12 Linear Programming

“ *Transportation systems rely upon linear programming for cost and time efficiency. They factor in scheduling travel time and passengers, optimize their profits according to different seat prices and customer demand, pilot/driver scheduling and routes, etc. This method of solving problems using linear programming has applications in a number of industries, such as business, supply-chain management, hospitality, cooking, farming, and crafting.*

Topic Notes

- *Basic Concepts of Linear Programming*

BASIC CONCEPTS OF LINEAR PROGRAMMING

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TOPIC 1

LINEAR INEQUALITIES

Linear Programming is a mathematical technique that determines the best way to use available resources. Managers use the process to help make decisions about the most efficient use of limited resources, like money, time, materials, and machinery etc.

The term 'programming' refers to 'planning', *i.e.*, a plan of action for maximizing or minimizing a function under given constraints. The term 'linear' means all constraints in the form of inequalities or equations must be linear.

In Class XI, we have studied linear inequalities and system of linear inequalities in two variables and their solution by graphical method. Now we shall apply the same in solving linear programming problems.

In this chapter, we shall restrict our study to Linear Programming Problems in two variables and up to three non-trivial constraints.

(1) Let a and b be two real numbers and x be a variable. Then, $ax < b$, $ax > b$, $ax \leq b$ and $ax \geq b$ are called linear inequalities in one variable.

E.g., $2x < 5$, $4x > 9$, $8x \leq 13$ and $7x \geq 8$ are linear inequalities in one variable.

(2) Let a, b and c be three real numbers and x, y be two variables. Then,

$ax + by < c$, $ax + by > c$, $ax + by \leq c$ and $ax + by \geq c$ are called linear inequalities in two variables. *E.g.*, $x + 2y < 8$, $2x + 3y > 7$, $2x + 3y \leq 10$ and $7x - 2y \geq 1$ are linear inequalities in two variables.

TOPIC 2

GRAPHICAL SOLUTION OF A LINEAR INEQUALITY IN TWO VARIABLES

A line $ax + by = c$ lying in a plane divides a plane into the following three disjoint sets.

(1) A half plane $ax + by < c$

(2) A half plane $ax + by > c$

(3) The line $ax + by = c$.

The half planes, are of two types, viz,. open half planes and closed half planes.

(1) The inequalities $ax + by < c$ and $ax + by > c$ represent open half planes. (The points lying on the line $ax + by = c$ are not included)

(2) The inequalities $ax + by \leq c$ and $ax + by \geq c$ represent closed half planes. (The points lying on the line $ax + by = c$ are included)

So, a linear inequality in two variables represents a half plane.

Now, to solve a linear inequality in two variables, we have the following working rule:

Step 1: Write the given inequality as an equation and sketch its graph which is a straight line.

Step 2: If the given inequality is a strict inequality (inequality containing < or >), then the points on the line are not included in the solution region. So, we represent this by drawing a dotted line.

If the given inequality is a slack inequality (inequality containing ≤ or ≥), then the points on the line are included in the solution region. So, we represent this by drawing a solid line.

Step 3: The line will divide the xy-plane into two half-planes.

Step 4: If the point (0, 0) satisfies the linear inequality, then the half plane containing the origin will be the solution region.

Step 5: If the point (0, 0) does not satisfy the linear inequality, then the other half plane not containing the origin will be the solution region.

We represent the solution region by shading it.

We can extend the above method to solve a system of linear inequalities in two variables, using the following working rule:

Shade the solution region of each of the linear inequality.

The common region will represent the solution of the system of linear inequalities.

Illustration: Solve the following system of linear inequalities graphically:

$$2x + 3y \leq 6,\ \ 3x - 2y \leq 6,\ y \leq 1,\ x \geq 0,\ y \geq 0$$

We consider the following equations:

$$2x + 3y = 6,\ \ 3x - 2y = 6,\ y = 1,\ x = 0,\ y = 0$$

$2x + 3y = 6$

x	3	0
y	0	2

$3x - 2y = 6$

x	2	0
y	0	-3

Solution region of $2x + 3y \le 6$

The line $2x + 3y = 6$ divides the xy-plane into two half planes.

The point (0, 0) does not lie on the line, but satisfy the inequality $2x + 3y \le 6$.

So, the half plane containing the origin is the solution region of the given inequality.

Solution region of $3x - 2y \le 6$

The line $3x - 2y = 6$ divides the xy-plane into two half planes.

The point (0, 0) does not lie on the line, but satisfy the inequality $3x - 2y \le 6$.

So, the half plane containing the origin is the solution region of the given inequality.

Solution region of $y \le 1$

The line $y = 1$ is a horizontal line and the solution region of the given inequality is the half plane below the line $y = 1$. Also, points on the line satisfy the given inequality.

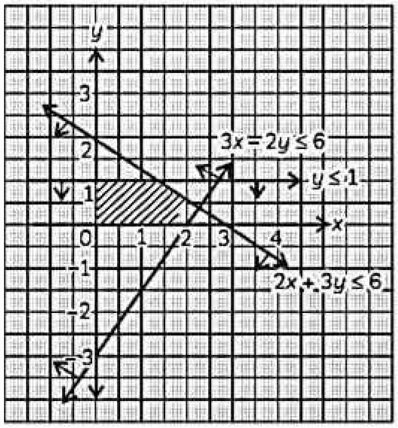
Solution region of $x \ge 0$

The line $x = 0$ is a vertical line and the solution region of the given inequality is the half plane to the right of the line $x = 0$. Also, points on the line satisfy the given inequality.

Solution region of $y \ge 0$

The line $y = 0$ is a horizontal line and the solution region of the given inequality is the half plane above the line $y = 0$ *i.e.*, x-axis. Also, points on the line satisfy the given inequality.

Hence the required solution region is shown shaded in the graph.



TOPIC 3

A LINEAR PROGRAMMING PROBLEM (LPP) AND A GRAPHICAL METHOD OF SOLVING IT

A Linear Programming Problem is concerned with finding the optimal value (maximum or minimum value) of a linear function (called objective function) of several variables (say x and y), subject to the conditions that the variables are non-negative and satisfy a set of linear inequalities.

Standard Form of a LPP

Here both objective function and constraints must be linear equations/inequalities. This means that power of any variable must not be greater than one.

Three Components in LPP

The three major components in LPP are decision variables, objective function and constraints.

A general LPP in two variables and up to three constraints is of the form:

Maximise (or Minimise) $Z = ax + by$} **Objective function**

subject to the constraints

$$\left.\begin{array}{l} a_1x + b_1y \{=, <, \le, >, \ge\}\ c_1 \\ a_2x + b_2y \{=, <, \le, >, \ge\}\ c_2 \\ a_3x + b_3y \{=, <, \le, >, \ge\}\ c_3 \end{array}\right\} \textbf{Constraints}$$

$x \ge 0, y \ge 0$ } **Non-negative restrictions**

How Do We maximize (or minimize) a Linear Programming Problem (LPP)?

(1) Write the objective function.

(2) Write the constraints.

(3) Graph the constraints and ascertain the valid side of all constraints.

(4) Identify the region of feasible solution and shade it.

(5) Identify the corner points.

(6) Determine the corner point(s) that gives the maximum (or minimum) value.

Before we proceed further, we now formally define some important terms (which have been used above) which we shall be using in the linear programming problems.

Objective Function

Linear function $Z = ax + by$, where a, b are constants, which has to be maximized or minimized is called objective function. Variables x and y are called decision variables.

Constraints

The linear inequalities or equations on the variables of a LPP are called constraints. The conditions $x \geq 0$, $y \geq 0$ are called non-negative restrictions.

Optimization Problem

A problem which seeks to maximize or minimize a linear function (say, of two variables x and y) subject to certain constraints as determined by a set of linear inequalities is called an optimization problem. Linear Programming Problems are special type of optimization problems.

Corner Point Method

This is the graphical method of solving a linear programming problem (LPP). In this method, we shall make use of the following terms. So, it is necessary that we have clear understanding of these terms before making attempt to solve a LPP.

(1) **Feasible Region:** The common region determined by the constraints and non-negative restrictions is called the feasible region.

(2) **Feasible Solution:** A set of values of decision variables of LPP satisfying the constraints and the non-negative restrictions is called feasible solution. Every point in the feasible region is a feasible solution of the given LPP.

(3) **Optimal Feasible Solution/Optimal Solution:** A feasible solution of LPP is said to be optimal feasible solution (or optimal solution) if it optimises (*i.e.,* maximises or minimises) the objective function.

(4) **Bounded Feasible Region:** A feasible region of a system of linear inequalities is said to be bounded feasible region if it can be enclosed within a circle.

(5) **Unbounded Feasible Region:** A feasible region of a system of linear inequalities is said to be unbounded feasible region if it cannot be enclosed within a circle, *i.e.,* if it extends indefinitely in any direction.

(6) **Corner Point:** A corner point of a feasible region is a point in the region which is the intersection of two boundary lines.

The following two theorems are fundamental in solving an LPP (The proofs of these theorems are beyond the scope of the present syllabus):

Theorem 1

Let R be the feasible region (convex region) for LPP and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2

Let R be the feasible region (convex region) for LPP and let $Z = ax + by$ be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum values on R and each of these occurs at corner point (vertex) of the feasible region.

