6 APPLICATION OF DERIVATIVES

There is no permanent place in the world of ugly mathematics

—*G.H. Hardy*

Objectives

After studying the material of this chapter, you should be able to :

- Understand to obtain the rate of change of quantities.
- Understand to find intervals in which the functions are : (I) Increasing (II) Strictly Increasing (III) Decreasing (IV) Strictly Decreasing.
- Understand to find the equations of tangents and normals.
- Understand to find differentials and approximations.
- Understand to find the absolute and local maxima and minima.

INTRODUCTION

So far we have discussed derivatives of various types of functions. Derivatives have a wide range of application in Science, Engineering and Social Sciences and many other fields. In this chapter, we shall consider a few applications of derivatives.

For instance, we shall learn how the derivatives are to be used in order to determine :

(*i*) rate of change of quantities (*ii*) intervals in which a function is increasing (or decreasing) (*iii*) equations of tangents and normals (*iv*) approximate values of certain quantities and (*v*) problems of maxima and minima.

SUB CHAPTER 6.1 Derivative As a Rate Measure

6.1. RATE OF CHANGE OF QUANTITIES

We know that the derivative $\frac{ds}{dt}$ represents the velocity, which is the rate of change of distance '*s*' with respect to time '*t*'. In the similar way, if one quantity '*y*' varies with another quantity '*x*' according to the law $y = f(x)$, then $f'(x_0)$ represents the rate of change of '*y*' with respect to '*x*' at $x = x_0$. In this section, we shall deal with the examples of this type.

Further, we also know that if both *x* and *y* vary with *t*, then by *Chain Rule*, we have :

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(x) \cdot \frac{dx}{dt}.$$

Hence, the rate of change of one variable can be calculated if the rate of change of the other variable is known.

Chapter at Glance

ILLUSTRATIVE EXAMPLES

Example 1. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm? *(A.I.C.B.S.E. 2015)*

Solution. Let 'x' be the side of an equilateral triangle.

Given : $\dfrac{dx}{dt} = 2$ cm/s.

If 'A' be the area of the equilateral triangle, then :

$$A = \sqrt{3}\frac{x^2}{4}.$$

$$\therefore \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4}(2x)\frac{dx}{dt} = \frac{\sqrt{3}x}{2}\frac{dx}{dt}.$$

When $x = 20$ cm, then $\dfrac{dA}{dt} = \dfrac{\sqrt{3}}{2}(20)(2) = 20\sqrt{3}$.

Hence, the rate of its area increasing = $20\sqrt{3}$ cm²/s.

Example 2. The length 'x' of a rectangle is decreasing at the rate of 3 cm/m and width 'y' is increasing at the rate of 2 cm/m. When x = 10 cm and y = 6 cm, find the rate of change of :

(a) the perimeter and (b) the area of the rectangle. *(N.C.E.R.T.)*

Solution. We have : $\dfrac{dx}{dt} = -3$ ...(1)

and $\dfrac{dy}{dt} = 2$ ...(2)

(*a*) Perimeter, $p = 2x + 2y$.

$$\therefore \quad \frac{dp}{dt} = 2\frac{dx}{dt} + 2\frac{dy}{dt}$$

$= 2(-3) + 2(2)$ [*Using (1) and (2)*]

$= -6 + 4 = -2.$

Hence, $\left.\dfrac{dp}{dt}\right]_{\substack{x=10\\y=6}} = -2$ cm/m.

(*b*) Area, $A = xy$.

$$\therefore \quad \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$= x(2) + y(-3).$ [*Using (1) and (2)*]

Hence, $\left.\dfrac{dA}{dt}\right]_{\substack{x=10\\y=6}} = 10(2) + 6(-3) = 20 - 18 = 2\text{ cm}^2/\text{m}.$

Exampe 3. The volume of a sphere is increasing at the rate of 3 cubic centimetre per second. Find the rate of increase of its surface area, when radius is 2 cm. *(C.B.S.E. 2017)*

Solution. Let r, S and V be the radius, surface area and volume of the sphere at time t.

Then, $\dfrac{dV}{dt} = 3$ cm³/s

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 3 \quad \Rightarrow \quad 4\pi r^2\frac{dr}{dt} = 3$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{3}{4\pi r^2} \quad ...(1)$$

Now, $S = 4\pi r^2$

$$\therefore \quad \frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$

$= 8\pi r.\dfrac{3}{4\pi r^2}$ [*Using (1)*]

$= \dfrac{6}{r}.$

Hence, $\left.\dfrac{dS}{dt}\right]_{r=2} = \dfrac{6}{2} = 3\text{cm}^2/\text{s}.$

Exampe 4. For the curve y = 5x – 2x³, if x increases at the rate of 2 units/sec., find the rate of change of the slope of the curve when x = 3. *(C.B.S.E. 2017)*

Solution. The given curve is $y = 5x - 2x^3$.

$$\therefore \quad \frac{dy}{dx} = 5 - 6x^2$$

i.e. $m = 5 - 6x^2$, where 'm' is the slope.

$$\therefore \quad \frac{dm}{dt} = -12x\frac{dx}{dt}$$

$= -12x(2) = -24x.$

$$\therefore \quad \left.\frac{dm}{dt}\right]_{x=3} = -24(3) = -72.$$

Hence, the rate of the change of the slope = – 72.

Example 5. A water tank has the shape of an inverted right-circular cone with its axis vertical and vertex lower most. Its semi-vertical angle is $\tan^{-1}\left(\frac{1}{2}\right)$. Water is poured into it at a constant rate of 5 cubic meter per minute. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 10 m.

Solution.

Let 'r' be the radius and 'h' the height of the right-circular cone.

If 'α' be the semi-vertical angle,

then $\tan\alpha = \dfrac{r}{h} \Rightarrow \tan\left(\tan^{-1}\dfrac{1}{2}\right) = \dfrac{r}{h}$

$$\Rightarrow \quad \frac{1}{2} = \frac{r}{h} \Rightarrow h = 2r \quad ...(1)$$

If 'V' be the volume at any time t,

then $\dfrac{dV}{dt} = 5$ ...(2)

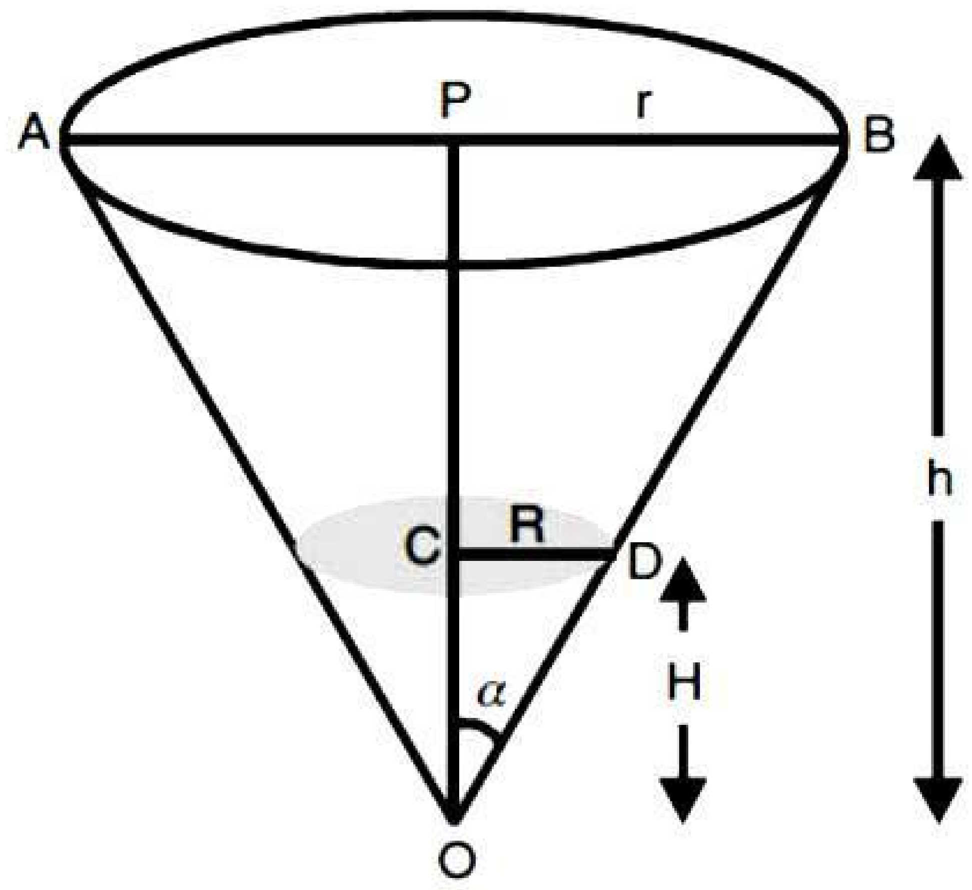


Fig.

At that instant, let the water form a cone of height OC (= H m) and radius CD (= R m).

Now ΔPBO and ΔCDO are similar.

$$\therefore \quad \frac{CD}{PB} = \frac{OC}{OP} \Rightarrow \frac{R}{r} = \frac{H}{h} \Rightarrow R = \frac{rH}{h}$$

$$\Rightarrow \quad R = \frac{1}{2}H \qquad \text{...(3) [Using (1)]}$$

$$\therefore \quad V = \frac{1}{3}\pi R^2 H$$

$$= \frac{1}{3}\pi\left(\frac{1}{4}H^2\right)H = \frac{1}{12}\pi H^3.$$

[*Using (3)*]

$$\therefore \quad \frac{dV}{dt} = \frac{1}{12}.3\pi H^2\frac{dH}{dt}$$

$$\Rightarrow \quad 5 = \frac{1}{4}\pi H^2\frac{dH}{dt} \qquad \text{[Using (2)]}$$

$$\Rightarrow \quad \frac{dH}{dt} = \frac{4}{\pi H^2}(5)$$

$$= \frac{20}{\pi H^2}.$$

When H = 10 m, $\frac{dH}{dt} = \frac{20}{\pi(100)} = \frac{1}{5\pi}$.

Hence, the rate of increase of water level = $\frac{1}{5\pi}$ m/m.

Example 6. Water is leaking from a conical funnel at the rate of 5 cm³/s. If the radius of the base of funnel is 5 cm and height 10 cm, find the rate at which the water level is dropping when it is 2·5 cm from the top.

Solution. Here 5 cm is the radius and 10 cm is the height of the conical funnel.

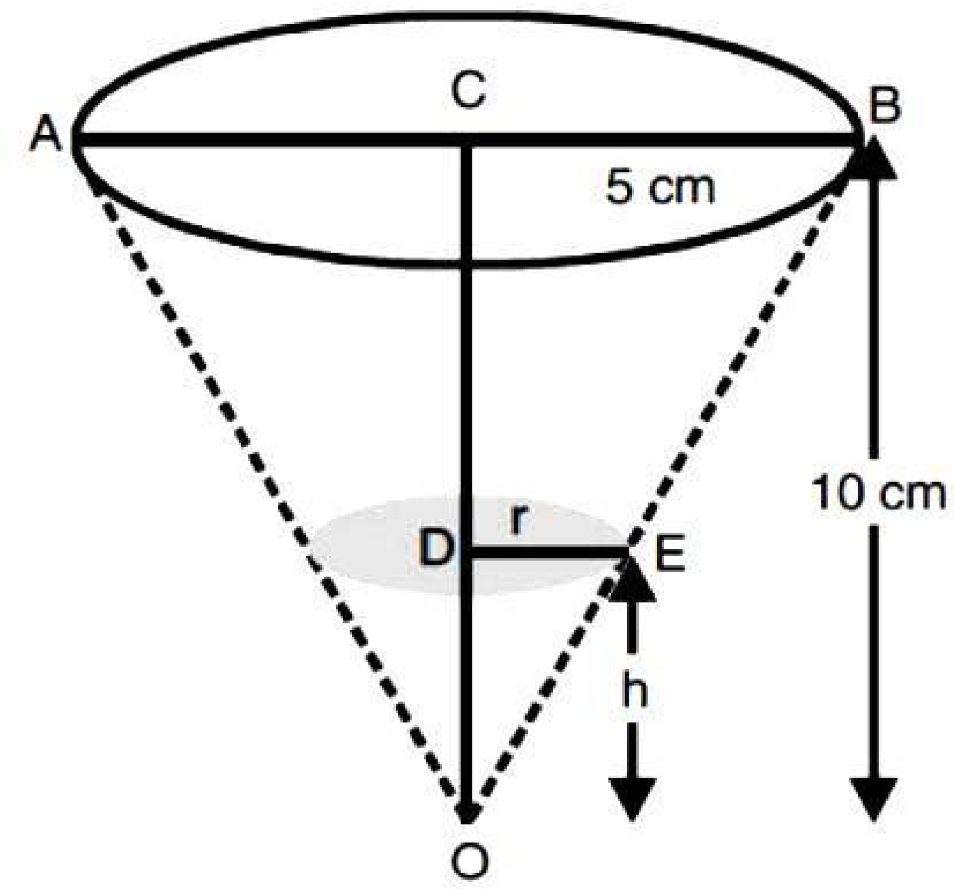


Fig.

