(c) 9 (d) 12

65. If 0 < x < 1, then

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots\infty =$$

(a) $\frac{1}{1-x}$ (b) $\frac{x}{1-x}$
(c) $\frac{x}{1+x}$ (d) $\frac{1-x}{1+x}$

66. If a function f (x) is continuous, f (1) > 0 and satisfies the relation f(x) < f(y) whenever x < y for all positive x and y, then for x ≥ 1, f (x) = 0 has

(a) exactly one root (b) exactly two roots
(c) more than two roots (d) no roots

67. Let g is the inverse function of f and $f'(x) = \frac{x^{10}}{(1+x^2)}$. If g(2) = a then g′ (2) is equal to

(a) $\frac{5}{2^{10}}$ (b) $\frac{1+a^2}{a^{10}}$
(c) $\frac{a^{10}}{1+a^2}$ (d) $\frac{1+a^{10}}{a^2}$

68. A non zero polynomial with real coefficients has the property that $f(x) = f'(x) \cdot f''(x)$. The leading coefficient of f(x) is

(a) $\frac{1}{6}$ (b) $\frac{1}{9}$

(c) $\frac{1}{12}$ (d) $\frac{1}{18}$

69. People living at Mars, instead of the usual definition of derivative D f(x), define a new kind of derivative, D*f(x) by the formula

$$D^*f(x) = \lim_{h\to 0}\frac{f^2(x+h) - f^2(x)}{h}$$ where $f^2(x)$ means $[f(x)]^2$.

If $f(x) = x\,ln x$ then

$D^*f(x)\big|_{x=e}$ has the value

(a) e (b) 2e

(c) 4e (d) 8 e

70. Suppose the function f(x) – f(2x) has the derivative 5 at x = 1 and derivative 7 at x = 2. The derivative of the function f(x) – f(4x) at x = 1, has the value equal to

(a) 19 (b) 9

(c) 17 (d) 14

71. If $y = \cos^{-1}\cos(|x| - f(x))$, where

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}, \text{ then } \left.\frac{dy}{dx}\right|_{x=\frac{5\pi}{4}} \text{ is}$$

(a) –1 (b) 1

(c) 0 (d) Indetermine

72. Let $f(x) = \lim_{h\to 0}\frac{(\sin(x+h))^{ln(x+h)} - (\sin x)^{ln x}}{h}$

then $f\left(\frac{\pi}{2}\right)$ is

(a) equal to 0 (b) equal to 1

(c) $ln\frac{\pi}{2}$ (d) non existent

Multiple Type Questions

73. If $f(x) = \sum_{k=0}^{n} a_k |x-1|^k$, where $a_i \in R$ then

(a) $f(x)$ is continuous at x = 1 for all $a_k \in R$

(b) $f(x)$ is differentiable at x = 1, if $a_1 = 0$

(c) $f(x)$ is differentiable at x = 1, if $a_{2k+1} = 0$

(d) $f(x)$ is continuous at x = 1, if & only if $a_{2k} = 0$

74. If $f(x) = \frac{1}{[\sin x]}$, where [.] denotes the greatest function, then

(a) Domain of $f(x)$ is $(2n\pi + \pi, 2n\pi + 2\pi) \cup \left\{2n\pi + \frac{\pi}{2}\right\}$

where $n \in I$

(b) $f(x)$ is continuous when $x \in (2n\pi + \pi, 2n\pi + 2\pi)$

(c) $f(x)$ is differentiable at $x = \pi/2$

(d) None of these

75. Let [x] denotes the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then $f(x)$ is

(a) continuous at x = 0 (b) continuous in (–1, 0)

(c) differentiable at x = 1 (d) differentiable in (–1, 1)

76. The function $f(x) = \max.\{(1-x), (1+x), 2\}$, $x \in (-\infty, \infty)$, is

(a) continuous at all points

(b) differentiable at all points

(c) differentiable at all points except at x = 1 and x = –1.

(d) continuous at all points except at x = 1 and x = –1, where it is discontinuous.

77. If $f(x) = \sqrt{|x-1|}$ and $g(x) = \sin x$, then

(a) $(fog)(x) = \sqrt{1-\sin x}$ for all x

(b) $(gof)(x) = \begin{cases} \sin\left(\sqrt{x-1}\right), & \text{if } x \ge 1 \\ \sin\left(\sqrt{1-x}\right), & \text{if } x < 1 \end{cases}$

(c) gof is differentiable at x = 1

(d) gof is not differentiable at x = 1

78. A function f (x) satisfies the relation

$f(x+y)=f(x)+f(y)+xy(x+y) \;\forall\; x, y \in R.$

If $f'(0)=-1$, then

(a) f (x) is a polynomial function

(b) f (x) is an exponential function

(c) f (x) is twice differentiable for all $x \in R$

(d) $f'(3)=8$

79. Let $f(x)=\frac{1}{[\sin x]}$, ([.] denotes the greatest integer function) then

(a) domain of $f(x)$ is $(2n\pi+\pi, 2n\pi+2\pi) \cup \{2n\pi+\pi/2\}$, where $n \in I$

(b) $f(x)$ is continuous, when $x \in (2n\pi+\pi, 2n\pi+2\pi)$, where $n \in I$

(c) $f(x)$ is differentiable at $x=\pi/2$

(d) none of the above

80. If $f(x)=\tan^{-1}\cot x$, then

(a) $f(x)$ is periodic with period π

(b) $f(x)$ is discontinuous at $x=\pi/2, 3\pi/2$

(c) $f(x)$ is not differentiable at $x=\pi, 99\pi, 100\pi$

(d) $f(x)=-1$, for $2n\pi \le x \le (2n+1)\pi$

81. If $y=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$, $a>b>0$, then

(a) $\frac{dy}{dx}=\frac{1}{a+b\cos x}$ (b) $\frac{d^2y}{dx^2}=\frac{b\sin x}{(a+b\cos x)^2}$

(c) $\frac{dy}{dx}=\frac{1}{a-b\cos x}$ (d) $\frac{d^2y}{dx^2}=\frac{-b\sin x}{(a-b\cos x)^2}$

82. If $f(x)+f(y)+f(z)+f(x).f(y).f(z)=14$ for all $x, y, z \in R$, then

(a) $f(0)=2$

(b) $f'(x)=0$, for all $x \in R$

(c) $f'(x)>0$, for all $x \in R$

(d) None of these

83. If $F(x)=f(x)\,g(x)$ and $f'(x)\,g'(x)=c$, then

(a) $F'=c\left[\frac{f}{f'}+\frac{g}{g'}\right]$ (b) $\frac{F''}{F}=\frac{f''}{f}+\frac{g''}{g}+\frac{2c}{fg}$

(c) $\frac{F'''}{F}=\frac{f'''}{f}+\frac{g'''}{g}$ (d) $\frac{F'''}{F''}=\frac{f'''}{f''}+\frac{g'''}{g''}$

