# JEE (ADVANCED) 2020

# DATE: 27-09-2020

# Questions & Solutions

## PAPER-1 | SUBJECT : MATHEMATICS

### PAPER-1 : INSTRUCTIONS TO CANDIDATES

- **Question Paper-1 has three (03) parts: Physics, Chemistry and Mathematics.**
- **Each part has a total eighteen (18) questions divided into three (03) sections (Section-1, Section-2 and Section-3)**
- **Total number of questions in Question Paper-1 are Fifty Four (54)** and Maximum Marks are **One Hundred Ninety Eight (198).**

### Type of Questions and Marking Schemes

**SECTION-1 (Maximum Marks : 18)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the correct option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

*Full Marks* : **+3** If ONLY the correct option is chosen ;

*Zero Marks* : 0 If none of the options is chosen (i.e. the question is unanswered).

*Negative Marks* : **–1** In all other cases.

**SECTION-2 (Maximum Marks : 24)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme.

*Full Marks* : **+4** If only (all) the correct option(s) is (are) chosen.

*Partial Marks* : **+3** If all the four options are correct but ONLY three options are chosen.

*Partial Marks* : **+2** If three or more options are correct but ONLY two options are chosen and both of which are correct.

*Partial Marks* : **+1** If two or more options are correct but ONLY one option is chosen and it is a correct option.

*Zero Marks* : **0** If none of the options is chosen (i.e. the question is unanswered).

*Negative Marks* : **–2** In all other cases.

**SECTION-3 (Maximum Marks : 24)**

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places **truncate/round-off** the value to **TWO** decimal placed.
- Answer to each question will be evaluated according to the following marking scheme :

*Full Marks* : **+4** If ONLY the correct numerical value is entered.

*Zero Marks* : **0** In all other cases.

# MATHEMATICS

**SECTION-1 (Maximum Marks : 18)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the correct option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

| | | |
|---|---|---|
| *Full Marks* | : | **+3** If ONLY the correct option is chosen ; |
| *Zero Marks* | : | 0 If none of the options is chosen (i.e. the question is unanswered). |
| *Negative Marks* | : | **–1** In all other cases. |

**1.** Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 - 20x + 2020$, then the value of

$$ac(a-c)+ad(a-d)+bc(b-c)+bd(b-d) \text{ is}$$

(A) 0 (B) 8000 (C) 8080 (D) 16000

**Ans.** **(D)**

**Sol.** Now $ac(a - c) + ad(a - d) + bc (b - c) + bd (b - d)$

$= a^2 (c + d) - a (c^2 + d^2) + b^2 (c + d) - b (c^2 + d^2)$

$= (a^2 + b^2) (c + d) - (a + b) (c^2 + d^2)$

$= \{(a + b)^2 - 2ab\} (c + d) - (a + b \{(c + d)^2 - 2cd\}$

$= 16000$

**2.** If the function $f : R \to R$ is defined by $f(x) = |x| (x - \sin x)$, then which of the following statements is TRUE ?

(A) $f$ is one-one, but NOT onto
(B) $f$ is onto, but NOT one-one
(C) $f$ is BOTH one-one and onto
(D) $f$ is NEITHER one-one NOR onto