Working Procedure: The corner Point Method for solving LPP involving two decision variables x and y comprises of the following steps:

Step 1: Find the feasible region of the LPP and determine its corner points (vertices).

Step 2: Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m be the greatest and the smallest values of Z at these corner points.

Step 3: When the feasible region is bounded, M and m are the maximum and minimum values of the objective function.

Step 4: When the feasible region is unbounded, then

- M is the maximum value of Z, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, Z has no maximum value.
- m is the minimum value of Z, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, Z has no minimum value.

Important

➥ *If Z has optimum (i.e., maximum or minimum) value at any two corner points, then it has the optimum value at all points on the line segment joining these two corner points.*

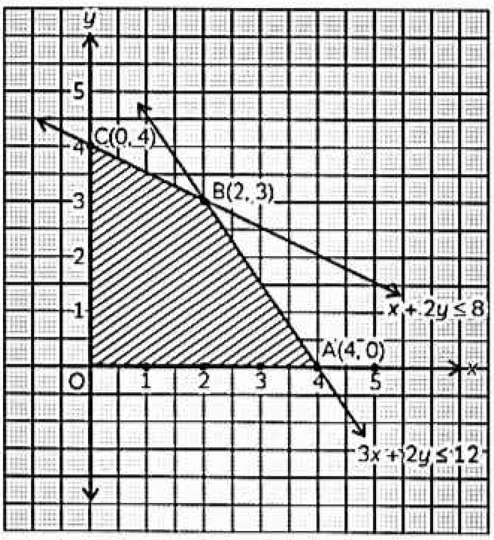
Illustration: Solve the following linear programming problem (LPP) graphically:

Maximise and Minimise $Z = -3x + 4y$

subject to the constraints

$x + 2y \leq 8$, $3x + 2y \leq 12$, $x \geq 0, y \geq 0$ **[NCERT]**

The shaded region in the following graph is the feasible region determined by the given system of constraints.



OABC is the bounded feasible region. The coordinates of O, A, B and C are (0, 0), (4, 0), (2, 3) and (0, 4) respectively.

Corner point	Corresponding value of Z
(0, 0)	0
(4, 0)	–12 (minimum)
(2, 3)	6
(0, 4)	14 (maximum)

Hence, maximum value of Z is 14 which occurs at (0, 4); and the minimum value of Z is –12 which occurs at (4, 0).

Example 1.1: Solve the following linear programming problem (LPP) graphically:

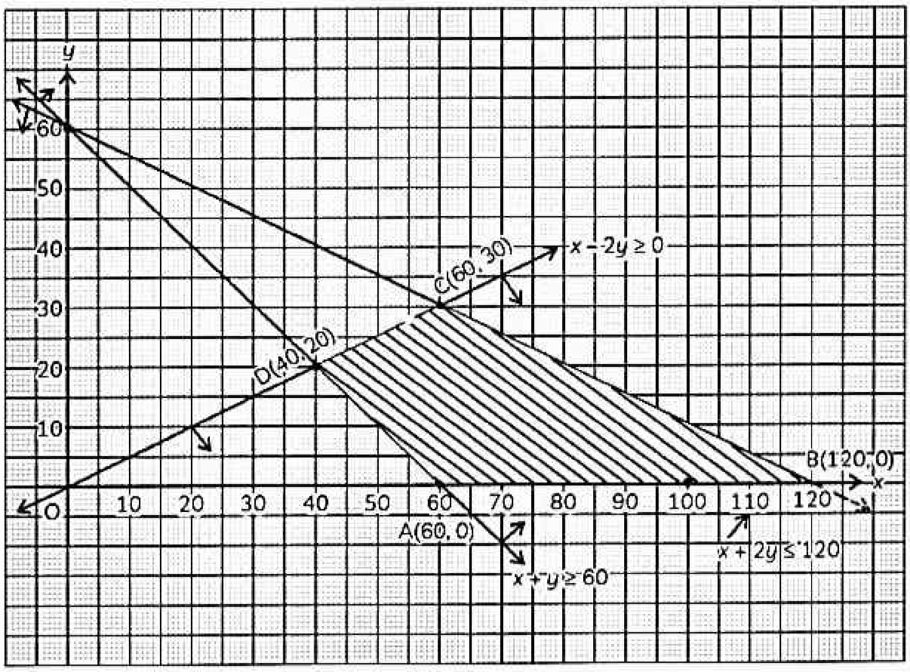
Maximise and Minimise Z = $5x + 10y$

subject to the constraints

$x + 2y \le 120, x + y \ge 60, x - 2y \ge 0, x \ge 0, y \ge 0$

[NCERT]

Ans. The shaded region in the following graph is the feasible region determined by the given system of constraints.



ABCD is the bounded feasible region. The coordinates of A, B, C and D are (60, 0), (120, 0), (60, 30) and (40, 20) respectively.

Corner point	Corresponding value of Z
(60, 0)	300 (minimum)
(120, 0)	600 (maximum)
(60, 30)	600 (maximum)
(40, 20)	400

Hence, the minimum value of Z is 300 which occurs at (60, 0).

But maximum value of Z is 600 which occurs at two points namely, (120, 0) and (60, 30). So, the maximum value of Z occurs at all points of the line segment joining the points (120, 0) and (60, 30).

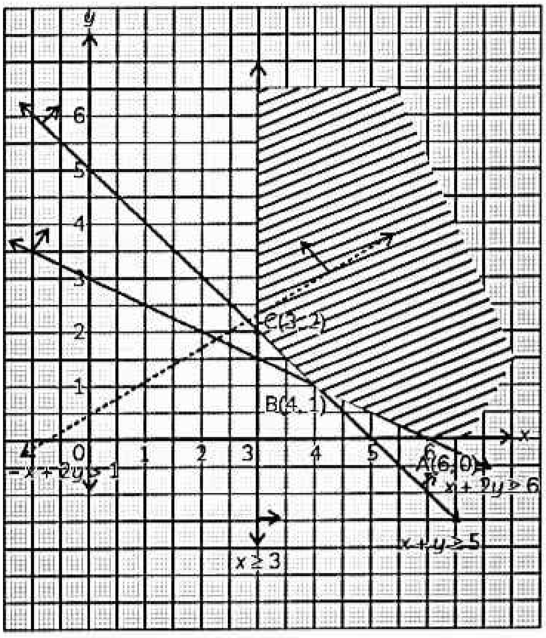
Example 1.2: Solve the following linear programming problem (LPP) graphically:

Maximise Z = $-x + 2y$

Subject to the constraints

$x \ge 3, x + y \ge 5, x + 2y \ge 6, y \ge 0$ **[NCERT]**

Ans. The shaded region in the following graph is the feasible region determined by the given system of constraints.



ABC is the unbounded feasible region. The coordinates of A, B and C are (6, 0), (4, 1) and (3, 2) respectively.

Corner point	Corresponding value of Z
(6, 0)	–6
(4, 1)	–2
(3, 2)	1

We see that maximum value of Z is 1 which occurs at (3, 2).

Since the feasible region is unbounded, 1 may or may not be the maximum value of Z.

We, therefore, first draw the graph of the inequality $-x + 2y > 1$.

The graph of $-x + 2y > 1$ is away from the origin.

Since the open half plane of $-x + 2y > 1$ has points in common with the feasible region, so, 1 is not the maximum value of Z.

Hence, Z has no maximum value.

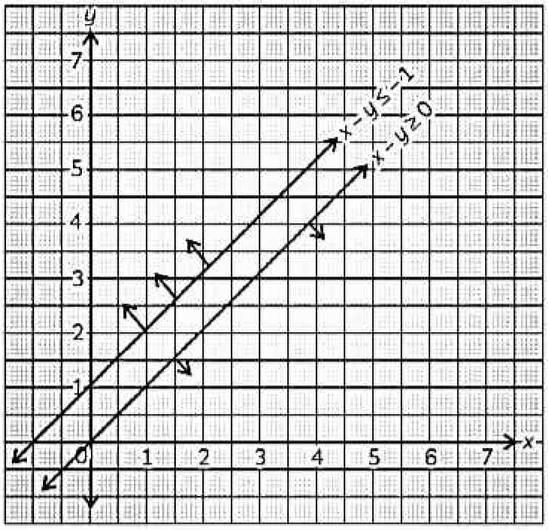
Example 1.3: Solve the following linear programming problem (LPP) graphically:

Maximise Z = $x + y$

subject to the constraints

$x - y \geq 0, x - y \leq -1, x, y \geq 0$ **[NCERT]**

Ans. The shaded region in the following graph is the feasible region determined by the given system of constraints.



There is no common feasible region.

Thus, there is no maximum value of Z.

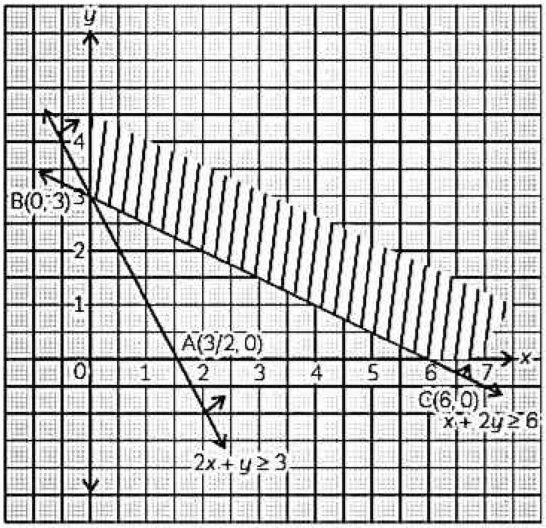
Example 1.4: Solve the following linear programming problem (LPP) graphically:

Minimise Z = $x + 2y$

subject to the constraints

$2x + y \geq 3, x + 2y \geq 6,\ x \geq 0, y \geq 0$. **[NCERT]**

Ans. The shaded region in the following graph is the feasible region determined by the given system of constraints.



