CHAPTER

08

Matrices

J. J. Sylvester was the first to use the word ‘‘Matrix” in 1850 and later on in 1858 Arthur Cayley developed the theory of matrices in a systematic way. ‘Matrices’ is a powerful tool in mathematics and its study is becoming important day by day due to its wide applications in almost every branch of science. This mathematical tool is not only used in certain branches of sciences but also in genetics, economics, sociology, modern psychology and industrial management.

Session 1

Definition, Types of Matrices, Difference Between a Matrix and a Determinant, Equal Matrices, Operations of Matrices, Various Kinds of Matrices

Definition

A set of mn numbers (real or complex) arranged in the form of a rectangular array having m rows and n columns is called a matrix of order $m \times n$ or an $m \times n$ matrix (which is read as m by n matrix).

An $m \times n$ matrix is usually written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{m_1} & a_{m_2} & a_{m_3} & \dots & a_{mn} \end{bmatrix}$$

In a compact form the above matrix is represented by $[a_{ij}]$, $i = 1, 2, 3, \dots, m,\ j = 1, 2, 3, \dots, n$ or simply by $[a_{ij}]_{m \times n}$, where the symbols a_{ij} represent any numbers (a_{ij} lies in the ith row (from top) and jth column (from left)).

Notations A matrix is denoted by capital letter such as A, B, C, ..., X, Y, Z.

Note

1. A matrix may be represented by the symbols $[a_{ij}], (a_{ij}), \| a_{ij} \|$ or by a single capital letter A (say)

   $A = [a_{ij}]_{m \times n}$ or $(a_{ij})_{m \times n}$ or $\| a_{ij} \|$

   Generally, the first system is adopted.
2. The numbers a_{11}, a_{12}, ..., etc., of rectangular array are called the elements or entries of the matrix.
3. A matrix is essentially an arrangement of elements and has no value.
4. The plural of ‘matrix’ is ‘matrices’.

Example 1. If a matrix has 12 elements, what are the possible orders it can have? What will be the possible orders if it has 7 elements?

Sol. We know that, if a matrix is of order $m \times n$, it has mn elements. Thus, to find all possible orders of a matrix with 12 elements, we will find all ordered pairs of natural numbers, whose product is 12.

Thus, all possible ordered pairs are (1, 12), (12, 1), (2, 6), (6, 2), (3, 4), (4, 3).

Hence, possible orders are 1×12, 12×1, 2×6, 6×2, 3×4 and 4×3.

If the matrix has 7 elements, then the possible orders will be 1×7 and 7×1.

Example 2. Construct a 2×3 matrix $A = [a_{ij}]$, whose elements are given by

(i) $a_{ij} = \dfrac{(i + 2j)^2}{2}$. (ii) $a_{ij} = \dfrac{1}{2} | 2i - 3j |$.

(iii) $a_{ij} = \begin{cases} i - j, & i \ge j \\ i + j, & i < j \end{cases}$.

(iv) $a_{ij} = \left[\dfrac{i}{j} \right]$,

where [.] denotes the greatest integer function.

(v) $a_{ij} = \left\{ \dfrac{2i}{3j} \right\}$,

where {.} denotes the fractional part function.

(vi) $a_{ij} = \left(\dfrac{3i + 4j}{2} \right)$,

where (.) denotes the least integer function.

Sol. We have, $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{bmatrix}_{2\times 3}$

(i) Since, $a_{ij}=\dfrac{(i+2j)^2}{2}$, therefore

$$a_{11}=\frac{(1+2)^2}{2}=\frac{9}{2}, a_{12}=\frac{(1+4)^2}{2}=\frac{25}{2},$$

$$a_{13}=\frac{(1+6)^2}{2}=\frac{49}{2}, a_{21}=\frac{(2+2)^2}{2}=8,$$

$$a_{22}=\frac{(2+4)^2}{2}=18 \text{ and } a_{23}=\frac{(2+6)^2}{2}=32$$

Hence, the required matrix is $A=\begin{bmatrix} \frac{9}{2} & \frac{25}{2} & \frac{49}{2} \\ 8 & 18 & 32\end{bmatrix}$.

(ii) Since, $a_{ij}=\dfrac{1}{2}|2i-3j|$, therefore

$$a_{11}=\frac{1}{2}|2-3|=\frac{1}{2}|-1|=\frac{1}{2},$$

$$a_{12}=\frac{1}{2}|2-6|=\frac{1}{2}|-4|=\frac{4}{2}=2,$$

$$a_{13}=\frac{1}{2}|2-9|=\frac{1}{2}|-7|=\frac{7}{2},$$

$$a_{21}=\frac{1}{2}|2-3|=\frac{1}{2}|-1|=\frac{1}{2},$$

$$a_{22}=\frac{1}{2}|2-6|=\frac{1}{2}|-4|=\frac{4}{2}=2$$

and $a_{23}=\dfrac{1}{2}|4-9|=\dfrac{1}{2}|-5|=\dfrac{5}{2}$

Hence, the required matrix is $A=\begin{bmatrix} \frac{1}{2} & 2 & \frac{7}{2} \\ \frac{1}{2} & 2 & \frac{5}{2}\end{bmatrix}$.

(iii) Since, $a_{ij}=\begin{cases} i-j, & i\ge j \\ i+j, & i<j\end{cases}$, therefore

$$a_{11}=1-1=0, a_{12}=1+2=3, a_{13}=1+3=4,$$
$$a_{21}=2-1=1, a_{22}=2-2=0 \text{ and } a_{23}=2+3=5$$

Hence, the required matrix is

$$A=\begin{bmatrix} 0 & 3 & 4 \\ 1 & 0 & 5\end{bmatrix}$$

(iv) Since, $a_{ij}=\left[\dfrac{i}{j}\right]$, therefore $\qquad [\because [x]\le x]$

$$a_{11}=\left[\frac{1}{1}\right]=[1]=1, a_{12}=\left[\frac{1}{2}\right]=[0.5]=0,$$

$$a_{13}=\left[\frac{1}{3}\right]=[0.33]=0, a_{21}=\left[\frac{2}{1}\right]=[2]=2$$

and $a_{22}=\left[\dfrac{2}{2}\right]=[1]=1$ and $a_{23}=\left[\dfrac{2}{3}\right]=[0.67]=0$

Hence, the required matrix is $A=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0\end{bmatrix}$

(v) Since, $a_{ij}=\left\{\dfrac{2i}{3j}\right\}$, therefore $\qquad [\because 0\le\{x\}<1]$

$$a_{11}=\left\{\frac{2}{3}\right\}=\frac{2}{3}, a_{12}=\left\{\frac{2}{6}\right\}=\left\{\frac{1}{3}\right\}=\frac{1}{3},$$

$$a_{13}=\left\{\frac{2}{9}\right\}=\frac{2}{9}, a_{21}=\left\{\frac{4}{3}\right\}=\left\{1+\frac{1}{3}\right\}=\frac{1}{3},$$

$$a_{22}=\left\{\frac{4}{6}\right\}=\left\{\frac{2}{3}\right\}=\frac{2}{3} \text{ and } a_{23}=\left\{\frac{4}{9}\right\}=\frac{4}{9}$$

Hence, the required matrix is $A=\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{9} \\ \frac{1}{3} & \frac{2}{3} & \frac{4}{9}\end{bmatrix}$.