**Ans.** **(C)**

**Sol.** $f(x)=|x|(x-\sin x)$ is odd function

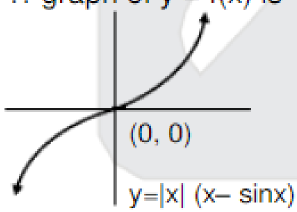
$\because f(-x)=-f(x)$

Now $f(x) = x^2 - x \sin x \quad x \geq 0$

$f'(x) = 2x - x \cos x - \sin x$

$f'(x) = (x - \sin x) + x(1 - \cos x) > 0$

$\therefore$ graph of $y = f(x)$ is




one-one and onto

**3.** Let the functions $: R \longrightarrow R$ and $g : R \longrightarrow R$ be defined by

$f(x)=e^{x-1}-e^{-|x-1|}$ and $g(x)=\frac{1}{2}\left(e^{x-1}+e^{1-x}\right)$

Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is

(A) $\left(2-\sqrt{3}\right)+\frac{1}{2}\left(e-e^{-1}\right)$

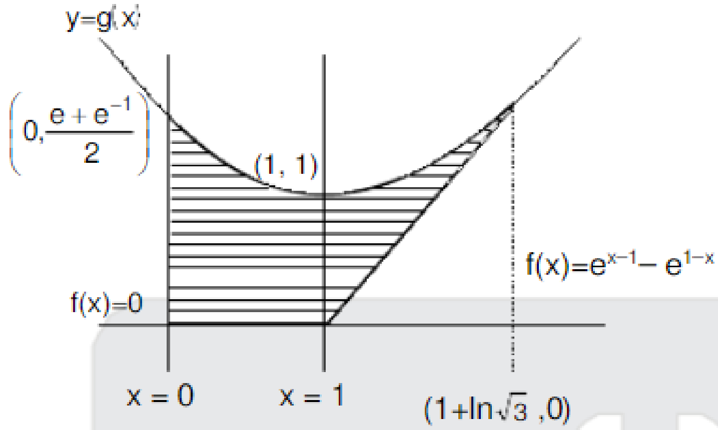
(B) $\left(2+\sqrt{3}\right)+\frac{1}{2}\left(e-e^{-1}\right)$

(C) $\left(2-\sqrt{3}\right)+\frac{1}{2}\left(e+e^{-1}\right)$

(D) $\left(2+\sqrt{3}\right)+\frac{1}{2}\left(e+e^{-1}\right)$

**Ans. (A)**

**Sol.**




$$= \int_0^1 g(x)dx + \int_1^{1+\ln\sqrt{3}} \{g(x) - f(x)\}dx$$

$$= \int_0^1 \frac{1}{2}\left(e^{x-1} + e^{1-x}\right)dx + \int_1^{1+\ln\sqrt{3}} \left\{\frac{1}{2}\left(e^{x-1} + e^{1-x}\right) - \left(e^{x-1} - e^{1-x}\right)\right\}dx$$

$$= \frac{1}{2}\int_0^1 (e^{x-1} + e^{1-x})dx + \frac{1}{2}\int_1^{1+\ln\sqrt{3}} (3e^{1-x} - e^{x-1})dx$$

$$\frac{1}{2}\left[e^{x-1} - e^{1-x}\right]_0^1 - \frac{1}{2}\left[3e^{1-x} + e^{x-1}\right]_1^{1+\ln\sqrt{3}}$$

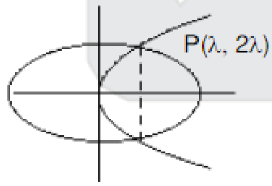
$$\frac{1}{2}(e - e^{-1}) - \frac{1}{2}[2\sqrt{3} - 4] = \frac{e - e^{-1}}{2} + 2 - \sqrt{3}$$

**4.** Let $a$, $b$ and $\lambda$ be positive real numbers. Suppose P is an end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and suppose the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point $P$. If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{5}$

**Ans.** **(A)**

**Sol.**




$y^2 = 4\lambda x \Rightarrow \left(\frac{dy}{dx}\right)_A = 1 = m_1$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \left(\frac{dy}{dx}\right)_A = \frac{-b^2}{2a^2} = m_2$

$\Rightarrow m_1.m_2 = -1 \Rightarrow b^2 = 2a^2$

and $a^2 = b^2 (1 - e^2)$

$\Rightarrow 1 = 2 (1 - e^2)$

$e = \frac{1}{\sqrt{2}}$

**5.** Let $C_1$ and $C_2$ be two biased coins such that the probabilities of getting head in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose $\alpha$ is the number of heads that appear when $C_1$ is tossed twice, independently, and suppose $\beta$ is the number of heads that appear when $C_2$ is tossed twice, independently. Then the probability that the roots of the quadratic polynomial $x^2 - \alpha x + \beta$ are real and equal, is

(A) $\frac{40}{81}$ (B) $\frac{20}{81}$ (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

**Ans.** **(B)**

**Sol.** Roots of equation $x^2 - \alpha x + \beta = 0$ are real and equal

when $D = 0$

$\alpha^2 - 4\beta = 0$

$\alpha^2 = 4\beta$

$(\alpha = 0, \beta = 0)$ or $(\alpha = 2, \beta = 1)$

prob. ${}^2C_0\left(\frac{1}{3}\right)^2 . {}^2C_0\left(\frac{2}{3}\right)^2 + {}^2C_2\left(\frac{2}{3}\right)^2 {}^2C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)$