Let '*r*' be the radius of the base and '*h*' the height at any stage.

$$\therefore \text{ V, volume of water in conical funnel } = \frac{1}{3}\pi r^2 h \qquad \text{...(1)}$$

Now Δs OBC and OED are similar.

$$\therefore \quad \frac{CB}{DE} = \frac{OC}{OD} \Rightarrow \frac{DE}{OD} = \frac{CB}{OC}$$

$$\Rightarrow \quad \frac{r}{h} = \frac{5}{10} \Rightarrow \frac{r}{h} = \frac{1}{2}$$

$$\Rightarrow \quad r = \frac{h}{2}.$$

∴ From (1), V, volume of water in conical funnel

$$= \frac{1}{3}\pi\left(\frac{h^2}{4}\right)h$$

$$= \frac{\pi h^3}{12}.$$

$$\therefore \quad \frac{dV}{dt} = \frac{3\pi h^2}{12}.\frac{dh}{dt} = \frac{\pi h^2}{4}.\frac{dh}{dt}.$$

Given : $\frac{dV}{dt} = -5$.

$$\therefore \quad \frac{\pi h^2}{4}.\frac{dh}{dt} = -5 \Rightarrow \frac{dh}{dt} = \frac{-20}{\pi h^2}.$$

When the water level is 2·5 cm from the top, then
$h = 10 - 2{\cdot}5 = 7{\cdot}5$ cm.

$$\therefore \quad \frac{dh}{dt} = -\frac{20}{\pi(7\cdot5)^2} = -\frac{20(2)^2}{\pi(15)^2}$$

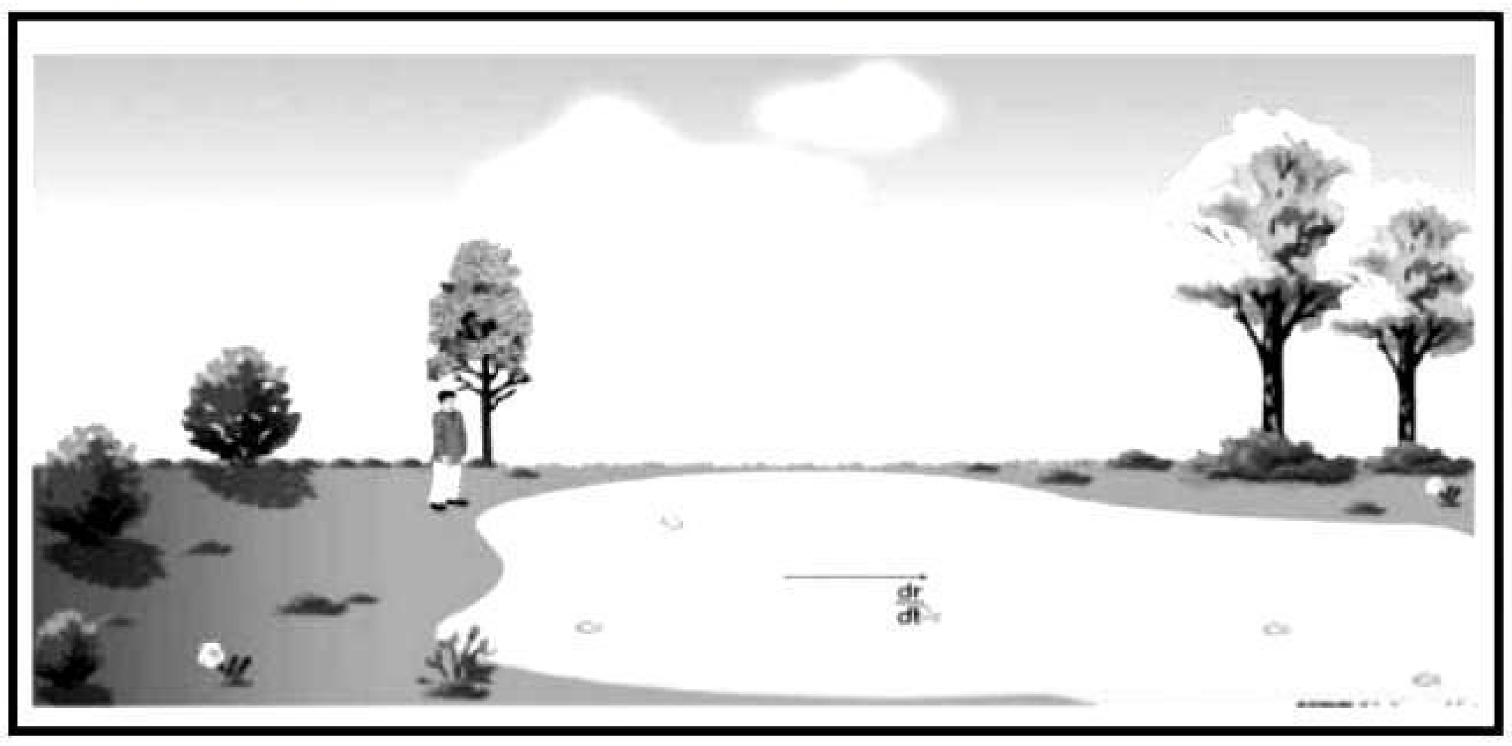
$$= -\frac{80}{225\pi} = -\frac{16}{45\pi}.$$

Hence, the rate at which the water level is dropping is $\frac{16}{45\pi}$ cm/s.

Example 7. A stone is dropped into a quiet lake and waves move in circles at a speed of 4 cm per second. At the instant when the radius of the circular wave is 10 cm, how fast is the enclosed area increasing ?

(N.C.E.R.T. ; H.P. B. 2015)

Solution. Let 'r' be the radius of the circular wave.



Then $A = \pi r^2$, where A is the enclosed area at time t.

Differentiating w.r.t. t, we have :

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r (4) \qquad \left[\because \frac{dr}{dt} = 4\ cm/s\right]$$

$$= 8\pi r.$$

When $r = 10$ cm, $\dfrac{dA}{dt} = 8\pi(10) = 80\pi\ \text{cm}^2/\text{s}$.

Hence, the enclosed area is increasing at the rate of 80π cm^2/s. when r = 10 cm.

Example 8. A man 2 m high walks at a uniform speed of 6 km/h away from a lamp post 6 metres high. Find the rate at which the length of his shadow increases.

Solution. Let AB be the lamp post 6 m high and MN = (2 m) the man.

Let MS be the shadow of the man at any time t, where S is the end of his shadow.

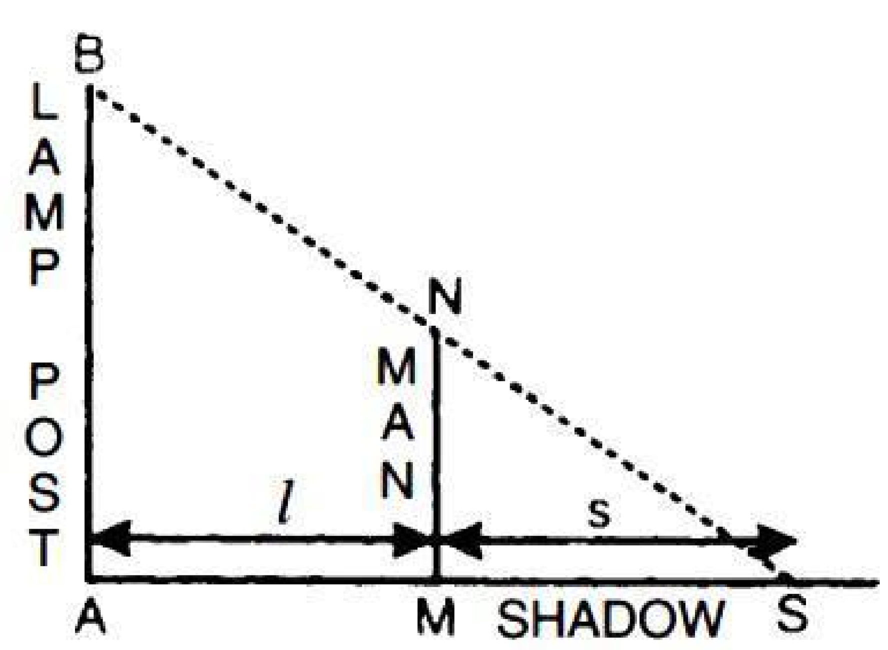


Fig.

Let AM = l and MS = s.

Now Δs ASB and MSN are similar.

$$\therefore \quad \frac{MS}{AS} = \frac{MN}{AB}$$

$$\Rightarrow \quad \frac{s}{l+s} = \frac{2/1000}{6/1000}$$

$$\Rightarrow \quad \frac{s}{l+s} = \frac{1}{3} \Rightarrow 3s = l + s$$

$$\Rightarrow \quad l = 2s. \therefore \frac{dl}{dt} = 2\frac{ds}{dt} \qquad ...(1)$$

But $\dfrac{dl}{dt} = 6$ km/h. [*Given*]

$\therefore$ From (1), $\dfrac{ds}{dt} = \dfrac{6}{2} = 3$ km/h.

Hence, the shadow increases at the rate of 3 km/h.

Example 9. A man is moving away from a tower 41·6 m high at the rate of 2 m/s. Find the rate at which the angle of elevation of the top of tower is changing when he is at a distance of 30 m from the foot of the tower. Assume that the eye level of the man is 1·6 m from the ground.

Solution. Let AB (= 41.6 m) be the tower.

Let the man be at a distance of 'x' metres from the tower AB at any time t. If 'θ' be the angle of elevation at time t, then

$$\tan\theta = \frac{LB}{QL} = \frac{AB - AL}{x} = \frac{AB - PQ}{x} = \frac{41\cdot6 - 1\cdot6}{x}$$

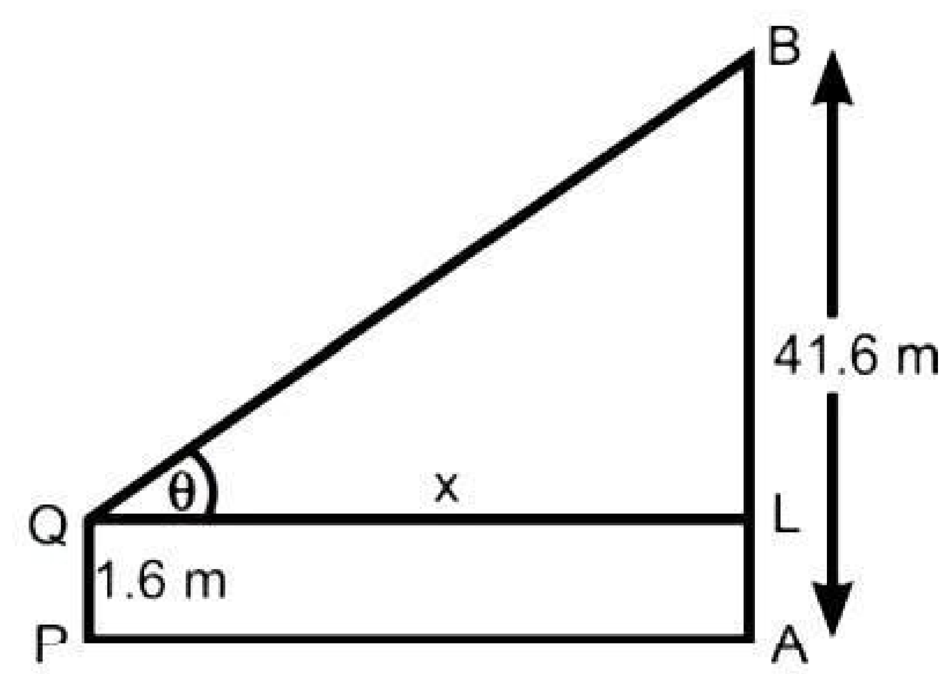


Fig.

$$\Rightarrow \tan\theta = \frac{40}{x} \Rightarrow x = 40\cot\theta \qquad ...(1)$$

$$\therefore \quad \frac{dx}{dt} = -40\,\text{cosec}^2\theta \cdot \frac{d\theta}{dt}$$

$$\Rightarrow \quad 2 = -40\,\text{cosec}^2\theta \cdot \frac{d\theta}{dt} \qquad \left[\because \frac{dx}{dt} = 2\ (given)\right]$$

$$\Rightarrow \frac{d\theta}{dt} = -\frac{1}{20\,\text{cosec}^2\theta} \qquad ...(2)$$

When $x = 30$, then from (1), $30 = 40\cot\theta$

$$\Rightarrow \cot\theta = \frac{3}{4}$$

so that $\text{cosec}^2\theta = 1 + \cot^2\theta = 1 + \dfrac{9}{16} = \dfrac{25}{16}$.

Putting in (2), $\dfrac{d\theta}{dt} = -\dfrac{1}{20\left(\dfrac{25}{16}\right)} = -\dfrac{4}{125}$ radians/s.