(vi) Since, $a_{ij}=\left(\dfrac{3i+4j}{2}\right)$, therefore $\qquad [\because (x)\ge x]$

$$a_{11}=\left(\frac{3+4}{2}\right)=\left(\frac{7}{2}\right)=(3.5)=4,$$

$$a_{12}=\left(\frac{3+8}{2}\right)=\left(\frac{11}{2}\right)=(5.5)=6,$$

$$a_{13}=\left(\frac{3+12}{2}\right)=\left(\frac{15}{2}\right)=(7.5)=8,$$

$$a_{21}=\left(\frac{6+4}{2}\right)=\left(\frac{10}{2}\right)=(5)=5,$$

$$a_{22}=\left(\frac{6+8}{2}\right)=\left(\frac{14}{2}\right)=(7)=7$$

and $a_{23}=\left(\dfrac{6+12}{2}\right)=\left(\dfrac{18}{2}\right)=(9)=9$

Hence, the required matrix is

$$A=\begin{bmatrix} 4 & 6 & 8 \\ 5 & 7 & 9\end{bmatrix}$$

Types of Matrices

1. Row Matrix or Row Vector

A matrix is said to be row matrix or row vector, if it contains only one row, i.e. a matrix $A=[a_{ij}]_{m\times n}$ is said to be row matrix, if $m=1$.

For example,

(i) $A=[a_{11} \quad a_{12} \quad a_{13} \quad \dots \quad a_{1n}]_{1\times n}$

(ii) $B=[3 \quad 5 \quad -7 \quad 9]_{1\times 4}$

are called row matrices.

2. Column Matrix or Column Vector

A matrix is said to be column matrix or column vector, if it contains only one column, i.e., a matrix $A=[a_{ij}]_{m\times n}$ is said to be column matrix, if $n=1$. *For example,*

$$\text{(i) } A=\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{mn} \end{bmatrix}_{m\times 1} \quad \text{(ii) } B=\begin{bmatrix} 7 \\ 0 \\ -8 \\ 2 \\ 1 \end{bmatrix}_{5\times 1}$$

are called column matrices.

3. Rectangular Matrix

A matrix is said to be rectangular matrix, if the number of rows and the number of columns are not equal i.e., a matrix $A=[a_{ij}]_{m\times n}$ is called a rectangular matrix, iff $m \neq n$. *For example,*

$$\text{(i) } A=\begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 0 & -3 & 8 \\ 7 & 4 & 2 & 5 \end{bmatrix}_{3\times 4} \quad \text{(ii) } B=\begin{bmatrix} 2 & -3 \\ 3 & 0 \\ 4 & 8 \end{bmatrix}_{3\times 2}$$

are called rectangular matrices.

4. Square Matrix

A matrix is said to be a square matrix, if the number of rows and the number of columns are equal i.e., a matrix $A=[a_{ij}]_{m\times n}$ is called a square matrix, iff $m=n$.

For example,

$$\text{(i) } A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \quad \text{(ii) } B=\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$$

are called square matrices.

Remark

If $A=[a_{ij}]$ is a square matrix of order n, then elements (entries) $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$ are said to constitute the *diagonal* of the matrix A. The line along which the diagonal elements lie is called principal or leading diagonal. Thus, if $A=\begin{bmatrix} 1 & 4 & 0 \\ 8 & 3 & -2 \\ 9 & 2 & 5 \end{bmatrix}$, then the elements of the diagonal of A are 1, 3, 5.

5. Diagonal Matrix

A square matrix is said to be a diagonal matrix, if all its non-diagonal elements are zero. Thus, $A=[a_{ij}]_{n\times n}$ is called a diagonal matrix, if $a_{ij}=0$, when $i \neq j$.

For example,

$$\text{(i) } A=[2] \quad \text{(ii) } B=\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{(iii) } C=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

are diagonal matrices of order 1, 2 and 3, respectively. A diagonal matrix of order n having $d_1, d_2, d_3, \dots, d_n$ as diagonal elements may be denoted by diag $(d_1, d_2, d_3, \dots, d_n)$.

Thus, $A=$ diag (2), $B=$ diag (-1, 2) and $C=$ diag (3, 5, 7).

Remark

(i) No element of principal diagonal in a diagonal matrix is zero.

(ii) Minimum number of zero in a diagonal matrix is given by $n(n-1)$, where n is order of matrix.

6. Scalar Matrix

A diagonal matrix is said to be a scalar matrix, if its diagonal elements are equal. Thus, $A=[a_{ij}]_{n\times n}$ is called scalar matrix, if

$$a_{ij}=\begin{cases} 0, \text{ if } i \neq j \\ k, \text{ if } i=j \end{cases}, \text{ where } k \text{ is scalar.}$$

For example,

$$\text{(i) } [7] \quad \text{(ii) } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{(iii) } \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

are scalar matrices of order 1, 2 and 3, respectively. They can be written as diag (7), diag (2, 2) and diag (5, 5, 5), respectively.

7. Unit or Identity Matrix

A diagonal matrix is said to be an identity matrix, if its diagonal elements are equal to 1.

Thus, $A=[a_{ij}]_{n\times n}$ is called unit or identity matrix, if

$$a_{ij}=\begin{cases} 0, \text{ if } i \neq j \\ 1, \text{ if } i=j \end{cases}$$

A unit matrix of order n is denoted by I_n or I. *For example,*

$$\text{(i) } I_1=[1] \quad \text{(ii) } I_2=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(iii) } I_3=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are identity matrices of order 1, 2 and 3, respectively.

8. Singleton Matrix

A matrix is said to be singleton matrix, if it has only one element i.e. a matrix $A=[a_{ij}]_{m\times n}$ is said to be singleton matrix, if $m=n=1$.

For example, $[3], [k], [-2]$ are singleton matrices.