$= \frac{1}{9} \times \frac{4}{9} + \frac{4}{9} \times \frac{4}{9} = \frac{20}{81}$

**6.** Consider all rectangles lying in the region

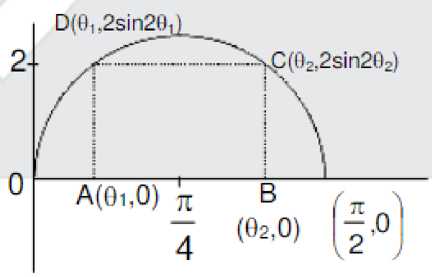
$$\left\{(x,y)\in R\times R: 0\le x\le \frac{\pi}{2} \text{ and } 0\le y\le 2\sin(2x)\right\}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

(A) $\frac{3\pi}{2}$ (B) $\pi$ (C) $\frac{\pi}{2\sqrt{3}}$ (D) $\frac{\pi\sqrt{3}}{2}$

**Ans. (C)**

**Sol.**




$2\sin 2\theta_1 = 2\sin 2\theta_2$

$2\theta_1 = \pi - 2\theta_2$

$\theta_2 = \frac{\pi}{2} - \theta_1$ ....(1)

Now perimeter $p(\theta_1, \theta_2) = 2\{(\theta_2 - \theta_1) + 2\sin 2\theta_1\}$

$p(\theta_1) = 2\left[\frac{\pi}{2} - 2\theta_1 + 2\sin 2\theta_1\right]$

$p'(\theta_1) = 2(-2 + 4\cos 2\theta_1)$

$p''(\theta_1) = 2(-8\sin 2\theta_1)$

for maximum perimeter

$p'(\theta_1) = 0$ and $P''(\theta_1) < 0$

$\cos 2\theta_1 = \frac{1}{2} \Rightarrow 2\theta_1 = \frac{\pi}{3} \Rightarrow \theta_1 = \frac{\pi}{6}$

Now area at $\theta_1 = \frac{\pi}{6}$

$= (\theta_2 - \theta_1) \times 2\sin 2\theta_1$

$= \left(\frac{\pi}{2} - 2\theta_1\right).2\sin 2\theta_1$

$= \left(\frac{\pi}{2} - \frac{\pi}{3}\right) \times 2\sin\frac{\pi}{3} = \frac{\pi}{6}.\sqrt{3} = \frac{\pi}{2\sqrt{3}}$

**SECTION-2 (Maximum Marks : 24)**

- This section contains **SIX (06)** questions.
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme.

| | |
|---|---|
| *Full Marks* : | **+4** If only (all) the correct option(s) is (are) chosen. |
| *Partial Marks* : | **+3** If all the four options are correct but ONLY three options are chosen. |
| *Partial Marks* : | **+2** If three or more options are correct but ONLY two options are chosen and both of which are correct. |
| *Partial Marks* : | **+1** If two or more options are correct but ONLY one option is chosen and it is a correct option. |
| *Zero Marks* : | **0** If none of the options is chosen (i.e. the question is unanswered). |
| *Negative Marks* : | **–2** In all other cases. |

**7.** Let the function $f: R \to R$ be defined by $f(x) = x^3 - x^2 + (x - 1)\sin x$ and let $g: R \to R$ be an arbitrary function. Let $fg: R \to R$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE?

(A) If g is continuous at $x = 1$, then $fg$ is differentiable at $x = 1$

(B) If $fg$ is differentiable at $x = 1$, then g is continuous at $x = 1$

(C) If g is differentiable at $x = 1$, then f g is differentiable at $x = 1$

(D) If $fg$ is differentiable at $x = 1$, then g is differentiable at $x = 1$

**Ans.** **(A,C)**

**Sol.** Differentiability of fg at x = 1

$$(fg)'(1) = \lim_{h\to 0} \frac{fg(1+h) - fg(1)}{h}$$

$$\lim_{h\to 0} \frac{\{(1+h)^3 (1+h)^2 + h\sin(1+h)\}g(1+h) - 0}{h}$$

$$\lim_{h\to 0} \{(1+h)^2 + \sin(1+h)\}\, g(1+h)$$

If g(x) is continuous at x = 1

then $\lim_{h\to 0} g(1+h) = g(1)$

so $\lim_{h\to 0} (fg)'(1) = (1+\sin 1)\, g(1)$

8. Let M be a 3 × 3 invertible matrix with real entries and let I denote the 3 × 3 identity matrix. If $M^{-1} = \text{adj}(\text{adj}M)$, then which of the following statements is/are ALWAYS TRUE?

