# Session 5

## Mean

## Mean

### Arithmetic Mean

If three terms are in AP, then the middle term is called the **Arithmetic Mean** (or shortly written as AM) between the other two, so if $a, b, c$ are in AP, then $b$ is the AM of $a$ and $c$.

#### (i) Single AM of $n$ Positive Numbers

Let $n$ positive numbers be $a_1, a_2, a_3, \ldots, a_n$ and $A$ be the AM of these numbers, then

$$A = \frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}$$

**In particular** Let $a$ and $b$ be two given numbers and $A$ be the AM between them, then $a, A, b$ are in AP.

$$\therefore \quad A = \frac{a+b}{2}$$

***Remark***

1. AM of $2a, 3b, 5c$ is $\frac{2a + 3b + 5c}{3}$.
2. AM of $a_1, a_2, a_3, \ldots, a_{n-1}, 2a_n$ is $\frac{a_1 + a_2 + a_3 + \ldots + a_{n-1} + 2a_n}{n}$.

#### (ii) Insert $n$-Arithmetic Mean Between Two Numbers

Let $a$ and $b$ be two given numbers and $A_1, A_2, A_3, \ldots, A_n$ are AM's between them.

Then, $a, A_1, A_2, A_3, \ldots, A_n, b$ will be in AP.

Now, $b = (n+2)$ th term $= a + (n + 2 - 1)\, d$

$$\therefore \quad d = \left(\frac{b-a}{n+1}\right)$$

[Remember] [where, $d$ = common difference] ...(i)

$\therefore \quad A_1 = a + d, A_2 = a + 2d, \ldots, A_n = a + nd$

$$\Rightarrow \quad A_1 = a + \left(\frac{b-a}{n+1}\right), A_2 = a + 2\left(\frac{b-a}{n+1}\right), \ldots, A_n = a + n\left(\frac{b-a}{n+1}\right)$$

***Corollary* I The sum of $n$ AM's between two given quantities is equal to $n$ times the AM between them.**

Let two numbers be $a$ and $b$ and $A_1, A_2, A_3, \ldots, A_n$ are $n$ AM's between them.

Then, $a, A_1, A_2, A_3, \ldots, A_n, b$ will be in AP.

$\therefore$ Sum of $n$ AM's between $a$ and $b$

$= A_1 + A_2 + A_3 + \ldots + A_n$

$= \frac{n}{2}(A_1 + A_n)$ [$\because A_1, A_2, A_3, \ldots, A_n$ are in AP]

$= \frac{n}{2}(a + d + a + nd) = \frac{n}{2}[2a + (n+1)\, d]$

$= \frac{n}{2}(2a + b - a)$ [from Eq. (i)]

$= n\left(\frac{a+b}{2}\right) = n$ [AM between $a$ and $b$] [Remember]

**Aliter** $A_1 + A_2 + A_3 + \ldots + A_n$

$= (a + A_1 + A_2 + A_3 + \ldots + A_n + b) - (a + b)$

$= \frac{(n+2)}{2}(a+b) - (a+b) = n\left(\frac{a+b}{2}\right)$

$= n$ [AM of $a$ and $b$]

**Aliter**

[This method is applicable only when $n$ is even]

$A_1 + A_2 + A_3 + \ldots + A_{n-2} + A_{n-1} + A_n$

$= (A_1 + A_n) + (A_2 + A_{n-1}) + (A_3 + A_{n-2}) + \ldots$ upto $\frac{n}{2}$ terms

$= (a+b) + (a+b) + (a+b) + \ldots$ upto $\frac{n}{2}$ times [$\because T_n + T'_n = a + l$]

$= \frac{n}{2}(a+b) = n\left(\frac{a+b}{2}\right) = n$ [AM of $a$ and $b$]

***Corollary* II The sum of $m$ AM's between any two numbers is to the sum of $n$ AM's between them as $m : n$.**

Let two numbers be $a$ and $b$.

$\therefore$ Sum of $m$ AM's between $a$ and $b = m$ [AM of $a$ and $b$] ...(i)

Similarly, sum of $n$ AM's between $a$ and $b = n$ [AM of $a$ and $b$] ...(ii)

$$\therefore \quad \frac{\text{Sum of } m \text{ AM's}}{\text{Sum of } n \text{ AM's}} = \frac{m(\text{AM of } a \text{ and } b)}{n(\text{AM of } a \text{ and } b)} = \frac{m}{n}$$

**Example 62.** ***If $a, b, c$ are in AP and $p$ is the AM between $a$ and $b$ and $q$ is the AM between $b$ and $c$, then show that $b$ is the AM between $p$ and $q$.***

***Sol.*** $\because$ $a, b, c$ are in AP.

$$\therefore \quad 2b = a + c \quad \text{...(i)}$$

$\because$ $p$ is the AM between $a$ and $b$.

$$\therefore \quad p = \frac{a+b}{2} \quad \text{...(ii)}$$

$\because$ $q$ is the AM between $b$ and $c$.

$$\therefore \quad q = \frac{b+c}{2} \quad \text{...(iii)}$$

On adding Eqs. (ii) and (iii), then

$$p + q = \frac{a+b}{2} + \frac{b+c}{2} = \frac{a+c+2b}{2} = \frac{2b+2b}{2} \quad \text{[using Eq. (i)]}$$

$$\therefore \quad p + q = 2b \text{ or } b = \frac{p+q}{2}$$

Hence, $b$ is the AM between $p$ and $q$.

**Example 63.** Find $n$, so that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ $(a \neq b)$ be the AM between $a$ and $b$.