Hence, the angle of elevation of the top of the tower is changing at the rate of $\dfrac{4}{125}$ radians/s.

Example 10. Two equal sides of an isosceles triangle with fixed base 'a' are decreasing at the rate of 9 cm/s. How fast is the area of the triangle decreasing when the two sides are equal to 'a'.

Solution. Let 'A' be the area of ΔPQR is which :

$PQ = PR = x$ and $QR = a$.

$$\therefore \quad A = \frac{1}{2} QR \times PL$$

$$= \frac{1}{2} a \sqrt{x^2 - \left(\frac{a}{2}\right)^2} = \frac{a}{4}\sqrt{4x^2 - a^2}.$$

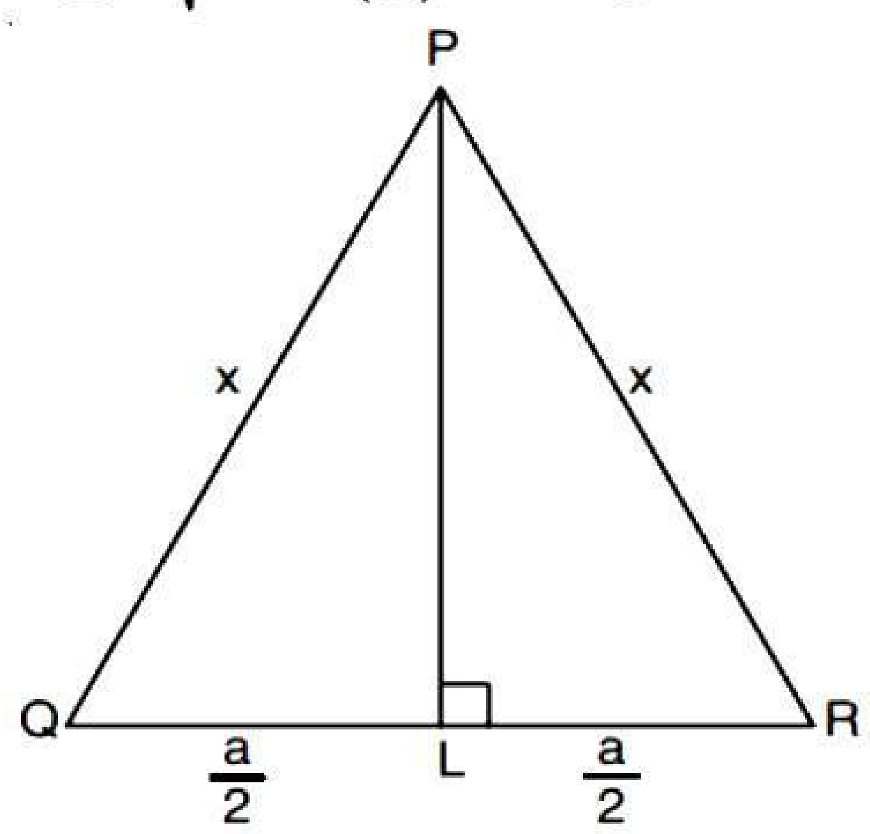


Fig.

$$\therefore \quad \frac{dA}{dt} = \frac{a}{4} \cdot \frac{1}{2\sqrt{4x^2 - a^2}} (8x) \frac{dx}{dt}$$

$$= \frac{ax}{\sqrt{4x^2 - a^2}} (-9)$$

$$\left[\because \frac{dx}{dt} = -9 \text{ (given)}\right]$$

$$= \frac{-9ax}{\sqrt{4x^2 - a^2}}.$$

$$\therefore \quad \left.\frac{dA}{dt}\right]_{x=a} = \frac{-9a(a)}{\sqrt{4a^2 - a^2}}$$

$$= -\frac{9a^2}{\sqrt{3}a} = -3\sqrt{3}\,a.$$

Hence, area of the triangle is decreasing at the rate of $3\sqrt{3}\,a$ cm²/s.

Definition

1. **Marginal cost** *is the instantaneous rate of change of total cost at any level of output.*
   Thus if $C(x)$ is the total cost of 'x' units, then $C'(x)$ is the marginal cost.
2. **Marginal Revenue** *is the rate of change of total revenue with respect to the number of items sold at an instant.*
   Thus if $R(x)$ is the total revenue from the sale of 'x' units, then $R'(x)$ is the marginal revenue.

Example 11. The amount of pollution control in air in a city due to x-diesel vehicles is given by :

$$\mathbf{P(x) = 0.005\,x^3 + 0.02x^2 + 30x.}$$

Find the marginal increase in pollution control when 3-diesel vehicles are added. *(C.B.S.E. 2013)*

Solution. We have :

$$P(x) = 0.005x^3 + 0.02x^2 + 30x.$$

$$\therefore \quad P'(x) = 0.015x^2 + 0.04x + 30.$$

$$\therefore \quad P'(3) = 0.015(9) + 0.04(3) + 30$$

$$= 0.135 + 0.12 + 30 = 30.255,$$

which is the marginal increase in pollution control when 3-diesel vehicles are added.

Example 12. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in ₹) received from the rate of 'x' units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when x = 5, and write which value does the question indicate.

(A.I.C.B.S.E. 2013)

Solution. We have : $R(x) = 3x^2 + 36x + 5$.

$$\therefore \quad MR = \frac{dR}{dx} = 6x + 36.$$

Hence, $\quad MR]_{x=5} = 6(5) + 36 = 30 + 36 = 66.$

EXERCISE 6 (a)

Fast Track Answer Type Questions

FTATQ

1. Find the rate of change of the area of a circle with respect to its radius 'r' when $r = 5$ cm. *(N.C.E.R.T.)*

2. Find the rate of change of the volume of a ball with respect to its radius 'r'.

3. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing, when the edge is 10 cm long ? *(Jammu B. 2016; H.P.B. 2015)*

4. The radius of a soap-bubble is increasing at the rate of 0·2 cm/s. Find the rate of increase of :

its volume when the radius is 5 cm.

5. (*i*) The radius of a circle is increasing at the rate of 0·7 cm/s. What is the rate of increase of its circumference ?

(N.C.E.R.T.)

(*ii*) The radius of a circle is increasing uniformly at the rate of 4 cm per second. Find the rate at which the area of the circle is increasing when the radius is 8 cm.

(*iii*) If the area of a circle increases uniformly, then show that the rate of increment of its circumference is inversely proportional to its radius. *(W. Bengal B. 2016)*

6. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

(N.C.E.R.T.; Uttarakhand B. 2015 ; H.P.B. 2010)

7. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm ? *(N.C.E.R.T.)*

8. The radius of spherical balloon is increasing at the rate 5 cm per second. At what rate is the surface of the balloon increasing, when the radius is 10 cm ? *(P.B. 2010 S)*

9. The radius of an air bubble is increasing at the rate of 0·5 cm/s. Find the rate of change of its volume, when the radius is 1 cm. *(Meghalaya B. 2017)*

Very Short Answer Type Questions

VSATQ

10. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to 'x'. *(N.C.E.R.T.; H.P.B. 2018)*

11. (*i*) A balloon, which always remains spherical, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm.

(N.C.E.R.T.; H.P.B. 2018)

(*ii*) A balloon, which always remains spherical, has a variable radius. Find the rate at which its volume is increasing with the radius when the radius is 10 cm. *(H.P.B. 2018)*

12. (*a*) The volume of a cube is increasing at the rate of 9 cubic centimetres per second. How fast is surface area increasing when the length of an edge is 10 centimetres ?

(N.C.E.R.T. ; A.I.C.B.S.E. 2017; H.P.B. 2015 ; Kashmir B. 2011)

(*b*) The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm ? *(N.C.E.R.T.; H.P.B. 2010)*

(*c*) The volume of a cube is increasing at the rate of 7 cubic centimeters per second. How fast is its surface area increasing at the instant when the length of an edge of the cube is 24 cm ? *(Meghalaya B. 2014)*

13. (*i*) A particle moves along the curve $y = \frac{4}{3}x^3 + 5$. Find the points on the curve at which the y-coordinate changes as fast as the x-coordinate.

(*ii*) A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the y-coordinate is changing 8 times as fast as x-coordinate.

(N.C.E.R.T. ; Meghalaya B. 2015)

(*iii*) A particle moves along the parabola $y^2 = 4x$. Find the co-ordinates of the point on the parabola where the rate of increment of abscissa is twice the rate of increment of the ordinate. *(W. Bengal B. 2018)*

Short Answer Type Questions

SATQ

14. The radius of a cylinder increases at the rate of 1 cm/s and its height decreases at the rate of 1 cm/s. Find the rate of change of its volume when the radius is 5 cm and the height is 5 cm.

If the volume should not change even when the radius and height are changed, what is the relation between the radius and height ? *(Kerala B. 2015)*

15. The total cost C (x) associated with product of 'x' units of an item is given by :

$$C(x) = 0{\cdot}005x^3 - 0{\cdot}02x^2 + 30x + 5000.$$

Find the marginal cost when 3 units are produced.

(N.C.E.R.T. ; C.B.S.E. 2018)

16. The total revenue in rupees received from the sale of 'x' units of a product is given by :

$$R(x) = 7x^2 + 50x + 119.$$

Find the marginal revenue when 10 units of the product are sold.

17. Total revenue from the sale of 'x' units of a product is given by :

$$R(x) = 40x - \frac{x^2}{2}.$$

Find the marginal revenue when $x = 6$ and interpret it.

18. (*i*) A man 160 cm tall, walks away from a source of light situated at the top of a pole 6 m high at the rate of 1·1 m/s. How fast is the length of his shadow increasing when he is 1 m away from the pole ?

(*ii*) A man 2 metres high walks at a uniform speed of 5 km/hour away from a lamp-post 6 metres high. Find the rate at which the length of his shadow increases.

(N.C.E.R.T.; Mizoram B. 2016; H.P.B. 2014)

19. The length x, of a rectangle is decreasing at the rate of 5 cm/ minute and the width y, is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rate of change of the area of the rectangle. *(A.I.C.B.S.E. 2017)*

20. The volume of a sphere is increasing at the rate of 8 cm^3/s. Find the rate at which its surface area is increasing when the radius of the sphere is 12 cm.

(A.I.C.B.S.E. 2017)

21. The length 'x' of a rectangle is decreasing at the rate of 3 cm/m and the width 'y' is increasing at the rate of 2 cm/m. Find the rates of change of :

(*a*) the perimeter (*b*) the area of the rectangle when $x = 8$ cm and $y = 6$ cm. *(P.B. 2014 S)*

22. An edge of a variable cube is increasing at the rate of 3 cm per second. How fast is the volume of the cube increasing when the edge is 10 cm long ?

(N.C.E.R.T.; Kashmir B. 2016)

23. (*i*) Water is dripping out from a conical funnel at the uniform rate of 2 cm^3/s through a tiny hole at the vertex at the bottom. When the slant height of the water is 5 cm, find the rate of decrease of the slant height of the water.

(*ii*) An inverted conical vessel whose height is 10 cm and the radius of whose base is 5 cm is being filled with water at the uniform rate of 1· 5 cu cm/m. Find the rate at which the level of water in the vessel is rising when the depth is 4 cm.

24. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 m/sec. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?

(N.C.E.R.T.; Jharkhand B. 2016; A.I.C.B.S.E. 2012)

25. The radius of a circular soap bubble is increasing at the rate of 0·2 cm/s. Find the rate of change of its :

(I) Volume (II) Surface area

when the radius is 4 cm.

Long Answer Type Questions

LATQ

26. Water is running into a conical vessel, 15 cm deep and 5 cm in radius at the rate of 0·1 cu. cm/s. When the water is 6 cm deep, find at what rate is :

(*a*) the water level rising ?

(*b*) the water surface area increasing ?

(*c*) the wetted surface of the vessel increasing ?

Answers

1. 10π cm^2/cm.

2. $4\pi r^2$.

3. 900 cm^3/s.

4. 20π cm^3/s.

5. (*i*) $1{\cdot}4\pi$ cm/s. (*ii*) 64π cm^2/s.

6. 60π cm^2/s.

7. 2π cm^3/s.

8. 400π cm^2/s

9. $0{\cdot}4\pi$ cm^3/s.

10. $\frac{27\pi}{8}(2x+1)^2$.

11. (*i*) $\frac{1}{\pi}$ cm/s (*ii*) 400π cm^3/r.

12. (*a*) 3·6 cm^2/s. (*b*) $\frac{8}{3}$ cm^2/s. (*c*) $\frac{7}{6}$ cm^2/s.

13. (*i*) $\left(\frac{1}{2}, \frac{31}{6}\right), \left(-\frac{1}{2}, \frac{29}{6}\right)$ (*ii*) $(4, 11), \left(-4, -\frac{31}{3}\right)$.

(*iii*) $(-4, 4)$.

14. 125π cm^3/s ; $r = 2h$.

15. 30.02 units,

16. 190 units.

