Work, Energy, Power and Circular Motion

6

6.1 WORK DONE

6.1.1 By a Constant Force

If force displaces the particle from its initial position $\vec{r}_i$ to final postion $\vec{r}_f$ then displacement vector $\vec{s} = \vec{r}_f - \vec{r}_i$.

$$W = \vec{F} \cdot \vec{s} = \vec{F} \cdot (\vec{r}_f - \vec{r}_i) = Fs \cos\theta$$

$$= (\text{Force}) \times (\text{component of displacement in the direction of force})$$

or $$W = \vec{F} \cdot \vec{s} = (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k})$$

or $$W = F_x x + F_y y + F_z z$$

6.1.2 By a Variable Force

$$W = \int_{x_i}^{x_f} F\,dx, \text{where } F = f(x)$$

$$= \int_{x_i}^{x_f} (F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz$$

6.1.3 By Area Under *F*-x Graph

If force is a function of x, we can find work done by area under F-x graph with projection along x-axis. In this method, magnitude of work done can be obtained by area under F-x graph, but sign of work done should be decided by you. If force and displacement both are positive or negative, work done will be positive. If one is positive and other is negative then work done will be negative.

$$\text{Work done by the spring on the external agent} = -\frac{1}{2}kx^2$$

$$\text{Work done by the external agent on the spring} = +\frac{1}{2}kx^2$$

Quantities like mass, time, acceleration and force in Newotonian mechanics are invariant i.e., these have same numerical values in different inertial frames. Quantities like velocity, kinetic energy and work done have different values in different inertial frames.

6.2 POWER OF A FORCE

1. **Average power:**

$$P_{\text{av}} = \frac{\text{Total work done}}{\text{Total time taken}} = \frac{W_{\text{Total}}}{t}$$

2. **Instantaneous power:**

$$P_{\text{ins.}} = \text{Rate of doing work done} = \frac{dW}{dt} = \vec{F} \cdot \vec{v} = Fv\cos\theta$$

Power of pump required to just lift the water, $v = 0$

$$\therefore \qquad P = \left(\frac{dm}{dt}\right)gh$$

If efficiency of pump is η, then $\eta = \frac{P_{out}}{P_{in}}$

6.2.1 Conservative and Non-conservative Forces

In case of conservative forces work done is path independent and in a round trip net work done is zero.

Examples: Gravitational force, electrostatic force and elastic force.

If work done by a force in displacing a particle depends on path, the force is said to be non-conservative or dissipative forces.

Examples: Frictional force and viscous force.

Potential energy is defined only for conservative forces. If only conservative forces are acting on a system, its mechanical energy should remain constant.

6.3 POTENTIAL ENERGY

The energy associated due to interaction between the particles of same body or between particles of different bodies or the energy associated with the configuration of a system in which conservative force acts is called potential energy. Energy due to interaction between particles of same body is called *self-energy* or *internal potential energy* U_i. Energy due to interaction between particles of different bodies is called *external potential energy* U_e or simply *potential energy*.

In a conservative force field, difference in potential energy between two points is the negative of work done by conservative forces in displacing the body (or system) from some initial position to final position. Hence,

$$\Delta U = -W \text{ or } U_B - U_A = -W_{A \to B}$$

Absolute potential energy at a point can be defined with respect to a reference point where potential energy is assumed to be zero. Negative of work done in displacement of body from reference point (say O) to the point under consideration (say P) is called absolute potential energy at P. Thus, $U_p = -W_{0 \to P}$.

6.3.1 Relation Between Potential Energy (U) and Conservative Force ($\vec{F}$)

1. If U is a function of only one variable, then

$$F = \frac{dU}{dr} = -\text{slope of } U\text{-}r \text{ graph}$$

2. If U is a function of three coordinate variables x, y and z, then

$$\vec{F} = -\left[\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right]$$

The sum of the kinetic energy and potential energy of the body is called mechanical energy. Thus,

$$\text{M.E.} = \text{K.E.} + \text{P.E.}$$

6.3.2 Principle of Conservation of Mechanical Energy

$$W_{\text{NC}} + W_{\text{Other}} = \Delta\text{M.E.}$$

If only conservative forces act on the particle, we have

$$W_{\text{NC}} = 0 \text{ and } W_{\text{Other}} = 0$$

$$\therefore \quad 0 = \Delta\text{M.E. or M.E.} = \text{Constant}$$

6.3.3 Work-energy Theorem

Work done by net force is equal to the change in kinetic energy of the body. This is called **work-energy theorem**.

$$W_{\text{net force}} = K_f - K_i = \Delta\text{K.E.}$$

The work-energy theorem is not independent of Newton's second law. It may be viewed as scalar form of second law.