***Sol.*** $\because$

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n} = \frac{a+b}{2}$$

$$\Rightarrow \quad \frac{b^{n+1}\left[\left(\frac{a}{b}\right)^{n+1}+1\right]}{b^n\left[\left(\frac{a}{b}\right)^{n}+1\right]} = \frac{b}{2}\left[\left(\frac{a}{b}\right)+1\right]$$

$$\Rightarrow \quad 2\left[\left(\frac{a}{b}\right)^{n+1}+1\right] = \left[\left(\frac{a}{b}\right)^{n}+1\right]\left(\frac{a}{b}+1\right)$$

Let $\dfrac{a}{b} = \lambda$

$$\therefore \quad 2\lambda^{n+1} + 2 = (\lambda^n + 1)(\lambda + 1)$$

$$\Rightarrow \quad 2\lambda^{n+1} + 2 = \lambda^{n+1} + \lambda^n + \lambda + 1$$

$$\Rightarrow \quad \lambda^{n+1} - \lambda^n - \lambda + 1 = 0 \Rightarrow (\lambda^n - 1)(\lambda - 1) = 0$$

$$\lambda - 1 \neq 0 \quad [\because a \neq b]$$

$$\therefore \quad \lambda^n - 1 = 0 \Rightarrow \lambda^n = 1 = \lambda^0$$

$$\Rightarrow \quad n = 0$$

**Example 64.** There are $n$ AM's between 3 and 54 such that 8th mean is to $(n-2)$th mean as 3 to 5. Find $n$.

***Sol.*** Let $A_1, A_2, A_3, \ldots, A_n$ be $n$ AM's between 3 and 54.

If $d$ be the common difference, then

$$d = \frac{54-3}{n+1} = \frac{51}{n+1} \quad \text{...(i)}$$

According to the example,

$$\frac{A_8}{A_{n-2}} = \frac{3}{5}$$

$$\Rightarrow \quad 5(3 + 8d) = 3[3 + (n-2)d] \Rightarrow 6 = d(3n - 46)$$

$$\Rightarrow \quad 6 = (3n - 46)\frac{51}{(n+1)} \quad \text{[from Eq. (i)]}$$

$$\Rightarrow \quad 6n + 6 = 153n - 2346 \Rightarrow 147n = 2352$$

$$\therefore \quad n = 16$$

**Example 65.** If 11 AM's are inserted between 28 and 10, then find the three middle terms in the series.

***Sol.*** Let $A_1, A_2, A_3, \ldots, A_{11}$ be 11 AM's between 28 and 10.

If $d$ be the common difference, then

$$d = \frac{10-28}{12} = -\frac{3}{2}$$

Total means = 11 (odd)

$$\therefore \quad \text{Middle mean} = \left(\frac{11+1}{2}\right)\text{th} = 6\text{th} = A_6$$

Then, three middle terms are $A_5$, $A_6$ and $A_7$.

$$\therefore \quad A_5 = 28 + 5d = 28 - \frac{15}{2} = \frac{41}{2}$$

$$A_6 = 28 + 6d = 28 - 9 = 19$$

$$\text{and} \quad A_7 = 28 + 7d = 28 - \frac{21}{2} = \frac{35}{2}$$

**Example 66.** If $a, b, c$ are in AP, then show that $a^2(b+c) + b^2(c+a) + c^2(a+b) = \dfrac{2}{9}(a+b+c)^3$.

***Sol.*** $\because$ $a, b, c$ are in AP.

$$\therefore \quad b = \frac{a+c}{2} \text{ i.e., } 2b = a + c \quad \text{...(i)}$$

$$\begin{aligned} \text{LHS} &= a^2(b+c) + b^2(c+a) + c^2(a+b) \\ &= (a^2b + a^2c) + b^2(2b) + (c^2a + c^2b) \\ &= b(a^2 + c^2) + ac(a+c) + 2b^3 \\ &= b[(a+c)^2 - 2ac] + ac(2b) + 2b^3 \\ &= b(a+c)^2 + 2b^3 = b(2b)^2 + 2b^3 = 6b^3 \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \frac{2}{9}(a+b+c)^3 = \frac{2}{9}(2b+b)^3 \\ &= \frac{2}{9} \times 27b^3 = 6b^3 \end{aligned}$$

Hence, LHS = RHS

## Geometric Mean

If three terms are in GP, then the middle term is called the **Geometric Mean** (or shortly written as GM) between the other two, so if $a, b, c$ are in GP, then $b$ is the GM of $a$ and $c$.

## (i) Single GM of $n$ Positive Numbers

Let $n$ positive numbers be $a_1, a_2, a_3, \ldots, a_n$ and $G$ be the GM of these numbers, then $G = (a_1 a_2 a_3 \ldots a_n)^{1/n}$

**In particular** Let $a$ and $b$ be two numbers and $G$ be the GM between them, then $a, G, b$ are in GP.
Hence, $G = \sqrt{ab}$ ; $a > 0, b > 0$

***Remark***

1. If $a < 0$, $b < 0$, then $G = -\sqrt{ab}$
2. If $a < 0$, $b > 0$ or $a > 0$, $b < 0$, then GM between $a$ and $b$ does not exist.

### Example

(i) The GM between 4 and 9 is given by

$$G = \sqrt{4 \times 9} = 6$$

(ii) The GM between $-4$ and $-9$ is given by

$$G = \sqrt{-4 \times -9} = -6$$

(iii) The GM between $-4$ and 9 or 4 and $-9$ does not exist.

i.e. $\sqrt{(-4) \times 9} = \sqrt{-1}\sqrt{36} = 6i$

and $\sqrt{4 \times (-9)} = \sqrt{-1}\sqrt{36} = 6i$

## (ii) Insert $n$-Geometric Mean Between Two Numbers

Let $a$ and $b$ be two given numbers and $G_1, G_2, G_3, \ldots, G_n$ are $n$ GM's between them.

Then, $a, G_1, G_2, G_3, \ldots, G_n, b$ will be in GP.

Now, $b = (n+2)$ th term $= ar^{n+2-1}$

$$\therefore \quad r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \text{ [where } r = \text{common ratio] [Remember]} \quad \ldots\text{(i)}$$

$$\therefore \quad G_1 = ar, G_2 = ar^2, \ldots, G_n = ar^n$$

$$\Rightarrow \quad G_1 = a\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, G_2 = a\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \ldots, G_n = a\left(\frac{b}{a}\right)^{\frac{n}{n+1}}$$

***Corollary*** **The product of $n$ geometric means between $a$ and $b$ is equal to the $n$th power of the geometric mean between $a$ and $b$.**

Let two numbers be $a$ and $b$ and $G_1, G_2, G_3, \ldots, G_n$ are $n$ GM's between them.

Then, $a, G_1, G_2, G_3, \ldots, G_n, b$ will be in GP.