17. MR = 34 units, MR increases when an additional unit beyond 6 units is sold.

18. (*i*) 0·4 m/s. (*ii*) $\frac{5}{2}$ km/h.

19. 2 cm^2/minute

20. 768π cm^2/s

21. (*a*) – 2 cm/m. (*b*) – 2 cm^2/m.

22. 900 cm^3/s

23. (*i*) $\frac{2}{25\pi \sin^2 \alpha \cos \alpha}$ cm/s,

where 'α' is the semivertical angle of the conical funnel

(*ii*) $\frac{3}{8\pi}$ cm/m.

24. $\frac{8}{3}$ m/s.

25. (I) $\frac{64\pi}{5}$ (II) $\frac{32\pi}{5}$.

26. (*a*) $\frac{1}{40\pi}$ cm/s. (*b*) $\frac{1}{30}$cm^2 / s.

(*c*) $\frac{\sqrt{10}}{30}$cm^2 / s.

Hints to Selected Questions

5. (*i*) $C = 2\pi r, \frac{dr}{dt} = 0{\cdot}7.$

$$\therefore \frac{dC}{dt} = 2\pi \frac{dr}{dt} = 2\pi\,(0{\cdot}7) = 1{\cdot}4\pi \text{ cm/s}.$$

7. Here $V = \frac{4}{3}\pi r^3$ so that $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

Find $\frac{dV}{dt}$ when $r = 1$ and $\frac{dr}{dt} = \frac{1}{2}$.

8. Here $S = 4\pi r^2$ so that $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$.

Find $\frac{dS}{dt}$ when $r = 10$ and $\frac{dr}{dt} = 5$.

10. $V = \frac{4}{3}\pi\left(\frac{3}{4}(2x+1)\right)^3 = \frac{9}{16}\pi(2x+1)^3$.

$$\therefore \frac{dV}{dx} = \frac{9}{16}\pi.3(2x+1)^2.2 = \frac{27}{8}\pi(2x+1)^2.$$

12. (*a*) $\frac{d}{dt}(x^3) = 9 \Rightarrow 3x^2\frac{dx}{dt} = 9$

$$\Rightarrow \frac{dx}{dt} = \frac{9}{3x^2} = \frac{3}{x^2} \qquad ...(1)$$

Now $S = 6x^2$.

$$\therefore \frac{dS}{dt} = 12x\frac{dx}{dt} = 12x\left(\frac{3}{x^2}\right) = \frac{36}{x}.$$

Hence, $\left.\frac{dS}{dt}\right]_{x=10} = \frac{36}{10} = 3.6\,\text{cm}^2/\text{s}.$

13. (*i*) Here $y = \frac{4}{3}x^3 + 5$ so that $\frac{dy}{dx} = 4x^2$.

Now $\frac{dy}{dt} = \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = 1 \Rightarrow 4x^2 = 1 \Rightarrow x \pm \frac{1}{2}$; etc.

16. $R(x) = 7x^2 + 50x + 119.$

$\therefore \quad R'(x) = 14x + 50.$

$\therefore \quad R'(10) = 14\,(10) + 50 = 190$ units.

23. (*i*) $V = \frac{1}{3}\pi\,(l^2 \sin^2 \alpha)\,(l \cos \alpha)$, where 'α' is the semi–vertical angle, and '*l*' is the slant height

so that $\frac{dV}{dt} = \frac{1}{3}\pi \sin^2 \alpha \cos \alpha.\, 3l^2 \frac{dl}{dt}$.

Find $\frac{dl}{dt}$ when $l = 5$ and $\frac{dV}{dt} = -2$.

SUB CHAPTER 6.2 Increasing and Decreasing Functions

6.2. INCREASING AND DECREASING FUNCTIONS

Now we shall use differentiation to determine whether the function is increasing or decreasing or none.

(a) (i) INCREASING FUNCTION

Definition

A function f (x) is said to be an increasing function of x if :
f (x) increases as x increases
or f (x) decreases as x decreases.

Symbolically : $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2)$

or $x_1 \ge x_2 \Rightarrow f(x_1) \ge f(x_2)$.

KEY POINT

If $f(x)$ is an increasing function, then x and $f(x)$ both increase or decrease simultaneously.

For Example : $f(x) = x^2,\ x \ge 0$ *is an increasing function of* x because $f(x)$ increases as x increases.

(See the following table) :

$x =$	0	1	2	3	...
$f(x) =$	0	1	4	9	...

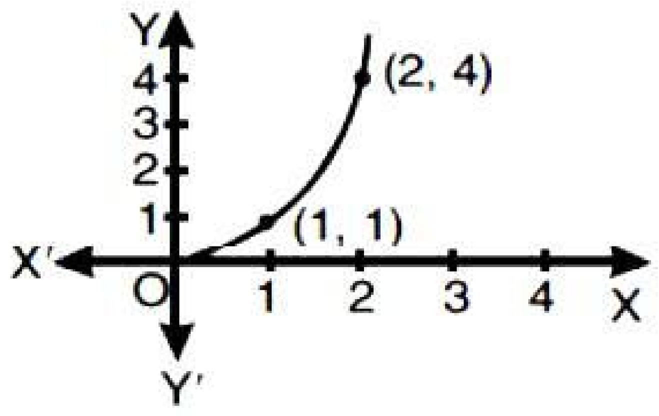


Fig.

Theorem. *For a function f(x) continuous in* [a, b] *if* $f'(x) \ge 0$ *for all x in* (a, b), *then f(x) is an increasing function in* [a, b].

Proof. Let $\delta x \ge 0$ and $x \in (a, b)$.

Since 'f' is an increasing function of x in $[a, b]$,

$\therefore \quad x + \delta x \ge x \Rightarrow f(x + \delta x) \ge f(x)$ [*Def.*]

$\Rightarrow \quad f(x + \delta x) - f(x) \ge 0$

$$\Rightarrow \quad \frac{f(x+\delta x) - f(x)}{\delta x} \ge 0 \qquad [\because \delta x > 0]$$

$$\Rightarrow \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \ge 0 \quad \Rightarrow f'(x) \ge 0.$$

(ii) STRICTLY INCREASING FUNCTION

Definition

A function f(x) is said to be strictly increasing function of x if :

$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

or $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$.

Theorem. *For a function f (x) continuous in [a, b] if $f'(x) > 0$ for all x in (a, b), then f (x) is strictly increasing function in (a, b).*

This can be proved as above.

KEY POINT

There can be functions, which are increasing but not strictly increasing.

(b) (i) DECREASING FUNCTION

Definition

A function f (x) is said to be a decreasing function of x if :

f (x) increases as x decreases

or f (x) decreases as x increases.

Symbolically : $x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2)$ *or* $x_1 \ge x_2 \Rightarrow f(x_1) \le f(x_2)$.

For Example : $f(x) = \cos x, 0 \le x \le \frac{\pi}{2}$ *is a decreasing function* because *f*(*x*) decreases as *x* increases.

(See the following table) :

$x =$	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$f(x) =$	1	0·87	0·5	0

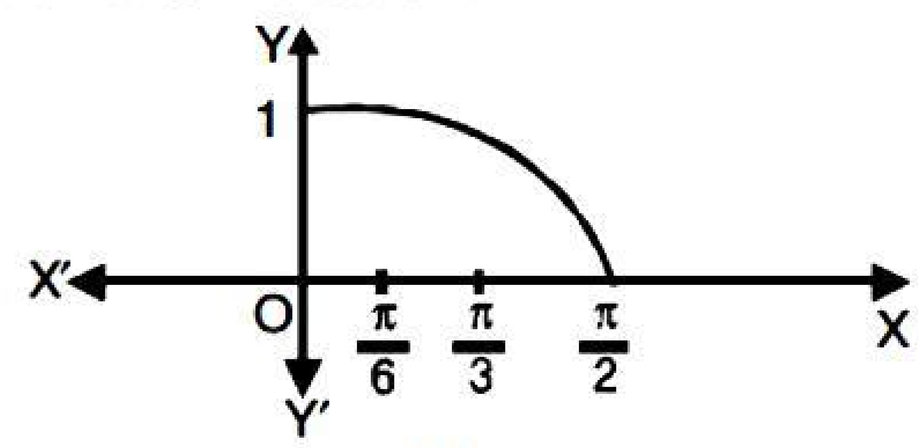


Fig.

Theorem. *For a function f (x) continuous in [a, b] if $f'(x) \le 0$ for all x in (a, b), then f (x) is a decreasing function in [a, b].*

Proof. Let $\delta x \ge 0$ and $x \in (a, b)$.

Since '*f*' is a decreasing function of *x* in [*a*, *b*],

$\therefore \quad x + \delta x \ge x \Rightarrow f(x + \delta x) \le f(x)$ [*Def.*]

$\Rightarrow \quad f(x + \delta x) - f(x) \le 0$

$$\Rightarrow \quad \frac{f(x+\delta x) - f(x)}{\delta x} \le 0 \qquad [\because \delta x > 0]$$

$$\Rightarrow \quad \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} \le 0$$

$\Rightarrow \quad f'(x) \le 0.$

(ii) STRICTLY DECREASING FUNCTION

Definition

A function f (x) is said to be strictly decreasing function of x if :

$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

or $x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$.

Theorem. *For a function f (x) continuous in [a, b], if $f'(x) < 0$ for all x in (a, b), then f (x) is strictly decreasing function in (a, b).*

This can be proved as above.

KEY POINT

There can be functions, which are decreasing but not strictly decreasing.

(c) MIXED FUNCTION

A function may be increasing in a certain interval and decreasing in another interval.

The function is neither wholly increasing nor wholly decreasing.

Such functions are called **mixed functions.**

In the adjoining figure, $f(x)$ is increasing in $[a, c]$ and decreasing in $[c, b]$.

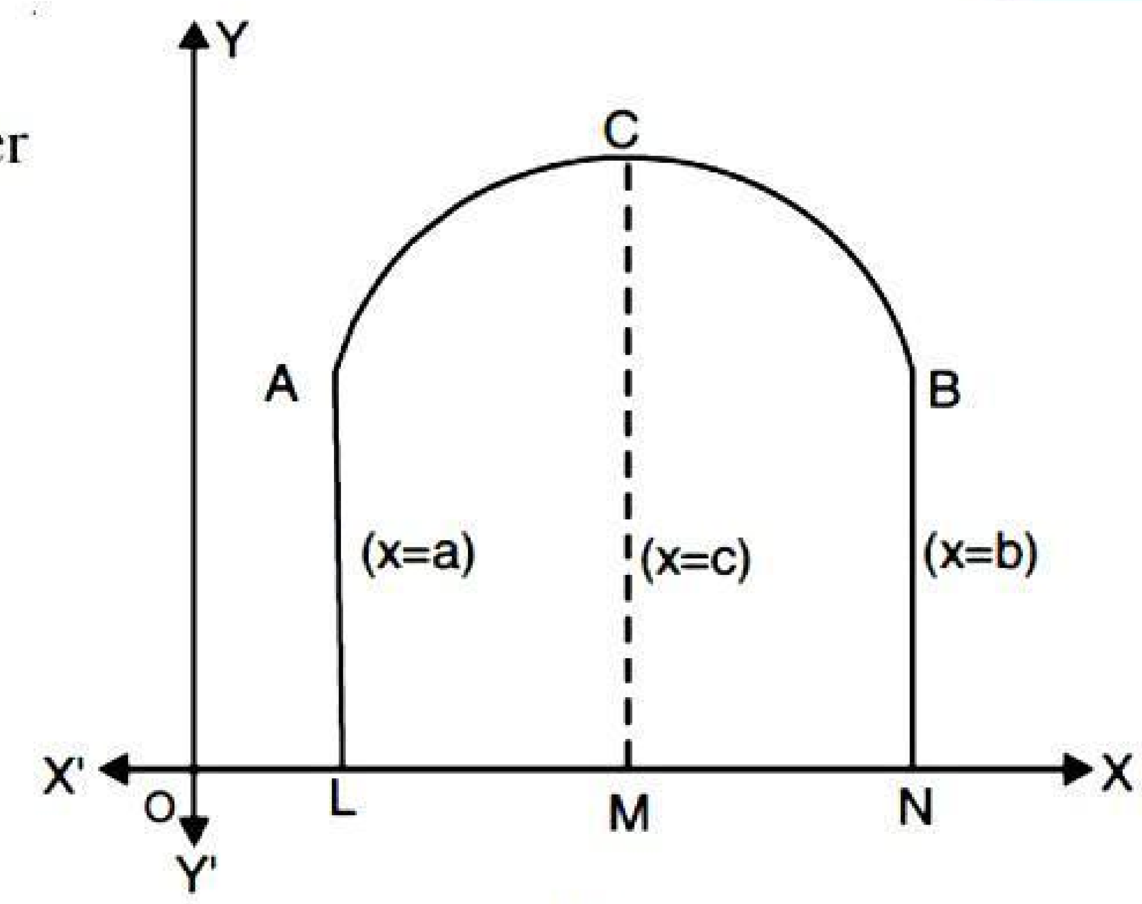


Fig.