9. Triangular Matrix

A square matrix is called a triangular matrix, if its each element above or below the principal diagonal is zero. It is of two types:

(a) **Upper Triangular Matrix** A square matrix in which all elements below the principal diagonal are zero is called an upper triangular matrix i.e., a matrix $A=[a_{ij}]_{n\times n}$ is said to be an upper triangular matrix, if $a_{ij}=0$, when $i > j$.

For example,

(i) $\begin{bmatrix} 3 & -2 & 4 & 1 \\ 0 & 2 & -3 & 2 \\ 0 & 0 & 7 & 5 \\ 0 & 0 & 0 & 8 \end{bmatrix}_{4 \times 4}$

(ii) $\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}_{5 \times 5}$

are upper triangular matrices.

(b) **Lower Triangular Matrix** A square matrix in which all elements above the principal diagonal are zero is called a lower triangular matrix i.e., a matrix $A = [a_{ij}]_{n \times n}$ is said to be a lower triangular matrix, if $a_{ij} = 0$, when $i < j$. *For example,*

(i) $\begin{bmatrix} 7 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & 3 & 4 \end{bmatrix}_{3 \times 3}$ (ii) $\begin{bmatrix} 10 & 0 & 0 & 0 \\ 8 & 9 & 0 & 0 \\ 5 & 6 & 7 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}_{4 \times 4}$

are lower triangular matrices.

Note

Minimum number of zeroes in a triangular matrix is given by $\frac{n(n-1)}{2}$, where n is order of matrix.

10. Horizontal Matrix

A matrix is said to be horizontal matrix, if the number of rows is less than the number of columns i.e., a matrix $A = [a_{ij}]_{m \times n}$ is said to horizontal matrix, iff $m < n$.

For example, $A = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 8 & 9 & 7 & -2 \\ 2 & -2 & -3 & 4 \end{bmatrix}_{3 \times 4}$ is a horizontal matrix. [$\because$ number of rows (3) < number of columns (4)]

11. Vertical Matrix

A matrix is said to be vertical matrix, if the number of rows is greater than the number of columns i.e., a matrix $A = [a_{ij}]_{m \times n}$ is said to vertical matrix, iff $m > n$.

For example, $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 7 \\ 3 & 5 & 4 \\ 2 & 7 & 9 \\ -1 & 2 & -5 \end{bmatrix}_{5 \times 3}$ is a vertical matrix.

[$\because$ number of rows (5) > number of columns (3)]

12. Null Matrix or Zero Matrix

A matrix is said to be null matrix or zero matrix, if all elements are zero i.e., a matrix $A = [a_{ij}]_{m \times n}$ is said to be a zero or null matrix, iff $a_{ij} = 0, \forall\, i, j$. It is denoted by O.

For example,

(i) $O_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $O_{3 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

are called the null matrices.

13. Sub-Matrix

A matrix which is obtained from a given matrix by deleting any number of rows and number of columns is called a sub-matrix of the given matrix.

For example, $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$ is a sub-matrix of $\begin{bmatrix} 8 & 9 & 5 \\ 2 & 3 & 4 \\ 3 & -2 & 5 \end{bmatrix}$.

14. Trace of a Matrix

The sum of all diagonal elements of a square matrix $A = [a_{ij}]_{n \times n}$ (say) is called the **trace** of a matrix A and is denoted by Tr (A).

Thus, $$\text{Tr}(A) = \sum_{i=1}^{n} a_{ii}$$

For example, If $A = \begin{bmatrix} 2 & -7 & 9 \\ 0 & 3 & 2 \\ 8 & 9 & 4 \end{bmatrix}$, then

$$\text{Tr}(A) = 2 + 3 + 4 = 9$$

Properties of Trace of a Matrix

Let $A = [a_{ij}]_{n \times n}$, $B = [b_{ij}]_{n \times n}$ and k is a scalar, then

(i) $\text{Tr}(kA) = k \cdot \text{Tr}(A)$

(ii) $\text{Tr}(A \pm B) = \text{Tr}(A) \pm \text{Tr}(B)$

(iii) $\text{Tr}(AB) = \text{Tr}(BA)$

(iv) $\text{Tr}(A) = \text{Tr}(A')$

(v) $\text{Tr}(I_n) = n$

(vi) $\text{Tr}(AB) \neq \text{Tr}(A)\, \text{Tr}(B)$

(vii) $\text{Tr}(A) = \text{Tr}(C\, A\, C^{-1})$,

where C is a non-singular square matrix of order n.

15. Determinant of Square Matrix

Let $A = [a_{ij}]_{n \times n}$ be a matrix. The determinant formed by the elements of A is said to be the determinant of matrix A. This is denoted by $|A|$.

For example,

If $A = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 2 & -3 & 5 \end{bmatrix}$, then $|A| = \begin{vmatrix} 3 & 4 & 5 \\ 6 & 7 & 8 \\ 2 & -3 & 5 \end{vmatrix} = -39$.

Remark

1. If $A_1, A_2, A_3, \ldots, A_n$ are square matrices of the same order, then $|A_1 A_2 A_3 \ldots A_n| = |A_1||A_2||A_3| \ldots |A_n|$.
2. If k is a scalar and A is a square matrix of order n, then

$$|kA| = k^n |A|$$

16. Comparable Matrices

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are said to be comparable, if $m = p$ and $n = q$.

For example,

The matrices $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ and $\begin{bmatrix} p & q & r \\ s & t & u \end{bmatrix}$ are comparable but the matrices $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 & 6 \\ 5 & 3 & 1 \end{bmatrix}$ are not comparable.

Difference Between a Matrix and a Determinant

(i) A matrix cannot be reduced to a number but determinant can be reduced to a number.

(ii) The number of rows may or may not be equal to the number of columns in matrices but in determinant the number of rows is equal to the number of columns.

(iii) On interchanging the rows and columns, a different matrix is formed but in determinant it does not change the value.

(iv) A square matrix A such that $|A| \neq 0$, is called a non-singular matrix. If $|A| = 0$, then the matrix A is called a singular matrix.

(v) Matrices represented by [ ], ( ), $\| \ \|$ but determinant is represented by $| \ |$.

Equal Matrices

Two matrices are said to be equal, if

(i) they are of the same order i.e., if they have same number of rows and columns.

(ii) the elements in the corresponding positions of the two matrices are equal.