(A) $M = I$ (B) $\det M = 1$ (C) $M^2 = I$ (D) $(\text{adj } M)^2 = I$

**Ans.** **(BCD)**

**Sol.** $M^{-1} = \text{Adj (Adj M)}$

$\text{Adj M}.M^{-1} = \text{Adj M. Adj (AdjM)}$

$\text{Adj M}. M^{-1} = |\text{Adj M}|\, I$

$\text{Adj M} = |M|^2 M$ ……………..(1)

$|\text{Adj M}| = ||M|^2M| = |M|^6 |M|$

$|M|^2 = |M|^7 \Rightarrow |M| \neq 0, |M| = 1$ ……………..(2)

by equation (1)

$\text{Adj M} = M$

$M.\text{AdjM} = M^2$

$|M|\, I = M^2 \Rightarrow M^2 = I$

again by (1) (2) Adj M = M

$(\text{Adj M})^2 = M^2 = I$

9. Let S be the set of all complex numbers Z satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE ?

(A) $\left|z+\frac{1}{2}\right| \leq \frac{1}{2}$ for all $z \in S$ (B) $|z| \leq 2$ for all $z \in S$

(C) $\left|z+\frac{1}{2}\right| \geq \frac{1}{2}$ for all $z \in S$ (D) The set S has exactly four elements.

**Ans.** **(BC)**

**Sol.** $Z^2 + Z + 1 = e^{i\theta}$ $\theta \in (-\pi, \pi]$

$Z^2 + Z + 1 - e^{i\theta} = 0$

$$Z = \frac{-1 \pm \sqrt{4e^{i\theta} - 3}}{2} \quad .....(i)$$

$$Z + \frac{1}{2} = \pm \sqrt{(4\cos\theta - 3) + i4\sin\theta}$$

$$\left|Z + \frac{1}{2}\right| = [(4\cos\theta - 3)^2 + (4\sin\theta)^2]^{1/4}$$

Now $|25 - 24\cos\theta|^{1/4} \in [1, \sqrt{7}]$

$\left|Z + \frac{1}{2}\right| \in [1, \sqrt{7}]$ option (C) correct

By equation (i)

$|2Z| \leq 1 + \sqrt{|4e^{i\theta} - 3|}$

$|2Z| \leq 1 + (25 - 24\cos\theta)^{1/4}$

$|2Z| \leq 1 + \sqrt{7} < 4$

$|Z| \leq 2$ option (B) is correct

**10.** Let x, $y$ and z be positive real numbers. Suppose $x$, $y$ and z are the lengths of the sides of a triangle opposite to its angles $X$, $Y$ and Z, respectively. If

$$\tan\frac{X}{2}+\tan\frac{Z}{2}=\frac{2y}{x+y+z}$$

then which of the following statements is/are TRUE?

(A) 2Y= X + Z (B) Y= X + Z (C) $\tan\frac{X}{2}=\frac{x}{y+z}$ (D) $x^2+z^2-y^2=xz$

**Ans.** **(BC)**

**Sol.** $\tan\frac{X}{2}+\tan\frac{Z}{2}=\frac{2y}{x+y+z}$

$\frac{\Delta}{s(s-x)}+\frac{\Delta}{s(s-z)}=\frac{y}{s}\Rightarrow \Delta=(s-x)(s-z)$

$\Delta^2=s(s-x)(s-y)(s-z)=(s-x)^2(s-z)^2$

$\Rightarrow y^2=x^2+z^2\Rightarrow \angle Y=90^\circ$

$\angle Y=\angle X+\angle Z$ option B is correct

Now $\tan\frac{X}{2}=\frac{\Delta}{s(s-x)}=\frac{4\Delta}{(y+z+x)(y+z-x)}=\frac{4\times\frac{1}{2}xz}{(y+z)^2-x^2}$