$\therefore$ Product of $n$ GM's between $a$ and $b$

$$= G_1 G_2 G_3 \ldots G_n = (ar)(ar^2)(ar^3)\ldots(ar^n)$$

$$= a^{1+1+1+\ldots+1} \cdot r^{1+2+3+\ldots+n}$$

$$= a^n \cdot r^{\frac{n(n+1)}{2}} = a^n \cdot \left[\left(\frac{b}{a}\right)^{\frac{1}{n+1}}\right]^{\frac{n(n+1)}{2}} \quad \text{[from Eq. (i)]}$$

$$= a^n \left(\frac{b}{a}\right)^{\frac{n}{2}} = a^{n/2} b^{n/2} = (\sqrt{ab})^n$$

$$= [\text{GM of } a \text{ and } b]^n \quad \text{[Remember]}$$

**Aliter** [This method is applicable only when $n$ is even]

$$G_1 G_2 G_3 \ldots G_{n-2}\, G_{n-1}\, G_n = (G_1\, G_n)(G_2 G_{n-1})(G_3 G_{n-2}) \ldots \frac{n}{2} \text{ factors}$$

$$= (ab)(ab)(ab) \ldots \frac{n}{2} \text{ factors} \quad [\because T_n \times T'_n = a \times l]$$

$$= (ab)^{n/2} = (\sqrt{ab})^n = [\text{GM of } a \text{ and } b]^n$$

**Example 67.** ***If $a$ be one AM and $G_1$ and $G_2$ be two geometric means between $b$ and $c$, then prove that $G_1^3 + G_2^3 = 2abc$.***

***Sol.*** Given, $a$ = AM between $b$ and $c$

$$\Rightarrow \quad a = \frac{b+c}{2} \Rightarrow 2a = b + c \quad \ldots\text{(i)}$$

Again, $b, G_1, G_2, c$ are in GP.

$$\therefore \quad \frac{G_1}{b} = \frac{G_2}{G_1} = \frac{c}{G_2} \Rightarrow b = \frac{G_1^2}{G_2}, c = \frac{G_2^2}{G_1}$$

and $\quad G_1 G_2 = bc \quad \ldots$(ii)

From Eqs. (i) and (ii),

$$2a = \frac{G_1^2}{G_2} + \frac{G_2^2}{G_1} = \frac{G_1^3 + G_2^3}{G_1 G_2} = \frac{G_1^3 + G_2^3}{bc} \quad [\because G_1 G_2 = bc]$$

$$\Rightarrow \quad G_1^3 + G_2^3 = 2abc$$

**Example 68.** ***If one geometric mean $G$ and two arithmetic means $p$ and $q$ be inserted between two quantities, then show that $G^2 = (2p-q)(2q-p)$.***

***Sol.*** Let the two quantities be $a$ and $b$, then

$$G^2 = ab \quad \ldots\text{(i)}$$

Again, $a, p, q, b$ are in AP.

$$\therefore \quad p - a = q - p = b - q$$

$$\Rightarrow \quad a = 2p - q$$

$$b = 2q - p \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$G^2 = (2p-q)(2q-p)$$

**Example 69.** ***Find $n$, so that $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ $(a \neq b)$ be the GM between $a$ and $b$.***

***Sol.*** $\because \quad \dfrac{a^{n+1} + b^{n+1}}{a^n + b^n} = \sqrt{ab}$

$$\Rightarrow \frac{b^{n+1}\left[\left(\frac{a}{b}\right)^{n+1}+1\right]}{b^n\left[\left(\frac{a}{b}\right)^n+1\right]} = b\sqrt{\frac{a}{b}} \Rightarrow \frac{\left(\frac{a}{b}\right)^{n+1}+1}{\left(\frac{a}{b}\right)^n+1} = \left(\frac{a}{b}\right)^{\frac{1}{2}}$$

Let $\frac{a}{b} = \lambda$

$$\Rightarrow \frac{\lambda^{n+1}+1}{\lambda^n+1} = \lambda^{\frac{1}{2}} \Rightarrow \lambda^{n+1}+1 = \lambda^{n+\frac{1}{2}} + \lambda^{\frac{1}{2}}$$

$$\Rightarrow \lambda^{n+\frac{1}{2}}(\lambda^{\frac{1}{2}}-1) - (\lambda^{\frac{1}{2}} - 1) = 0$$

$$\Rightarrow (\lambda^{\frac{1}{2}}-1)(\lambda^{n+\frac{1}{2}}-1) = 0$$

$$\Rightarrow \lambda^{\frac{1}{2}} - 1 \neq 0 \qquad [\because a \neq b]$$

$$\therefore \lambda^{n+\frac{1}{2}} - 1 = 0$$

$$\Rightarrow \lambda^{n+\frac{1}{2}} = 1 = \lambda^0$$

$$\Rightarrow n + \frac{1}{2} = 0 \text{ or } n = -\frac{1}{2}$$

**Example 70.** Insert five geometric means between $\frac{1}{3}$ and 9 and verify that their product is the fifth power of the geometric mean between $\frac{1}{3}$ and 9.

***Sol.*** Let $G_1, G_2, G_3, G_4, G_5$ be 5 GM's between $\frac{1}{3}$ and 9.

Then, $\frac{1}{3}, G_1, G_2, G_3, G_4, G_5, 9$ are in GP.

Here, $r$ = common ratio $= \left(\frac{9}{\frac{1}{3}}\right)^{1/6} = 3^{\frac{1}{2}} = \sqrt{3}$

$$\therefore G_1 = ar = \frac{1}{3}\cdot\sqrt{3} = \frac{1}{\sqrt{3}}$$

$$G_2 = ar^2 = \frac{1}{3}\cdot 3 = 1$$

$$G_3 = ar^3 = \frac{1}{3}\cdot 3\sqrt{3} = \sqrt{3}$$

$$G_4 = ar^4 = \frac{1}{3}\cdot 9 = 3$$

$$G_5 = ar^5 = \frac{1}{3}\cdot 9\sqrt{3} = 3\sqrt{3}$$

Now, Product $= G_1 \times G_2 \times G_3 \times G_4 \times G_5$

$$= \frac{1}{\sqrt{3}} \times 1 \times \sqrt{3} \times 3 \times 3\sqrt{3} = 9\sqrt{3} = (3)^{\frac{5}{2}} = \left(\sqrt{\frac{1}{3} \times 9}\right)^5$$

$$= \left[\text{GM of } \frac{1}{3} \text{ and } 9^{\,5}\right]$$

## An Important Theorem

Let $a$ and $b$ be two real, positive and unequal numbers and $A, G$ are arithmetic and geometric means between them, then