Thus, if $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{p \times q}$, then $A = B$, iff

(i) $m = p, n = q$ (ii) $a_{ij} = b_{ij}, \forall\ i, j$

For example, If $A = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 0 & 5 \end{bmatrix}_{2 \times 3}$ and

$B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{2 \times 3}$ are equal matrices, then

$a = -1, b = 2, c = 4 d = 3, e = 0, f = 5$

Example 3. If $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4w-8 \end{bmatrix} = \begin{bmatrix} -x-1 & 0 \\ 3 & 2w \end{bmatrix}$, then find the value of $|x+y| + |z+w|$.

Sol. As the given matrices are equal so their corresponding elements are equal.

$$x + 3 = -x - 1 \Rightarrow 2x = -4 \quad \text{...(i)}$$

$\therefore \quad x = -2$

$2y + x = 0$

$\Rightarrow \quad 2y - 2 = 0$ [from Eq. (i)]

$\Rightarrow \quad y = 1$ ...(ii)

$z - 1 = 3$

$\Rightarrow \quad z = 4$ ...(iii)

$4w - 8 = 2w$

$\Rightarrow \quad 2w = 8$

$\therefore \quad w = 4$ ...(iv)

Hence, $|x + y| + |z + w| = |-2 + 1| + |4 + 4| = 1 + 8 = 9$

Example 4. If $\begin{bmatrix} 2\alpha + 1 & 3\beta \\ 0 & \beta^2 - 5\beta \end{bmatrix} = \begin{bmatrix} \alpha + 3 & \beta^2 + 2 \\ 0 & -6 \end{bmatrix}$, find the equation whose roots are α and β.

Sol. The given matrices will be equal, iff

$2\alpha + 1 = \alpha + 3 \Rightarrow \alpha = 2$

$3\beta = \beta^2 + 2 \Rightarrow \beta^2 - 3\beta + 2 = 0$

$\therefore \quad \beta = 1, 2$ and $\beta^2 - 5\beta = -6$ ...(i)

$\Rightarrow \quad \beta^2 - 5\beta + 6 = 0$

$\therefore \quad \beta = 2, 3$ ...(ii)

From Eqs. (i) and (ii), we get $\beta = 2$

$\Rightarrow \quad \alpha = 2, \beta = 2$

$\therefore$ Required equation is $x^2 - (2 + 2)x + 2 \cdot 2 = 0$

$\Rightarrow \quad x^2 - 4x + 4 = 0$

Operations of Matrices

Addition of Matrices

Let A, B be two matrices, each of order $m \times n$. Then, their sum $A + B$ is a matrix of order $m \times n$ and is obtained by adding the corresponding elements of A and B.

Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$, then $A+B=[a_{ij}+b_{ij}]_{m\times n}, \forall\, i, j$

Example 5. Given, $A=\begin{bmatrix}1 & 3 & 5\\ -2 & 0 & 2\\ 0 & 4 & -3\end{bmatrix}, B=\begin{bmatrix}0 & 3\\ -2 & 0\\ 0 & -4\end{bmatrix}$

and $C=\begin{bmatrix}4 & 1 & -2\\ 3 & 2 & 1\\ 2 & -1 & 7\end{bmatrix}$. Find (whichever defined)

(i) $A+B$. (ii) $A+C$.

Sol. (i) Given, A is a matrix of the type 3×3 and B is a matrix of the type 3×2.
Since, A and B are not of the same type.
$\therefore$ Sum $A+B$ is not defined.

` (ii) As A and C are two matrices of the same type, therefore the sum $A+C$ is defined.

$$\therefore\ A+C=\begin{bmatrix}1 & 3 & 5\\ -2 & 0 & 2\\ 0 & 4 & -3\end{bmatrix}+\begin{bmatrix}4 & 1 & -2\\ 3 & 2 & 1\\ 2 & -1 & 7\end{bmatrix}$$

$$=\begin{bmatrix}1+4 & 3+1 & 5-2\\ -2+3 & 0+2 & 2+1\\ 0+2 & 4-1 & -3+7\end{bmatrix}=\begin{bmatrix}5 & 4 & 3\\ 1 & 2 & 3\\ 2 & 3 & 4\end{bmatrix}$$

Example 6. If $a,b;b,c$ and c,a are the roots of $x^2-4x+3=0$, $x^2-8x+15=0$ and $x^2-6x+5=0$, respectively. Compute $\begin{bmatrix}a^2+c^2 & a^2+b^2\\ b^2+c^2 & a^2+c^2\end{bmatrix}+\begin{bmatrix}2ac & -2ab\\ -2bc & -2ac\end{bmatrix}$.

Sol. $\because$ $x^2-4x+3=0$

$\Rightarrow$ $(x-1)(x-3)=0$ $\therefore$ $x=1, 3$

$x^2-8x+15=0$

$\Rightarrow$ $(x-3)(x-5)=0$ $\therefore$ $x=3, 5$

and $x^2-6x+5=0$

$\Rightarrow$ $(x-5)(x-1)=0$ $\therefore$ $x=5, 1$

It is clear that $a=1$, $b=3$ and $c=5$

Now, $\begin{bmatrix}a^2+c^2 & a^2+b^2\\ b^2+c^2 & a^2+c^2\end{bmatrix}+\begin{bmatrix}2ac & -2ab\\ -2bc & -2ac\end{bmatrix}$

$$=\begin{bmatrix}a^2+c^2+2ac & a^2+b^2-2ab\\ b^2+c^2-2bc & a^2+c^2-2ac\end{bmatrix}=\begin{bmatrix}(a+c)^2 & (a-b)^2\\ (b-c)^2 & (a-c)^2\end{bmatrix}$$

$$=\begin{bmatrix}(1+5)^2 & (1-3)^2\\ (3-5)^2 & (1-5)^2\end{bmatrix}=\begin{bmatrix}36 & 4\\ 4 & 16\end{bmatrix}$$

Properties of Matrix Addition

Property 1 Addition of matrices is commutative,

i.e. $A+B=B+A$

where A and B are any two $m\times n$ matrices, i.e. matrices of the same order.

Property 2 Addition of matrices is associative

i.e. $(A+B)+C=A+(B+C)$

where A, B and C are any three matrices of the same order $m\times n$ (say).

Property 3 Existence of additive identity

i.e. $A+O=A=O+A$

where A be any $m\times n$ matrix and O be the $m\times n$ null matrix. The null matrix O is the identity element for matrix addition.

Property 4 Existence of additive inverse

If A be any $m\times n$ matrix, then there exists another $m\times n$ matrix B, such that $A+B=O=B+A$

where O is the $m\times n$ null matrix.

Here, the matrix B is called the additive inverse of the matrix A or the negative of A.