$=\frac{2xz}{2z^2+2yz}=\frac{x}{z+y}$ option C is correct

**11.** Let $L_1$ and $L_2$ be the following straight lines.

$L_1:\frac{x-1}{1}=\frac{y}{-1}=\frac{z-1}{3}$ and $L_2:\frac{x-1}{-3}=\frac{y}{-1}=\frac{z-1}{1}$

Suppose the straight line $L:\frac{x-\alpha}{\ell}=\frac{y-1}{m}=\frac{z-\gamma}{-2}$ lies in the plane containing $L_1$ and $L_2$, and passes through the point of intersection of $L_1$ and $L_2$. If the line L bisects the acute angle between the lines $L_1$ and $L_2$, then which of the following statements is/are TRUE?

(A) $\alpha-\gamma=3$ (B) $\ell+m=2$ (C) $\alpha-\gamma=1$ (D) $\ell+m=0$

**Ans.** **(AB)**

**Sol.** $L_1:\frac{x-1}{1}=\frac{y}{-1}=\frac{z-1}{3}=\lambda \Rightarrow (\lambda+1,-\lambda\,3\lambda+1)$

& $\frac{x-1}{-3}=\frac{y}{-1}=\frac{z-1}{1}=\mu\Rightarrow(-3\mu+1,-\mu,\mu+1)$

Both interacts $\Rightarrow (\lambda+1,-\lambda,3\lambda+1)=(-3\mu+1,-\mu,\mu+1)$

$\Rightarrow \lambda+3\mu=0$

$\lambda=\mu$ $\Rightarrow \lambda=\mu=0$ and $3\lambda=\mu$

Both line passes through (1,0,1)

Direction ratio of the acute angle bisector between two lines is (–1, –1 , –2)

Hence equation of acute angle bisector between two lines $L_1$ & $L_2$

$\frac{x-1}{-1}=\frac{y-0}{-1}=\frac{z-1}{2}\Rightarrow \frac{x-\alpha}{\ell}=\frac{y-1}{m}=\frac{z-\gamma}{-2}$

$\Rightarrow \alpha=2$ & $\gamma=-1$

and $\ell = 1$, $m = 1 \Rightarrow \alpha - \gamma = 3$, $\ell + m = 2$

A & B correct.

**12.** Which of the following inequalities is/are TRUE ?

(A) $\int_0^1 x\cos x\,dx \geq \frac{3}{8}$

(B) $\int_0^1 x\sin x\,dx \geq \frac{3}{10}$

(C) $\int_0^1 x^2\cos x\,dx \geq \frac{1}{2}$

(D) $\int_0^1 x^2\sin x\,dx \geq \frac{2}{9}$

**Ans.** **(ABD)**

**Sol.** $\cos x \approx 1-\frac{x^2}{2}+\frac{x^4}{4}$ ..............

$\Rightarrow \cos x \geq 1-\frac{x^2}{2}$

$x\cos x \geq x-\frac{x^3}{2}$

$\int_0^1 x\cos x\,dx \geq \frac{1}{2}-\frac{1}{8}=\frac{3}{8}$ (A) correct

Now

$\sin x \cong x-\frac{x^3}{3!}+\frac{x^5}{5!}$ ..................

$\sin x \geq x-\frac{x^3}{3!}$

$x\sin x \geq x^2-\frac{x^4}{6}$

$\int_0^1 x\sin x\,dx \geq \frac{1}{3}-\frac{1}{6}\frac{1}{5}=\frac{9}{30}=\frac{3}{10}$ (B) correct

$$x^2 \cos x \geq x^2\left(1-\frac{x^2}{2}\right)$$

$$x^2 \cos x \geq x^2 - \frac{1}{2}x^4$$

$$\int_0^1 x^2 \cos x\, dx \geq \frac{1}{3} - \frac{1}{2}\frac{1}{5} = \frac{7}{30} \quad \text{(C) Wrong}$$

Now $x^2 \sin x \geq x^2\left(x-\frac{x^3}{3!}\right)$

$$\int_0^1 x^2 \sin x\, dx \geq \frac{1}{4} - \frac{1}{6}\frac{1}{6} = \frac{8}{36} = \frac{2}{9} \quad \text{(D) Correct s}$$

**SECTION-3 (Maximum Marks : 24)**

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places **truncate/round-off** the value to **TWO** decimal placed.
- Answer to each question will be evaluated according to the following marking scheme :

*Full Marks* : **+4** If ONLY the correct numerical value is entered.