(i) $a$ and $b$ are the roots of the equation

$$x^2 - 2Ax + G^2 = 0 \qquad \text{[Remember]}$$

(ii) $a$ and $b$ are given by $A \pm \sqrt{(A+G)(A-G)}$ [Remember]

(iii) $A > G$ [Remember]

**Proof** $\because A$ is the AM between $a$ and $b$, then

$$A = \frac{a+b}{2} \Rightarrow a + b = 2A \qquad \text{...(i)}$$

and $G$ is the GM between $a$ and $b$, then

$$G = \sqrt{ab} \Rightarrow ab = G^2 \qquad \text{...(ii)}$$

$\therefore a$ and $b$ are the roots of the equation, then

$x^2 -$ (sum of roots) $x +$ product of roots $= 0$

$$\Rightarrow x^2 - (a+b)x + ab = 0$$

i.e. $x^2 - 2Ax + G^2 = 0$ is the required equation.

$$\Rightarrow x = \frac{2A \pm \sqrt{(-2A)^2 - 4\cdot 1\cdot G^2}}{2\cdot 1} = A \pm \sqrt{(A^2 - G^2)}$$

$$\therefore x = A \pm \sqrt{(A+G)(A-G)}$$

Now, for real, positive and unequal numbers of $a$ and $b$,

$$(A+G)(A-G) > 0 \Rightarrow (A-G) > 0$$

$$\therefore A > G$$

***Remark***

1. If $a$ and $b$ are real and positive numbers, then $A \geq G$
2. If $a_1, a_2, a_3, \ldots, a_n$ are $n$ positive numbers, then AM $\geq$ GM i.e.,

$$\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n} \geq (a_1 a_2 a_3 \ldots a_n)^{1/n}$$

3. (i) If $a > 0, b > 0$ or $a < 0, b < 0$ and $\lambda_1 > 0, \lambda_2 > 0$, then

$$\lambda_1 \frac{a}{b} + \lambda_2 \frac{b}{a} \geq 2\sqrt{\lambda_1 \lambda_2}$$

if $\frac{a}{b} = x > 0$ and $\lambda_1 = \lambda_2 = 1$, then $x + \frac{1}{x} \geq 2$

(ii) If $a > 0, b < 0$ or $a < 0, b > 0$ and $\lambda_1 > 0, \lambda_2 > 0$, then

$$\lambda_1 \frac{a}{b} + \lambda_2 \frac{b}{a} \leq -2\sqrt{\lambda_1 \lambda_2}$$

if $\frac{a}{b} = x < 0$ and $\lambda_1 > 0, \lambda_2 > 0$ then, $x + \frac{1}{x} \leq -2$

**Example 71.** AM between two numbers whose sum is 100 is to the GM as 5 : 4, find the numbers.

***Sol.*** Let the numbers be $a$ and $b$.

Then, $a + b = 100$

or $2A = 100$

$$\Rightarrow \qquad A = 50 \qquad \text{...(i)}$$

and given, $\dfrac{A}{G} = \dfrac{5}{4} \Rightarrow \dfrac{50}{G} = \dfrac{5}{4}$ [from Eq. (i)]

$$\therefore \qquad G = 40 \qquad \text{...(ii)}$$

From important theorem $a, b = A \pm \sqrt{(A+G)(A-G)}$

$$= 50 \pm \sqrt{(50+40)(50-40)}$$

$$= 50 \pm 30 = 80, 20$$

$\therefore$ $a = 80, b = 20$

or $a = 20, b = 80$

**Example 72.** If $a_1, a_2, \ldots, a_n$ are positive real numbers whose product is a fixed number $c$, then find the minimum value of $a_1 + a_2 + \ldots + a_{n-1} + 3a_n$.

***Sol.*** $\because$ AM $\geq$ GM

$$\therefore \frac{a_1 + a_2 + \ldots + a_{n-1} + 3a_n}{n} \geq (a_1 a_2 \ldots a_{n-1} 3a_n)^{1/n} = (3c)^{1/n}$$

$$\Rightarrow a_1 + a_2 + \ldots + a_{n-1} + 3a_n \geq n\,(3c)^{1/n}$$

Hence, the minimum value of $a_1 + a_2 + \ldots + a_{n-1} + 3a_n$ is $n\,(3c)^{1/n}$.

# Harmonic Mean

If three terms are in HP, then the middle term is called the **Harmonic Mean** (or shortly written as HM) between the other two, so if $a, b, c$ are in HP, then $b$ is the HM of $a$ and $c$.

## (i) Single HM of $n$ Positive Numbers

Let $n$ positive numbers be $a_1, a_2, a_3, \ldots, a_n$ and $H$ be the HM of these numbers, then

$$H = \frac{n}{\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \ldots + \dfrac{1}{a_n}\right)}$$

**In particular** Let $a$ and $b$ be two given numbers and $H$ be the HM between them $a, H, b$ are in HP.

Hence, $H = \dfrac{2}{\dfrac{1}{a} + \dfrac{1}{b}}$ i.e., $H = \dfrac{2ab}{(a+b)}$

***Remark***

HM of $a, b, c$ is $\dfrac{3}{\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}}$ or $\dfrac{3abc}{ab + bc + ca}$.

***Caution*** The AM between two numbers $a$ and $b$ is $\dfrac{a+b}{2}$. It does not follow that HM between the same numbers is $\dfrac{2}{a+b}$. The HM is the reciprocals of $\dfrac{\dfrac{1}{a} + \dfrac{1}{b}}{2}$ i.e., $\dfrac{2ab}{(a+b)}$.

## (ii) Insert $n$-Harmonic Mean Between Two Numbers

Let $a$ and $b$ be two given numbers and $H_1, H_2, H_3, \ldots, H_n$ are $n$ HM's between them.

Then, $a, H_1, H_2, H_3, \ldots, H_n, b$ will be in HP, if $D$ be the common difference of the corresponding AP.