84. If $f(x)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then

(a) f is derivable for all x, with $|x|<1$

(b) f is not derivable at $x=1$

(c) f is not derivable at $x=-1$

(d) f is derivable for all x, with $|x|>1$

85. If $f_n(x)=e^{f_{n-1}(x)}$ for all $n \in N$ and $f_0(x)=x$, then $\frac{d}{dx}\{f_n(x)\}$ is equal to

(a) $f_n(x).\frac{d}{dx}\{f_{n-1}(x)\}$ (b) $f_n(x).f_{n-1}(x)$

(c) $f_n(x).f_{n-1}(x)...f_2(x).f_1(x)$ (d) $\prod_{i=1}^{n} f_i(x)$

86. Let $f(x)=e^{ax}\sin(bx+c)$ and $f''(x)=r^2e^{ax}\sin(bx+\theta)$ then

(a) $r=a^2+b^2$ (b) $r=\sqrt{a^2+b^2}$

(c) $\theta=c+2\tan^{-1}(b/a)$ (d) $\theta=2a\tan^{-1}(b/a)$

87. Suppose f and g are functions having second derivatives f″ and g″ everywhere, if $f(x).g(x)=1$ for all x and f′ and g′ are never zero, then $\frac{f''(x)}{f'(x)}-\frac{g''(x)}{g'(x)}$ equals

(a) $-2\frac{f'(x)}{f(x)}$ (b) $-\frac{2g'(x)}{g(x)}$

(c) $-\frac{f'(x)}{f(x)}$ (d) $2\frac{f'(x)}{f(x)}$

88. If $x=\varphi(t)$ and $y=\psi(t)$ then $\frac{d^2y}{dx^2}$ is equal to

(a) $\frac{\varphi'\psi''-\psi'\varphi''}{\varphi'^2}$ (b) $\frac{\varphi'\psi''-\psi'\varphi''}{\varphi'^3}$

(c) $\frac{\varphi''}{\psi''}$ (d) $\frac{\psi''}{\varphi'^2}-\frac{\psi'.\varphi''}{\varphi'^3}$

89. If $x^3 - 2x^2y^2 + 5x + y - 5 = 0$ and $y(1) = 1$ then

(a) $y'(1) = 4/3$ (b) $y''(1) = -4/3$

(c) $y''(1) = -8\frac{22}{27}$ (d) $y'(1) = 2/3$

90. If $f_n(x) = e^{f_{n-1}(x)}$ for all $n \in N$ and $f_0(x) = x$, then $\frac{d}{dx}\{f_n(x)\}$ is equal to

(a) $f_n(x).\frac{d}{dx}\{f_{n-1}(x)\}$ (b) $f_n(x).f_{n-1}(x)$

(c) $f_n(x).f_{n-1}(x)...f_2(x).f_1(x)$ (d) $\prod_{i=1}^{n} f_i(x)$

91. Choose the correct statement :

(a) If u(x) is differentiable then $\frac{d}{dx}|u| = \frac{uu'}{|u|}$, $u \neq 0$

(b) If $u(x) = \sin bx$ then $u''(x) + b^2u(x) = 0$

(c) If $g(x) = \sqrt{x(x+n)}$ and $a = \frac{2x+n}{2}$, then $\frac{dg}{dx} = \frac{a}{g}$

(d) none of these

92. Which of the following statements are true ?

(a) If $xe^{xy} = y + \sin^2 x$, then at $y'(0) = 1$

(b) If $f(x) = a_0x^{2m+1} + a_1x^{2m} + a_3x^{2m-1} + ... + a_{2m+1} = 0$ $(a_0 \neq 0)$ is a polynomial equation with rational coefficients then the equation $f'(x) = 0$ must have a real root. $(m \in N)$

(c) If $(x - r)$ is a factor of the polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_0$ repeated m times where $1 \leq m \leq n$ then r is a root of the equation $f'(x) = 0$ repeated $(m-1)$ times

(d) If $y = \sin^{-1}(\cos \sin^{-1} x) + \cos^{-1}(\sin \cos^{-1} x)$ then $\frac{dy}{dx}$ is independent on x.

93. Let $y = \sqrt{(\sin x + \sin 2x + \sin 3x)^2 + (\cos x + \cos 2x + \cos 3x)^2}$ then which of the following is correct ?

(a) $\frac{dy}{dx}$ when $x = \frac{\pi}{2}$ is -2

(b) value of y when $x = \frac{\pi}{5}$ is $\frac{3+\sqrt{5}}{2}$

(c) value of y when $x = \frac{\pi}{12}$ is $\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}}{2}$

(d) y simplifies to $(1 + 2\cos x)$ in $[0, \pi]$

94. Let $f(x) = \frac{1-x^{n+1}}{1-x}$ and $g(x) = 1 - \frac{2}{x} + \frac{3}{x^2} - .... + (-1)^n \frac{n+1}{x^n}$.
Then the constant term in $f'(x) \times g(x)$ is equal to

(a) $\frac{n(n^2-1)}{6}$ when n is even

(b) $\frac{n(n+1)}{2}$ when n is odd

(c) $-\frac{n}{2}(n+1)$ when n is even

(d) $\frac{n(n-1)}{2}$ when n is odd

Paragraph Type Questions

Passage–1

A curve is represented parametrically by the equations $x = f(t) = a^{\ln(b^t)}$ and $y = g(t) = b^{-\ln(a^t)}$ $a, b > 0$ and $a \neq 1$, $b \neq 1$ where $t \in R$.

95. Which of the following is not a correct expression for $\frac{dy}{dx}$?

(a) $\frac{-1}{f(t)^2}$ (b) $-(g(t))^2$

(c) $\frac{-g(t)}{f(t)}$ (d) $\frac{-f(t)}{g(t)}$

96. The value of $\frac{d^2y}{dx^2}$ at the point where f (t) = g (t) is

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) 2

97. The value of $\frac{f(t)}{f'(t)}.\frac{f''(-t)}{f'(-t)}+\frac{f(-t)}{f'(-t)}.\frac{f''(t)}{f'(t)}\ \forall\, t \in R$, is equal to

(a) –2 (b) 2

(c) –4 (d) 4

Passage – 2

A curve is represented parametrically by the equations $x = e^t \cos t$ and $y = e^t \sin t$ where t is a parameter. Then

98. The relation between the parameter 't' and the angle α between the tangent to the given curve and the x-axis is given by, 't' equals.

(a) $\frac{\pi}{2}-\alpha$ (b) $\frac{\pi}{4}+\alpha$

(c) $\alpha-\frac{\pi}{4}$ (d) $\frac{\pi}{4}-\alpha$

99. The value of $\frac{d^2y}{dx^2}$ at the point where t = 0 is

(a) 1 (b) 2

(c) –2 (d) 3

100. If $F(t) = \int (x + y)\, dt$ then the value of $F\left(\frac{\pi}{2}\right) - F(0)$ is

(a) 1 (b) –1

(c) $e^{\pi/2}$ (d) 0

ASSERTION REASON

(A) ASSERTION is true, REASON is true, REASON is a correct explanation for ASSERTION.

(B) ASSERTION is true, REASON is true, REASON is not a correct explanation for ASSERTION.

(C) ASSERTION is true, REASON is false.

(D) ASSERTION is false, REASON is true.

(E) Both ASSERTION and REASON are false.