BC is the unbounded feasible region. The coordinates of B and C are (0, 3) and (6, 0) respectively.

Corner point	Corresponding value of Z
(0, 3)	6 (minimum)
(6, 0)	6 (minimum)

Hence, Z is minimum at (6, 0) and (0, 3) and minimum value is 6.

Since the feasible region is unbounded, 6 may or may not be the minimum value of Z.

We, therefore, first draw the graph of the inequality $x + 2y < 6$.

Since the open half plane of $x + 2y < 6$ has no points in common with the feasible region, 6 is the minimum value of Z which occurs at all points on the line segment joining (6, 0) and (0, 3).

OBJECTIVE Type Questions

[**1** mark]

Multiple Choice Questions

1. The solution set of the inequality $3x + 5y < 4$ is:

(a) open half plane not containing the origin.

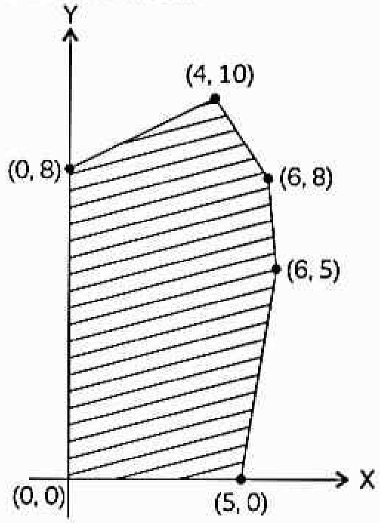
(b) whole xy-plane except the points lying on the line $3x + 5y = 4$.

(c) open half plane containing the origin.

(d) none of these.

Ans. *(c) open half plane containing the origin.*

2. In the given graph, the feasible region for a LPP is shaded.

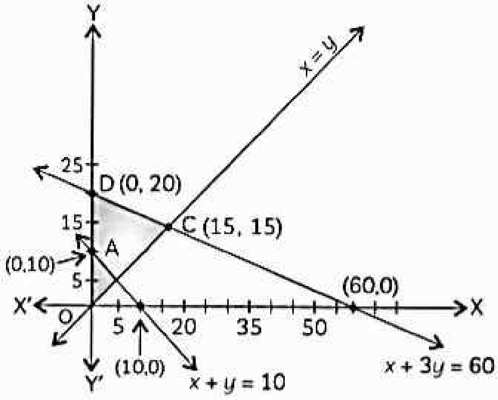


The objective function $Z = 2x - 3y$, will be minimum at:

(a) (4, 10) (b) (6, 8)
(c) (0, 8) (d) (6, 5)

[CBSE Term-1 SQP 2021]

3. Based on the given shaded region as the feasible region in the graph, at which point(s) is the objective function $Z = 3x + 9y$ maximum?



(a) Point B
(b) Point C
(c) Point D
(d) Every point on the line segment CD

[CBSE Term-1 SQP 2021]

Ans. (d) *every point on the line segment CD*

Z is maximum 180 at points C (15, 15) and D (0, 20).

⇒ Z is maximum at every point on the line segment CD

[CBSE Marking Scheme Term-1 SQP 2021]

Explanation: The corner points of the feasible region are A(0, 10), B(5, 5), C(15, 15) and D(0, 20).

Corner Points	Objective Function $Z = 3x + 9y$
A (0, 10)	90
B (5, 5)	60
C (15, 15)	180 (maximum)
D (0, 20)	180 (maximum)

Since maximum value of Z occurs at two points, namely C and D, so every point on the line segment joining the points C and D gives the maximum value.

4. Corner points of the feasible region for an LPP are (0, 2), (3, 0), (6, 0), (6, 8) and (0, 5). Let $F = 4x + 6y$ be objective function. The Minimum value of F occurs at:

(a) (0, 2) only
(b) (3, 0) only
(c) the mid point of the line segment joining the points (0, 2) and (3, 0) only.
(d) any point on the line segment joining the points (0, 2) and (3, 0). [NCERT Exemplar]

Ans. (d) *any point on the line segment joining the points (0, 2) and (3, 0).*

Explanation: Here, objective function is

$$F = 4x + 6y$$

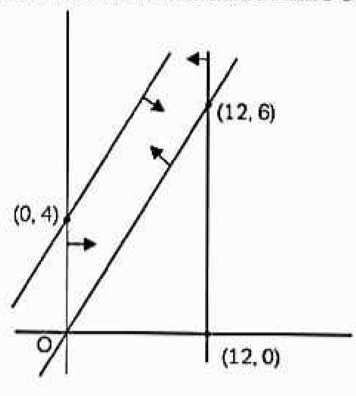
Corner points	$F = 4x + 6y$
(0, 2)	12 (Minimum)
(3, 0)	12 (Minimum)
(6, 0)	24
(6, 8)	72 (Maximum)
(0, 5)	30

Hence the minimum value of F occurs at points on the line segment joining the points (0, 2) and (3, 0).

⚠ Caution

- *There could be more than one point at which minimum value occur.*

5. The feasible region for an LPP is shown in the figure below. Let $F = 3x - 4y$ be the objective function. Maximum value of F is:



(a) 0 (b) 8
(c) 12 (d) –18

[NCERT Exemplar]

6. In a linear programming problem, the constraints on the decision variables x and y are $x - 3y \geq 0$, $y \geq 0$, $0 \leq x \leq 3$. The feasible region:

(a) is not in the first quadrant.
(b) is bounded in the first quadrant.
(c) is unbounded in the first quadrant.
(d) does not exist.

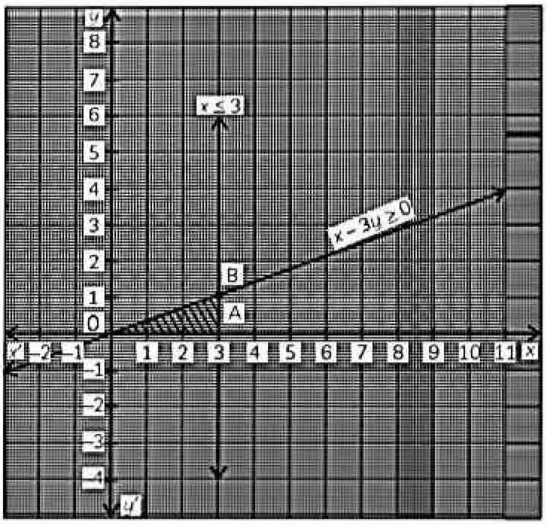
[CBSE Term-1 SQP 2021]

Ans. *(b) is bounded in the first quadrant*

Feasible region is bounded in the first quadrant.

[CBSE Marking Scheme Term-1 SQP 2021]

Explanation: The graphical representation of linear inequations $x - 3y \geq 0$, $y \geq 0$ and $0 \leq x \leq 3$ is shown below.



So, the feasible region is bounded and lies in first quadrant.

7. The feasible region, for the inequalities $x + 2y \leq 6$, $y \geq 0$, $0 \leq x$ lies in:

(a) First Quadrant (b) Second Quadrant
(c) Third Quadrant (d) Fourth Quadrant

[Delhi Gov. 2022]

8. For an objective function $Z = ax + by$, where $a, b > 0$; the corner points of the feasible region determined by a set of constraints (linear inequalities) are (0, 20), (10, 10), (30, 30) and (0, 40). The condition on a and b such that the maximum Z occurs at both the points (30, 30) and (0, 40) is:

(a) $b - 3a = 0$ (b) $a = 3b$
(c) $a + 2b = 0$ (d) $2a - b = 0$

[CBSE Term-1 SQP 2021]

9. The corner points of feasible region for a linear Programming problem are P(0, 5), Q(1, 5), R(4, 2) and S(12, 0). The minimum value of the objective function $Z = 2x + 5y$ is at the point:

(a) P (b) Q
(c) R (d) S

[CBSE Term-1 2021]

10. If the objective function $z = ax + y$ is minimum at (1, 4) and its minimum value is 13, then value of a is:

(a) 1 (b) 4
(c) 9 (d) 13

[Delhi Gov. 2022]

Ans. *(c) 9*

Explanation: At (1, 4), $13 = a(1) + 4 \Rightarrow a = 9$

11. A linear programming problem is as follows:

Minimize $Z = 30x + 50y$

subject to the constraints,

$3x + 5y \geq 15$
$2x + 3y \leq 18$
$x \geq 0, y \geq 0$

In the feasible region, the minimum value of Z occurs at:

(a) a unique point
(b) no point
(c) infinitely many poins
(d) two points only

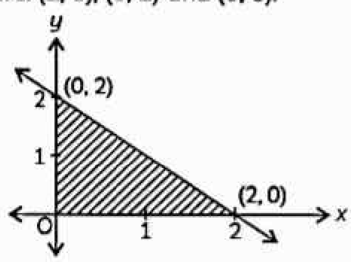
[CBSE Term-1 SQP 2021]

12. If $x + y \leq 2$, $x, y \geq 0$, the point at which maximum value of $3x + 2y$ attained, will be:

(a) (0, 2) (b) (0, 0)
(c) (2, 0) (d) $\left(\frac{1}{2}, \frac{1}{2}\right)$

Ans. *(c) (2, 0)*

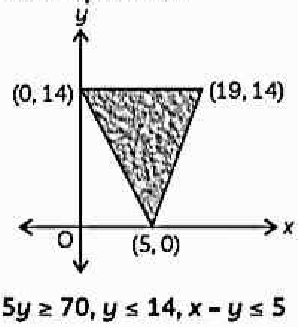
Explanation: The corner points of the feasible region are: (2, 0), (0, 2) and (0, 0).