$\therefore b = (n+2)$ th term of HP.

$$= \frac{1}{(n+2)\text{th term of corresponding AP}}$$

$$= \frac{1}{\dfrac{1}{a} + (n+2-1)\,D}$$

$$\Rightarrow \qquad D = \frac{\dfrac{1}{b} - \dfrac{1}{a}}{(n+1)} \qquad \text{[Remember]}$$

$$\therefore \qquad \frac{1}{H_1} = \frac{1}{a} + D,\ \frac{1}{H_2} = \frac{1}{a} + 2D, \ldots, \frac{1}{H_n} = \frac{1}{a} + nD$$

$$\Rightarrow \qquad \frac{1}{H_1} = \frac{1}{a} + \frac{(a-b)}{ab\,(n+1)}, \frac{1}{H_2} = \frac{1}{a} + \frac{2\,(a-b)}{ab\,(n+1)}, \ldots, \frac{1}{H_n} = \frac{1}{a} + \frac{n\,(a-b)}{ab\,(n+1)}$$

***Corollary*** **The sum of reciprocals of $n$ harmonic means between two given numbers is $n$ times the reciprocal of single HM between them.**

Let two numbers be $a$ and $b$ and $H_1, H_2, H_3, \ldots, H_n$ are $n$ HM's between them. Then, $a, H_1, H_2, H_3, \ldots, H_n, b$ will be in HP.

$$\therefore \qquad \frac{1}{H_1} + \frac{1}{H_2} + \frac{1}{H_3} + \ldots + \frac{1}{H_n} = \frac{n}{2}\left(\frac{1}{H_1} + \frac{1}{H_n}\right) \qquad \left[\because S_n = \frac{n}{2}(a+l)\right]$$

$$= \frac{n}{2}\left(\frac{1}{a} + D + \frac{1}{b} - D\right) = \frac{n}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

$$= \frac{n}{\left(\dfrac{2}{\dfrac{1}{a} + \dfrac{1}{b}}\right)} = \frac{n}{[\text{HM of } a \text{ and } b]}$$

**Aliter** [This method is applicable only when $n$ is even]

$$\frac{1}{H_1} + \frac{1}{H_2} + \frac{1}{H_3} + \ldots + \frac{1}{H_{n-2}} + \frac{1}{H_{n-1}} + \frac{1}{H_n}$$

$$= \left(\frac{1}{H_1} + \frac{1}{H_n}\right) + \left(\frac{1}{H_2} + \frac{1}{H_{n-1}}\right) + \left(\frac{1}{H_3} + \frac{1}{H_{n-2}}\right) + \ldots \text{upto } \frac{n}{2} \text{ terms}$$

$$=\left(\frac{1}{a}+D+\frac{1}{b}-D\right)+\left(\frac{1}{a}+2D+\frac{1}{b}-2D\right)+\left(\frac{1}{a}+3D+\frac{1}{b}-3D\right)+\ldots\text{upto } \frac{n}{2} \text{ terms}$$

$$=\left(\frac{1}{a}+\frac{1}{b}\right)+\left(\frac{1}{a}+\frac{1}{b}\right)+\left(\frac{1}{a}+\frac{1}{b}\right)+\ldots \text{ upto } \frac{n}{2} \text{ terms}$$

$$=\frac{n}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=\frac{n}{\left(\dfrac{2}{\frac{1}{a}+\frac{1}{b}}\right)}=\frac{n}{(\text{HM of } a \text{ and } b)}$$

**Example 73.** If $H$ be the harmonic mean between $x$ and $y$, then show that $\frac{H+x}{H-x}+\frac{H+y}{H-y}=2$

***Sol.*** We have, $H=\frac{2xy}{x+y}$

$$\therefore \quad \frac{H}{x}=\frac{2y}{x+y} \text{ and } \frac{H}{y}=\frac{2x}{x+y}$$

By componendo and dividendo, we have

$$\frac{H+x}{H-x}=\frac{2y+x+y}{2y-x-y}=\frac{x+3y}{y-x}$$

and $$\frac{H+y}{H-y}=\frac{2x+x+y}{2x-x-y}=\frac{3x+y}{x-y}$$

$$\therefore \quad \frac{H+x}{H-x}+\frac{H+y}{H-y}=\frac{x+3y}{y-x}+\frac{3x+y}{x-y}$$

$$=\frac{x+3y-3x-y}{y-x}=\frac{2(y-x)}{(y-x)}=2$$

**Aliter** $\frac{H+x}{H-x}+\frac{H+y}{H-y}=2$

$$\Rightarrow \left(\frac{H+x}{H-x}-1\right)=\left(1-\frac{H+y}{H-y}\right) \Rightarrow \frac{2x}{H-x}=\frac{-2y}{H-y}$$

i.e. $\quad Hx-xy=-Hy+xy \Rightarrow H(x+y)=2xy$

i.e. $\quad H=\frac{2xy}{(x+y)}$

which is true as, $x, H, y$ are in HP. Hence, the required result.

**Example 74.** ***If $a_1, a_2, a_3, \ldots, a_{10}$ be in AP and $h_1, h_2, h_3, \ldots, h_{10}$ be in HP. If $a_1=h_1=2$ and $a_{10}=h_{10}=3$, then find the value of $a_4 h_7$.***

***Sol.*** $\because$ $a_1, a_2, a_3, \ldots, a_{10}$ are in AP.

If $d$ be the common difference, then

$$d=\frac{a_{10}-a_1}{9}=\frac{3-2}{9}=\frac{1}{9}$$

$$\therefore \quad a_4=a_1+3d=2+\frac{3}{9}=2+\frac{1}{3}=\frac{7}{3} \qquad \ldots\text{(i)}$$

and given $h_1, h_2, h_3, \ldots, h_{10}$ are in HP.

If $D$ be common difference of corresponding AP.

Then, $$D=\frac{\frac{1}{h_{10}}-\frac{1}{h_1}}{9}=\frac{\frac{1}{3}-\frac{1}{2}}{9}=-\frac{1}{54}$$

$$\therefore \quad \frac{1}{h_7}=\frac{1}{h_1}+6D=\frac{1}{2}-\frac{6}{54}=\frac{1}{2}-\frac{1}{9}=\frac{7}{18} \Rightarrow h_7=\frac{18}{7}$$

Hence, $$a_4 \cdot h_7=\frac{7}{3}\times\frac{18}{7}=6$$

**Example 75.** Find $n$, so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ $(a \neq b)$ be HM between $a$ and $b$.