101. Assertion : Let $p(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$. If $|p(x)| \le |e^{x-1} - 1|$ for all $x \ge 0$ then $|a_1 + 2a_2 + ... + na_n| \le 1$.

Reason : $|p(x)| \le |e^{x-1} - 1|$ $\Rightarrow p(1) = 0$ and

$$p'(1) = \lim_{h\to 0}\frac{p(1+h)-p(1)}{h}$$

(a) A (b) B

(c) C (d) D

(e) E

102. Assertion : The function $f(t) = \frac{1-\cos(1-\cos t)}{t^4}$ is continuous every where if $f(0) = \frac{1}{8}$.

Reason : For continuous function $f(0) = \underset{t\to 0}{\text{Lt}}\, f(t)$

(a) A (b) B

(c) C (d) D

(e) E

103. Assertion : Let $f(x) = [\cos x + \sin x]$, $0 < x < 2\pi$, where [x] denotes the integral part of x then $f(x)$ is discontinuous at 5 points.

Reason : for $x = \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}, \frac{3\pi}{2}$, right hand limit not equal to left hand limit.

(a) A (b) B

(c) C (d) D

(e) E

104. Assertion : $\frac{d}{dx}\{\tan^{-1}(\sec x + \tan x)\}$

$= \frac{d}{dx}\{\cot^{-1}(\operatorname{cosec} x + \cot x)\}, x \in \left(0, \frac{\pi}{4}\right)$.

Reason : $\sec^2 x - \tan^2 x = 1 = \operatorname{cosec}^2 x - \cot^2 x$.

(a) A (b) B (c) C

(d) D (e) E

105. **Assertion :** $f(x)=\cos^2 x+\cos^2\left(x+\frac{\pi}{3}\right)-\cos x\cos\left(x+\frac{\pi}{3}\right)$

then $f'(x)=0$

Reason : Derivative of constant function is zero.

(a) A (b) B (c) C
(d) D (e) E

106. **Assertion :** Derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ is 1 for $0<x<1$.

Reason : $\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ for $-1\le x\le 1$

(a) A (b) B (c) C
(d) D (e) E

107. **Assertion :** If $e^{xy}+\text{In}(xy)+\cos(xy)+5=0$, then $\frac{dy}{dx}=-\frac{y}{x}$.

Reason : $\frac{d}{dx}(xy)=0\Rightarrow\frac{dy}{dx}=-\frac{y}{x}$

(a) A (b) B (c) C
(d) D (e) E

Match The Column

108. **Column–I** **Column–II**

(A) If the function (P) 6

$$f(x)=\begin{cases}\frac{\sin 3x}{x}, & x\ne 0\\ \frac{K}{2}, & x=0\end{cases}$$

is continuous at x = 0, then k =

(B) If $f(x)=\frac{x^2-10x+25}{x^2-7x+10}$ for $x\ne 5$ (Q) $2\log|a|$

& it is continuous at x = 5 then $f(5)=$

(C) If f R → R defined by (R) 3

$$f(x)=\begin{cases}a^2\cos^2 x+b^2\sin^2 x\,(x\le 0)\\ e^{ax+b}\,(x>0)\end{cases}$$

is continuous function then b =

(S) $\log|a|$

(T) 0

109. **Column - I** **Column - II**

(A) If $f'(x)=\sqrt{3x^2+6}$ & $y=f(x^3)$ (P) –2

then at x = 1, $\frac{dy}{dx}=$

(B) If f be a diffr. fun. such that (Q) –1

$f(xy)=f(x)+f(y);\ \forall\ x,y\in R$

then $f(e)+f(1/e)=$

(C) If f be a twice diffr. fun. such that (R) 0

$f''(x)=-f(x)\ \&\ f'(x)=g(x);$

If $h(x)=(f(x))^2+(g(x))^2\ \&\ h(s)=9$

then $h(10)=?$

(D) $y=\tan^{-1}(\cot x)+\cot^{-1}(\tan x)$, (S) 9

$\frac{\pi}{2}<x<\pi$ then $\frac{dy}{dx}$

Subjective Type Questions

110. The function given by

$$f(x)=\begin{cases}\frac{\sqrt{\pi}-\sqrt{\cos^{-1}x}}{\sqrt{x+1}}, & x\ne -1\\ \frac{1}{\sqrt{\lambda\pi}}, & x=-1\end{cases}$$

The value of λ for which the function $f(x)$ is continuous at x = –1 from the right, must be

111. If $f(x)=\begin{cases}\frac{36^x-9^x-4^x+1}{\sqrt{2}-\sqrt{(1+\cos x)}}, & x\ne 0\\ \lambda, & x=0\end{cases}$ is continuous at x = 0, then $\lambda=\sqrt{\mu}\ \ln 2.\ \ln 3$ then the value of μ must be

112. If $f(x)=\begin{cases}\frac{8^x-4^x-2^x+1^x}{x^2}, & x>0\\ e^x\sin x+\pi x+\lambda\ln 4, & x\le 0\end{cases}$ is continuous at x = 0, then the value of 1000 e^{λ} must be

113. Let $f(x)=\dfrac{\cos^{-1}(1-\{x\})\,.\,\sin^{-1}(1-\{x\})}{\sqrt{2\{x\}}\,.\,(1-\{x\})}$, then the value of $\dfrac{2008\sqrt{2}}{\pi}\lim\limits_{x\to 0^-} f(x)$ must be (where $\{x\}$ denotes the fractional part of x).

114. If the third derivative of $\dfrac{x^4}{(x-1)(x-2)}$ is $\dfrac{-k}{(x-2)^4}+\dfrac{6}{(x-1)^4}$ then the numerical quantity k must be equal to

115. If $f(x)=\dfrac{1}{\sin x-\sin a}-\dfrac{1}{(x-a)\cos x}$ then

$$\frac{d}{da}\lim_{x\to a} f(x)=\frac{-1}{k}\sec a-\sec a\tan^2 a.$$

The numerical quantity k should be equal to

116. If $y=\tan(x+y)$, then $\dfrac{d^n y}{dx^n}=-\dfrac{6y^4+16y^2+10}{y^8}$ n must be equal to

117. Let f, g and h are differentiable functions. If f (0) = 1; g (0) = 2; h (0) = 3 and the derivatives of their pair wise products at x = 0 are

$(fg)'(0)=6;\ (gh)'(0)=4$ and $(hf)'(0)=5$

then compute the value of $(fgh)'(0)$.

118. Let P (x) be a polynomial of degree 4 such that $P(1)=P(3)=P(5)=P'(7)=0$. If the real number $x\neq 1, 3, 5$ is such that $P(x)=0$ can be expressed as $x=p/q$ where 'p' and 'q' are relatively prime, then find $(p+q)$.

EXERCISE - 4 : PREVIOUS YEAR JEE ADVANCED QUESTIONS

Single Answer Type Questions

1. For real number y, let [y] denotes the greatest integer less than or equal to y. Then the function.

$f(x)=\dfrac{\tan \pi[(x-\pi)]}{1+[x]^2}$ is. **(1981)**

(a) discontinuous at some x

(b) continuous at all x, but the derivative $f'(x)$ does not exist for some x.