The values of Z at these corner points are 6, 4 and 0, respectively.

Thus, $Z_{max.} = 6$ at (2, 0).

13. **The shaded region shown in the figure, is given by the inequalities:**



(a) $14x + 5y \geq 70, y \leq 14, x - y \leq 5$

(b) $14x + 5y \geq 70, y \leq 14, x - y \geq 5$

(c) $14x + 5y \leq 70, y \leq 14, x - y \geq 5$

(d) $14x + 5y \geq 70, y \geq 14, x - y \geq 5$

Ans. *(a) $14x + 5y \geq 70, y \leq 14, x - y \leq 5$*

Explanation: Let A(5, 0), B(19, 14) and C(0, 14).

So, equation of line AB is:

$$(y - 0) = \left(\frac{14 - 0}{19 - 5}\right)(x - 5)$$

$\Rightarrow \quad y = x - 5$

$\Rightarrow \quad x - y = 5$

Since, the feasible region of the line AB contains the origin, so its equality is $x - y \leq 5$.

Similarly, inequalities of lines BC and AC are $y \leq 14$ and $14x + 5y \geq 70$, respectively.

14. **Which one of the following is correct?**

(a) Every LPP admits an optimal solution.

(b) A LPP admits a unique solution.

(c) The optimal value occurs at a corner point of the feasible region.

(d) If a LPP admits two optimal solutions, then it has infinite optimal solutions.

Ans. *(d) If a LPP admits two optimal solutions, then it has infinite optimal solutions.*

15. **The optimal value of the objective function is attained at the points:**

(a) given by the intersection of inequalities with x-axis.

(b) given by corner points of the feasible region.

(c) given by the intersection of inequalities with axes only.

(d) none of these

Ans. *(b) given by corner points of the feasible region.*

16. **In a linear programming problem, If the feasible region is bounded then objective function $Z = px + qy$ has:**

(a) Maximum value only

(b) Minimum value only

(c) Maximum and minimum value both

(d) Neither maximum nor minimum value

[Delhi Gov. 2022]

Ans. *(c) Maximum and minimum value both*

Explanation: In a linear programming problem, if the feasible region is bounded then objective function $Z = px + qy$ has maximum and minimum values both.

17. **Which of the following is false?**

(a) The feasible region of a LPP is always a convex polygon.

(b) In a LPP, the constraints are always given by the inequalities.

(c) The minimum value of an objective function $Z = ax + by$ in a LPP always occurs only at the corner point of the feasible region.

(d) If the feasible region of a LPP is bounded, then the objective function $Z = ax + by$ has both maximum and minimum.

18. **Sam manufactures two types of lamp shades, A and B. The maximum number of hours a day required to manufacture two lamp shades of type A and one lamp shade of type B is 10 hours, whereas, the maximum number of hours a day required to manufacture one lamp shade of type A and three lamp shades of type B is 15 hours.**

The corner points of the feasible region determined by a system of linear inequalities are (0, 0), (5, 0), (3, 4) and (0, 5).

Let $Z = px + qy$, where $p, q > 0$

Condition on p and q so that the maximum of Z occurs at both (3, 4) and (0, 5) is:

(a) $p = q$ (b) $p = 2q$

(c) $p = 3q$ (d) $q = 3p$

Ans. *(d) $q = 3p$*

Explanation: Let, the maximum value of Z be M. Then,

At (3, 4), $M = 3p + 4q$ ...(i)

At (0, 5), $M = 5q$ ...(ii)

From (i) and (ii), we have

$3p + 4q = 5q$

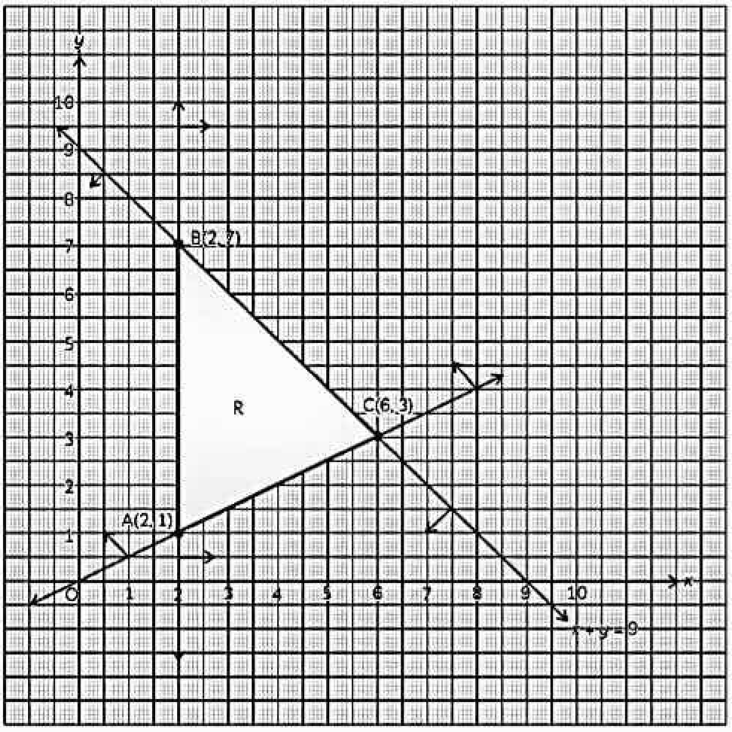
$\Rightarrow \quad 3p = q$

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

19. *In the figure given below, O is the origin, A is the point (2, 1), B is the point (2, 7) and C is the point (6, 3). The shaded region R is defined by three inequalities. One of these three inequalities is $x + y \le 9$.*



(A) **The other two inequalities are:**

(a) $x \ge 2, 2y \ge x$ (b) $x \le 2, 2y \le x$

(c) $x \ge -2, y \ge 2x$ (b) $x \ge -2, 2y \le x$

(B) **Given that the point (x, y) is in the region R, the maximum value of $Z = 3x + y$ is**

(a) 7 (b) 13

(c) 21 (d) 26

(C) **Given that the point (x, y) is in the region R, the minimum value of $P = 5x - 2y$ is**

(a) –8 (b) –4

(c) –24 (d) 8

(D) **Area of the region R is:**

(a) 12 sq. units (b) 15 sq. units

(c) 18 sq. units (d) 24 sq. units

(E) **Which of the following points lie inside the region R?**

(a) (0, 2) (b) (4, 3)

(c) (3, 1) (d) (2, 0)

Ans. (B) *(c) 21*

Explanation: The corner points of the feasible region R are: A(2, 1), B(2, 7) and C(6, 3).

So, $Z_A = 7, Z_B = 13, Z_C = 21$

$\therefore Z_{max} = 21$

(D) *(a) 12 sq. units*

Explanation: Area of region R

$$= \text{Area of } \Delta ABC$$

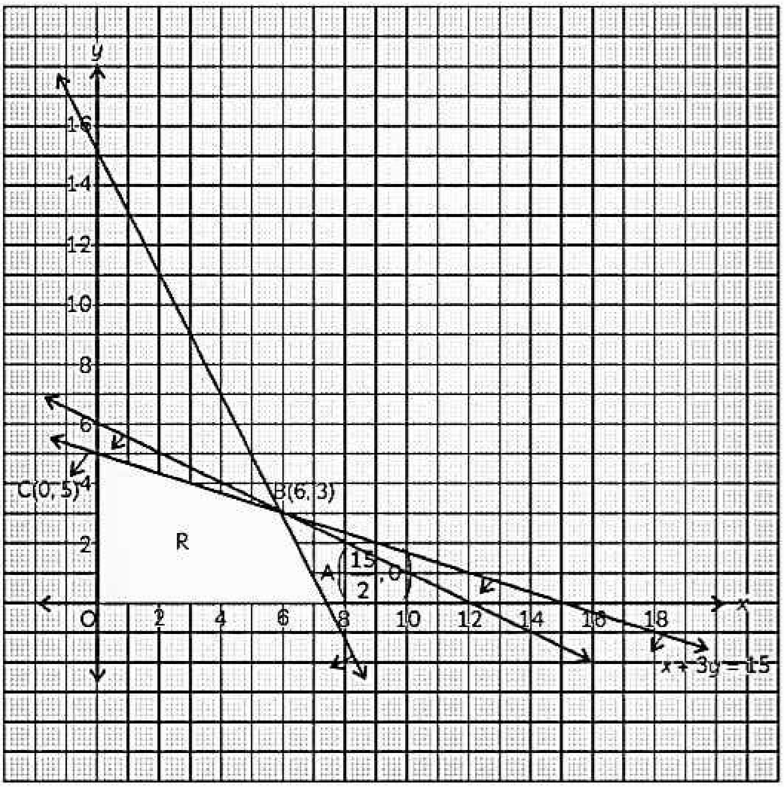
$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ 6 & 3 & 1 \\ 2 & 7 & 1 \end{vmatrix}$$

$$= \frac{1}{2} |2(3-7) - 1(6-2) + 1(42-6)|$$

$$= 12 \text{ sq. units}$$

20. *In the figure, O is the origin, A is the point $\left(\frac{15}{2}, 0\right)$, B(6, 3) and C(0, 5). The shaded feasible region is defined by three non-trivial inequalities and two trivial inequalities. One of the three non-trivial inequalities is $x + 3y \le 15$.*



(A) Write the other two non-trivial inequalities plotted on the graph.