***Sol.*** $$\because \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\frac{2ab}{a+b}$$

$$\Rightarrow \quad \frac{b^{n+1}\left[\left(\frac{a}{b}\right)^{n+1}+1\right]}{b^n\left[\left(\frac{a}{b}\right)^n+1\right]}=\frac{b^2\left[2\left(\frac{a}{b}\right)\right]}{b\left(\frac{a}{b}+1\right)}$$

$$\Rightarrow \quad \frac{\left(\frac{a}{b}\right)^{n+1}+1}{\left(\frac{a}{b}\right)^n+1}=\frac{2\left(\frac{a}{b}\right)}{\left(\frac{a}{b}\right)+1}$$

Let $$\frac{a}{b}=\lambda$$

Then, $$\frac{\lambda^{n+1}+1}{\lambda^n+1}=\frac{2\lambda}{\lambda+1}$$

$$\Rightarrow \quad (\lambda+1)(\lambda^{n+1}+1)=2\lambda(\lambda^n+1)$$

$$\Rightarrow \quad \lambda^{n+2}+\lambda+\lambda^{n+1}+1=2\lambda^{n+1}+2\lambda$$

$$\Rightarrow \quad \lambda^{n+2}-\lambda^{n+1}-\lambda+1=0$$

$$\Rightarrow \quad \lambda^{n+1}(\lambda-1)-1(\lambda-1)=0$$

$$\Rightarrow \quad (\lambda-1)(\lambda^{n+1}-1)=0$$

$$\Rightarrow \quad \lambda-1 \neq 0 \qquad [\because a \neq b]$$

$$\therefore \quad \lambda^{n+1}-1=0$$

$$\Rightarrow \quad \lambda^{n+1}=1=\lambda^0$$

$$\Rightarrow \quad n+1=0 \text{ or } n=-1$$

**Example 76.** Insert 6 harmonic means between 3 and $\frac{6}{23}$.

***Sol.*** Let $H_1, H_2, H_3, H_4, H_5, H_6$ be 6 HM's between 3 and $\frac{6}{23}$.

Then, 3, $H_1, H_2, H_3, H_4, H_5, H_6, \frac{6}{23}$ are in HP.

$$\Rightarrow \quad \frac{1}{3}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \frac{1}{H_4}, \frac{1}{H_5}, \frac{1}{H_6}, \frac{23}{6} \text{ are in AP.}$$

Let common difference of this AP be $D$.

$$\therefore \quad D = \frac{\frac{23}{6} - \frac{1}{3}}{7} = \frac{(23-2)}{7 \times 6} = \frac{21}{7 \times 6} = \frac{1}{2}$$

$$\therefore \quad \frac{1}{H_1} = \frac{1}{3} + D = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

$$\Rightarrow \quad H_1 = \frac{6}{5} = 1\frac{1}{5}$$

$$\frac{1}{H_2} = \frac{1}{3} + 2D = \frac{1}{3} + 1 = \frac{4}{3} \Rightarrow H_2 = \frac{3}{4}$$

$$\frac{1}{H_3} = \frac{1}{3} + 3D = \frac{1}{3} + \frac{3}{2} = \frac{11}{6} \Rightarrow H_3 = \frac{6}{11}$$

$$\frac{1}{H_4} = \frac{1}{3} + 4D = \frac{1}{3} + 2 = \frac{7}{3} \Rightarrow H_4 = \frac{3}{7}$$

$$\frac{1}{H_5} = \frac{1}{3} + 5D = \frac{1}{3} + \frac{5}{2} = \frac{17}{6} \Rightarrow H_5 = \frac{6}{17}$$

and $$\frac{1}{H_6} = \frac{1}{3} + 6D = \frac{1}{3} + 3 = \frac{10}{3} \Rightarrow H_6 = \frac{3}{10}$$

$\therefore$ HM's are $1\frac{1}{5}, \frac{3}{4}, \frac{6}{11}, \frac{3}{7}, \frac{6}{17}, \frac{3}{10}$.

## Important Theorem 1

Let $a$ and $b$ be two real, positive and unequal numbers and $A, G$ and $H$ are arithmetic, geometric and harmonic means respectively between them, then

**(i)** $A, G, H$ form a GP i.e., $G^2 = AH$ [Remember]

**(ii)** $A > G > H$ [Remember]

**Proof**

(i) $\because \quad A = \frac{a+b}{2}, G = \sqrt{ab}$ and $H = \frac{2ab}{a+b}$

Now, $$AH = \left(\frac{a+b}{2}\right)\left(\frac{2ab}{a+b}\right) = ab = G^2$$

Therefore, $G^2 = AH$ i.e. $A, G, H$ are in GP.

### *Remark*

The result $AH = G^2$ will be true for $n$ numbers, if they are in GP.

(ii) $\because \quad A > G$ [from important theorem of GM] ...(i)

or $\quad \frac{A}{G} > 1$

$\Rightarrow \quad \frac{G}{H} > 1 \quad \left[\because \frac{A}{G} = \frac{G}{H} \Rightarrow G^2 = AH\right]$

$\Rightarrow \quad G > H$ ...(ii)

From Eqs. (i) and (ii), we get

$$A > G > H$$

### *Remark*

If $a_1, a_2, a_3, \ldots, a_n$ are $n$ positive numbers, then AM $\geq$ GM $\geq$ HM i.e.,

$$\frac{a_1 + a_2 + \ldots + a_n}{n} \geq (a_1 a_2 \ldots a_n)^{1/n} \geq \frac{n}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)}$$

Sign of equality (AH = GM = HM) holds when numbers are equal i.e., $a_1 = a_2 = \ldots = a_n$.