(d) GRAPHICAL REPRESENTATIONS

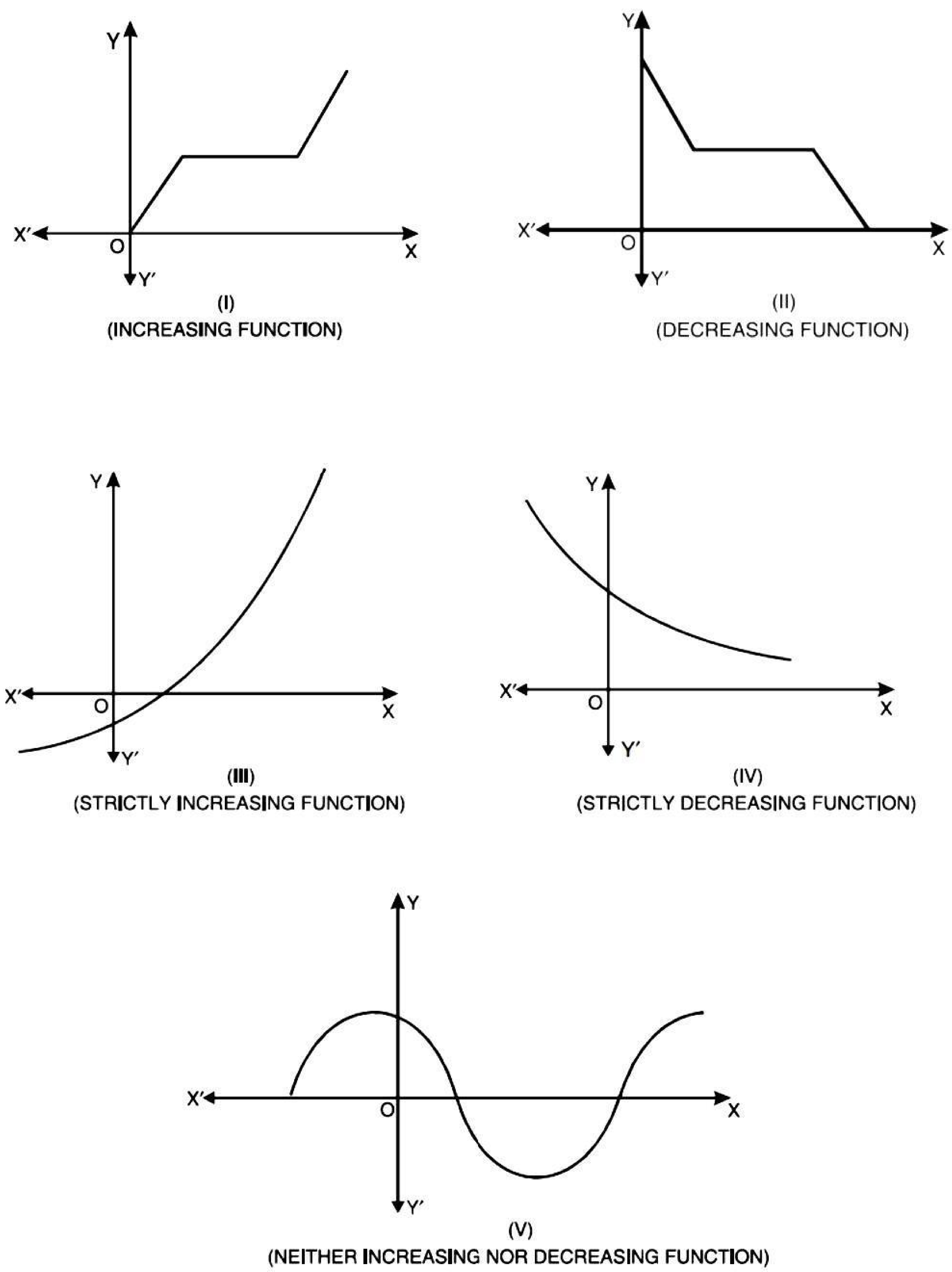


Fig.

(e) MONOTONE FUNCTION

Definition

A function f(x) is said to be monotone if it is either increasing or decreasing.

SOME USEFUL RESULTS

(1) If $|x| < l$, then $-l < x < l$.

(1)′ If $|x| \le l$, then $-l \le x \le l$.

(2) If $|x| > l$, then either $x < -l$ or $x > l$.

(2)′ If $|x| \ge l$, then either $x \le -l$ or $x \ge l$.

(3) If $(x - \alpha)(x - \beta) < 0$, then x lies between α and β, where $\beta > \alpha$.

(4) If $(x - \alpha)(x - \beta) > 0$, then x does not lie between α and β, where $\beta > \alpha$.

Frequently Asked Questions

FAQs

Example 1. Without using the derivative, show that the function f (x) = 7x – 3 is a strictly increasing function in R. (N.C.E.R.T.)

Solution. Let x_1 and $x_2 \in$ **R**.

Now $x_1 > x_2 \Rightarrow 7x_1 > 7x_2$

$\Rightarrow 7x_1 - 3 > 7x_2 - 3$

$\Rightarrow f(x_1) > f(x_2)$.

Hence, 'f' is strictly increasing function in **R**.

Exampe 2. Show that function :

$$\mathbf{f(x) = 4x^3 - 18x^2 - 27x - 7}$$

is always increasing in R. (C.B.S.E. 2017)

Solution. We have : $f(x) = 4x^3 - 18x^2 + 27x - 7$.

$\therefore \quad f'(x) = 12x^2 - 36x + 27$

$= 12(x^2 - 3x) + 27$

$= 12\left(x^2 - 3x + \frac{9}{4}\right) + 27 - 27$

$= 12\left(x - \frac{3}{2}\right)^2 \ge 0 \ \forall \ x \in \mathbf{R}$.

Hence, $f(x)$ is always increasing in **R**.

Example 3. Find the intervals in which the function f (x) is (i) strictly increasing (ii) strictly decreasing :

$$\mathbf{f(x) = x^3 - 12x^2 + 36x + 17.}$$ (C.B.S.E. 2009 C)

Solution. We have : $f(x) = x^3 - 12x^2 + 36x + 17$.

$\therefore \quad f'(x) = 3x^2 - 24x + 36$.

(I) *For f (x) to be strictly increasing function of x :*

$f'(x) > 0$

$\Rightarrow 3x^2 - 24x + 36 > 0$

$\Rightarrow x^2 - 8x + 12 > 0$

$\Rightarrow (x - 2)(x - 6) > 0$

$\Rightarrow x < 2$ or $x > 6$.

Hence, $f(x)$ is increasing in the interval $(-\infty, 2) \cup (6, \infty)$.

Fig.

(II) *For f (x) to be strictly decreasing function of x :*

$f'(x) < 0$

$\Rightarrow 3x^2 - 24x + 36 < 0$

$\Rightarrow x^2 - 8x + 12 < 0$

$\Rightarrow (x - 2)(x - 6) < 0$

$\Rightarrow 2 < x < 6$.

Hence, $f(x)$ is decreasing in the interval (2, 6).

Fig.

Example 4. Find the intervals in which the function

$$\mathbf{f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12} \text{ is :}$$

(*a*) strictly increasing (*b*) strictly decreasing. (C.B.S.E. 2018)

Solution. The given function is

$$f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12.$$

$$\therefore \quad f'(x) = \frac{4x^3}{4} - 3x^2 - 10x + 24$$

$= x^3 - 3x^2 - 10x + 24$.

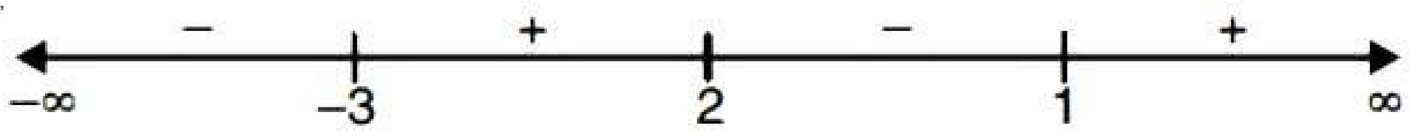
Now, $f'(x) = 0$

$\Rightarrow x^3 - 3x^2 - 10x + 24 = 0$

$\Rightarrow \quad (x-2)\ (x^2 - x - 12) = 0$

$\Rightarrow \ (x-2)\ (x-4)\ (x+3) = 0$

$\Rightarrow \qquad x = -3, 2, 4.$



When $x < -3$, then $f'(x)$ is $(-)\ (-)\ (-)$ *i.e.* –ve

When $-3 < x < 2$, then $f'(x)$ is $(-)\ (-)\ (+)$ *i.e.* +ve

When $2 < x < 4$, then $f'(x)$ is $(+)\ (-)\ (+)$ *i.e.* –ve

When $x > 4$, then $f'(x)$ is $(+)\ (+)\ (+)$ *i.e.* +ve.

Hence, $f(x)$ is strictly increasing in $(-3, 2) \cup (4, \infty)$ and strictly decreasing in $(-\infty, -3) \cup (2, 4)$.

Example 5. Find the intervals in which the function :

$$\mathbf{f(x) = \sin^4 x + \cos^4 x, 0 \le x \le \frac{\pi}{2}}$$

is strictly increasing or decreasing.

Solution. We have : $f(x) = \sin^4 x + \cos^4 x$.

$\therefore \quad f'(x) = 4 \sin^3 x \,.\, \cos x + 4 \cos^3 x\ (-\sin x)$

$= 4 \sin^3 x \cos x - 4 \cos^3 x \sin x$

$= -4 \sin x \cos x\ (\cos^2 x - \sin^2 x)$

$= -2 \sin 2x \cos 2x = -\sin 4x.$

(I) *For $f(x)$ to be stricitly increasing function of x* :

$f'(x) > 0 \ \Rightarrow \ -\sin 4x > 0$

$\Rightarrow \sin 4x < 0 \ \Rightarrow \ \pi < 4x < 2\pi$

$\Rightarrow \dfrac{\pi}{4} < x < \dfrac{\pi}{2}.$

Hence, $f(x)$ is strictly increasing in $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$.

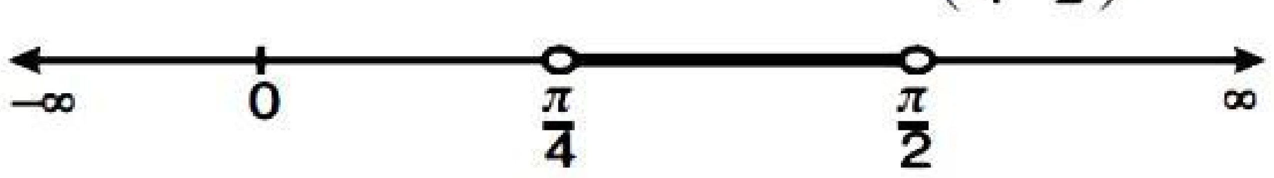


Fig.

(II) *For $f(x)$ to be strictly decreasing function of x* :

$f'(x) < 0 \ \Rightarrow \ -\sin 4x < 0$

$\Rightarrow \sin 4x > 0 \ \Rightarrow \ 0 < 4x < \pi$

$\Rightarrow \qquad 0 < x < \dfrac{\pi}{4}.$

Hence, $f(x)$ is strictly decreasing in $\left(0, \dfrac{\pi}{4}\right)$.

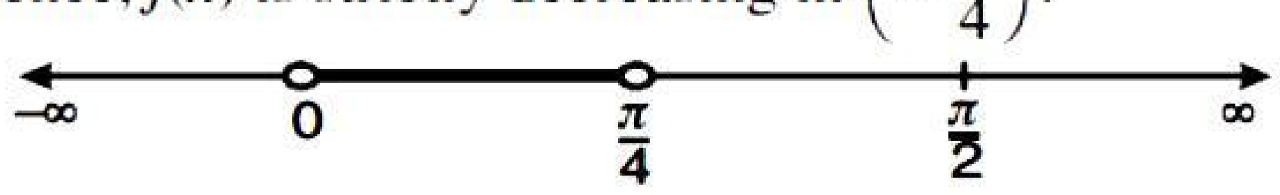


Fig.

Example 6. Find the intervals in which :

$\mathbf{f(x) = \sin 3x - \cos 3x, 0 < x < \pi}$, is strictly increasing or strictly decreasing. *(C.B.S.E. 2016)*

Solution. We have : $f(x) = \sin 3x - \cos 3x$.

$\therefore f'(x) = 3 \cos 3x + 3 \sin 3x.$

Now $f'(x) = 0 \Rightarrow 3 \cos 3x + 3 \sin 3x = 0$

$\Rightarrow \cos 3x = -\sin 3x$

$\Rightarrow \tan 3x = -1 \Rightarrow 3x = \dfrac{3\pi}{4}$ or $\dfrac{7\pi}{4}$ or $\dfrac{11\pi}{4}$

$[\because 0 < x < \pi]$

$\Rightarrow x = \dfrac{\pi}{4}, \dfrac{7\pi}{12}$ or $\dfrac{11\pi}{12}$.