(B) Given that the point (x, y) is in the region R, find the minimum value of $(6x - 5y)$.

Ans. (A) Equation of a line joining the points (0, 6) and (12, 0) is:

$$y - 6 = \frac{0-6}{12-0}(x - 0)$$

$\Rightarrow \quad 2y - 12 = -x$

$\Rightarrow \quad x + 2y = 12$

And, equation of a line joining the points (0, 15) and $\left(\frac{15}{2}, 0\right)$ is:

$$y - 0 = \frac{0-15}{\frac{15}{2}-0}\left(x - \frac{15}{2}\right)$$

$\Rightarrow \quad \frac{15}{2}y = -15\left(x - \frac{15}{2}\right)$

$\Rightarrow \quad y = -2x + 15$

$\Rightarrow \quad 2x + y = 15$

By observing the feasible region R, the inequations of these two equations are $x + 2y \le 12$ and $2x + y \le 15$.

(B) The corner points of the feasible region are $A\left(\frac{15}{2}, 0\right)$, B(6, 3), C(0, 5) and O(0, 0).

Now, $(6x - 5y)_A = 45$

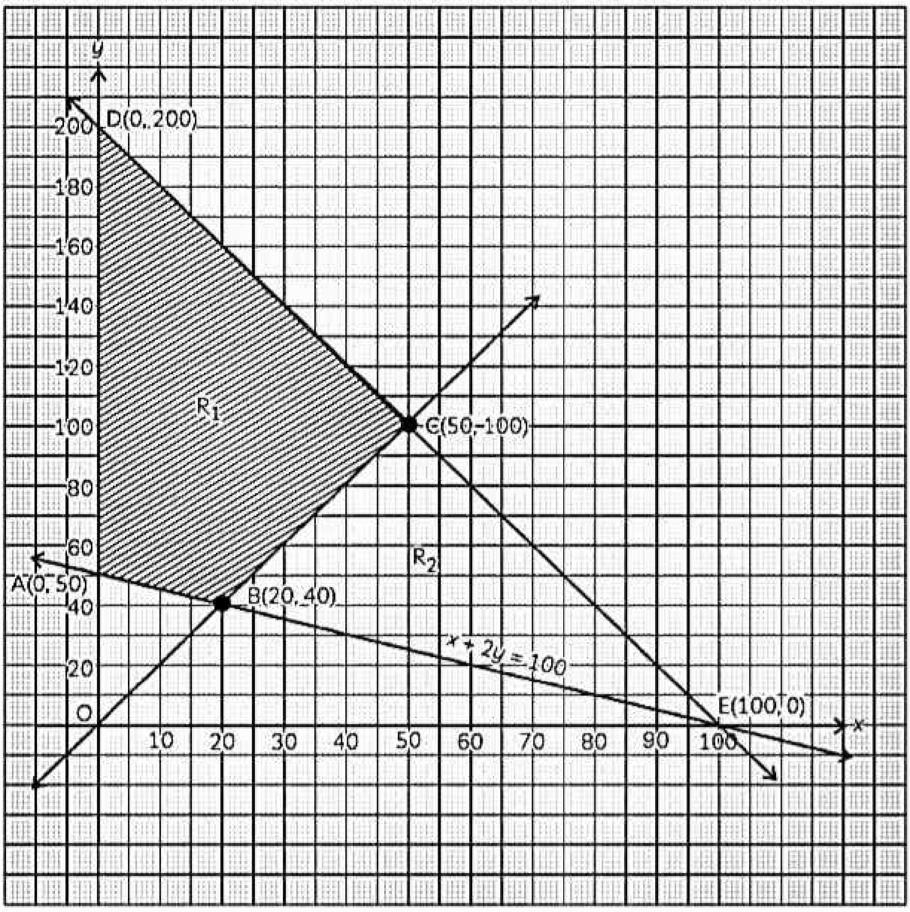
$(6x - 5y)_B = 15$

$(6x - 5y)_C = -25$

$(6x - 5y)_O = 0$

So, $(6x - 5y)_{min} = -25$

21. *Look at the graph below. Here, A(0, 50), B(20, 40), C(50, 100), D(0, 200) and E(100, 0).*



This graph is constructed using three non-trivial constraints and two trivial constraints. One of the non-trivial constraints is $x + 2y \geq 100$.

(A) **What are the two trivial constraints?**

(a) $x \leq 0, y \geq 0$ (b) $y \geq 0, 0$

(c) $x \geq 0, y \geq 0$ (d) $x \geq 0, y \leq 0$

(B) **Considering R_1 as the feasible region, find the other two non-trivial constraints.**

(a) $2x - y \leq 0, 2x + y \leq 200$

(b) $2x + y \geq 0, 2x + y = 200$

(c) $3x + y \geq 0, 3x - y \leq 200$

(d) $4x + 2y \geq 0, 5x + y = 200$

(C) **Considering R_2 as the feasible region, find the other two non-trivial constraints.**

(a) $2x - y \geq 0$ and $2x + y \leq 200$

(b) $2x + y \geq 0$ and $2x - y \leq 200$

(c) $2x - y \leq 0$ and $2x + y \geq 200$

(d) $3x + 2y \leq 0$ and $3x - y \geq 200$

(D) **Given that the point (x, y) is in the region R_1, find the maximum value of $5x + 2y$.**

(a) 400 (b) 450

(c) 500 (d) 600

(E) **Given that the point (x, y) is in the region R_2, find the maximum value of $5x + 2y$.**

(a) 400 (b) 300

(c) 500 (d) 600

Ans. (B) *(a) $2x - y \leq 0, 2x + y \leq 200$*

Explanation: From the figure, the region R_1 is bounded by the inequality $x + 2y \geq 100$ and the lines BC and DC.

Now, equation line BC is:

$$y - 40 = \frac{100 - 40}{50 - 20}(x - 20),$$

i.e., $2x - y = 0$; and

Equation of line DC is:

$$y - 200 = \left(\frac{100 - 200}{50 - 0}\right)(x - 0)$$

i.e., $2x + y = 200$.

To find the sign of inequality of these two lines, take any point from the feasible region R_1, say (20, 60).

Now, at (20, 60)

$$2x - y = 2 \times 20 - 60 = -20 \leq 0$$

$\Rightarrow \quad 2x - y \leq 0$

and $\quad 2x + y - 200 = 2 \times 20 + 60 - 200 = 100 \leq 0$

$\therefore$ $2x + y - 200 \leq 0$, or $2x + y \leq 200$

So, the other two inequalities are: $2x - y \leq 0$, $2x + y \leq 200$.

(E) *(c) 500*

Explanation: The corner points of the feasible region R_2 are: B(20, 40), C(50, 100) and E(100, 0).

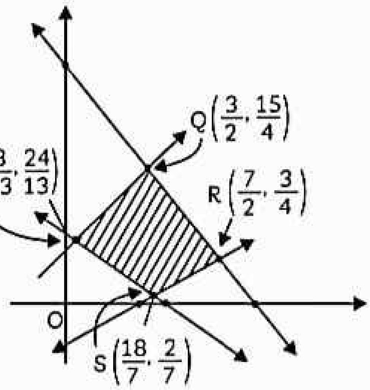
Now, $(5x + 2y)_B = 180$, $(5x + 2y)_C = 450$, $(5x + 2y)_E = 500$

So, $(5x + 2y)_{max} = 500$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

22. In the figure, given below, the feasible region (shaded) for a LPP is shown. Determine the maximum and minimum values of Z = x + $2y$. **[NCERT Exemplar]**



Ans. From the shaded region, it is clear that the coordinates of corner points are $\left(\frac{3}{13}, \frac{24}{13}\right)$, $\left(\frac{18}{7}, \frac{2}{7}\right)$, $\left(\frac{7}{2}, \frac{3}{4}\right)$ and $\left(\frac{3}{2}, \frac{15}{4}\right)$.

Here, objective function is

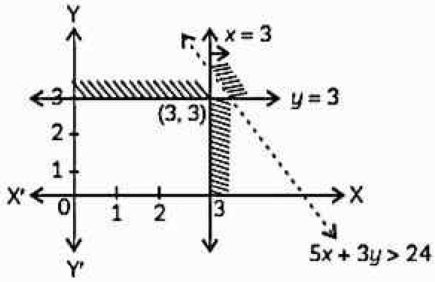
$$Z = x + 2y$$

Corner points	$Z = x + 2y$
$\left(\frac{3}{13}, \frac{24}{13}\right)$	$\frac{3}{13} + \frac{48}{13} = \frac{51}{13}$
$\left(\frac{18}{7}, \frac{2}{7}\right)$	$\frac{18}{7} + \frac{4}{7} = \frac{22}{7}$ (Minimum)
$\left(\frac{7}{2}, \frac{3}{4}\right)$	$\frac{7}{2} + \frac{6}{4} = \frac{20}{4} = 5$
$\left(\frac{3}{2}, \frac{15}{4}\right)$	$\frac{3}{2} + \frac{30}{4} = \frac{36}{4} = 9$ (maximum)

Hence, the maximum and minimum values of Z are 9 and $\frac{22}{7}$, respectively.