## Important Theorem 2

If $A, G, H$ are arithmetic, geometric and harmonic means of three given numbers $a, b$ and $c$, then the equation having $a, b, c$ as its roots is

$$\boldsymbol{x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0} \quad \text{[Remember]}$$

**Proof** $\because A =$ AM of $a, b, c = \frac{a+b+c}{3}$

i.e., $\quad a + b + c = 3A$ ...(i)

$G =$ GM of $a, b, c = (abc)^{1/3}$

i.e. $\quad abc = G^3$ ...(ii)

and $\quad H =$ HM of $a, b, c$

$$= \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} = \frac{3abc}{ab + bc + ca} = \frac{3G^3}{ab + bc + ca}$$

[from Eq. (ii)]

i.e. $\quad ab + bc + ca = \frac{3G^3}{H}$ ...(iii)

$\therefore a, b, c$ are the roots of the equation

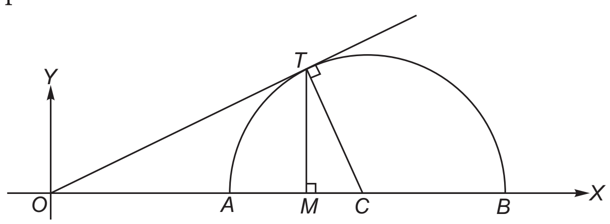
$$x^3 - (a + b + c)\, x^2 + (ab + bc + ca)\, x - abc = 0$$

i.e., $$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

[from Eqs. (i), (ii) and (iii)]

**Geometrical Proof of $A > G > H$**

Let $OA = a$ unit and $OB = b$ unit and $AB$ be a diameter of semi-circle. Draw tangent $OT$ to the circle and $TM$ perpendicular to $AB$.




Let $C$ be the centre of the semi-circle.

$$\because \quad \frac{OA + OB}{2} = \frac{(OC - AC) + (OC + CB)}{2}$$

$$= \frac{2\,OC}{2} = OC \quad [\because AC = CB = \text{radius of circle}]$$

$\therefore \quad OC = \dfrac{a+b}{2}$ [i.e. $OC$ = arithmetic mean]

$\Rightarrow \quad A = \dfrac{a+b}{2}$

Now, from geometry

$$(OT)^2 = OA \times OB = ab = G^2$$

$\therefore \quad OT = G$, the geometric mean

Now, from similar $\Delta OCT$ and $\Delta OMT$, we have

$$\frac{OM}{OT} = \frac{OT}{OC} \text{ or } OM = \frac{(OT)^2}{OC} = \frac{ab}{\frac{a+b}{2}} = \frac{2ab}{a+b}$$

$\therefore \quad OM = H$, the harmonic mean

Also, it is clear from the figure, that

$$OC > OT > OM \text{ i.e. } A > G > H$$

**Example 77.** If $A^x = G^y = H^z$, where $A, G, H$ are AM, GM and HM between two given quantities, then prove that $x, y, z$ are in HP.

***Sol.*** Let $A^x = G^y = H^z = k$

Then, $\quad A = k^{1/x}, G = k^{1/y}, H = k^{1/z}$

$\because \quad G^2 = AH \Rightarrow (k^{1/y})^2 = k^{1/x} \cdot k^{1/z}$

$\Rightarrow \quad k^{2/y} = k^{1/x + 1/z} \Rightarrow \dfrac{2}{y} = \dfrac{1}{x} + \dfrac{1}{z} \Rightarrow \dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ are in AP.

Hence, $x, y, z$ are in HP.

**Example 78.** The harmonic mean of two numbers is 4, their arithmetic mean $A$ and geometric mean $G$ satisfy the relation $2A + G^2 = 27$. Find the numbers.

***Sol.*** Let the numbers be $a$ and $b$.

Given, $\quad H = 4$

$\because \quad G^2 = AH = 4A$ ...(i)

and given $\quad 2A + G^2 = 27$

$\Rightarrow \quad 2A + 4A = 27$ [from Eq. (i)]

$\therefore \quad A = \dfrac{9}{2}$

From Eq. (i), $G^2 = 4 \times \dfrac{9}{2} = 18$

Now, from important theorem of GM

$$a, b = A \pm \sqrt{(A^2 - G^2)} = \frac{9}{2} \pm \sqrt{\left(\frac{81}{4} - 18\right)}$$

$$= \frac{9}{2} \pm \frac{3}{2} = 6, 3 \text{ or } 3, 6$$

**Example 79.** If the geometric mean is $\dfrac{1}{n}$ times the harmonic mean between two numbers, then show that the ratio of the two numbers is

$$1 + \sqrt{(1-n^2)} : 1 - \sqrt{(1-n^2)}.$$

***Sol.*** Let the two numbers be $a$ and $b$.

Given, $\quad G = \dfrac{1}{n} H$ ...(i)

Now, $\quad G^2 = AH$

$\Rightarrow \quad \dfrac{H^2}{n^2} = AH$ [from Eq. (i)]

$\therefore \quad A = \dfrac{H}{n^2}$ ...(ii)

Now, from important theorem of GM

$$a, b = A \pm \sqrt{(A^2 - G^2)} = \frac{H}{n^2} \pm \sqrt{\left(\frac{H^2}{n^4} - \frac{H^2}{n^2}\right)}$$

$$= \frac{H}{n^2}[1 \pm \sqrt{(1-n^2)}]$$

$$\therefore \quad \frac{a}{b} = \frac{\frac{H}{n^2}[1 + \sqrt{(1-n^2)}]}{\frac{H}{n^2}[1 - \sqrt{(1-n^2)}]}$$

$\therefore \quad a : b = 1 + \sqrt{(1-n^2)} : 1 - \sqrt{(1-n^2)}$

**Example 80.** If three positive unequal quantities $a, b, c$ be in HP, then prove that $a^n + c^n > 2b^n, n \in N$

***Sol.*** $\because \quad G > H$

$\therefore \quad \sqrt{ac} > b$

$\Rightarrow \quad (ac)^{\frac{n}{2}} > b^n$ or $a^{\frac{n}{2}} c^{\frac{n}{2}} > b^n$ ...(i)

Also, $\quad (a^{\frac{n}{2}} - c^{\frac{n}{2}})^2 > 0 \Rightarrow a^n + c^n - 2a^{\frac{n}{2}} c^{\frac{n}{2}} > 0$

$\Rightarrow \quad a^n + c^n > 2a^{\frac{n}{2}} c^{\frac{n}{2}} > 2b^n$ [from Eq. (i)]