$\therefore$ The points $x = \dfrac{\pi}{4}, \dfrac{7\pi}{12}, \dfrac{11\pi}{12}$ divide the interval into four parts:

$$\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{7\pi}{12}\right), \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right) \left(\frac{11\pi}{12}, \pi\right).$$

In $\left(0, \dfrac{\pi}{4}\right)$, $\quad f'(x) > 0 \Rightarrow \quad f$ is strictly increasing.

In $\left(\dfrac{\pi}{4}, \dfrac{7\pi}{12}\right)$, $\quad f'(x) < 0 \Rightarrow \quad f$ is strictly decreasing.

In $\left(\dfrac{7\pi}{12}, \dfrac{11\pi}{12}\right)$, $f'(x) > 0 \Rightarrow \quad f$ is strictly increasing.

In $\left(\dfrac{11\pi}{12}, \pi\right)$, $\quad f'(x) < 0 \Rightarrow \quad f$ is strictly decreasing.

Hence, 'f' is strictly increasing in $\left(0, \dfrac{\pi}{4}\right) \cup \left(\dfrac{7\pi}{12}, \dfrac{11\pi}{12}\right)$

and strictly decreasing in $\left(\dfrac{\pi}{4}, \dfrac{7\pi}{12}\right) \cup \left(\dfrac{11\pi}{12}, \pi\right)$.

Example 7. Find the values of 'x' for which $\mathbf{f(x) = x^x}$; $\mathbf{x > 0}$ is strictly increasing or decreasing.

Solution. We have : $f(x) = x^x$, which is defined for $x > 0$.

$\therefore \qquad D_f = (0, \infty).$

Now $\qquad f(x) = x^x \ \Rightarrow \ f(x) = e^{\log x^x}$

$\Rightarrow \qquad f(x) = e^{x \log x}.$

$\therefore \qquad f'(x) = e^{x \log x} . \dfrac{d}{dx}(x \log_e x)$

$= e^{x \log x} . \left(x . \dfrac{1}{x} + \log_e x . 1\right)$

$= e^{\log x^x}\ (1 + \log_e x)$

$= x^x\ (1 + \log_e x).$

(I) *For f(x) to be strictly increasing function of x* :

$$f'(x) > 0 \quad \Rightarrow \quad x^x(1+\log_e x) > 0$$

$\Rightarrow \ 1 + \log_e x > 0$ $\qquad [\because x^x > 0 \text{ for } x > 0]$

$\Rightarrow \ \log_e x > -1 \Rightarrow x > e^{-1}$

$\Rightarrow \ x > \frac{1}{e}.$

Hence, $f(x)$ is strictly increasing on $\left(\frac{1}{e}, \infty\right)$.

(II) *For f(x) to be strictly decreasing function of x* :

$$f'(x) < 0 \quad \Rightarrow \quad x^x(1+\log_e x) < 0$$

$\Rightarrow \ 1 + \log_e x < 0$ $\qquad [\because x^x > 0 \text{ for } x > 0]$

$\Rightarrow \ \log_e x < -1 \Rightarrow x < e^{-1}$

$\Rightarrow \ x < \frac{1}{e}.$

Hence, $f(x)$ is strictly decreasing on $\left(0, \frac{1}{e}\right)$.

$[\because D_f = (0, \infty)]$

Example 8. If a, b, c are real numbers, then find the intervals in which :

$$\mathbf{f(x)} = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$$

is strictly increasing or decreasing.

Solution. We have :

$$f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}.$$

$$\therefore \ f'(x) = \begin{vmatrix} 1 & 0 & 0 \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ 0 & 1 & 0 \\ ac & bc & x+c^2 \end{vmatrix} + \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (x+b^2)(x+c^2) - b^2c^2 + (x+a^2)(x+c^2) - a^2c^2 + (x+a^2)(x+b^2) - a^2b^2$$

$$= 3x^2 + 2(a^2+b^2+c^2)x.$$

(I) *For f(x) to be strictly increasing function of x* :

$$f'(x) > 0 \ \Rightarrow \ 3x^2 + 2(a^2+b^2+c^2)x > 0$$

$\Rightarrow \ 3x\left(x + \frac{2}{3}(a^2+b^2+c^2)\right) > 0$

$\Rightarrow \ \left(x + \frac{2}{3}(a^2+b^2+c^2)\right)x > 0$

$\Rightarrow \ \left(x - \left(-\frac{2}{3}(a^2+b^2+c^2)\right)\right)(x-0) > 0$

$\Rightarrow \ x < -\frac{2}{3}(a^2+b^2+c^2)$ or $x > 0.$

Hence, $f(x)$ is strictly increasing in :

$$\left(-\infty, \frac{-2}{3}(a^2+b^2+c^2)\right) \cup (0, \infty).$$

(II) *For f (x) to be strictly decreasing function of x* :

$$f'(x) < 0 \quad \Rightarrow \quad 3x^2 + 2x(a^2+b^2+c^2) < 0$$

$\Rightarrow \ 3x\left(x + \frac{2}{3}(a^2+b^2+c^2)\right) < 0$

$\Rightarrow \ \left(x + \frac{2}{3}(a^2+b^2+c^2)\right)x < 0$

$\Rightarrow \ \left(x - \left(-\frac{2}{3}(a^2+b^2+c^2)\right)\right)(x-0) < 0$

$\Rightarrow \ \frac{-2}{3}(a^2+b^2+c^2) < x < 0.$

Hence, $f(x)$ is strictly decreasing in $\left(-\frac{2}{3}(a^2+b^2+c^2), 0\right)$.

Example 9. Show that :

$$\mathbf{\frac{x}{1+x} < \log(1+x) < x \text{ for } x > 0.}$$

Solution. Let $f(x) = \log(1+x) - \frac{x}{1+x}$.

$\therefore \ f'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2} > 0$ for $x > 0$.

$\therefore \ f'(x) > 0$ for all $x > 0$ and $f'(0) = 0$.

Thus $f(x)$ is increasing in $(0, \infty)$.

Also $f(0) = 0$.

Now $x > 0 \ \Rightarrow \ f(x) > f(0)$

$\Rightarrow \ f(x) > 0$

$\Rightarrow \ \left(\log(1+x) - \frac{x}{1+x}\right) > 0$

$\Rightarrow \ \log(1+x) > \frac{x}{1+x}$ $\qquad$ ...(1)

Again let $g(x) = x - \log(1 + x)$.

$\therefore\ g'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x} > 0$ for $x > 0$.

$\therefore\ g'(x) > 0$ for all $x > 0$ and $g'(0) = 0$.

Thus $g(x)$ is increasing in $(0, \infty)$.

Also $g(0) = 0$.

Now $x > 0 \Rightarrow g(x) > g(0) \Rightarrow g(x) > 0$

$\Rightarrow [x - \log(1 + x)] > 0$

$\Rightarrow x > \log(1 + x)$ ...(2)

Combining (1) and (2), $\dfrac{x}{1+x} < \log(1 + x) < x$ for $x > 0$, which is true.

EXERCISE 6 (b)

Fast Track Answer Type Questions

FTATQ

1. Show that the following functions are strictly increasing on **R** :

(*a*) $f(x) = 3x + 17$ (*b*) (*i*) $f(x) = e^x$ (*ii*) $f(x) = e^{2x}$. *(N.C.E.R.T.)*

2. Without using the dervative, show that $f(x) = |x|$ is :

(*a*) strictly increasing in $(0, \infty)$

(*b*) strictly decreasing in $(-\infty, 0)$.

3. (*i*) Show that the function $f(x) = x^3 - 3x^2 + 4x$, $x \in \mathbf{R}$ is strictly increasing on **R**.

(N.C.E.R.T.; Kashmir B. 2016, 11 ; H.P.B. 2011)

(*ii*) Show that the function $f(x) = x^3 - 6x^2 + 15x + 4$ is strictly increasing in **R.** *(Kerala B. 2013)*

Prove the following (4–5) :

4. (*i*) $f(x) = x^2$ is a decreasing function for $x < 0$, where $x \in \mathbf{R}$

(*ii*) $f(x) = x^2 - 8x,\ x \le 4$ is a decreasing function

(*iii*) $f(x) = \dfrac{3}{x} + 7$ is strictly decreasing for $x \in \mathbf{R}\ (x \neq 0)$.

5. (*i*) $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing function on **R** *(N.C.E.R.T.)*

(*ii*) $f(x) = x^3 - 3x^2 + 4x,\ x \in \mathbf{R}$ is strictly increasing. *(Kerala B. 2018)*

Very Short Answer Type Questions

VSATQ

Prove the following (6 – 10) :

6. (*i*) $f(x) = \cos x$ is :

(I) strictly decreasing in $(0, \pi)$

(II) strictly increasing in $(\pi, 2\pi)$

(III) neither increasing nor decreasing in $(0, 2\pi)$

(N.C.E.R.T. ; Uttarakhand B. 2013)

(*ii*) $f(x) = \cos^2 x$ is strictly decreasing function in $\left(0, \dfrac{\pi}{2}\right)$.

7. (*i*) $f(x) = \sin x$ is :

(I) strictly increasing in $\left(0, \dfrac{\pi}{2}\right)$

(II) strictly decreasing in $\left(\dfrac{\pi}{2}, \pi\right)$

(III) neither increasing nor decreasing in $(0, \pi)$ *(N.C.E.R.T.)*

(*ii*) $f(x) = 2\sin x + 1$ is an increasing function on $\left[0, \dfrac{\pi}{2}\right]$.

(J. & K. B. 2011, 10)

8. $f(x) = \tan^{-1}(\sin x + \cos x)$ is strictly decreasing function on $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$.

9. The logarithmic function is strictly increasing on $(0, \infty)$. *(N.C.E.R.T.)*

10. (*i*) $f(x) = \log \sin x$ is strictly increasing on $\left(0, \dfrac{\pi}{2}\right)$ and strictly decreasing on $\left(\dfrac{\pi}{2}, \pi\right)$.

(N.C.E.R.T.; H.P.B. 2014, 11)

(*ii*) $f(x) = \log \cos x$ is strictly decreasing on $\left(0, \dfrac{\pi}{2}\right)$ and strictly increasing on $\left(\dfrac{\pi}{2}, \pi\right)$.

(N.C.E.R.T.; Jammu B. 2018; H.P.B. 2011)

11. Show that the function :

$f(x) = x^3 - 3x^2 + 6x - 100$

is increasing on **R.** *(A.I.C.B.S.E. 2017)*

12. (*a*) Find the intervals in which the following functions are increasing :

(*i*) $2x^3 - 3x$ *(H.B. 2011)*

(*ii*) $10 - 6x - 2x^2$. *(H.B. 2011)*

(*b*) Find the interval in which $2x^3 + 9x^2 + 12x - 1$ is strictly increasing. *(Kerala B. 2017)*

(*c*) Find the intervals in which the functions :

(*i*) $f(x) = x^3 + 2x^2 - 1$

(*ii*) $30 - 24x + 15x^2 - 2x^3$ *(P.B. 2011)*

are strictly decreasing.

13. Prove that $f(x) = x^2 - x + 1$ is neither increasing nor decreasing strictly on $(-1, 1)$. *(N.C.E.R.T.)*

14. (*i*) Find the values of 'a' for which the function : $f(x) = x^2 - 2ax + 6$ is increasing when $x > 0$.

(*ii*) Find the values of 'a' for which $f(x) = \sin x - ax + b$ is decreasing function on **R**.

15. Find the values of 'x' for which $y = [x(x-2)]^2$ is an increasing function. *(N.C.E.R.T.; A.I.C.B.S.E. 2014)*

Determine for which values of x, the following functions (16 – 18) are increasing or decreasing :

16. $f(x) = -3x^2 + 12x + 8$.

17. (*i*) $f(x) = x^3 - 12x$ (*ii*) $f(x) = 2x^3 - 24x + 107$.

18. (*i*) $f(x) = x^4 - 2x^2$

(*ii*) $f(x) = x^3 + \dfrac{1}{x^3}, x \neq 0$. *(N.C.E.R.T.)*

19. For what values of 'x' are the following functions increasing or decreasing ?

(*i*) $y = x + \dfrac{1}{x}, x \neq 0$

(*ii*) $y = 5x^{3/2} - 3x^{5/2}, x > 0$. *(P.B. 2010)*

Short Answer Type Questions

SATQ

Determine the intervals in which the following functions (20–29) are strictly increasing or strictly decreasing :

20. (*i*) $f(x) = 2x^2 - 3x$ *(Kashmir B. 2016)*

(*ii*) $f(x) = x^2 + 2x - 5$

(N.C.E.R.T.; Kashmir B. 2016; H.P.B. 2013, 10)

(*iii*) $f(x) = 10 - 6x - 2x^2$

(N.C.E.R.T.; H.P.B. 2016, 13, 12, 10; Mizoram B. 2015 ; P.B. 2010)

(*iv*) $f(x) = 6 - 9x - x^2$

(N.C.E.R.T.; H.P.B. 2012, 10 ; P.B. 2010)

(*v*) $f(x) = x^2 - 4x + 6$

(N.C.E.R.T.; Kerala B. 2016; Karnataka B. 2014)

(*vi*) $f(x) = \dfrac{1}{4}x^4 + \dfrac{2}{3}x^3 - \dfrac{5}{2}x^2 - 6x + 7$ *(J. & K.B. 2011)*

(*vii*) $f(x) = -3\log(1+x) + 4\log(2+x) - \dfrac{4}{2+x}$.