Property 5 Cancellation laws

If A, B and C are matrices of the same order $m\times n$ (say),

then $A+B=A+C\Rightarrow B=C$ [left cancellation law]

and $B+A=C+A\Rightarrow B=C$ [right cancellation law]

Scalar Multiplication

Let $A=[a_{ij}]_{m\times n}$ be a matrix and k be any number called a scalar. Then, the matrix obtained by multiplying every element of A by k is called the scalar multiple of A by k and is denoted by kA.

Thus, $kA=[ka_{ij}]_{m\times n}$

Properties of Scalar Multiplication

If $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$ are two matrices and k, l are scalars , then

(i) $k(A+B)=kA+kB$ (ii) $(k+l)\,A=kA+lA$

(iii) $(kl)A=k(lA)=l(kA)$

(iv) $(-k)A=-(kA)=k(-A)$

(v) $1A=A, (-1)A=-A$

Example 7. Determine the matrix A,

when $A=4\begin{bmatrix}1 & 2 & 3\\ -1 & -2 & -3\\ 4 & 2 & 6\end{bmatrix}+2\begin{bmatrix}5 & 4 & 1\\ 3 & 2 & 4\\ 3 & 8 & 2\end{bmatrix}$.

Sol. $A=\begin{bmatrix}4 & 8 & 12\\-4 & -8 & -12\\16 & 8 & 24\end{bmatrix}+\begin{bmatrix}10 & 8 & 2\\6 & 4 & 8\\6 & 16 & 4\end{bmatrix}$

$$=\begin{bmatrix}4+10 & 8+8 & 12+2\\-4+6 & -8+4 & -12+8\\16+6 & 8+16 & 24+4\end{bmatrix}=\begin{bmatrix}14 & 16 & 14\\2 & -4 & -4\\22 & 24 & 28\end{bmatrix}$$

Example 8. If $A=\begin{bmatrix}0 & 2\\3 & -4\end{bmatrix}$ and $kA=\begin{bmatrix}0 & 3a\\2b & 24\end{bmatrix}$, then find the value of $b-a-k$.

Sol. We have, $A=\begin{bmatrix}0 & 2\\3 & -4\end{bmatrix}\Rightarrow kA=\begin{bmatrix}0 & 2k\\3k & -4k\end{bmatrix}$

But $kA=\begin{bmatrix}0 & 3a\\2b & 24\end{bmatrix}$

$\therefore$ $\begin{bmatrix}0 & 2k\\3k & -4k\end{bmatrix}=\begin{bmatrix}0 & 3a\\2b & 24\end{bmatrix}$

$\Rightarrow$ $2k=3a, 3k=2b, -4k=24$

$\Rightarrow$ $k=-6, a=-4, b=-9$

Hence, $b-a-k=-9-(-4)-(-6)=-9+4+6=1$

Subtraction of Matrices

Let A, B be two matrices, each of order $m\times n$. Then, their subtraction $A-B$ is a matrix of order $m\times n$ and is obtained by subtracting the corresponding elements of A and B. Thus, if $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{m\times n}$,

then $A-B=[a_{ij}-b_{ij}]_{m\times n}, \forall\ i, j$

For example, If $A=\begin{bmatrix}2 & 3\\4 & 5\\6 & 7\end{bmatrix}$ and $B=\begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}$,

then $A-B=\begin{bmatrix}2 & 3\\4 & 5\\6 & 7\end{bmatrix}-\begin{bmatrix}a & b\\c & d\\e & f\end{bmatrix}=\begin{bmatrix}2-a & 3-b\\4-c & 5-d\\6-e & 7-f\end{bmatrix}$

Example 9. Given, $A=\begin{bmatrix}1 & 2 & -3\\5 & 0 & 2\\1 & -1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}3 & -1 & 2\\4 & 2 & 5\\2 & 0 & 3\end{bmatrix}$. Find the matrix C such that $A+2C=B$.

Sol. Given, $A+2C=B$

$$2C=B-A=\begin{bmatrix}3 & -1 & 2\\4 & 2 & 5\\2 & 0 & 3\end{bmatrix}-\begin{bmatrix}1 & 2 & -3\\5 & 0 & 2\\1 & -1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}3-1 & -1-2 & 2+3\\4-5 & 2-0 & 5-2\\2-1 & 0+1 & 3-1\end{bmatrix}$$

$\therefore$ $2C=\begin{bmatrix}2 & -3 & 5\\-1 & 2 & 3\\1 & 1 & 2\end{bmatrix}\Rightarrow C=\frac{1}{2}\begin{bmatrix}2 & -3 & 5\\-1 & 2 & 3\\1 & 1 & 2\end{bmatrix}$

$$=\begin{bmatrix}1 & -3/2 & 5/2\\-1/2 & 1 & 3/2\\1/2 & 1/2 & 1\end{bmatrix}$$

Example 10. Solve the following equations for X and Y. $2X-Y=\begin{bmatrix}3 & -3 & 0\\3 & 3 & 2\end{bmatrix}, 2Y+X=\begin{bmatrix}4 & 1 & 5\\-1 & 4 & -4\end{bmatrix}$

Sol. Given, $2X-Y=\begin{bmatrix}3 & -3 & 0\\3 & 3 & 2\end{bmatrix}$

On multiplying both sides by 2, we get

$4X-2Y=2\begin{bmatrix}3 & -3 & 0\\3 & 3 & 2\end{bmatrix}$; $4X-2Y=\begin{bmatrix}6 & -6 & 0\\6 & 6 & 4\end{bmatrix}$ ...(i)

also given $X+2Y=\begin{bmatrix}4 & 1 & 5\\-1 & 4 & -4\end{bmatrix}$ ...(ii)

Adding Eqs. (i) and (ii), we get

$$5X=\begin{bmatrix}6 & -6 & 0\\6 & 6 & 4\end{bmatrix}+\begin{bmatrix}4 & 1 & 5\\-1 & 4 & -4\end{bmatrix}$$

$$=\begin{bmatrix}6+4 & -6+1 & 0+5\\6-1 & 6+4 & 4-4\end{bmatrix}=\begin{bmatrix}10 & -5 & 5\\5 & 10 & 0\end{bmatrix}$$

$\therefore$ $X=\frac{1}{5}\begin{bmatrix}10 & -5 & 5\\5 & 10 & 0\end{bmatrix}\Rightarrow X=\begin{bmatrix}2 & -1 & 1\\1 & 2 & 0\end{bmatrix}$

Putting the value of X in Eq. (ii), we get

$$\begin{bmatrix}2 & -1 & 1\\1 & 2 & 0\end{bmatrix}+2Y=\begin{bmatrix}4 & 1 & 5\\-1 & 4 & -4\end{bmatrix}$$