Work-energy theorem holds in all types of frames; inertial or non-inertial. In non-inertial frame, we have to include the pseudo force in the calculation of the net force.

$$W_{\text{external}} + W_{\text{internal}} + W_{\text{pseudo}} + W_{\text{other}} = \Delta\text{K.E.}$$

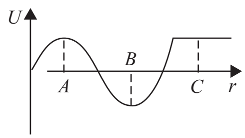
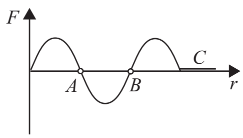
When both external and internal forces act on the system, we can write

$$W_{\text{external}} + W_{\text{internal}} = \Delta\text{K.E.}$$

6.3.4 Types of Equilibrium

For the equilibrium of any body, the net force on it must be zero, that is, $\vec{F}_{\text{net}} = 0$. For the equilibrium of body under conservative forces, we have

$$[F_C]_{\text{net}} = \frac{-dU}{dx} = 0 \quad \text{or} \quad \frac{dU}{dx} = 0$$

Physical Situation	Stable Equilibrium	Unstable Equilibrium	Neutral Equilibrium
(a) Net force	Zero	Zero	Zero
(b) Potential energy	Minimum	Maximum	Constant
(c) When displaced from mean (equilibrium) position.	A restoring nature of force will act on the body, which brings the body back towards mean position.	A force will act which moves the body away from mean position.	Force is again zero
(d) In U-r graph 	At point B	At point A	At point C
(e) In F-r graph 	At point A	At point B	At point C

6.3.5 Circular Motion

In uniform circular motion, a particle has only one acceleration called as centripetal acceleration and in non-uniform circular motion, a particle has two components of particle acceleration:

1. Centripetal acceleration
2. Tangential acceleration

Also, the cause of acceleration is the force and the direction of acceleration is along the direction of the force. Hence, the cause of centripetal acceleration is called as centripetal force (mv^2/R) and the cause of tangential acceleration is called as tangential force (= mdv/dt)

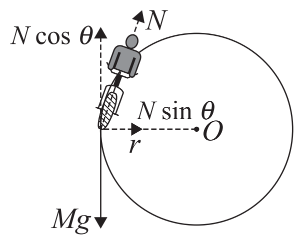
In uniform circular motion, the only force is centripetal force, which acts perpendicular to the velocity. Thus the rate of doing work i.e., power is equal to zero.

In non-uniform circular motion, there are normal and tangential forces.

The rate of doing work, $P = \dfrac{dW}{dt} = (\vec{F}_c + \vec{F}_t) \cdot \vec{v} = F_t v$

If a system is observed w.r.t. rotating N.I.F. and the system is found to be in equilibrium, then a pseudo force is to be applied (It is called centrifugal force). But if the system is found to be in motion with constant speed then two pseudo forces are to be applied—one is called centrifugal force and the other is called Coriolis force.

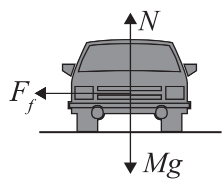
6.3.6 Turning of a Cyclist Around a Corner on the Road



1. When a cyclist turns around a corner on the road, he needs a centripetal force (Mv^2/r). The forces acting on the cyclist are
   (a) Weight Mg
   (b) Normal force N
2. In order to generate the necessary centripetal force, the cyclist bends inwards by an angle θ w.r.t. vertical.
3. In equilibrium,

$$N \cos\theta = Mg \qquad \text{and } N \sin\theta = \frac{Mv^2}{r} \qquad \text{So, } \tan\theta = \frac{v^2}{rg}$$