(c) $f'(x)$ exist for all x but the derivative $f''(x)$ does not exist for some x.

(d) $f''(x)$ exists for all x.

2. If $G(x)=-\sqrt{25-x^2}$, then $\lim\limits_{x\to 1}\dfrac{G(x)-G(1)}{x-1}$ has the value **(1983)**

(a) $\dfrac{1}{\sqrt{24}}$ (b) $\dfrac{1}{5}$

(c) $-\sqrt{24}$ (d) None of these

3. The function $f(x)=\dfrac{\log(1+ax)-\log(1-bx)}{x}$ is not defined at x = 0. The value which should be assigned to f at x = 0, so that it is continuous at x = 0, is **(1983)**

(a) a – b (b) a + b

(c) log a + log b (d) None of these

4. If $f(x)=x(\sqrt{x}+\sqrt{(x+1)})$, then **(1985)**

(a) f (x) is continuous but not differentiable at x = 0

(b) f (x) is differentiable at x = 0

(c) f (x) is not differentiable at x = 0

(d) None of the above

5. The set of all points where the function,

$f(x)=\dfrac{x}{(1+|x|)}$ is differentaible, is **(1987)**

(a) $(-\infty, \infty)$ (b) $[0, \infty)$

(c) $(-\infty, 0)\cup(0, \infty)$ (d) $(0, \infty)$

6. If $f(x)=\dfrac{1}{2}x-1$, then on the interval $[0, \pi]$ **(1989)**

(a) tan [f (x)] and 1/f (x) are both continuous

(b) tan [f (x)] and 1/f (x) are both discontinuous

(c) tan [f (x)] and $f^{-1}(x)$ are both continuous

(d) tan [f (x)] is continuous but 1/f (x) is not continuous

7. The function $f(x)=[x]\cos\left(\dfrac{2x-1}{2}\right)\pi$, [.] denotes the greatest integer function, is discontinuous at **(1993)**

(a) all x (b) all integer points

(c) no x (d) x which is not an integer

8. Let [.] denotes the greatest integer function and $f(x)=[\tan^2 x]$, then : **(1993)**

(a) $\lim\limits_{x\to 0} f(x)$ does not exist $x\to 0$

(b) $f(x)$ is continuous at x = 0

(c) $f(x)$ is not differentiable at x = 0

(d) $f'(0)=1$

9. If $y=(\sin x)^{\tan x}$, then $\dfrac{dy}{dx}$ is equal to **(1994)**

(a) $(\sin x)^{\tan x}(1+\sec^2 x\,\log\sin x)$

(b) $\tan x(\sin x)^{\tan x-1}\cos x$

(c) $(\sin x)^{\tan x}\sec^2 x\,\log\sin x$

(d) $\tan x(\sin x)^{\tan x-1}$

10. Let $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ where p is a constant. Then

$\dfrac{d^3}{dx^3}(f(x))$ at x = 0 is **(1997)**

(a) p (b) $p+p^2$

(c) $p+p^3$ (d) Independent of p

11. The function f (x) = $[x]^2 - [x^2]$ (where [x] is the greatest integer less than or equal to x), is discontinuous at **(1999)**

(a) all integers

(b) all integers except 0 and 1

(c) all integers except 0

(d) all integers except 1

12. The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos(|x|)$ is not differentiable at x : **(1999)**

(a) –1 (b) 0

(c) 1 (d) 2

13. Let $f(x)$ be defined for all x > 0 and be differentiable. $f(x)$ satisfy $f\left(\frac{x}{y}\right) = f(x) - f(y)\ \forall\ x, y \in R^+$ and $f(e) = 1$, then **(1999)**

(a) $f(x)$ is bounded (b) $f\left(\frac{1}{x}\right) \to 0$ as $x \to 0$

(c) $x f(x) \to 1$ as $x \to 0$ (d) $f(x) = \ln x$

14. Let $f : R \to R$ be any function. Define $g : R \to R$ by $g(x) = |f(x)|$ for all x. Then g is : **(2000)**

(a) onto if f is onto

(b) one-one if f is one-one

(c) continuous if f is continuous

(d) differentiable if f is differentiable

15. Let $f : R \to R$ be a function defined by $f(x) = \max\{x, x^3\}$. The set of all points where $f(x)$ is not differentiable is : **(2001)**

(a) {–1, 1} (b) {–1, 0}

(c) {0, 1} (d) {–1, 0, 1}

16. The left hand derivative of $f(x) = [x]\sin(\pi x)$ at x = k, k is an integer is **(2001)**

(a) $(-1)^k (k-1)\pi$ (b) $(-1)^{k-1}(k-1)\pi$

(c) $(-1)^k k\pi$ (d) $(-1)^{k-1} k\pi$

17. Which of the following functions is differentiable at x = 0? **(2001)**

(a) $(\cos|x|) + |x|$ (b) $\cos(|x|) - |x|$

(c) $\sin(|x|) + |x|$ (d) $\sin(|x|) - |x|$

18. The domain of the derivative of the functions

$$f(x) = \begin{cases} \tan^{-1} x, & \text{if } |x| \le 1 \\ \frac{1}{2}(|x| - 1), & \text{if } |x| > 1 \end{cases} \text{ is}$$ **(2002)**

(a) R – {0} (b) R – {1}

(c) R – {–1} (d) R – {–1, 1}

19. $\lim\limits_{h \to 0} \frac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)}$, given that $f'(2) = 6$ and $f'(1) = 4$: **(2003)**

(a) does not exist (b) is equal to –3/2

(c) is equal to 3/2 (d) is equal to 3

20. If y is a function of x and $\log(x + y) - 2xy = 0$, then the value of y′ (0) is equal to **(2004)**

(a) 1 (b) – 1

(c) 2 (d) 0

21. If y = y (x) and it follows the relation $x\cos y + y\cos x = \pi$, then y″ (0) **(2005)**

(a) – 1 (b) π

(c) – π (d) 1

22. Let $f(x) = ||x| - 1|$, then points where $f(x)$, is not differentiable is/(are) : **(2005)**

(a) 0 (b) 1

(c) ± 1 (d) 0, ± 1

23. If $F(x) = \left(f\left(\frac{x}{2}\right)\right)^2 + \left(g\left(\frac{x}{2}\right)\right)^2$ where $f''(x) = -f(x)$ and $g(x) = f'(x)$ and given that $F(5) = 5$, then $F(10)$ is equal to **(2006)**

(a) 5 (b) 10

(c) 0 (d) 15

24. $\lim\limits_{x \to \frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t)\,dt}{x^2 - \frac{\pi^2}{16}}$ equals **(2007)**

(a) $\frac{8}{\pi} f(2)$ (b) $\frac{2}{\pi} f(2)$

(c) $\frac{2}{\pi} f\left(\frac{1}{2}\right)$ (d) $4 f(2)$

25. $\frac{d^2x}{dy^2}$ equals **(2007)**

(a) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(c) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (d) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

26. Let $g(x) = \log f(x)$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x + 1) = x f(x)$. Then, for N = 1, 2, 3, ..., **(2008)**