$\Rightarrow$ $2Y=\begin{bmatrix}4 & 1 & 5\\-1 & 4 & -4\end{bmatrix}-\begin{bmatrix}2 & -1 & 1\\1 & 2 & 0\end{bmatrix}$

$$=\begin{bmatrix}4-2 & 1+1 & 5-1\\-1-1 & 4-2 & -4-0\end{bmatrix}=\begin{bmatrix}2 & 2 & 4\\-2 & 2 & -4\end{bmatrix}$$

$\therefore$ $Y=\begin{bmatrix}1 & 1 & 2\\-1 & 1 & -2\end{bmatrix}$

Hence, $X=\begin{bmatrix}2 & -1 & 1\\1 & 2 & 0\end{bmatrix}$ and $Y=\begin{bmatrix}1 & 1 & 2\\-1 & 1 & -2\end{bmatrix}$

Remark

If two matrices A and B are of the same order, then only their addition and subtraction is possible and these matrices are said to be conformable for addition or subtraction. On the other hand, if the matrices A and B are of different orders, then their addition and subtraction is not possible and these matrices are called non-conformable for addition and subtraction.

Multiplication

Conformable for Multiplication

If A and B be two matrices which are said to be conformable for the product AB. If the number of columns in A (called the pre-factor) is equal to the number of rows in B (called the post-factor) otherwise non-conformable for multiplication. Thus,

(i) AB is defined, if number of columns in A = number of rows in B.

(ii) BA is defined, if number of columns in B = number of rows in A.

Multiplication of Matrices

Let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ be two matrices, then the product AB is defined as the matrix $C=[C_{ij}]_{m\times p}$,

where $C_{ij}=\sum_{j=1}^{n} a_{ij}\, b_{jk}, 1\le i\le m, 1\le k\le p$

$$=a_{i1}b_{1k}+a_{i2}b_{2k}+a_{i3}b_{3k}+\ldots+a_{in}b_{nk}$$

i.e., (i,k) th entry of the product AB is the sum of the product of the corresponding elements of the ith row of A (pre-factor) and kth column of B (post-factor).

Note

In the product AB, $\begin{cases} A = \text{Pre-factor} \\ B = \text{Post-factor}\end{cases}$

Example 11. If $A=\begin{bmatrix}0&1&2\\1&2&3\\2&3&4\end{bmatrix}$ and $B=\begin{bmatrix}1&-2\\-1&0\\2&-1\end{bmatrix}$, obtain the product AB and explain why BA is not defined?

Sol. Here, the number of columns in $A=3=$ the number of rows in B. Therefore, the product AB is defined.

$$AB=\begin{bmatrix}0&1&2\\1&2&3\\2&3&4\end{bmatrix}\begin{matrix}R_1\\R_2\\R_3\end{matrix}\times\overset{\begin{matrix}C_1 & C_2\end{matrix}}{\begin{bmatrix}1&-2\\-1&0\\2&-1\end{bmatrix}}$$

R_1, R_2, R_3 are rows of A and C_1, C_2 are columns of B.

$$\therefore\quad AB=\begin{bmatrix}R_1C_1 & R_1C_2\\ R_2C_1 & R_2C_2\\ R_3C_1 & R_3C_2\end{bmatrix}_{3\times 2}$$

$$\begin{bmatrix}\boxed{0\ 1\ 2}\begin{bmatrix}1\\-1\\2\end{bmatrix} & \boxed{0\ 1\ 2}\begin{bmatrix}-2\\0\\-1\end{bmatrix}\\ \boxed{1\ 2\ 3}\begin{bmatrix}1\\-1\\2\end{bmatrix} & \boxed{1\ 2\ 3}\begin{bmatrix}-2\\0\\-1\end{bmatrix}\\ \boxed{2\ 3\ 4}\begin{bmatrix}1\\-1\\2\end{bmatrix} & \boxed{2\ 3\ 4}\begin{bmatrix}-2\\0\\-1\end{bmatrix}\end{bmatrix}_{3\times 2}$$

For convenience of multiplication we write columns in horizontal rectangles.

$$\begin{bmatrix}\begin{Bmatrix}\boxed{0\ 1\ 2}\\ \boxed{1\ -1\ 2}\end{Bmatrix} & \begin{Bmatrix}\boxed{0\ 1\ 2}\\ \boxed{-2\ 0\ -1}\end{Bmatrix}\\ \begin{Bmatrix}\boxed{1\ 2\ 3}\\ \boxed{1\ -1\ 2}\end{Bmatrix} & \begin{Bmatrix}\boxed{1\ 2\ 3}\\ \boxed{-2\ 0\ -1}\end{Bmatrix}\\ \begin{Bmatrix}\boxed{2\ 3\ 4}\\ \boxed{1\ -1\ 2}\end{Bmatrix} & \begin{Bmatrix}\boxed{2\ 3\ 4}\\ \boxed{-2\ 0\ -1}\end{Bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}0\times1+1\times(-1)+2\times2 & 0\times(-2)+1\times0+2\times(-1)\\ 1\times1+2\times(-1)+3\times2 & 1\times(-2)+2\times0+3\times(-1)\\ 2\times1+3\times(-1)+4\times2 & 2\times(-2)+3\times0+4\times(-1)\end{bmatrix}_{3\times 2}$$

$$=\begin{bmatrix}0-1+4 & 0+0-2\\ 1-2+6 & -2+0-3\\ 2-3+8 & -4+0-4\end{bmatrix}_{3\times 2}=\begin{bmatrix}3&-2\\5&-5\\7&-8\end{bmatrix}_{3\times 2}$$

Since, the number of columns of B is 2 and the number of rows of A is 3, BA is not defined ($\because 2\ne 3$).

Remark

Verification for the product to be correct .

