Session 4

Harmonic Sequence or Harmonic Progression (HP)

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A Harmonic Progression (HP) is a sequence, if the reciprocals of its terms are in Arithmetic Progression (AP) i.e., $t_1, t_2, t_3, \ldots$ is HP if and only if $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}, \ldots$ is an AP.

For example, The sequence

(i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \ldots$ (ii) $2, \frac{5}{2}, \frac{10}{3}, \ldots$

(iii) $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \ldots$ are HP's.

Remark

1. No term of HP can be zero.
2. The most general or standard HP is

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \ldots$$

Example 52. ***If*** a, b, c ***are in HP, then show that*** $\frac{a-b}{b-c} = \frac{a}{c}$.

Sol. Since, a, b, c are in HP, therefore

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in AP}$$

i.e. $$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

or $$\frac{a-b}{ab} = \frac{b-c}{bc} \text{ or } \frac{a-b}{b-c} = \frac{a}{c}$$

Remark

A HP may also be defined as a series in which every three consecutive terms (say I, II, III) satisfy $\frac{\text{I}-\text{II}}{\text{II}-\text{III}} = \frac{\text{I}}{\text{III}}$ this relation.

Example 53. Find the first term of a HP whose second term is $\frac{5}{4}$ and the third term is $\frac{1}{2}$.

Sol. Let a be the first term. Then, $a, \frac{5}{4}, \frac{1}{2}$ are in HP.

Then, $$\frac{a - \frac{5}{4}}{\frac{5}{4} - \frac{1}{2}} = \frac{a}{\frac{1}{2}} \quad \text{[from above note]}$$

$$\Rightarrow \frac{4a-5}{5-2} = 2a$$

$$\Rightarrow 4a - 5 = 6a \text{ or } 2a = -5$$

$$\therefore a = -\frac{5}{2}$$

(i) nth Term of HP from Beginning

Let a be the first term, d be the common difference of an AP. Then, nth term of an AP from beginning $= a + (n-1)d$

Hence, the nth term of HP from beginning

$$= \frac{1}{a + (n-1)d}, \forall n \in N$$

(ii) nth Term of HP from End

Let l be the last term, d be the common difference of an AP. Then,

nth term of an AP from end $= l - (n-1)d$

Hence, the nth term of HP from end $= \frac{1}{l - (n-1)d}, \forall n \in N$

Remark

1. $\frac{1}{n\text{th term of HP from beginning}} + \frac{1}{n\text{th term of HP from end}}$
$= a + l = \frac{1}{\text{first term of HP}} + \frac{1}{\text{last term of HP}}$
2. There is no general formula for the sum of any number of quantities in HP are generally solved by inverting the terms and making use of the corresponding AP.

Example 54. ***If*** $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$, ***then prove that*** a, b, c ***are in HP, unless*** $b = a + c$.

Sol. We have, $$\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{c-b} = 0$$

$$\Rightarrow \left(\frac{1}{a} + \frac{1}{c-b}\right) + \left(\frac{1}{c} + \frac{1}{a-b}\right) = 0$$

$$\Rightarrow \frac{(c-b+a)}{a(c-b)} + \frac{(a-b+c)}{c(a-b)} = 0$$

$$\Rightarrow (a+c-b)\left[\frac{1}{a(c-b)} + \frac{1}{c(a-b)}\right] = 0$$

$$\Rightarrow (a+c-b)[2ac - b(a+c)] = 0$$

If $\quad a+c-b\neq 0$, then $2ac-b(a+c)=0$

or $$b=\frac{2ac}{a+c}$$

Therefore, a, b, c are in HP and if $2ac-b(a+c)\neq 0$, then $a+c-b=0$ i.e., $b=a+c$.

Example 55. If $a_1, a_2, a_3, \ldots, a_n$ are in HP, then prove that $a_1a_2+a_2a_3+a_3a_4+\ldots+a_{n-1}a_n=(n-1)a_1a_n$

Sol. Given, $a_1, a_2, a_3, \ldots, a_n$ are in HP.