$g''\left(N+\frac{1}{2}\right)-g''\left(\frac{1}{2}\right)=$

(a) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N-1)^2}\right\}$

(b) $4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N-1)^2}\right\}$

(c) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N+1)^2}\right\}$

(d) $4\left\{1+\frac{1}{9}+\frac{1}{25}+...+\frac{1}{(2N+1)^2}\right\}$

27. Let f be a real-valued function defined on the interval (–1,1) such that $e^{-x}f(x) = 2+\int_0^x \sqrt{t^4+1}\,dt$, for all $x \in (-1, 1)$ and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to **(2010)**

(a) 1 (b) $\frac{1}{3}$

(c) $\frac{1}{2}$ (d) $\frac{1}{e}$

28. Let $f(x)=\begin{cases} x^2\left|\cos\frac{\pi}{x}\right|, & x \neq 0, x \in R, \\ 0, & x = 0 \end{cases}$ then f is **(2012)**

(a) differentiable both at x = 0 and x = 2

(b) differentiable at x = 0 but not differentiable at x = 2

(c) not differentiable at x = 0 but differentiable at x = 2

(d) differentiable neither at x = 0 nor at x = 2

29. Let $f_1 : R \to R$, $f_2 : [0, \infty) \to R$, $f_3 : R \to R$ and $f_4 : R \to [0, \infty)$ be defined by

$f_1(x) = \begin{cases} |x| & \text{if } x < 0, \\ e^x & \text{if } x \geq 0; \end{cases}$

$f_2(x) = x^2;$

$f_3(x) = \begin{cases} \sin x & \text{if } x < 0, \\ x & \text{if } x \geq 0 \end{cases}$

and

$f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0, \\ f_2(f_1(x)) - 1 & \text{if } x \geq 0. \end{cases}$ **(2014)**

	List I		List II
P.	f_4 is	1.	onto but not one-one
Q.	f_3 is	2.	neither continuous nor one-one
R.	f_2 of$_1$ is	3.	differentiable but not one-one
S.	f_2 is	4.	continuous and one-one

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

Multiple Answers Questions

30. If x + |y| = 2y, then y as a function of x is **(1984)**

(a) defined for all real x

(b) continuous at x = 0

(c) differentiable for all x

(d) such that $\frac{dy}{dx}=\frac{1}{3}$ for x < 0

31. The function $f(x) = 1 + |\sin x|$ is **(1986)**

(a) continuous no where

(b) continuous everywhere

(c) differentiable at $x = 0$

(d) not differentiable at infinite number of points

32. Let $[x]$ denote the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then $f(x)$ is **(1986)**

(a) continuous at $x = 0$ (b) continuous in $(-1, 0)$

(c) differentiable at $x = 1$ (d) differentiable in $(-1, 1)$

33. The function $f(x) = \begin{cases} |x-3|, & x \ge 1 \\ \frac{x^2}{4} - \frac{3x}{2} + \frac{13}{4}, & x < 1 \end{cases}$ is **(1988)**

(a) continuous at $x = 1$ (b) differentiable at $x = 1$

(c) discontinuous at $x = 1$ (d) differentiable at $x = 3$

34. The following functions are continuous on $(0, \pi)$ **(1991)**

(a) $\tan x$

(b) $\int_0^x t \sin \frac{1}{t}\, dt$

(c) $\begin{cases} 1, 0 \le x \le \frac{3\pi}{4} \\ 2\sin \frac{2}{9}x, \frac{3\pi}{4} < x < \pi \end{cases}$

(d) $\begin{cases} x \sin x, 0 < x \le \frac{\pi}{2} \\ \frac{\pi}{2} \sin (\pi + x), \frac{\pi}{2} < x < \pi \end{cases}$

35. Let $f(x) = \begin{cases} 0, & x < 0 \\ x^2, & x \ge 0 \end{cases}$ then for all x **(1994)**

(a) f' is differentiable (b) f is differentiable

(c) f' is continuous (d) f is continuous

36. Let $g(x) = x\, f(x)$, where $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$.

At $x = 0$ **(1994)**

(a) g is differentiable but g' is not continuous

(b) g is differentiable while f is not

(c) both f and g are differentiable

(d) g is differentiable and g' is continuous

37. The function

$f(x) = \max \{(1-x), (1+x), 2\}, x \in (-\infty, \infty)$ is **(1995)**

(a) continuous at all points

(b) differentiable at all points

(c) differentiable at all points except at $x = 1$ and $x = -1$

(d) continuous at all points except at $x = 1$ and $x = -1$, where it is discontinuous

38. Let $h(x) = \min \{x, x^2\}$ for every real number of x. Then **(1998)**

(a) h is continuous for all x

(b) h is differentiable for all x

(c) $h'(x) = 1$, for all $x > 1$

(d) h is not differentiable at two values of x.

39. If $f(x) = \min \{1, x^2, x^3\}$, then **(2006)**

(a) $f(x)$ is continuous $\forall\ x \in R$

(b) $f(x) > 0,\ \forall\ x > 1$

(c) $f(x)$ is continuous but not differentiable $\forall\ x \in R$

(d) $f(x)$ is not differentiable at two points.

40. Let $f : R \to R$ be a function such that $f(x + y) = f(x) + f(y)$, $\forall\ x, y \in R$. If $f(x)$ is differentiable at $x = 0$, then **(2011)**

(a) $f(x)$ is differentiable only in a finite interval containing zero

(b) $f(x)$ is continuous $\forall\ x \in R$

(c) $f'(x)$ is constant $\forall\ x \in R$

(d) $f(x)$ is differentiable except at finitely many points

41. If $f(x)=\begin{cases}-x-\frac{\pi}{2}, & x\le -\frac{\pi}{2}\\ -\cos x, & -\frac{\pi}{2}<x\le 0\\ x-1, & 0<x\le 1\\ \ln x, & x>1\end{cases}$, then **(2011)**

(a) $f(x)$ is continuous at $x=-\frac{\pi}{2}$

(b) $f(x)$ is not differentiable at $x=0$

(c) $f(x)$ is differentiable at $x=1$

(d) $f(x)$ is differentiable at $x=-\frac{3}{2}$

42. For every integer n, let a_n and b_n be real numbers. Let function $f: R\to R$ be given by

$$f(x)=\begin{cases}a_n+\sin \pi x, & \text{for } x\in[2n, 2n+1]\\ b_n+\cos \pi x, & \text{for } x\in(2n-1, 2n)\end{cases},$$ for all integers n.