23. Find the maximum value of the function Z = $5x + 3y$ subjected to the constraints $x \geq 3$ and $y \geq 3$.

Ans. Here, value of Z at the corner point (3, 3) of the feasible region is 24.



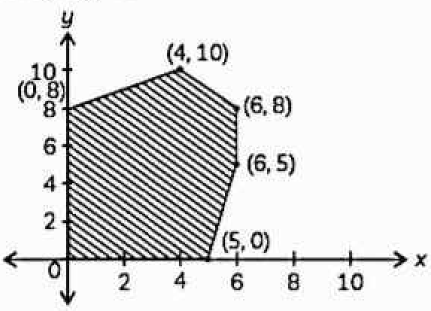
Since, region is unbounded, so we draw the graph of inequality $5x + 3y > 24$.

Since, open half plane of $5x + 3y > 24$ has common points with feasible region.

So, maximum value does not exist.

24. What type of polygon is formed by the feasible region for an L.P.P.?

25. The feasible solution for a LPP is shown in the following figure.



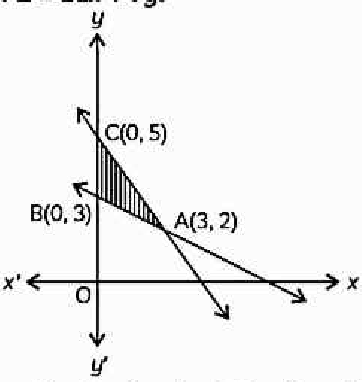
Let Z = 3x – 4y be the objective function. Then, at what point minimum value of Z occurs.

Ans.

Corner Points	Corresponding Value of Z = 3x – 4y
(0, 0)	0
(5, 0)	15
(6, 5)	–2
(6, 8)	–14
(4, 10)	–28
(0, 8)	–32 → Minimum

Hence, minimum value of Z occurs at (0, 8).

26. The feasible region for an LPP is shown in the following figure. Then, find the minimum value of $Z = 11x + 7y$.



Ans. Here, the objective function is $Z = 11x + 7y$.
And the corner points of the feasible region are A(3, 2), B(0, 3) and C(0, 5).

∴ At A(3, 2), $Z = 11 \times 3 + 7 \times 2$
$= 33 + 14 = 47$

At B(0, 3), $Z = 11 \times 0 + 7 \times 3$
$= 21$

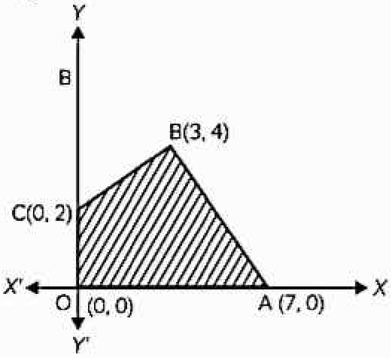
At C(0, 5), $Z = 11 \times 0 + 7 \times 5$
$= 35$

Hence, the minimum value of Z is 21 which occurs at point B(0, 3).

27. What is the condition $x \geq 0$, $y \geq 0$ in linear programming called?

Ans. This condition is called the non-negative restriction *i.e.*, x and y could not have the negative values.

28. Feasible region (shaded) for a LPP is shown in the figure below. Maximize $Z = 5x + 7y$.



[NCERT Exemplar]

Ans. The shaded region is bounded and has coordinates of corner points as (0, 0), (7, 0), (3, 4) and (0, 2).

Also, $Z = 5x + 7y$

Corner points	Z = 5x + 7y
(0, 0)	0
(7, 0)	35
(3, 4)	43 (Maximum)
(0, 2)	14

Hence, the maximum value of Z is 43 which occurs at point (3, 4).

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

29. Maximise $Z = 3x + 4y$, subject to the constraints: $x + y \leq 1$, $x \geq 0$, $y \geq 0$.

[NCERT Exemplar]

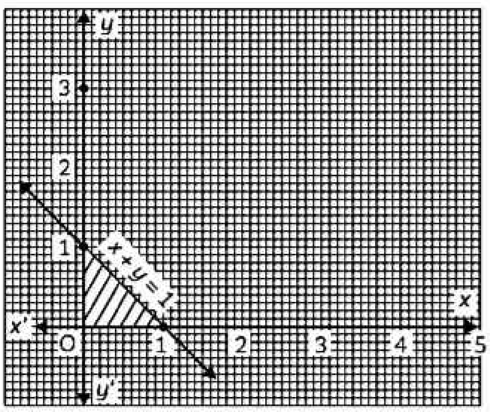
Ans. Maximize $Z = 3x + 4y$

Subject to constraints

$$x + y \leq 1$$
$$x \geq 0,\ y \geq 0$$

For $x + y = 1$

x	1	0
y	0	1



Corner points	$Z = 3x + 4y$
(0, 0)	0
(1, 0)	3
(0, 1)	4 (Maximum)

Hence the maximum value of Z is 4 at (0, 1).

SHORT ANSWER Type-II Questions (SA-II)

[**3** marks]

30. Solve the following LPP graphically:

Minimize, $Z = 5x + 7y$

Subject to the constraints

$$2x + y \geq 8$$
$$x + 2y \geq 10$$
$$x, y \geq 0$$

[CBSE 2020]

Ans. Given, we have to minimize

$$Z = 5x + 7y$$

Subject to the constraints,

$$2x + y \geq 8$$
$$x + 2y \geq 10$$
$$x, y \geq 0$$

Convert the inequations to equations, to plot them on the graph.

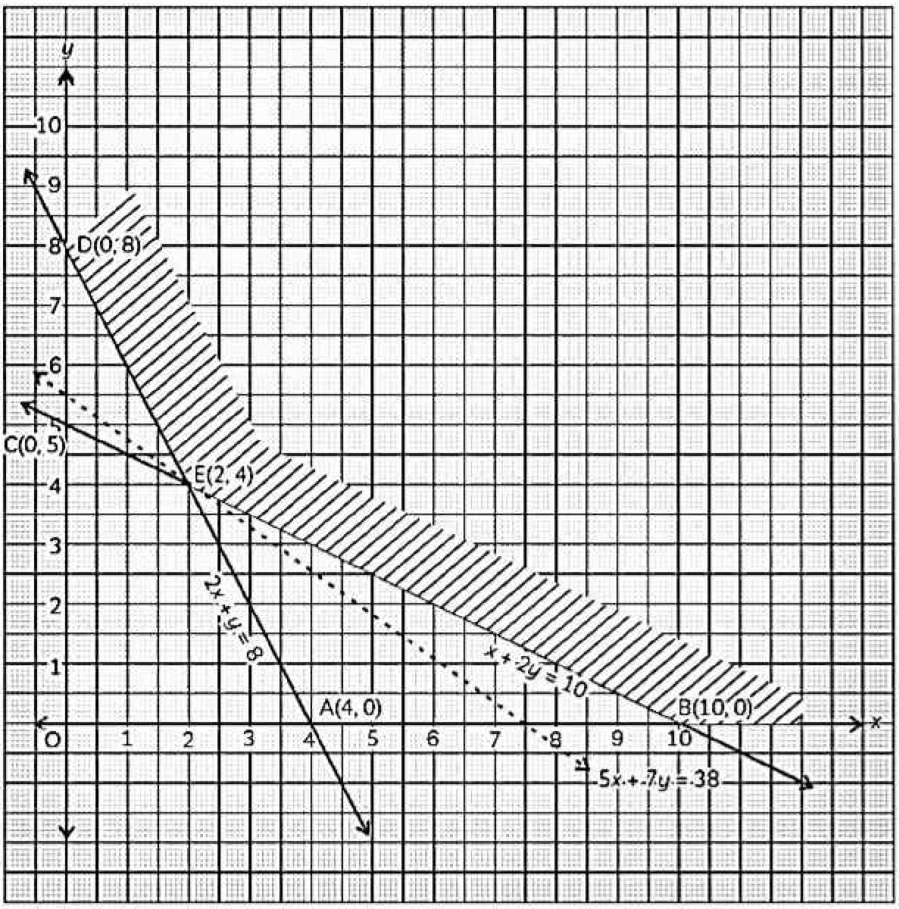
For the line $2x + y = 8$ ...(i)

x	0	4
y	8	0

And, for the line $x + 2y = 10$ ...(ii)

x	0	10
y	5	0

Plotting these points on a graph, we get a feasible region (shaded) which is unbounded.



The corner points of feasible region are D(0, 8), B(10, 0) and E(2, 4).

Corner Points	$Z = 5x + 7y$
D(0, 8)	56
B(10, 0)	50
E(2, 4)	38 (Minimum)

Here, 38 is the minimum value at E(2, 4). But the region is unbounded, so we draw the graph of inequality $5x + 7y < 38$.