$\therefore \quad \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ are in AP.

Let D be the common difference of the AP, then

$$\frac{1}{a_2}-\frac{1}{a_1}=\frac{1}{a_3}-\frac{1}{a_2}=\frac{1}{a_4}-\frac{1}{a_3}=\ldots=\frac{1}{a_n}-\frac{1}{a_{n-1}}=D$$

$$\Rightarrow \quad \frac{a_1-a_2}{a_1a_2}=\frac{a_2-a_3}{a_2a_3}=\frac{a_3-a_4}{a_3a_4}=\ldots=\frac{a_{n-1}-a_n}{a_{n-1}a_n}=D$$

$$\Rightarrow \quad a_1a_2=\frac{a_1-a_2}{D}, a_2a_3=\frac{a_2-a_3}{D}, a_3a_4=\frac{a_3-a_4}{D}, \ldots, a_{n-1}a_n=\frac{a_{n-1}-a_n}{D}$$

On adding all such expressions, we get

$$a_1a_2+a_2a_3+a_3a_4+\ldots+a_{n-1}a_n=\frac{a_1-a_n}{D}=\frac{a_1a_n}{D}\left(\frac{1}{a_n}-\frac{1}{a_1}\right)$$

$$=\frac{a_1a_n}{D}\left[\frac{1}{a_1}+(n-1)D-\frac{1}{a_1}\right]=(n-1)a_1a_n$$

Hence, $a_1a_2+a_2a_3+a_3a_4+\ldots+a_{n-1}a_n=(n-1)a_1a_n$

Remark

In particular case,

1. when $n=4$ $a_1a_2+a_2a_3+a_3a_4=3a_1a_4$
2. when $n=6$ $a_1a_2+a_2a_3+a_3a_4+a_4a_5+a_5a_6=5a_1a_6$

Example 56. ***The sum of three numbers in HP is 37 and the sum of their reciprocals is $\frac{1}{4}$. Find the numbers.***

Sol. Three numbers in HP can be taken as $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$.

Then, $$\frac{1}{a-d}+\frac{1}{a}+\frac{1}{a+d}=37 \quad \ldots(i)$$

and $$a-d+a+a+d=\frac{1}{4}$$

$$\therefore \quad a=\frac{1}{12}$$

From Eq. (i), $\frac{12}{1-12d}+12+\frac{12}{1+12d}=37$

$$\Rightarrow \quad \frac{12}{1-12d}+\frac{12}{1+12d}=25 \Rightarrow \frac{24}{1-144d^2}=25$$

$$\Rightarrow \quad 1-144d^2=\frac{24}{25} \text{ or } d^2=\frac{1}{25\times 144}$$

$$\therefore \quad d=\pm\frac{1}{60}$$

$\therefore \quad a-d, a, a+d$ are $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$ or $\frac{1}{10}, \frac{1}{12}, \frac{1}{15}$.

Hence, three numbers in HP are 15, 12, 10 or 10, 12, 15.

Example 57. ***If pth, qth and rth terms of a HP be respectively a, b and c, then prove that $(q-r)bc+(r-p)ca+(p-q)ab=0$.***

Sol. Let A and D be the first term and common difference of the corresponding AP. Now, a, b, c are respectively the p th, q th and r th terms of HP.

$\therefore \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ will be respectively the p th, q th and r th terms of the corresponding AP.

$$\Rightarrow \quad \frac{1}{a}=A+(p-1)D \quad \ldots(i)$$

$$\frac{1}{b}=A+(q-1)D \quad \ldots(ii)$$

$$\frac{1}{c}=A+(r-1)D \quad \ldots(iii)$$

On subtracting Eq. (iii) from Eq. (ii), we get

$$\frac{1}{b}-\frac{1}{c}=(q-r)D \Rightarrow bc(q-r)=\frac{(c-b)}{D}=-\frac{(b-c)}{D}$$

So, LHS $=(q-r)bc+(r-p)ca+(p-q)ab$

$$=-\frac{1}{D}\{b-c+c-a+a-b\}=0=\text{RHS}$$

Theorem Relating to the Three Series

If a, b, c are three consecutive terms of a series, then

if $\frac{a-b}{b-c}=\frac{a}{a}$, then a, b, c are in AP.

if $\frac{a-b}{b-c}=\frac{a}{b}$, then a, b, c are in GP and if $\frac{a-b}{b-c}=\frac{a}{c}$, then a, b, c are in HP.