6.3.7 A Car Taking a Turn on a Level Road



1. When a car takes a turn on a level road, the portion of the turn can be approximated by an arc of a circle of radius r. If the car makes the turn at a constant speed v, then there must be some centripetal force acting on the car. This force is generated by the friction between the tyres and the road.
2. The maximum frictional force is: $F_f = \mu_s N$, where μ_s is the coefficient of static friction. Then, the maximum safe velocity v is such that

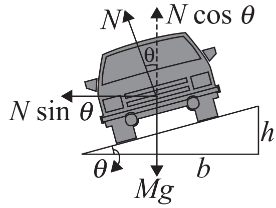
$$\left(\frac{mv^2}{r}\right) = \mu_s N \text{ or } \mu_s = \left(\frac{v^2}{rg}\right) \text{ or } v = \sqrt{\mu_s rg}$$

3. It is important to note that safe velocity is independent of the mass of the car.

6.3.8 Banking of Tracks

1. In order that a vehicle may make a safe and easier turn without depending on friction, roads on large highways are generally banked, i.e., road bend at the curved path is raised a little on the side away from the centre of the curved path.
2. By banking the road, a component of the normal force points towards the centre of curvature of the road. This component supplies the necessary centripetal force required for circular

motion. The vertical component of the normal force is balanced by the weight of the vehicle, i.e.,



$$N\cos\theta = Mg \text{ and } N\sin\theta = \frac{Mv^2}{r}$$

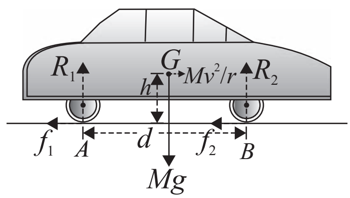
$$\therefore \qquad \tan\theta = \frac{v^2}{rg} \qquad (\text{where } \tan\theta = h/b)$$

3. For a road with angle of banking θ, the speed v at which minimum wear away of tyre takes place is given by

$$v = \sqrt{rg\tan\theta}$$

6.3.9 Stability of a Vehicle on a Horizontal Turn

1. From the point of view of non-inertial frame, if the vehicle does not overturn, then balancing the force, we get



$$R_1 + R_2 = Mg$$

2. Now balancing torques about point B and then about point A we have

$$Mg\frac{d}{2} + \frac{Mv^2}{r}h = R_2 d \text{ and } Mg\frac{d}{2} - \frac{Mv^2}{r}h = R_1 d$$

Thus, normal reaction at the inner wheel (i.e., R_1) is always less than that at the outer wheel (i.e., R_2) when making the circular turn.

3. Further, if v is such that R_1 becomes zero, then the vehicle has a tendency to overturn, i.e., the inner wheel loses contact and the vehicle overturns outwards. Thus, the maximum safe velocity for not overturning is

$$v = \sqrt{\left(\frac{grd}{2h}\right)}$$

4. The frictional forces f_1 and f_2 provide the necessary centripetal force, i.e., $f_1 + f_2 = \left(\frac{Mv^2}{r}\right)$. The safe speed for not skidding is such that

$$f_1 + f_2 \le \mu(R_1 + R_2) \text{ or } v < \sqrt{\mu rg}$$

6.3.10 Conical Pendulum

1. If a small body of mass m tied to a string is whirled in a horizontal circle, the string will not remain horizontal [as a vertical force mg cannot be balanced by a horizontal force (T)] but

the string becomes inclined to the vertical and sweeps a cone while the body moves on a horizontal circle with uniform speed. Such an arrangement is called conical pendulum.

2. In case of conical pendulum, the vertical component of tension balances the weight while its horizontal component provides the necessary centripetal force, i.e.,

$$T\cos\theta = mg \text{ and } T\sin\theta = \frac{mv^2}{r} \text{ or } \tan\theta = \frac{v^2}{rg} \tag{1}$$

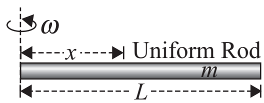
Also,
$$T = m\sqrt{g^2 + \left(\frac{v^2}{r}\right)^2} \tag{2}$$

Hence, $v = \sqrt{rg\tan\theta}$ i.e., $\omega = \sqrt{\left(\frac{g\tan\theta}{r}\right)}$ (3)

Hence, time period
$$t = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{g\tan\theta}} = 2\pi\sqrt{\frac{L\cos\theta}{g}} \tag{4}$$

3. Time period t is independent of the mass of the body and depends on $L\cos\theta$ (= h), i.e., distance between point of suspension and centre of circle.
4. If $\theta = 90°$, the pendulum becomes horizontal and it follows from equations (1), (2) and (4) that $v = \infty$, $T = \infty$ and $t = 0$ which is practically impossible.