(C.B.S.E. Sample Paper 2018)

21. (*i*) $f(x) = 20 - 9x + 6x^2 - x^3$ *(P.B. 2012)*

(*ii*) $f(x) = 2x^3 - 15x^2 + 36x + 17$.

22. (*i*) $f(x) = 2x^3 - 9x^2 + 12x + 15$ *(C.B.S.E. 2010)*

(*ii*) $f(x) = 2x^3 - 3x^2 - 36x + 7$.

(N.C.E.R.T.; H.P.B. 2017, Kashmir B. 2017; H.B. 2017, Jammu B. 2015 ; Meghalaya B. 2013)

23. (*i*) $f(x) = x^3 - 6x^2 + 9x + 8$ *(P.B. 2010)*

(*ii*) $f(x) = 4x^3 - 6x^2 - 72x + 30$.

(N.C.E.R.T. ; Kashmir B. 2017; H.P.B. 2017, 16)

24. (*i*) $f(x) = 2x^3 - 12x^2 + 18x + 5$ *(P.B. 2010)*

(*ii*) $f(x) = \dfrac{4x^2 + 1}{x}$.

25. (*i*) $f(x) = -2x^3 - 9x^2 - 12x + 1$

(N.C.E.R.T.; H.P.B. 2017, 16, 13, 12; Jammu B. 2016)

(*ii*) $f(x) = x^3 + 3x^2 - 4$.

26. (*i*) $f(x) = 2x^3 - 15x^2 + 36x + 6$

(*ii*) $f(x) = (x-1)(x-2)^2$.

(Bihar B. 2014; H.B. 2011; A.I.C.B.S.E. 2009 C)

27. (*i*) $f(x) = \dfrac{3}{10}x^4 - \dfrac{4}{5}x^3 - 3x^2 + \dfrac{36}{5}x + 11$

(N.C.E.R.T.)

(*ii*) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$. *(C.B.S.E. 2014)*

28. $f(x) = (x+1)^3(x-3)^3$. *(N.C.E.R.T. ; Jammu B. 2015)*

29. $f(x) = x^8 + 6x^2$.

30. On which of the following intervals is the function 'f' given by $f(x) = x^{100} + \sin x - 1$ strictly increasing ?

(*a*) $(-1, 1)$ (*b*) $(0, 1)$ (*c*) $\left(\dfrac{\pi}{2}, \pi\right)$ (*d*) $\left(0, \dfrac{\pi}{2}\right)$.

(N.C.E.R.T.)

31. Find the intervals in which $f(x) = \sin x - \cos x$, where $0 < x < 2\pi$, is strictly increasing or decreasing.

32. Find the intervals in which the function 'f' given by :

(*i*) $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$

(N.C.E.R.T.; H.B. 2017; P.B. 2015 ; Kashmir B. 2012 ; C.B.S.E. (F) 2011; A.I.C.B.S.E. 2009)

(*ii*) $f(x) = \sin x - \cos x$, $0 \le x \le 2\pi$

(Kashmir B. 2011 ; J. & K.B. 2011; C.B.S.E. 2010)

is strictly increasing or strictly decreasing.

33. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x + 29$ is :

(*i*) monotonic increasing (*ii*) monotonic decreasing.

34. Find the intervals in which the function given by $f(x) = \sin 3x,\ x \in \left[0, \frac{\pi}{2}\right]$ is :

(*a*) increasing (*b*) decreasing.

(N.C.E.R.T.; Assam B. 2018)

35. Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$?

(*i*) $\cos x$ (*ii*) $\cos 2x$

(*iii*) $\cos 3x$ (*iv*) $\tan x$. *(N.C.E.R.T.)*

Long Answer Type Questions

LATQ

36. If $x > -1$, show that : $\frac{x}{\sqrt{1+x}} - \log(1+x) + 9$ is an increasing function of x.

Answers

12. (*a*) (*i*) $\left(-\infty, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, \infty\right)$

(*ii*) $\left(-\infty, -\frac{3}{2}\right)$

(*b*) $(-\infty, -2) \cup (-1, \infty)$

(*c*) (*i*) $\left(\frac{-4}{3}, 0\right)$ (*ii*) $(-\infty, 1) \cup (4, \infty)$.

14. (*i*) $a \le 0$ (*ii*) $a \ge 1$.

15. $x \in (0, 1)$ and $x \in (2, \infty)$.

	Increasing	**Decreasing**
16.	$(-\infty, 2)$	$(2, \infty)$.
17. (*i*)	– (*ii*) $(-\infty, -2) \cup (2, \infty)$	$(-2, 2)$.
18. (*i*)	$(-1, 0) \cup (1, \infty)$	$(-\infty, -1) \cup (0, 1)$
(*ii*)	$(-\infty, -1) \cup (1, \infty)$	$(-1, 1) - \{0\}$.
19. (*i*)	$(-\infty, -1) \cup (1, \infty)$	$(-1, 0) \cup (0, 1)$
(*ii*)	$(0, 1)$	$(1, \infty)$.

	Strictly Increasing	**Structly Decreasing**
20. (*i*)	$\left(\frac{3}{4}, \infty\right)$	$\left(-\infty, \frac{3}{4}\right)$
(*ii*)	$(-1, \infty)$	$(-\infty, -1)$
(*iii*)	$\left(-\infty, \frac{-3}{2}\right)$	$\left(\frac{-3}{2}, \infty\right)$
(*iv*)	$\left(-\infty, -\frac{9}{2}\right)$	$\left(-\frac{9}{2}, \infty\right)$
(*v*)	$(2, \infty)$	$(-\infty, 2)$
(*vi*)	$(-3, -1) \cup (2, \infty)$	$(-\infty, -3) \cup (-1, 2)$
(*vii*)	$(-\infty, -4) \cup (0, \infty)$	$(-4, 0)$.
21. (*i*)	$(1, 3)$	$(-\infty, 1) \cup (3, \infty)$
(*ii*)	$(-\infty, 2) \cup (3, \infty)$	$(2, 3)$.
22. (*i*)	$(-\infty, 1) \cup (2, \infty)$	$(1, 2)$
(*ii*)	$(-\infty, -2) \cup (3, \infty)$	$(-2, 3)$.
23. (*i*)	$(-\infty, 1) \cup (3, \infty)$	$(1, 3)$
(*ii*)	$(-\infty, -2) \cup (3, \infty)$	$(-2, 3)$.

	Strictly Increasing	**Strictly Decreasing**
24. (*i*)	$(-\infty, 1) \cup (3, \infty)$	$(1, 3)$
(*ii*)	$\left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$	$\left(-\frac{1}{2}, \frac{1}{2}\right)$.
25. (*i*)	$(-2, -1)$	$(-\infty, -2) \cup (-1, \infty)$
(*ii*)	$(-\infty, -2) \cup (0, \infty)$	$(-2, 0)$.
26. (*i*)	$(-\infty, 2) \cup (3, \infty)$	$(2, 3)$
(*ii*)	$\left(-\infty, \frac{4}{3}\right) \cup (2, \infty)$	$\left(\frac{4}{3}, 2\right)$.
27. (*i*)	$(-2, 1) \cup (3, \infty)$	$(-\infty, -1) \cup (0, 2)$
(*ii*)	$(-1, 0) \cup (2, \infty)$	$(-\infty, 0) \cup (1, 2)$.
28.	$(1, \infty)$	$(-\infty, 1)$.
29.	$(0, \infty)$	$(-\infty, 0)$.

30. (*b*), (*c*), (*d*).

31. Strictly Increasing in $\left(0, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$

Strictly Decreasing in $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

32. (*i*) Strictly increasing in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$;

Strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

(*ii*) Strictly increasing in $\left[0, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right]$

Strictly decreasing in $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

33. (*i*) $(-\infty, 1) \cup (2, \infty)$ (*ii*) $(1, 2)$.

34. (*a*) Increasing on $\left[0, \frac{\pi}{6}\right]$

(*b*) Decreasing on $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

35. (*i*), (*ii*).

Hints to Selected Questions

10. (*i*) Here $f(x) = \log \sin x$.

$\therefore \quad f'(x) = \frac{1}{\sin x}.\cos x = \cot x > 0$ for all $x \in \left(0, \frac{\pi}{2}\right)$

< 0 for all $x \in \left(\frac{\pi}{2}, \pi\right)$.

13. Here $f(x) = x^2 - x + 1$.

$\therefore \quad f'(x) = 2x - 1 = 2\left(x - \frac{1}{2}\right) > 0$ where $x > \frac{1}{2}$

< 0 where $x < \frac{1}{2}$.

Hence, 'f' is neither increasing nor decreasing in (0, 1).

30. $f(x) = x^{100} + \sin x - 1$.

$\therefore \quad f'(x) = 100x^{99} + \cos x$.

Hence, $f'(x) > 0$ in (*b*), (*c*) and (*d*).

34. $f'(x) = 3\cos 3x$ so that $f'(x) = 0$

$\Rightarrow \quad \cos 3x = 0 \Rightarrow 3x = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{6}$.

When $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$; $f'(x) < 0 \Rightarrow f(x)$ is decreasing.

SUB CHAPTER 6.3 Tangents and Normals

6.3. TANGENTS AND NORMALS

Now we shall use differentiation to find the equations of tangent and normal to a curve at a given point.

Let $y = f(x)$ be the curve.

Let KPT be the tangent to the curve at the point $P(x', y')$.

Let $\angle TKX = \psi$.

Since $\frac{dy}{dx}$ = Slope of the tangent at any point,

$\therefore \frac{dy}{dx}$ at P = $\tan \psi$, which is also called as **gradient**

or $\quad m = \tan \psi = \left(\frac{dy}{dx}\right)_P$.

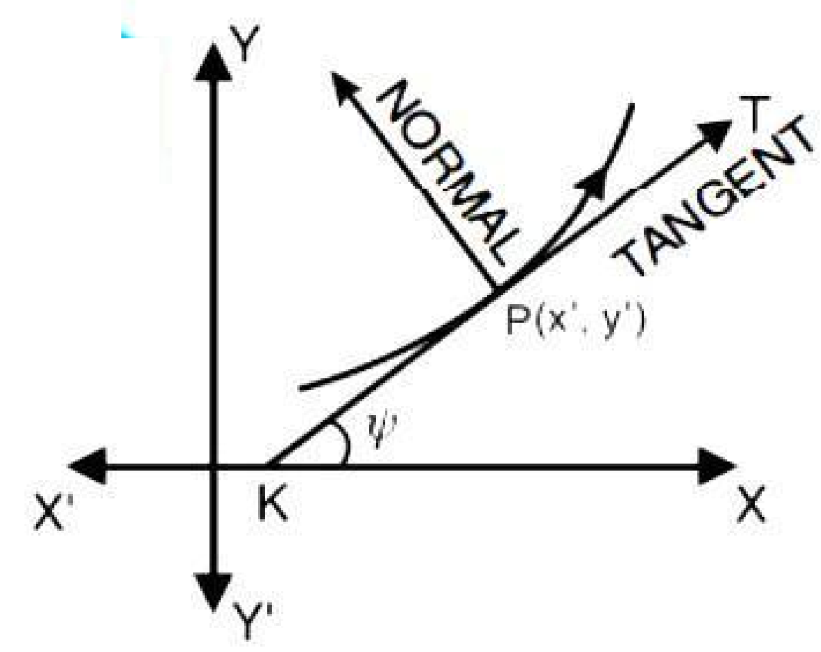


Fig.

(*i*) The **equation of the tangent** at $P(x', y')$ is $y - y' = m(x - x')$

i.e. $$y - y' = \left(\frac{dy}{dx}\right)_P (x - x').$$

(*ii*) Since the slope of the tangent at $P = \left(\frac{dy}{dx}\right)_P$,

$\therefore$ the slope of the normal at $P = -\frac{1}{\left(\frac{dy}{dx}\right)_P}$.