Now, inequality $5x + 7y < 38$ has no point in common with the feasible region. So, the minimum value of Z is obtained at E(2, 4) and the minimum value of Z is 38.

31. Determine the maximum value of $Z = 11x + 7y$ subject to the constraints:
$2x + y \le 6, x \le 2, x \ge 0, y \ge 0$. [NCERT Exemplar]

32. Maximise $Z = 2x + 3y$
subject to constraints

$$x + 2y \le 10$$
$$2x + y \le 14$$
and $$x \ge 0, y \ge 0$$

Ans. Given, maximise $Z = 2x + 3y$

subject to constraints

$x + 2y \le 10$ ...(i)

$2x + y \le 14$ ...(ii)

$x \ge 0, y \ge 0$ ...(iii)

Convert inequations to equations to plot them on the graph.

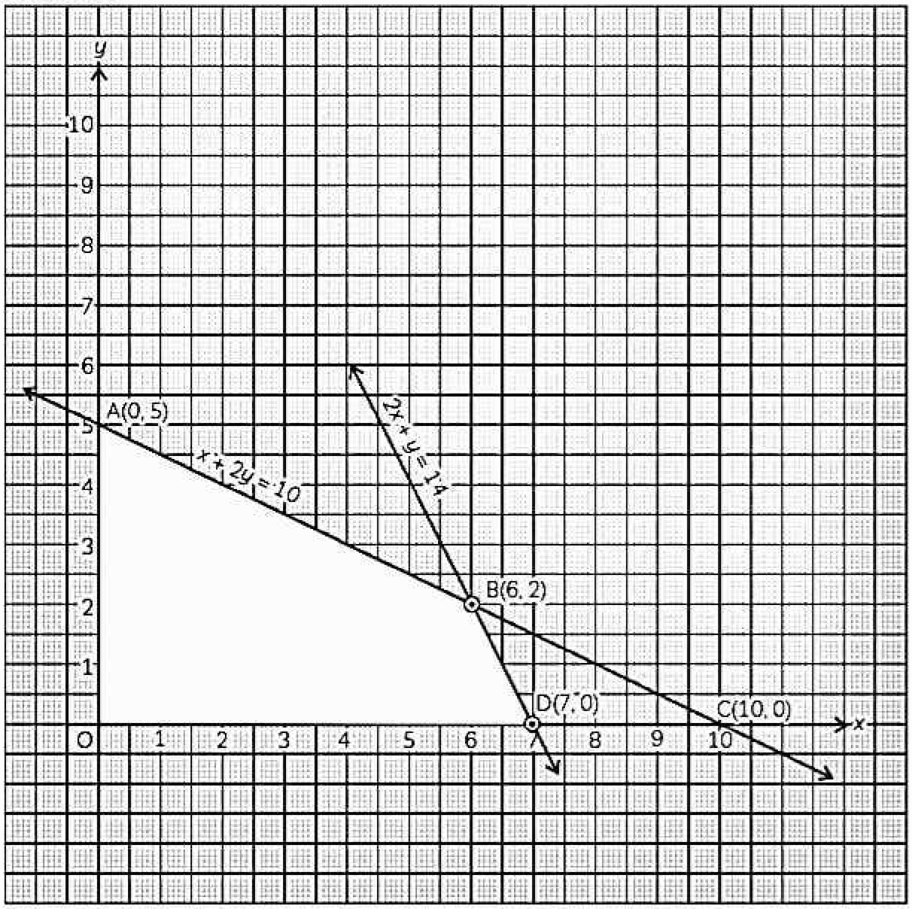
For the line $x + 2y = 10$

x	0	10
y	5	0

For the line $2x + y = 14$

x	0	7
y	14	0

Now, plotting these points on a graph, we get the feasible region (shaded) which is bounded.



The corner points of the feasible region are O (0, 0), A (0, 5), B (6, 2) and C (7, 0).

Corner Points	$Z = 2x + 3y$
O(0, 0)	$Z = 0$
A(0, 5)	$Z = 2 \times 0 + 3 \times 5 = 15$
B(6, 2)	$Z = 2 \times 6 + 3 \times 2 = 18$ →Maximum
D(7, 0)	$Z = 2 \times 7 + 3 \times 0 = 14$

Hence, the maximum value of Z is 18 at the point B(6, 2).

33. Maximise and minimise $Z = x + 2y$

subject to the constraints

$$x + 2y \geq 100$$
$$2x - y \leq 0$$
$$2x + y \leq 200$$
$$x, y \geq 0$$

Solve the above LPP graphically.[CBSE 2017]

Ans. Given, Max. and Min. $Z = x + 2y$

Subject to the constraints,

$x + 2y \geq 100$ ...(i)

$2x - y \leq 0$ ...(ii)

$2x + y \leq 200$ ...(iii)

and $x \geq 0, y \geq 0$ ...(iv)

Now, convert given inequalities to equalities, to plot them on the graph.

For the line $x + 2y = 100$

x	0	100
y	50	0

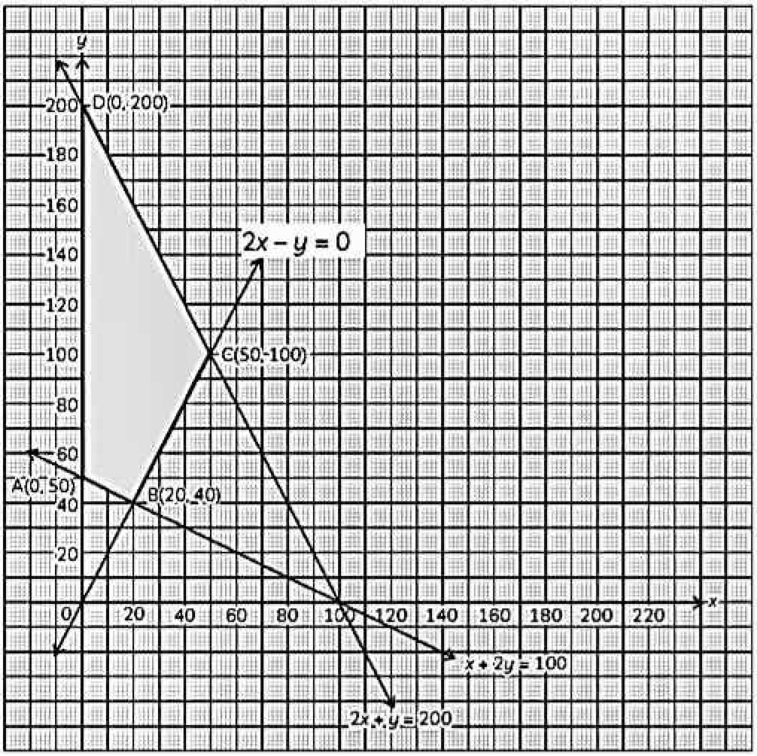
For the line $2x - y = 0$

x	0	10
y	0	20

And for the line $2x + y = 200$

x	0	100
y	200	0

Plotting these points on graph, we get a feasible region (shaded) which is bounded.

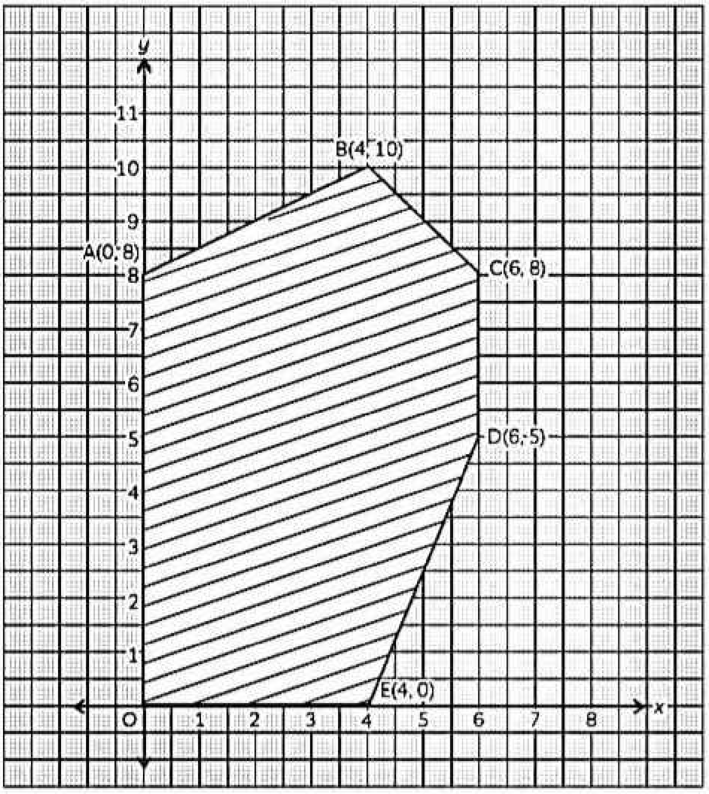


The corner points of the feasible region are A(0, 50), B (20, 40), C (50, 100) and D (0, 200).

Corner Points	$Z = x + 2y$
A(0, 50)	$Z = 0 + 2 \times 50 = 100$
B(20, 40)	$Z = 20 + 2 \times 40 = 100$
C(50, 100)	$Z = 50 + 2 \times 100 = 250$
D(0, 200)	$Z = 0 + 2 \times 200 = 400$

The maximum value of Z is 400 at D(0, 200) and the minimum value of Z is 100 at the points A(0, 50) and B(20, 40).