If f is continuous, then which of the following hold(s) for all n ? **(2012)**

(a) $a_{n-1}-b_{n-1}=0$ (b) $a_n-b_n=1$

(c) $a_n-b_{n+1}=1$ (d) $a_{n-1}-b_n=-1$

43. For every pair of continuous functions f, g: [0,1] → R such that

$\max\{f(x): x\in[0,1]\}=\max\{g(x): x\in[0,1]\}$,

the correct statement(s) is(are): **(2014)**

(a) $(f(c))^2+3f(c)=(g(c))^2+3g(c)$ for some $c\in[0,1]$

(b) $(f(c))^2+f(c)=(g(c))^2+3g(c)$ for some $c\in[0,1]$

(c) $(f(c))^2+3f(c)=(g(c))^2+g(c)$ for some $c\in[0,1]$

(d) $(f(c))^2=(g(c))^2$ for some $c\in[0,1]$

44. Let f: [a, b] → [1, ∞) be a continuous function and let g : R → R be defined as

$$g(x)=\begin{cases}0 & \text{if } x<a,\\ \int_a^x f(t)\,dt & \text{if } a\le x\le b,\\ \int_a^b f(t)\,dt & \text{if } x>b\end{cases}$$

Then **(2014)**

(a) g (x) is continuous but not differentiable at a

(b) g (x) is differentiable on R

(c) g (x) is continuous but not differentiable at b

(d) g (x) is continuous and differentiable at either a or b but not both.

45. Let g : R → R be a differentiable function with g(0) = 0, $g'(0)=0$ and $g'(1)\ne 0$.

Let $f(x)=\begin{cases}\frac{x}{|x|}g(x), & x\ne 0\\ 0, & x=0\end{cases}$ **(2015)**

(a) f is differentiable at x = 0

(b) h is differentiable at x = 0

(c) foh is differentiable at x = 0

(d) hof is differentiable at x = 0

46. Let $f:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ and $g:\left[-\frac{1}{2},2\right]\to\mathbb{R}$ be functions defined by $f(x)=[x^2-3]$ and $g(x)=|x|f(x)+|4x-7|f(x)$, where [y] denotes the greatest integer less than or equal to y for $y\in\mathbb{R}$. Then **(2016)**

(a) f is discontinuous exactly at three points in $\left[-\frac{1}{2},2\right]$

(b) f is discontinuous exactly at four points in

(c) g is **NOT** discontinuous exactly at four points in

(d) g is **NOT** discontinuous exactly at five points in

47. Let a, b ∈ $\mathbb{R}$ and f: $\mathbb{R}\to\mathbb{R}$, be defined by $f(x)=a\cos(|x^3-x|)+b|x|\sin(|x^3+x|)$. Then f is **(2016)**

(a) differentiable at x = 0 if a = 0 and b = 1

(b) differentiable at x = 1 if a = 1 and b = 0

(c) **NOT** differentiable at x = 0 if a = 1 and b = 0

(d) **NOT** differentiable at x = 1 if a = 1 and b = 1

48. Let f : $\mathbb{R}\to\mathbb{R}$, g : $\mathbb{R}\to\mathbb{R}$, and h : $\mathbb{R}\to\mathbb{R}$, be differentiable functions such that

$f(x)=x^3+3x+2$, $g(f(x))=x$ and $h(g(g(x)))=x$ for all $x\in\mathbb{R}$.

Then **(2016)**

(a) $g'(2)=\frac{1}{15}$ (b) $h'(1)=666$

(c) $h(0)=16$ (d) $h(g(3))=36$

Assertion and Reason

(A) If both assertion and reason are correct and reason is the correct explanation of assertion.

(B) If both assertion and reason are true but reason is not the correct explanation of assertion.

(C) If assertion is true but reason is false.

(D) If assertion is false but reason is true.

49. Let f and g be real valued functions defined on interval $(-1,1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$, and $f(x) = g(x)\sin x$.

Assertion : $\lim_{x\to 0} [g(x)\cot x - g(0)\operatorname{cosec} x] = f''(0)$.

and

Reason : $f'(0) = g(0)$. **(2008)**

Match the Columns

50. Match the conditions/expressions in Column I with statement in Column II. **(1992)**

	Column I	Column II
(A)	$\sin(\pi[x])$	(p) differentiable everywhere
(B)	$\sin\{\pi(x-[x])\}$	(q) nowhere differentiable
		(r) not differentiable at 1 and –1

51. In the following, [x] denotes the greatest integer less than or equal to x. **(2007)**

	Column I	Column II
(A)	$x\lvert x\rvert$	(p) continuous in (–1, 1)
(B)	$\sqrt{\lvert x\rvert}$	(q) differentiable in (–1, 1)
(C)	$x+[x]$	(r) strictly increasing (–1, 1)
		(s) not differentiable at least at
(D)	$\lvert x-1\rvert+\lvert x+1\rvert$	one point in (–1, 1)

Fill in the Blanks :

52. Let $f(x) = \begin{cases} (x^3+x^2-16x+20)/(x-2)^2, & \text{if } x \neq 2 \\ k, & \text{if } x = 2 \end{cases}$

If, f(x) is continuous for all x, then k = ... **(1981)**

53. Let $f(x) = \begin{cases} (x-1)^2 \sin\left(\dfrac{1}{x-1}\right) - \lvert x\rvert & \text{if } x \neq 1 \\ -1 & \text{if } x = 1 \end{cases}$

be a real valued function. Then the set of points where $f(x)$ is not differentiable is........... **(1981)**

54. A discontinuous function $y = f(x)$ satisfying $x^2 + y^2 = 4$ is given by $f(x) = \ldots$. **(1982)**

55. For the function $f(x) = \begin{cases} \dfrac{x}{1+e^{\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$;

the derivative from the right $f'(0^+) = \ldots\ldots\ldots$

and the derivative from the left $f'(0^-) = \ldots\ldots\ldots$ **(1983)**

56. If $f(9) = 9$, $f'(9) = 4$, then $\lim_{x\to 9} \dfrac{\sqrt{f(x)}-3}{\sqrt{x}-3}$ equals ... .**(1988)**

57. Let $f(x) = x\lvert x\rvert$. The set of points, where f (x) is twice differentiable is ... . **(1992)**

58. Let $f(x) = [x]\sin\left(\dfrac{\pi}{[x+1]}\right)$; where [.] denotes the greatest integer function. The domain of f is .... and the points of discontinuity of f in the domain are .... **(1996)**

59. $\lim_{x\to 0} \dfrac{\int_0^{x^2} \cos^2 t\, dt}{x \sin x} =$ **(1997)**

Subjective/Integer Type Questions

60. Find $f'(1)$ if $f(x) = \begin{cases} \dfrac{x-1}{2x^2-7x+5}, & \text{when } x \neq 1 \\ -\dfrac{1}{3}, & \text{when } x = 1 \end{cases}$ **(1979)**

61. If $f(x+y) = f(x) + f(y)$ for all x and y. If the function f is continuous at $x = 0$, then show that f is continuous for all x. **(1981)**

62. Determine the values a, b, c, for which the function

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{for } x < 0 \\ c, & \text{for } x = 0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}, & \text{for } x > 0 \end{cases}$$

is continuous at $x = 0$. **(1982)**

63. Let f be a twice differentiable function such that

$$f''(x) = -f(x) \quad \text{and} \quad f'(x) = g(x)$$

$$h(x) = [f(x)]^2 + [g(x)]^2$$

find $h(10)$ if $h(5) = 1$ **(1982)**

64. Let $f(x) = \begin{cases} 1+x, & 0 \le x \le 2 \\ 3-x, & 2 < x \le 3 \end{cases};$

Determine the form of $g(x) = f[f(x)]$ and hence find the points of discontinuity of g, if any. **(1983)**