From above example

$$\begin{bmatrix}0&1&2\\1&2&3\\2&3&4\end{bmatrix}\times\begin{bmatrix}1&-2\\-1&0\\2&-1\end{bmatrix}=\begin{bmatrix}3&-2\\5&-5\\7&-8\end{bmatrix}$$

Sum $\underline{3\quad 6\quad 9}$ $\qquad\qquad$ $\underline{15\ \ -15}$

Now, $\boxed{3\ 6\ 9}\ \boxed{\begin{matrix}1\\-1\\2\end{matrix}}$ $\begin{aligned}&=3\times1+6\times(-1)+9\times2\\&=3-6+18\\&=15\end{aligned}$

and $\boxed{3\ 6\ 9}\ \boxed{\begin{matrix}-2\\0\\-1\end{matrix}}$ $\begin{aligned}&=3(-2)+6\times0+9\times(-1)\\&=-6+0-9\\&=-15\end{aligned}$

Example 12. If $A=\begin{bmatrix}0 & -\tan(\alpha/2)\\ \tan(\alpha/2) & 0\end{bmatrix}$ and I is a 2×2 unit matrix, prove that

$$I+A=(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}.$$

Sol. Since, $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and given $A=\begin{bmatrix}0 & -\tan(\alpha/2)\\ \tan(\alpha/2) & 0\end{bmatrix}$

$$\therefore\quad I+A=\begin{bmatrix}1 & -\tan(\alpha/2)\\ \tan(\alpha/2) & 1\end{bmatrix}\quad\ldots\text{(i)}$$

$$\text{RHS}=(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

$$=\begin{bmatrix}1 & \tan(\alpha/2)\\ -\tan(\alpha/2) & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

$$= \begin{bmatrix} 1 & \tan(\alpha/2) \\ -\tan(\alpha/2) & 1 \end{bmatrix} \begin{bmatrix} \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} & \frac{-2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} \\ \frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)} & \frac{1-\tan^2(\alpha/2)}{1+\tan^2(\alpha/2)} \end{bmatrix}$$

Let $\tan(\alpha/2) = \lambda$, then

$$\text{RHS} = \begin{bmatrix} 1 & \lambda \\ -\lambda & 1 \end{bmatrix} \begin{bmatrix} \frac{1-\lambda^2}{1+\lambda^2} & \frac{-2\lambda}{1+\lambda^2} \\ \frac{2\lambda}{1+\lambda^2} & \frac{1-\lambda^2}{1+\lambda^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\lambda^2+2\lambda^2}{1+\lambda^2} & \frac{-2\lambda+\lambda(1-\lambda^2)}{1+\lambda^2} \\ \frac{-\lambda(1-\lambda^2)+2\lambda}{1+\lambda^2} & \frac{2\lambda^2+1-\lambda^2}{1+\lambda^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\lambda^2}{1+\lambda^2} & \frac{-\lambda(1+\lambda^2)}{1+\lambda^2} \\ \frac{\lambda(1+\lambda^2)}{1+\lambda^2} & \frac{1+\lambda^2}{1+\lambda^2} \end{bmatrix} = \begin{bmatrix} 1 & -\lambda \\ \lambda & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\tan(\alpha/2) \\ \tan(\alpha/2) & 1 \end{bmatrix} \quad [\because \lambda = \tan(\alpha/2)]$$

$= I + A$ [from Eq. (i)]

$=$ LHS

Pre-multiplication and Post-multiplication of Matrices

The matrix AB is the matrix B pre-multiplied by A and the matrix BA is the matrix B post-multiplied by A.

Properties of Multiplication of Matrices

Property 1 Multiplication of matrices is not commutative

i.e. $AB \neq BA$

Note

1. If $AB = -BA$, then A and B are said to anti-commute.
2. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, then $AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$.

Observe that multiplication of diagonal matrices of same order will be commutative.

Property 2 Matrix multiplication associative if conformability assumed.

i.e. $A(BC) = (AB)C$

Property 3 Matrix multiplication is distributive with respect to addition. i.e. $A(B+C) = AB + AC$, whenever both sides of equality are defined.

Property 4 If A is an $m \times n$ matrix, then $I_m A = A = A I_n$.

Property 5 If product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

For example,

(i) $$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1\cdot(-1)+1.1 & 1\cdot1+1\cdot(-1) \\ 2\cdot(-1)+2.1 & 2\cdot1+2\cdot(-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

(ii) $$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\cdot0+0\cdot0 & 0\cdot0+0\cdot1 \\ 1\cdot0+0\cdot0 & 1\cdot0+0\cdot1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

None of the matrices on the LHS is a null matrix whereas their product is a null matrix.

Note If A and B are two non-zero matrices such that $AB = 0$, then A and B are called the divisors of zero. Also, if

$AB = 0 \Rightarrow |AB| = 0 \Rightarrow |A||B| = 0$

$\Rightarrow$ $|A| = 0$ or $|B| = 0$ but not the converse.

Property 6 Multiplication of a matrix A by a null matrix conformable with A for multiplication.

For example, If $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}_{3\times2}$ and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}$,

then $AO = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$, which is a 3×3 null matrix.

Property 7 Multiplication of a matrix by itself

The product of $A\,A\,A \ldots m$ times $= A^m$ and $(A^m)^n = A^{mn}$

Note

1. If I be unit matrix, then $I^2 = I^3 = \ldots = I^m = I \; (m \in I_+)$
2. If A and B are two matrices of the same order, then
   (i) $(A+B)^2 = A^2 + AB + BA + B^2$
   (ii) $(A-B)^2 = A^2 - AB - BA + B^2$
   (iii) $(A-B)(A+B) = A^2 + AB - BA + B^2$
   (iv) $(A+B)(A-B) = A^2 - AB + BA - B^2$
   (v) $A(-B) = (-A)(B) = -AB$

Example 13. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$, verify that $(AB)C = A(BC)$ and $A(B+C) = AB + AC$.

Sol. We have, $AB = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1\cdot2+2\cdot2 & 1\cdot1+2\cdot3 \\ (-2)\cdot2+3\cdot2 & (-2)\cdot1+3\cdot3 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \times \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2\cdot(-3)+1\cdot 2 & 2\cdot 1+1\cdot 0 \\ 2\cdot(-3)+3\cdot 2 & 2\cdot 1+3\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6+2 & 2+0 \\ -6+6 & 2+0 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1\cdot(-3)+2\cdot 2 & 1\cdot 1+2\cdot 0 \\ (-2)\cdot(-3)+3\cdot 2 & (-2)\cdot 1+3\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3+4 & 1+0 \\ 6+6 & -2+0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2-3 & 1+1 \\ 2+2 & 3+0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix}$$

Now, $(AB)C = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} \times \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -18+14 & 6+0 \\ -6+14 & 2+0 \end{bmatrix}$

$$= \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix} \quad \text{...(i)}$$

$$A(BC) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+0 & 2+4 \\ 8+0 & -4+6 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix} \quad \text{...(ii)}$$

Thus, from Eqs. (i) and (ii), we get, $(AB)C = A(BC)$

Now, $A(B+C) = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} -1+8 & 2+6 \\ 2+12 & -4+9 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix} \quad \text{...(iii)}$$

and $AB + AC = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 12 & -2 \end{bmatrix} = \begin{bmatrix} 6+1 & 7+1 \\ 2+12 & 7-2 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 8 \\ 14 & 5 \end{bmatrix} \quad \text{...(iv)}$$

Thus, from Eqs. (iii) and (iv), we get

$$A(B+C) = AB + AC$$

Example 14. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$, show that $A^3 = pI + qA + rA^2$.