*Zero Marks* : **0** In all other cases.

**13.** Let $m$ be the minimum possible value of $\log_3(3^{y_1}+3^{y_2}+3^{y_3})$, where $y_1, y_2, y_3$ are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $(\log_3 x_1 + \log_3 x_2 + \log_3 x_3)$, where $x_1, x_2, x_3$ are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is ______.

**Ans.** 8

**Sol.** $\left(\dfrac{3^{y_1}+3^{y_2}+3^{y_3}}{3}\right) \ge \left(3^{y_1}.3^{y_2}.3^{y_3}\right)^{1/3} = \left(3^{y_1+y_2+y_3}\right)^{1/3}$

$\Rightarrow$ $3^{y_1+y_2+y_3} \ge 81$ so $m = \log_3(81) = 4$

$\Rightarrow$

$\log_3 x_1 + \log_3 x_2 + \log_3 x_3 = \log_3(x_1.x_2.x_3)$

$\because$ $\dfrac{x_1+x_2+x_3}{3} \ge (x_1 \,.\, x_2 \,.\, x_3)^{1/3} \Rightarrow x_1 x_2 x_3 \le 27$

$M = \log_3 27 = 3$

so $\log_2(m^3) + \log_3(M^2) = 8$

**14.** Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c, for which the equality

$$2(a_1 + a_2 + \ldots + a_n) = b_1 + b_2 + \ldots + b_n$$

holds for some positive integer n, is _____

**Ans.** 1

**Sol.** $2[a_1 + a_2 + \ldots + a_n] = b_1 + b_2 + \ldots + b_n$

$\Rightarrow$ $2\dfrac{n}{2}[2a_1 + (n-1).2] = \dfrac{b.(2^n-1)}{2-1}$

$\Rightarrow$ $n[2c + 2n - 2] = c(2^n - 1)$

$\Rightarrow$ $2n[c + n - 1] = c(2^n - 1)$

$\Rightarrow$ $c(2^n - 2n - 1) = 2n^2 - 2n$

$\Rightarrow$ $c = \dfrac{2n^2-2n}{2^n-2n-1} \ge 1$ ..................(1)

$\Rightarrow$ $2n(n-1) \ge 2^n - 2n - 1$

$\Rightarrow$ $2n^2 + 1 \ge 2^n$ $\Rightarrow$ $n \le 6$

now put n = 1, 2, ..6 in equation (1) and using $c \in I$

we get c = 12, when n = 3 (only one value of c)

**15.** Let $f : [0, 2] \to \mathbb{R}$ be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right).$$

If $\alpha, \beta \in [0, 2]$ are such that $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$, then the value of $\beta - \alpha$ is ______.

**Ans.** **1**

**Sol.** Let $\pi x - \frac{\pi}{4} = \theta$

$f(x) \geq 0 \Rightarrow \left[3 - \sin\left(2\theta + \frac{\pi}{2}\right)\right] \sin\theta - \sin(3\theta + \pi) \geq 0$

$(3 - \cos 2\theta) \sin\theta + \sin 3\theta \geq 0$

$(2 + 2\sin^2\theta)\sin\theta + 3\sin\theta - 4\sin^3\theta \geq 0$

$\sin\theta\,[5 - 2\sin^2\theta] \geq 0$

$\Rightarrow \sin\theta \geq 0 \Rightarrow \theta \in [0, \pi]$

(Now $x \in [0, 2] \Rightarrow \theta \in \left[\frac{-\pi}{4}, \frac{7\pi}{4}\right]$)

$\Rightarrow \sin\theta \geq 0 \Rightarrow \theta \in [0, \pi] \Rightarrow \pi x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$

$x \in \left[\frac{1}{4}, \frac{5}{4}\right] \Rightarrow \alpha = \frac{1}{4}, \beta = \frac{5}{4}$

Hence $x \in \left[\frac{1}{4}, \frac{5}{4}\right]$

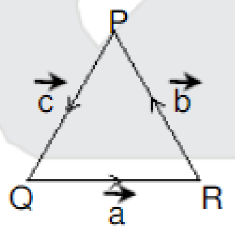
**16.** In a triangle PQR, let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$.