34. The corner points of the feasible region determined by a system of linear constraints are shown below:



Answer each of the following:

(A) Let $Z = 3x - 4y$ be the objective function. Find the maximum and minimum values of Z and also the corresponding points at which the maximum and minimum values occurs.

(B) Let $Z = px + qy$, where $p, q > 0$ be the objective function. Find the condition on p and q so that the maximum value of Z occurs at B(4, 10) and C(6, 8). Also, mention the number of optimal solutions in this case. [CBSE SQP 2020]

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

35. Maximize $Z = x + y$ subject to $x + 4y \leq 8$, $2x + 3y \leq 12$, $3x + y \leq 9$, ≥ 0, $y \geq 0$.

Ans. We have, LPP is

maximize $Z = x + y$

Subject to constraints

$x + 4y \leq 8$

$2x + 3y \leq 12$

$3x + y \leq 9$

$x \geq 0, y \geq 0$

For equation $x + 4y = 8$

x	0	8
y	2	0

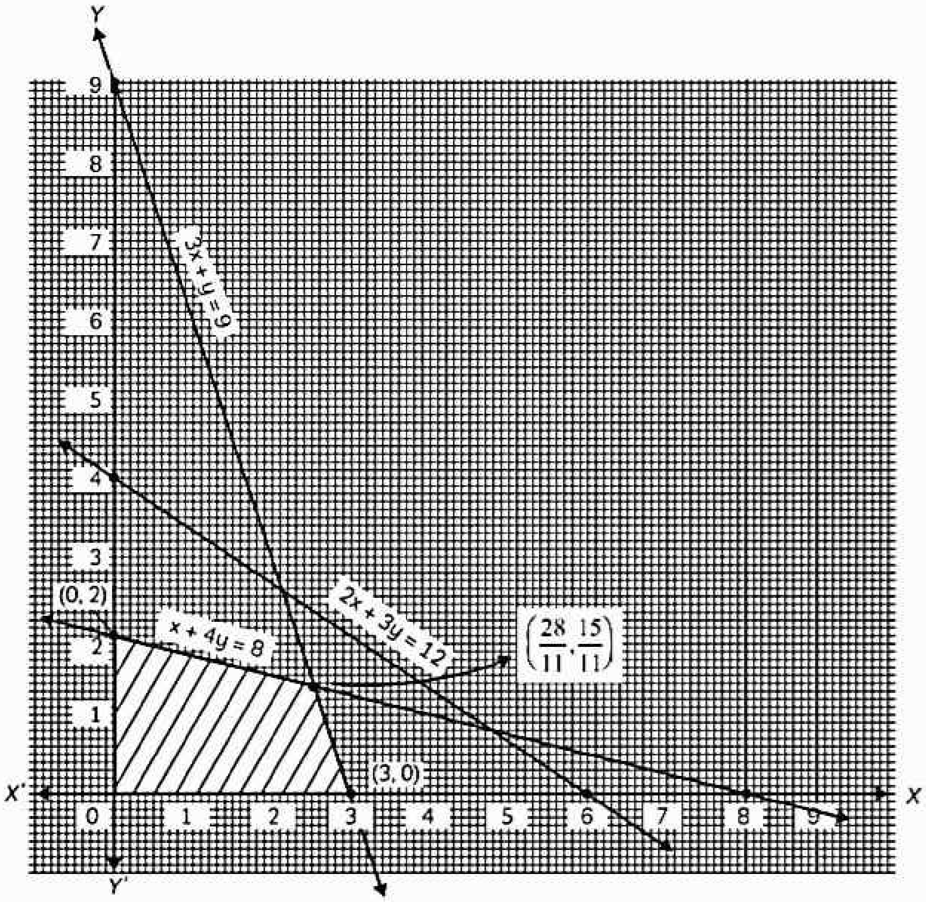
For equation $3x + y = 9$

x	0	3
y	9	0

For equation $2x + 3y = 12$

x	0	6
y	4	0

Plotting these points on the graph, we get the feasible region which is shaded.



Corner points	$Z = x + y$
(0, 0)	0
(3, 0)	3
$\left(\frac{28}{11}, \frac{15}{11}\right)$	$\frac{43}{11}$
(0, 2)	2

Hence, the maximum volume of Z is $\frac{43}{11}$ at point $\left(\frac{28}{11}, \frac{15}{11}\right)$.

36. Maximise and minimise $Z = 3x - 4y$

subject to $x - 2y \le 0$

$-3x + y \le 4$

$x - y \le 6$

$x, y \ge 0$

Ans. Given, LPP is

Maximize and minimize, $Z = 3x - 4y$

Subject to $x - 2y \le 0$

$-3x + y \le 4$

$x - y \le 6$

$x, y \ge 0$

For the equation $x - 2y = 0$

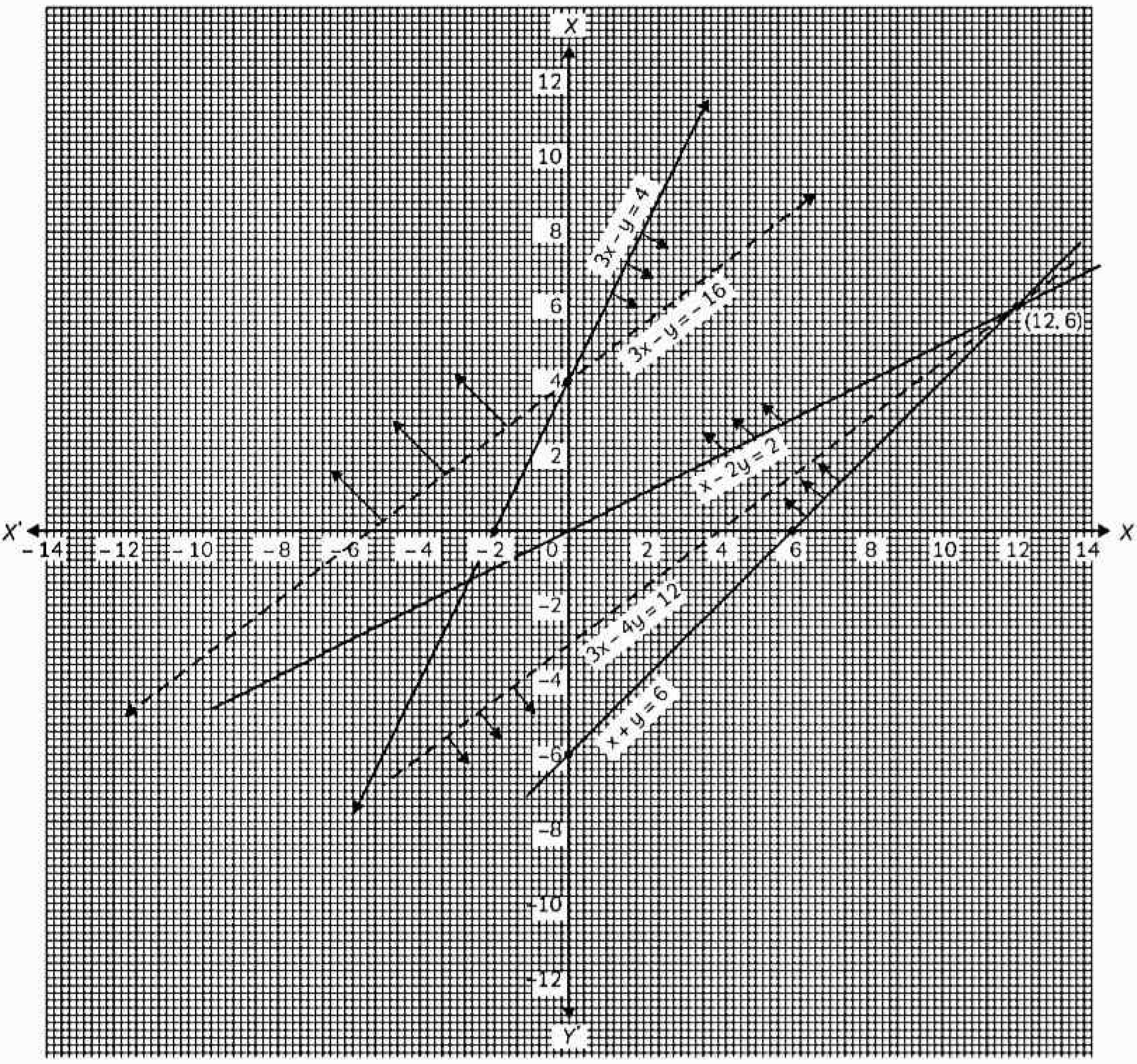
x	0	2
y	0	1

For equation $-3x + y = 4$

x	0	− 1
y	4	1

For equation $x - y = 6$

x	6	0
y	0	− 6



Plotting these points on a graph, we get a feasible region which is unbounded.

Corner points	$Z = 3x - 4y$
(0, 0)	0
(0, 4)	–16 (Minimum)
(12, 6)	12 (Maximum)

Now, to decide whether 12 and –6 are respectively the maximum and minimum values of Z or not, plot the inequalities $3x - 4y > 12$ and $3x - 4y < -6$.

From the graph, it is clear that the inequalities $3x - 4y > 12$ and $3x - 4y < -6$ has no point in common with the feasible region.

Hence, the maximum value of Z is 12 and its minimum value is –6.