Mixed Examples on AP, GP and HP

Example 58. ***If a, b, c are in AP and a^2, b^2, c^2 be in HP. Then, prove that $-\frac{a}{2}, b, c$ are in GP or else $a=b=c$.***

Sol. Given, a, b, c are in AP.

$\therefore \qquad b = \dfrac{a+c}{2} \qquad$ ...(i)

and a^2, b^2, c^2 are in HP.

$\therefore \qquad b^2 = \dfrac{2a^2c^2}{a^2+c^2} \qquad$ ...(ii)

From Eq. (ii) $b^2\{(a+c)^2 - 2ac\} = 2a^2c^2$

$\Rightarrow \qquad b^2\{(2b)^2 - 2ac\} = 2a^2c^2 \qquad$ [from Eq. (i)]

$\Rightarrow \qquad 2b^4 - acb^2 - a^2c^2 = 0$

$\Rightarrow \qquad (2b^2+ac)(b^2-ac) = 0$

$\Rightarrow \qquad 2b^2+ac = 0$ or $b^2 - ac = 0$

If $2b^2 + ac = 0$, then $b^2 = -\dfrac{1}{2}ac$ or $-\dfrac{a}{2}, b, c$ are in GP

and if $b^2 - ac = 0 \Rightarrow a, b, c$ are in GP.

But given, a, b, c are in AP.

Which is possible only when $a = b = c$

Example 59. ***If*** a, b, c ***are in HP,*** b, c, d ***are in GP and*** c, d, e ***are in AP, then show that*** $e = \dfrac{ab^2}{(2a-b)^2}$.

Sol. Given, a, b, c are in HP.

$\therefore \qquad b = \dfrac{2ac}{a+c}$ or $c = \dfrac{ab}{2a-b} \qquad$ ...(i)

Given, b, c, d are in GP.

$\therefore \qquad c^2 = bd \qquad$ ...(ii)

and given, c, d, e are in AP.

$\therefore \qquad d = \dfrac{c+e}{2}$

$\Rightarrow \qquad e = 2d - c$

$\qquad e = \left(\dfrac{2c^2}{b} - c\right) \qquad$ [from Eq. (ii)] ...(iii)

From Eqs. (i) and (iii), $e = \dfrac{2}{b}\left(\dfrac{ab}{2a-b}\right)^2 - \left(\dfrac{ab}{2a-b}\right)$

$$= \frac{ab}{(2a-b)^2}\{2a - (2a-b)\}$$

$$= \frac{ab^2}{(2a-b)^2}$$

Example 60. If a, b, c, d and e be five real numbers such that a, b, c are in AP; b, c, d are in GP; c, d, e are in HP. If $a = 2$ and $e = 18$, then find all possible values of b, c and d.

Sol. Given, a, b, c are in AP,

$\therefore \qquad b = \dfrac{a+c}{2} \qquad$ ...(i)

b, c, d are in GP,

$\therefore \qquad c^2 = bd \qquad$ ...(ii)

and c, d, e are in HP.