65. Let $f(x) = \begin{cases} \dfrac{x^2}{2}, & 0 \le x < 1 \\ 2x^2 - 3x + \dfrac{3}{2}, & 1 \le x \le 2 \end{cases}$

Discuss the continuity of f, f′ and and f″ on [0, 2]. **(1983)**

66. Let $f(x) = x^3 - x^2 - x + 1$ and

$$g(x) \begin{cases} = \max \{f(t); 0 \le t \le x\} \; 0 \le x \le 1 \\ = 3 - x, 1 < x \le 2 \end{cases}$$

Discuss the continuity and differentiability of the function g (x) in the interval (0, 2). **(1985)**

67. Let $f(x)$ be defined in the interval [–2, 2] such that

$$f(x) = \begin{cases} -1, & -2 \le x \le 0 \\ x - 1, & 0 < x \le 2 \end{cases}$$

$g(x) = f(|x|) + |f(x)|$

Test the differentiability of $g(x)$ in (–2, 2) **(1986)**

68. Let $g(x)$ be a polynomial of degree one and $f(x)$ be defined

by $f(x) = \begin{cases} g(x) & x \le 0 \\ \left[\dfrac{(1+x)}{(2+x)}\right]^{1/x} & x > 0 \end{cases}$

Find the continuous functions $f(x)$ satisfying.

$f'(1) = f(-1)$ **(1987)**

69. Let $f(x)$ be a function satisfying the condition

$f(-x) = f(x) \;\forall\; x$. If $f'(0)$ exists, find the value. **(1987)**

70. Let R be the set of real numbers and $f : R \to R$ be such that for all x and y in R in R $|f(x) - f(y)| \le (x - y)^3$. Prove that f (x) is a constant. **(1988)**

71. Draw the graph of the function $y = [x] + |1 - x|, -1 \le x \le 3$. Determine the points, if any, where this function is not differentiable, where [x] denotes greatest integer < x. **(1989)**

72. Find the values of a and b, so that the function

$$f(x) = \begin{cases} x + a\sqrt{2}\sin x, & 0 \le x < \pi/4 \\ 2x\cot x + b, & \pi/4 \le x \le \pi/2 \\ a\cos 2x - b\sin x, & \pi/2 < x \le \pi \end{cases}$$

is continuous for $0 \le x \le \pi$. **(1989)**

73. If $x = \sec\theta - \cos\theta$ and $y = \sec^n\theta - \cos^n\theta$, then show that

$$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(y^2+4)$$ **(1989)**

74. Let $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & x < 0 \\ a, & x = 0 \\ \dfrac{\sqrt{2}}{\sqrt{16+\sqrt{x}} - 4}, & x > 0 \end{cases}$

Determine the value of a, if possible, so that the function is continuous at x = 0 **(1990)**

75. Let $f(x) = \begin{cases} (1+|\sin x|)^{a/|\sin x|}, & -\dfrac{\pi}{6} < x < 0 \\ b, & x = 0 \\ e^{\tan 2x/\tan 3x}, & 0 < x < \dfrac{\pi}{6} \end{cases}$

Determine *a* & *b* so that *f* is continuous at x = 0 **(1994)**

76. Let $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \ne 0 \\ 0, & x = 0 \end{cases}$ Test whether

(a) f (x) is continuous at x = 0

(b) f (x) is differentiable at x = 0 **(1997)**

77. Determine the values of x for which the following function fails to be continuous or differentiable

$f(x) = \begin{cases} 1-x, & x < 1 \\ (1-x)(2-x), & 1 \le x \le 2 \\ 3-x, & x > 2 \end{cases}$ Justify your answer.

(1997)

78. Let $\alpha \in R$. Prove that a function $f : R \to R$ is differentiable at α if and only if there is a function $g : R \to R$ which is continuous at α and satisfies $f(x) - f(\alpha) = g(x)(x-\alpha)$ for all $x \in R$. **(2001)**

79. Let $f(x) = \begin{cases} x+a & \text{if } x < 0 \\ |x-1| & \text{if } x \geq 0 \end{cases}$

and $g(x) = \begin{cases} x+1 & \text{if } x < 0 \\ (x-1)^2 + b & \text{if } x \geq 0 \end{cases}$,

where a and b are non-negative real numbers. Determine the composite function *gof*. If (*gof*) (*x*) is continuous for all real x determine the values of a and b, is *gof* differentiable at x = 0 ? Justify your answer. **(2002)**

80. If a function $f : [-2a, 2a] \to R$ is an odd function such that $f(x) = f(2a - x)$ for $x \in [a, 2a]$ and the left hand derivative at x = a is 0 then find the left hand derivative at x = –a. **(2003)**

81. $f(x) = \begin{cases} b\sin^{-1}\left(\dfrac{x+c}{2}\right), & -\dfrac{1}{2} < x < 0 \\ \dfrac{1}{2}, & x = 0 \\ \dfrac{e^{ax/2}-1}{x}, & 0 < x < \dfrac{1}{2} \end{cases}$

If f (x) is differentiable at x = 0 and $|c| < \dfrac{1}{2}$, then find the value of a and prove that $64b^2 = (4 - c^2)$. **(2004)**

82. If $f : [-1, 1] \to R$ and $f(0) = 0$ then $f'(0) = \lim\limits_{n\to\infty} n f\left(\dfrac{1}{n}\right)$

Find the value of $\lim\limits_{n\to\infty} \dfrac{2}{\pi}(n+1)\cos^{-1}\left(\dfrac{1}{n}\right) - n$

Given that $0 < \left|\lim\limits_{n\to\infty} \cos^{-1}\left(\dfrac{1}{n}\right)\right| < \dfrac{\pi}{2}$ **(2004)**

83. If two functions '*f*' and '*g*' satisfying given conditions for $\forall$ x, y $\in$ R, $f(x-y) = f(x)\,g(y) - f(y)\,.\,g(x)$ and $g(x-y) = g(x)\,.\,g(y) + f(x)\,.\,f(y)$. If right hand derivative at x = 0 exists for $f(x)$ then find the derivative of $g(x)$ at x=0 **(2005)**

84. Let $f : R \to R$ and $g : R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define $h : R \to R$ by

$$h(x) = \begin{cases} \max\ \{f(x), g(x)\} & \text{if } x \leq 0, \\ \min\ \{f(x), g(x)\} & \text{if } x \geq 0. \end{cases}$$ **(2014)**