Sol. We have, $A^2 = A \cdot A$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix}$$

$$\therefore \quad A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$$

$$= \begin{bmatrix} p & q & r \\ pr & p+qr & q+r^2 \\ pq+r^2p & pr+q^2+qr^2 & p+2qr+r^3 \end{bmatrix} \quad \text{...(i)}$$

and $pI + qA + rA^2 = p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + q\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ p & q & r \end{bmatrix}$

$$+ r\begin{bmatrix} 0 & 0 & 1 \\ p & q & r \\ pr & p+qr & q+r^2 \end{bmatrix}$$

$$= \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} 0 & q & 0 \\ 0 & 0 & q \\ pq & q^2 & qr \end{bmatrix} + \begin{bmatrix} 0 & 0 & r \\ pr & qr & r^2 \\ pr^2 & pr+qr^2 & qr+r^3 \end{bmatrix}$$

$$= \begin{bmatrix} p+0+0 & 0+q+0 & 0+0+r \\ 0+0+pr & p+0+qr & 0+q+r^2 \\ 0+pq+pr^2 & 0+q^2+pr+qr^2 & p+qr+qr+r^3 \end{bmatrix}$$

$$= \begin{bmatrix} p & q & r \\ pr & p+qr & q+r^2 \\ pq+pr^2 & q^2+pr+qr^2 & p+2qr+r^3 \end{bmatrix} \quad \text{...(ii)}$$

Thus, from Eqs. (i) and (ii), we get $A^3 = pI + qA + rA^2$

Example 15. ***Find*** x***, so that*** $[1 \ x \ 1]\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = O$.

Sol. We have, $[1 \quad x \quad 1]\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = O$

$$\Rightarrow \quad [1 \quad 5x+6 \quad x+4]\begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = O$$

$$\Rightarrow \quad [1 + 5x + 6 + x^2 + 4x] = O$$

or $\quad x^2 + 9x + 7 = 0$

$$\therefore \quad x = \frac{-9 \pm \sqrt{(81-28)}}{2} \Rightarrow x = \frac{-9 \pm \sqrt{53}}{2}$$

Various Kinds of Matrices

Idempotent Matrix

A square matrix A is called idempotent provided it satisfies the relation $A^2 = A$.

Note

$A^n = A, \forall n \geq 2, n \in N.$

Example 16. Show that the matrix

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \text{ is idempotent.}$$

Sol. $A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 2\cdot 2 + (-2)\cdot(-1) + (-4)\cdot 1 & 2\cdot(-2) + (-2)\cdot 3 + (-4)\cdot(-2) & 2\cdot(-4) + (-2)\cdot 4 + (-4)\cdot(-3) \\ (-1)\cdot 2 + 3\cdot(-1) + 4\cdot 1 & (-1)\cdot(-2) + 3\cdot 3 + 4\cdot(-2) & (-1)\cdot(-4) + 3\cdot 4 + 4\cdot(-3) \\ 1.2 + (-2)\cdot(-1) + (-3)\cdot 1 & 1\cdot(-2) + (-2)\cdot 3 + (-3)\cdot(-2) & 1\cdot(-4) + (-2)\cdot 4 + (-3)\cdot(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence, the matrix A is idempotent.

Periodic Matrix

A square matrix A is called periodic, if $A^{k+1} = A$, where k is a positive integer. If k is the least positive integer for which $A^{k+1} = A$, then k is said to be **period** of A. For $k = 1$, we get $A^2 = A$ and we called it to be **idempotent matrix.**

Note

Period of an idempotent matrix is 1.

Nilpotent Matrix

A square matrix A is called nilpotent matrix of order m provided it satisfies the relation $A^k = O$ and $A^{k-1} \neq O$, where k is positive integer and O is null matrix and k is the order of the nilpotent matrix A.

Example 17. ***Show that*** $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ ***is nilpotent matrix of order 3.***

Sol. Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

$$\therefore A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore \quad A^3 = O$ i.e., $A^k = O$

Here, $\quad k = 3$

Hence, the matrix A is nilpotent of order 3.

Involutory Matrix

A square matrix A is called involutory provided it satisfies the relation $A^2 = I$, where I is identity matrix.

Note $A = A^{-1}$ for an involutory matrix.

Example 18. ***Show that the matrix***

$$A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \text{ is involutory.}$$

Sol. $A^2 = A \cdot A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \times \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, the given matrix A is involutory.

Exercise for Session 1

1. If $A = \begin{bmatrix} \alpha & 2 \\ 2 & \alpha \end{bmatrix}$ and $|A^3| = 125$, α is equal to

(a) ± 2
(b) ± 3
(c) ± 5
(d) 0

2. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$, the value of $a + b$ is

(a) 4
(b) 5
(c) 6
(d) 7

3. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $A^2 - \lambda A - I_2 = O$, then λ is equal to

(a) – 4
(b) – 2
(c) 2
(d) 4

4. Let $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ and $(A + I)^{50} - 50A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the value of $a + b + c + d$, is

(a) 1
(b) 2
(c) 4
(d) None of these

5. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, then $A^2 = I$ is true for

(a) $\theta = 0$
(b) $\theta = \frac{\pi}{4}$
(c) $\theta = \frac{\pi}{2}$
(d) None of these

6. If $\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}$ is to be the square root of two rowed unit matrix, then α, β and γ should satisfy the relation

(a) $1 - \alpha^2 + \beta\gamma = 0$
(b) $\alpha^2 + \beta\gamma - 1 = 0$
(c) $1 + \alpha^2 + \beta\gamma = 0$
(d) $1 - \alpha^2 - \beta\gamma = 0$

7. If $A = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$, then A^{100} is equal to

(a) $\begin{bmatrix} 1 & 0 \\ 25 & 0 \end{bmatrix}$
(b) $\begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 0 \\ (1/2)^{100} & 1 \end{bmatrix}$
(d) None of these

8. If the product of n matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\cdots\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is equal to the matrix $\begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$, the value of n is equal to

(a) 26
(b) 27
(c) 377
(d) 378

9. If A and B are two matrices such that $AB = B$ and $BA = A$, then $A^2 + B^2$ is equal to
(a) $2AB$
(b) $2BA$
(c) $A + B$
(d) AB

Answers

Exercise for Session 1

1. (b) **2.** (b) **3.** (d) **4.** (b) **5.** (a) **6.** (b)
7. (b) **8.** (b) **9.** (c)