$\therefore \qquad d = \dfrac{2ce}{c+e} \qquad$ ...(iii)

Now, substituting the values of b and d in Eq. (ii), then

$$c^2 = \left(\frac{a+c}{2}\right)\left(\frac{2ce}{c+e}\right)$$

$\Rightarrow \qquad c(c+e) = e(a+c)$

$\Rightarrow \qquad c^2 = ae \qquad$ ...(iv)

Given, $\qquad a = 2, e = 18$

From Eq. (iv), $\qquad c^2 = (2)(18) = 36$

$\therefore \qquad c = \pm 6$

From Eq. (i), $\qquad b = \dfrac{2 \pm 6}{2} = 4, -2$

and from Eq. (ii), $\qquad d = \dfrac{c^2}{b} = \dfrac{36}{b} = \dfrac{36}{4}$ or $\dfrac{36}{-2}$

$\therefore \qquad d = 9$ or -18

Hence, $\qquad c = 6, b = 4, d = 9$ or $c = -6, b = -2, d = -18$

Example 61. If three positive numbers a, b and c are in AP, GP and HP as well, then find their values.

Sol. Since a, b, c are in AP, GP and HP as well

$\therefore \qquad b = \dfrac{a+c}{2} \qquad$ ...(i)

$\qquad b^2 = ac \qquad$ ...(ii)

and $\qquad b = \dfrac{2ac}{a+c} \qquad$ ...(iii)

From Eqs. (i) and (ii), we have

$$\left(\frac{a+c}{2}\right)^2 = ac$$

or $\qquad (a+c)^2 = 4ac$

or $\qquad (a+c)^2 - 4ac = 0$

or $\qquad (a-c)^2 = 0$

$\therefore \qquad a = c \qquad$ ...(iv)

On putting $c = a$ in Eq. (i), we get $b = \dfrac{a+a}{2} = a \qquad$ ...(v)

From Eqs. (iv) and (v), $a = b = c$, thus the three numbers will be equal.

Remark

1. If three positive numbers are in any two of AP, GP and HP, then it will be also in third.
2. Thus, if three positive numbers are in any two of AP, GP and HP, then they will be in the third progression and the numbers will be equal.

Exercise for Session 4

1. If a, b, c are in AP and b, c, d be in HP, then

(a) $ab = cd$ (b) $ad = bc$ (c) $ac = bd$ (d) $abcd = 1$

2. If a, b, c are in AP, then $\frac{a}{bc}, \frac{1}{c}, \frac{2}{b}$ are in

(a) AP (b) GP (c) HP (d) None of these

3. If a, b, c are in AP and a, b, d are in GP, then $a, a - b, d - c$ will be in

(a) AP (b) GP (c) HP (d) None of these

4. If $x, 1, z$ are in AP and $x, 2, z$ are in GP, then $x, 4, z$ will be in

(a) AP (b) GP (c) HP (d) None of these

5. If a, b, c are in GP, $a - b, c - a, b - c$ are in HP, then $a + 4b + c$ is equal to

(a) 0 (b) 1 (c) -1 (d) None of these

6. If $(m + 1)\text{th}, (n + 1)\text{th}$ and $(r + 1)\text{th}$ terms of an AP are in GP and m, n, r are in HP, then the value of the ratio of the common difference to the first term of the AP is

(a) $-\frac{2}{n}$ (b) $\frac{2}{n}$ (c) $-\frac{n}{2}$ (d) $\frac{n}{2}$

7. If a, b, c are in AP and a^2, b^2, c^2 are in HP, then

(a) $a = b = c$ (b) $2b = 3a + c$ (c) $b^2 = \sqrt{\left(\frac{ac}{8}\right)}$ (d) None of these

8. If a, b, c are in HP, then $\frac{a}{b + c}, \frac{b}{c + a}, \frac{c}{a + b}$ are in

(a) AP (b) GP (c) HP (d) None of these

9. If $\frac{x + y}{2}, y, \frac{y + z}{2}$ are in HP, then x, y, z are in

(a) AP (b) GP (c) HP (d) None of these

10. If $\frac{a + b}{1 - ab}, b, \frac{b + c}{1 - bc}$ are in AP, then $a, \frac{1}{b}, c$ are in

(a) AP (b) GP (c) HP (d) None of these