The number of points at which h(x) is not differentiable is

ANSWER KEY

EXERCISE - 1 : BASIC OBJECTIVE QUESTIONS

1. (a)	2. (b)	3. (d)	4. (d)	5. (c)	6. (b)	7. (b)	8. (b)	9. (a)	10. (a)
11. (b)	12. (b)	13. (d)	14. (c)	15. (d)	16. (b)	17. (b)	18. (d)	19. (d)	20. (a)
21. (d)	22. (b)	23. (b)	24. (a)	25. (c)	26. (d)	27. (b)	28. (b)	29. (b)	30. (d)
31. (a)	32. (c)	33. (a)	34. (b)	35. (d)	36. (c)	37. (c)	38. (c)	39. (b)	40. (b)
41. (c)	42. (a)	43. (d)	44. (d)	45. (b)	46. (d)	47. (d)	48. (b)	49. (c)	50. (a)
51. (a)	52. (b)	53. (b)	54. (b)	55. (c)	56. (b)	57. (a)	58. (b)	59. (c)	60. (a)
61. (d)	62. (a)	63. (d)	64. (d)	65. (a)	66. (c)	67. (c)	68. (c)	69. (d)	70. (d)
71. (c)	72. (c)	73. (a)	74. (a)	75. (d)	76. (a)	77. (d)	78. (b)	79. (a)	80. (c)
81. (d)	82. (b)	83. (c)	84. (d)	85. (b)	86. (d)	87. (c)	88. (b)	89. (a)	90. (b)
91. (b)	92. (d)	93. (b)	94. (b)	95. (d)	96. (b)	97. (c)	98. (c)	99. (b)	100. (b)
101. (b)	102. (a)	103. (a)	104. (b)	105. (b)	106. (a)	107. (c)	108. (b)	109. (b)	110. (b)

EXERCISE - 2 : PREVIOUS YEAR JEE MAINS QUESTIONS

1. (a)	2. (a)	3. (b)	4. (d)	5. (c)	6. (b)	7. (a)	8. (a)	9. (b)	10. (c)
11. (c)	12. (b)	13. (c)	14. (d)	15. (b)	16. (c)	17. (d)	18. (d)	19. (d)	20. (a)
21. (b)	22. (c)	23. (b)	24. (d)	25. (a)	26. (c)	27. (d)	28. (c)	29. (b)	30. (b)
31. (d)	32. (b)	33. (c)	34. (c)	35. (a)	36. (d)	37. (b)	38. (b)	39. (c)	40. (b)
41. (c)									

EXERCISE - 3 : ADVANCED OBJECTIVE QUESTIONS

1. (a)	2. (b)	3. (c)	4. (c)	5. (d)	6. (b)	7. (d)	8. (c)	9. (b)	10. (b)
11. (a)	12. (d)	13. (a)	14. (c)	15. (b)	16. (c)	17. (b)	18. (c)	19. (c)	20. (c)
21. (d)	22. (d)	23. (a)	24. (a)	25. (c)	26. (c)	27. (a)	28. (b)	29. (c)	30. (a)
31. (c)	32. (a)	33. (c)	34. (b)	35. (a)	36. (b)	37. (c)	38. (c)	39. (a)	40. (c)
41. (c)	42. (b)	43. (b)	44. (b)	45. (d)	46. (b)	47. (c)	48. (b)	49. (a)	50. (a)
51. (b)	52. (a)	53. (b)	54. (c)	55. (b)	56. (d)	57. (c)	58. (c)	59. (c)	60. (d)
61. (c)	62. (a)	63. (a)	64. (d)	65. (a)	66. (d)	67. (b)	68. (d)	69. (c)	70. (a)
71. (b)	72. (a)	73. (a,b,c)	74. (a, b)	75. (a, b, d)	76. (a, c)	77. (a, b, d)	78. (a,c,d)	79. (a,b)	80. (a,c)
81. (a,b)	82. (a,b)	83. (a,b,c)	84. (a,b,c,d)	85. (a,c,d)	86. (b,c)	87. (b,d)	88. (b,d)	89. (a,c)	90. (a,c,d)
91. (a,b,c)	92. (a,c,d)	93. (a,b)	94. (b,c)	95. (d)	96. (d)	97. (b)	98. (c)	99. (b)	100. (c)
101. (a)	102. (a)	103. (a)	104. (b)	105. (a)	106. (c)	107. (a)	108. ($A \to P$, $B \to T$, $C \to Q$)		
109. (A–S; B–R; C–S; D–P)			110. 0002	111. 0512	112. 2000	113. 1004	114. (0096)	115. (0002)	116. (0003)
117. (0016)	118. (100)								

EXERCISE - 4 : PREVIOUS YEAR JEE ADVANCED QUESTIONS

1. (d) **2.** (a) **3.** (b) **4.** (b) **5.** (a) **6.** (b) **7.** (c) **8.** (b) **9.** (a) **10.** (d)
11. (d) **12.** (d) **13.** (d) **14.** (c) **15.** (d) **16.** (a) **17.** (d) **18.** (d) **19.** (d) **20.** (a)
21. (b) **22.** (d) **23.** (a) **24.** (a) **25.** (d) **26.** (a) **27.** (b) **28.** (b) **29.** (d) **30.** (a,b,d)
31. (b,d) **32.** (a,b,d) **33.** (a,b) **34.** (b,c) **35.** (b,c,d) **36.** (a,b) **37.** (a,c) **38.** (a,c,d) **39.** (a,b,c) **40.** (b,c)
41. (a,b,c,d) **42.** (b,d) **43.** (a,d) **44.** (a,c) **45.** (a, d) **46.** (b,c) **47.** (a,b) **48.** (b,c) **49.** (a) **50.** A–p; B–r
51. A – p,q,r; B–p,s; C–r,s; D–p,q **52.** 7 **53.** {0}

54. Although many such piecewise discontinuous functions are possible, one of them is $f(x)=\begin{cases}\sqrt{4-x^2} & ;-2\le x\le 0\\ -\sqrt{4-x^2} & ;0<x\le 2\end{cases}$

55. 0, 1 **56.** (4) **57.** $x\in R-\{0\}$ **58.** $(-\infty,-1)\cup[0,\infty)$, $I-\{0\}$ where I is the set of integer n except $n=-1$

59. (1) **60.** $f'(1)=-\frac{2}{9}$ **62.** $a=-\frac{3}{2}, c=\frac{1}{2}$ and $b\in R-\{0\}$ **63.** 1

64. $g(x)=\begin{cases}4-x, & 2<x\le 3\\ 2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\end{cases}$, discontinuous at x = {1, 2} **65.** f and f' are continuous and f'' is discontinuous at x = {1, 2}

66. Continuous and differentiable on (0, 2) – {1} **67.** not differentiable at x = 0, 1

68. $f(x)=\begin{cases}\left(\frac{2}{3}\ln\frac{2}{3}-\frac{1}{9}\right)x, & x\le 0\\ \left(\frac{1+x}{2+x}\right)^{1/2}, & x>0\end{cases}$ **69.** (0) **71.** $x=0,1,2,3$ **72.** $a=\frac{\pi}{6}, b=\frac{-\pi}{12}$

74. not possible **75.** $a=\frac{2}{3}, b=e^{2/3}$ **76.** (a) Yes (b) No

77. f is continuous and differentiable at all points except at x =2

79. $g(f(x))=\begin{cases}x+a+1 & \text{if } x<-a\\ (x+a-1)^2+b & \text{if } a\le x<0\\ x^2+b & \text{if } 0\le x\le 1\\ (x-2)^2+b & \text{if } x>1\end{cases}$ $a=1, b=0$, gof differentiable at $x=0$

80. 0 **81.** $a=1$ **82.** $1-\frac{2}{\pi}$ **83.** 0 **84.** (3)

Dream on !!