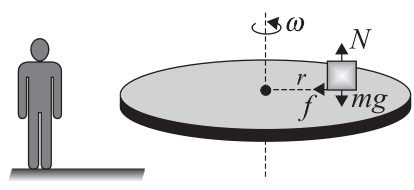
(a) The given rod is rotating uniformly about one end. The variation of tension along its length is $T = \frac{m\omega^2}{2L}(L^2 - x^2)$



(b) A metal ring of mass m and radius R is placed on a smooth horizontal table and is set rotating about its own axis in such a way that each part of the ring moves with a speed v. The tension in the ring is $T = \frac{mv^2}{2\pi R}$.

6.3.11 Centrifugal Force

Consider a block of mass m placed on the table at a distance r from its centre. Suppose the table rotates with constant angular velocity ω and block remains at rest with respect to table. Let us first analyse the motion of the block relative to an observer on the ground (inertial frame). In this frame, the block is moving in a circle of radius r. It therefore has an acceleration v^2/r towards the centre. The resultant force on the block must be towards the centre and its magnitude is mv^2/r. In this frame, the forces on the block are



1. Weight mg
2. Normal reaction N
3. Frictional force f by the table

Thus, we have

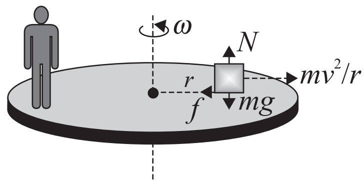
$$N = mg \quad \text{(i)}$$

for circular motion, $$f = \frac{mv^2}{r} \quad \text{(ii)}$$

Now observe the same block in a frame attached with the rotating table. The observer here finds that the block is at rest. Thus the net force on the block in this frame must be zero. The weight and normal reaction balance each other but frictional force, f acts on the block towards the centre of the table to make the resultant zero, a pseudo force must be assumed which acts on the block away radially outward and has a magnitude mv^2/r. This pseudo force is called centrifugal force.

In this frame, the forces on the block are

1. Weight mg
2. Normal reaction N
3. Frictional force f
4. Centrifugal force $\frac{mv^2}{r}$



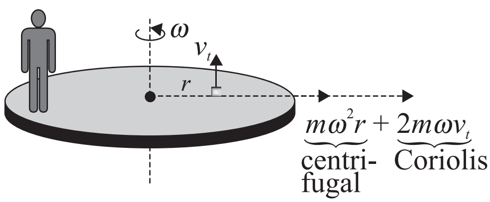
6.3.12 Coriolis Force

The force named after French mathematician G. Coriolis.

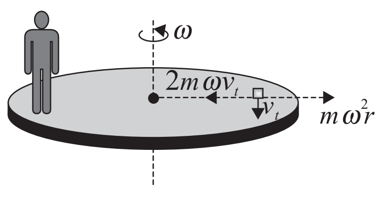
Consider a particle moving with a uniform tangential speed v with respect to a rotating table. The angular velocity of rotation of the table is ω and particle is at a distance r from the centre of the table.

1. If table was not rotating ($\omega = 0$) the particle has the only force, $F = \frac{mv_t^2}{r}$ in inertial frame.

   Thus due to rotation of table the particle experiences a pseudo force ($m\omega^2 r + 2\,m\omega v_t$).
   If particle is at rest w.r.t. table, $v_t = 0$. Then the only pseudo force is $m\omega^2 r$.



Thus on a moving particle on a rotating table an extra pseudo force $2m\omega v_t$ comes to act, is called Coriolis force. Its direction is perpendicular to the direction of v_t. As it is clear from the expression, $F_{\text{Coriolis}} = 2m\omega v$, Coriolis force does not depend on the position of particle but depends on its speed.



2. Particle moving with uniform radial velocity v_r with respect to rotating table. Here we have centrifugal force $m\omega^2 r$ radially outward and Coriolis force $2m\omega v_r$ perpendicular to v_r.