$\therefore \quad a^n + c^n > 2b^n$

**Example 81.**

(i) If $a, b, c, d$ be four distinct positive quantities in AP, then

(a) $bc > ad$

(b) $c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(ii) If $a, b, c, d$ be four distinct positive quantities in GP, then

(a) $a + d > b + c$

(b) $c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(iii) If $a, b, c, d$ be four distinct positive quantities in HP, then

(a) $a + d > b + c$ (b) $ad > bc$

***Sol.*** (i) $\because a, b, c, d$ are in AP.

(a) Applying AM > GM

For first three members, $b > \sqrt{ac}$

$\Rightarrow \quad b^2 > ac$ ...(i)

and for last three members, $c > \sqrt{bd}$

$\Rightarrow \quad c^2 > bd \quad$ ...(ii)

From Eqs. (i) and (ii), we get

$$b^2c^2 > (ac)(bd)$$

Hence, $\quad bc > ad$

(b) Applying AM > HM

For first three members,

$$b > \frac{2ac}{a+c}$$

$\Rightarrow \quad ab + bc > 2ac \quad$ ...(iii)

For last three members, $c > \dfrac{2bd}{b+d}$

$$bc + cd > 2bd \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get

$$ab + bc + bc + cd > 2ac + 2bd$$

or $\quad ab + cd > 2(ac + bd - bc)$

Dividing in each term by $abcd$, we get

$c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(ii) $\because$ $a, b, c, d$ are in GP.

(a) Applying AM > GM

For first three members,

$$\frac{a+c}{2} > b$$

$\Rightarrow \quad a + c > 2b \quad$ ...(v)

For last three members, $\dfrac{b+d}{2} > c$

$\Rightarrow \quad b + d > 2c \quad$ ...(vi)

From Eqs. (v) and (vi), we get

$a + c + b + d > 2b + 2c \;$ or $\; a + d > b + c$

(b) Applying GM > HM

For first three members, $b > \dfrac{2ac}{a+c}$

$\Rightarrow \quad ab + bc > 2ac \quad$ ...(vii)

For last three members, $c > \dfrac{2bd}{b+d}$

$\Rightarrow \quad bc + cd > 2bd \quad$ ...(viii)

From Eqs. (vii) and (viii), we get

$$ab + bc + bc + cd > 2ac + 2bd$$

or $\quad ab + cd > 2(ac + bd - bc)$

Dividing in each term by $abcd$, we get

$c^{-1}d^{-1} + a^{-1}b^{-1} > 2(b^{-1}d^{-1} + a^{-1}c^{-1} - a^{-1}d^{-1})$

(iii) $\because$ $a, b, c, d$ are in HP.

(a) Applying AM > HM

For first three members,

$$\frac{a+c}{2} > b$$

$\Rightarrow \quad a + c > 2b \quad$ ...(ix)

For last three members, $\dfrac{b+d}{2} > c$

$\Rightarrow \quad b + d > 2c \quad$ ...(x)

From Eqs. (ix) and (x), we get

$$a + c + b + d > 2b + 2c$$

or $\quad a + d > b + c$

(b) Applying GM > HM

For first three members, $\sqrt{ac} > b$

$\Rightarrow \quad ac > b^2 \quad$ ...(xi)

For last three members,

$$\sqrt{bd} > c$$

$\Rightarrow \quad bd > c^2 \quad$ ...(xii)

From Eqs. (xi) and (xii), we get

$$(ac)(bd) > b^2c^2$$

or $\quad ad > bc$

# *Exercise for Session 5*

**1.** If the AM of two positive numbers $a$ and $b$ $(a > b)$ is twice of their GM, then $a : b$ is

(a) $2 + \sqrt{3} : 2 - \sqrt{3}$
(b) $7 + 4\sqrt{3} : 7 - 4\sqrt{3}$
(c) $2 : 7 + 4\sqrt{3}$
(d) $2 : \sqrt{3}$

**2.** If $A_1, A_2; G_1, G_2$ and $H_1, H_2$ are two arithmetic, geometric and harmonic means, respectively between two quantities $a$ and $b$, then which of the following is not the value of $ab$ is?

(a) $A_1 H_2$
(b) $A_2 H_1$
(c) $G_1 G_2$
(d) None of these

**3.** The GM between $-9$ and $-16$, is

(a) 12
(b) $-12$
(c) $-13$
(d) None of these

**4.** Let $n \in N, n > 25$. If $A, G$ and $H$ denote the arithmetic mean, geometric mean and harmonic mean of 25 and $n$. Then, the least value of $n$ for which $A, G, H \in \{25, 26, \dots, n\}$, is

(a) 49
(b) 81
(c) 169
(d) 225

**5.** If 9 harmonic means be inserted between 2 and 3, then the value of $A + \dfrac{6}{H} + 5$ (where $A$ is any of the AM's and $H$ is the corresponding HM), is

(a) 8
(b) 9
(c) 10
(d) None of these

**6.** If $H_1, H_2, \dots, H_n$ be $n$ harmonic means between $a$ and $b$, then $\dfrac{H_1 + a}{H_1 - a} + \dfrac{H_n + b}{H_n - b}$ is

(a) $n$
(b) $n + 1$
(c) $2n$
(d) $2n - 2$

**7.** The AM of two given positive numbers is 2. If the larger number is increased by 1, the GM of the numbers becomes equal to the AM to the given numbers. Then, the HM of the given numbers is

(a) $\dfrac{3}{2}$
(b) $\dfrac{2}{3}$
(c) $\dfrac{1}{2}$
(d) 2

**8.** If $a, a_1, a_2, a_3, \dots, a_{2n}, b$ are in AP and $a, b_1, b_2, b_3, \dots, b_{2n}, b$ are in GP and $h$ is the HM of $a$ and $b$, then $\dfrac{a_1 + a_{2n}}{b_1 b_{2n}} + \dfrac{a_2 + a_{2n-1}}{b_2 b_{2n-1}} + \dots + \dfrac{a_n + a_{n+1}}{b_n b_{n+1}}$ is equal to

(a) $\dfrac{2n}{h}$
(b) $2nh$
(c) $nh$
(d) $\dfrac{n}{h}$