If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $\dfrac{\vec{a}.(\vec{c} - \vec{b})}{\vec{c}.(\vec{a} - \vec{b})} = \dfrac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}$,

then the value of $|\vec{a} \times \vec{b}|^2$ is ______.

**Ans.** **108**

**Sol.** $\vec{a} + \vec{b} + \vec{c} = 0$




$$\frac{\vec{a}.(\vec{c} - \vec{b})}{\vec{c}.(\vec{a} - \vec{b})} = \frac{-(\vec{b} + \vec{c}).(\vec{c} - \vec{b})}{-(\vec{a} + \vec{b}).(\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}$$

$\Rightarrow \quad \dfrac{|\vec{c}|^2 - |\vec{b}|^2}{|\vec{a}|^2 - |\vec{b}|^2} = \dfrac{3}{3 + 4} = \dfrac{3}{7}$

$\Rightarrow \quad \vec{c}|^2 = 13$

$\vec{a} + \vec{b} = -\vec{c} \quad \Rightarrow \quad |\vec{a} + \vec{b}|^2 = |-\vec{c}|^2$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{c}|^2$

$\Rightarrow \quad 9 + 16 + 2(\vec{a}.\vec{b}) = 13$

$\vec{a}.\vec{b} = -6$

$|\vec{a}\times\vec{b}|^2 + (\vec{a}.\vec{b}) = |\vec{a}|^2|\vec{b}|^2$

$|\vec{a}\times\vec{b}|^2 + 36 = (9)\ (16) \Rightarrow \quad |\vec{a}\times\vec{b}|^2 = 108$

**17.** For a polynomial g(x) with real coefficients, let $m_g$ denote the number of distinct real roots of g(x). Suppose S is the set of polynomials with real coefficients defined by
$S = \{(x^2 - 1)^2 (a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$
For a polynomial f, let f′ and f″ denote its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is _____ .

**Ans.** 5

**Sol.** f(x) is a polynomial in x such that $f(x) = \{(x^2 - 1)^2 (a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$
$f'(x) = (x^2 - 1)\ [q(x)]$ where q(x) is polynomial
roots of f'(x) = –1, 1 and by Roll's theorem because $f(1) = 0 = f(-1) \Rightarrow f'(\alpha) = 0$ where $\alpha \in (-1, 1)$
$\Rightarrow$ minimum $m_{f'} = 3$
Now by Roll's theorem at least one real root lies in $(-1, \alpha)$ and at least one real root lies in $(\alpha, 1)$ of f''(x)
so minimum $m_{f''} = 2$
$\Rightarrow$ minimum possible value of $(m_{f'} + m_{f''}) = 5$

**18.** Let e denote the base of the natural logarithm. The value of the real number $a$ for which the right hand limit

$$\lim_{x\to 0^+}\frac{(1-x)^{1/x} - e^{-1}}{x^a}$$

is equal to a nonzero real number, is _____.

**Ans.** 1

**Sol.**

$$\lim_{x\to 0^+}\frac{(1-x)^{1/x}}{ax^{a-1}}\frac{d\left[\frac{\ell n(1-x)}{x}\right]}{dx} = \lim_{x\to 0^+}\frac{(1-x)^{1/x}}{ax^{a-1}}\left[+\frac{1}{(x)(x-1)} - \frac{\ell n(1-x)}{x^2}\right]$$

$$= \lim_{x\to 0^+}\frac{(1-x)^{1/x}}{ax^{a-1}}\left[\frac{x-(x-1)\ell n(1-x)}{x^2(x-1)}\right]$$

$$\lim_{x\to 0^+}\frac{(1-x)^{1/x}\left\{x+(x-1)(x+\frac{x^2}{2}+\frac{x^3}{3}......\right\}}{-a(1-x)x^{a+1}}$$

$$= \lim_{x\to 0}\frac{(1-x)^{1/x}\left\{\frac{x^2}{2}+x^3\left(\frac{1}{2}-\frac{1}{3}\right)+x^4\left(\frac{1}{3}-\frac{1}{4}\right)+.......\right\}}{-a(1-x)x^{a+1}}$$

which is real number iff $a + 1 = 2 \Rightarrow a = 1$