$\therefore$ The **equation of the normal** at $P(x', y')$ is :

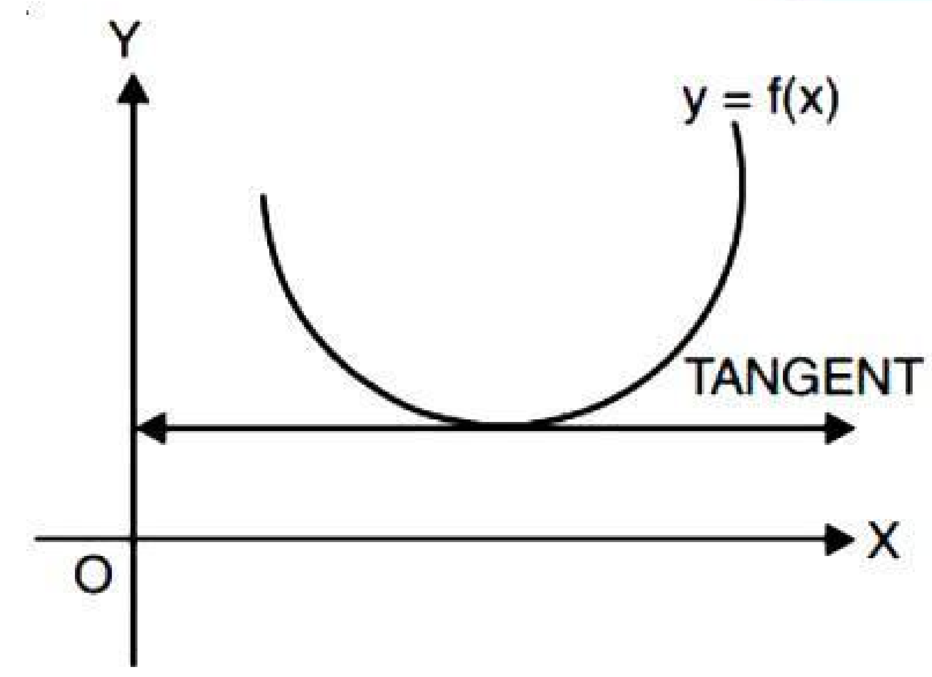
$$y - y' = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(x - x')$$

i.e. $(y - y')\left(\frac{dy}{dx}\right)_P + (x - x') = 0.$

Here P is the **point of contact.**

Particular Cases :

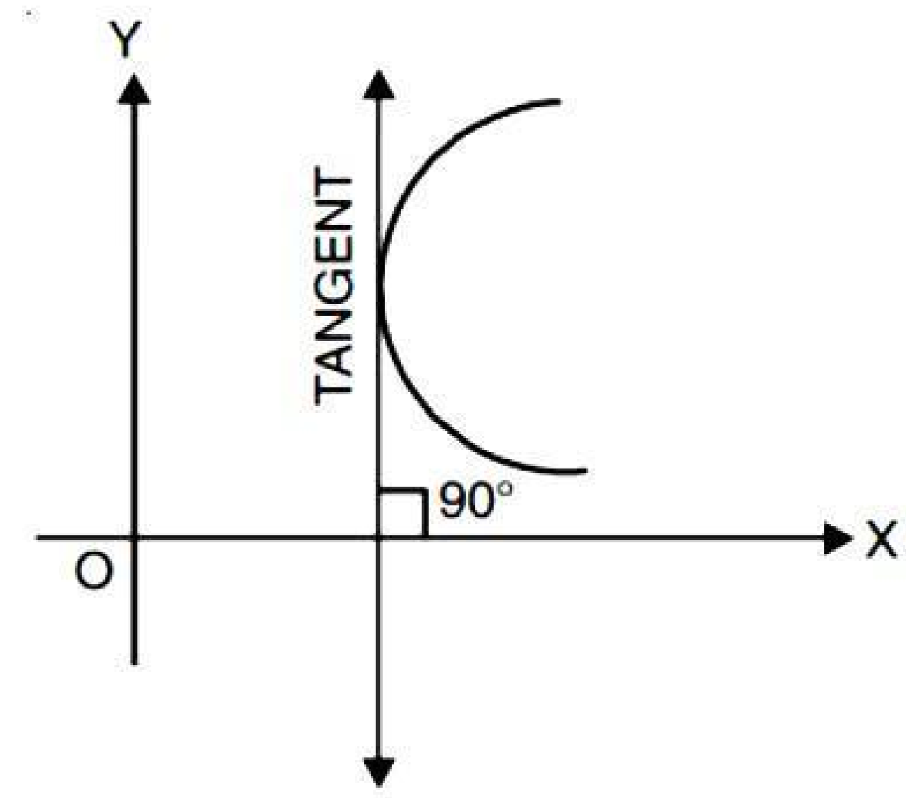
Case I. *When the tangent is parallel to the x–axis.*



Here $\psi = 0°$ $\quad \therefore \tan\psi = 0 \quad \therefore \dfrac{dy}{dx} = 0.$

Thus the equation of the tangent at (x', y') is $y = y'$.

Case II. *When the tangent is perpendicular to the x–axis.*



Here $\psi = \pm\dfrac{\pi}{2} \quad \therefore \cot\psi = 0 \quad \therefore \dfrac{dx}{dy} = 0.$

Thus the equation of the tangent at (x', y') is $x = x'$.

6.4. ANGLE OF INTERSECTION BETWEEN TWO CURVES

Definition

The angle of intersection between two curves is the angle between the tangents to the two curves at their point of intersection.

To find the angle between two curves C_1 and C_2.

Let the two curves C_1 and C_2 meet at P. Let PT_1 and PT_2 be the tangents at P to the two curves C_1 and C_2 respectively.

$\therefore m_1$, slope of $PT_1 = \tan\psi_1$ and m_2, slope of $PT_2 = \tan\psi_2$.

If 'θ' be the angle between the two tangents,

then $\quad \theta = \psi_1 - \psi_2. \quad \therefore \quad \tan\theta = \tan(\psi_1 - \psi_2)$

$$\Rightarrow \tan\theta = \frac{\tan\psi_1 - \tan\psi_2}{1 + \tan\psi_1 \tan\psi_2}.$$

$$\text{Hence, } \tan\theta = \left| \frac{\left(\dfrac{dy}{dx}\right)_{C_1} - \left(\dfrac{dy}{dx}\right)_{C_2}}{1 + \left(\dfrac{dy}{dx}\right)_{C_1}\left(\dfrac{dy}{dx}\right)_{C_2}} \right|.$$

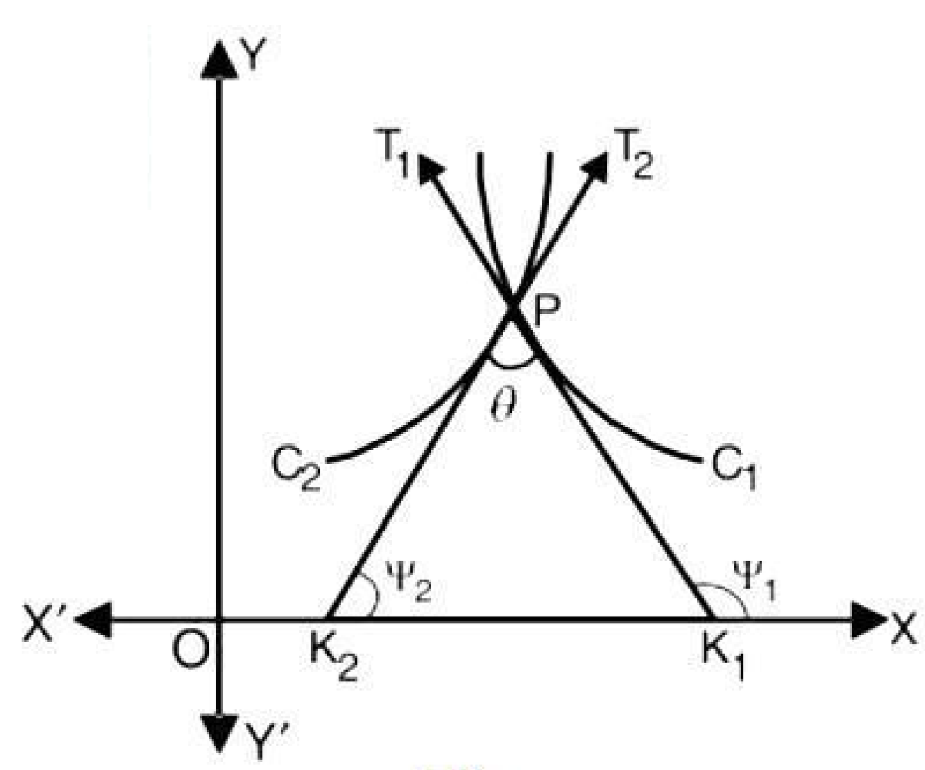


Fig.

Note. The other angle between the tangents = 180° – θ.

KEY POINT

Generally, the smaller of the two angles is the angle of intersection.

6.5. ORTHOGONAL CURVES

If the angle between the two curves is a right angle, the curves are said to be orthogonal curves.

Here $\quad \theta = \dfrac{\pi}{2}.$

$$\therefore \quad m_1 m_2 = -1 \quad \Rightarrow \quad \left(\frac{dy}{dx}\right)_{C_1}\left(\frac{dy}{dx}\right)_{C_2} = -1.$$

Frequently Asked Questions

FAQs

Example 1. Find the slope of the tangent to the curve : $x = at^2$, $y = 2at$ at $t = 2$. *(J. & K.B. 2010)*

Solution. The given curve is $x = at^2$, $y = 2at$.

$$\therefore \quad \frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}.$$

Hence, slope of the tangent at $t = 2 = \left.\frac{dy}{dx}\right]_{t=2} = \frac{1}{2}$.

Example 2. Find the point at which the tangent to the curve $y = \sqrt{4x-3} - 1$ has its slope $\frac{2}{3}$. *(N.C.E.R.T.)*

Solution. The given curve is $y = \sqrt{4x-3} - 1$ ....(1)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(4x-3)^{-1/2}(4) = \frac{2}{\sqrt{4x-3}}.$$

By the question, slope $= \frac{2}{3}$

$$\Rightarrow \quad \frac{2}{\sqrt{4x-3}} = \frac{2}{3} \quad \Rightarrow \quad \sqrt{4x-3} = 3.$$

Squaring, $4x - 3 = 9 \quad \Rightarrow \quad 4x = 12 \quad \Rightarrow \quad x = 3$.

Putting in (1), $y = \sqrt{4(3)-3} - 1 = \sqrt{9} - 1 = 3 - 1 = 2$.

Hence, the reqd. point is (3, 2).

Example 3. Find the equations of all lines having slope 2 and being tangents to the curve $y + \frac{2}{x-3} = 0$. *(N.C.E.R.T.; Assam B. 2018)*

Solution. The given curve is $y + \frac{2}{x-3} = 0$

$$\Rightarrow \quad y = \frac{-2}{x-3} \quad ...(1)$$

$$\therefore \quad \frac{dy}{dx} = \frac{2}{(x-3)^2}.$$

By the question, $\frac{2}{(x-3)^2} = 2 \quad \Rightarrow (x-3)^2 = 1$

$\Rightarrow x - 3 = \pm 1 \quad \Rightarrow \quad x = 2, 4.$

When $x = 2$, then from (1), $y = \frac{-2}{2-3} = \frac{-2}{-1} = 2$.

When $x = 4$, then from (1), $y = \frac{-2}{4-3} = \frac{-2}{1} = -2$.

Thus there are two tangents to the given curve with slope 2 and passing through (2, 2) and (4, – 2).

∴ The equation of the tangent through (2, 2) is :

$$y - 2 = 2(x-2) \quad \Rightarrow \quad y - 2x + 2 = 0$$

and the equation of the tangent through (4, – 2) is :

$$y - (-2) = 2(x-4) \quad \Rightarrow \quad y - 2x + 10 = 0.$$

Example 4. Find the equation of the tangent to the curve $y = \frac{x-7}{(x-2)(x-3)}\left(y = \frac{x-7}{x^2-5x+6}\right)$ at the point, where it cuts the x-axis. *(N.C.E.R.T.; C.B.S.E. 2010 C, 10)*

Solution. The given curve is :

$$y = \frac{x-7}{x^2-5x+6} = \frac{x-7}{(x-2)(x-3)} \quad ...(1)$$

This cuts x-axis ($y = 0$), where :

$$0 = \frac{x-7}{(x-2)(x-3)} \quad \Rightarrow \quad x - 7 = 0 \quad \Rightarrow \quad x = 7.$$

Thus (1) cuts x-axis at (7, 0).

(1) can be written as $(x-2)(x-3)y = x - 7$.

Diff. w.r.t. x, $(x-2)(x-3)\frac{dy}{dx} + (x-2)(1)y + (1)(x-3)y = 1$

$$\Rightarrow \quad (x-2)(x-3)\frac{dy}{dx} = 1 - y(2x-5)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1 - y(2x-5)}{(x-2)(x-3)}.$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{(7,0)} = \frac{1-0}{(7-2)(7-3)} = \frac{1}{20}.$$

∴ The equation of the tangent is $y - 0 = \frac{1}{20}(x-7)$

$$\Rightarrow \quad 20y = x - 7 \quad \Rightarrow x - 20y - 7 = 0.$$

Example 5. Find the equations of the tangent and the normal to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1), where $x_1 = 2$ and $y_1 > 0$. *(C.B.S.E. 2018)*

Solution. The given curve is $16x^2 + 9y^2 = 145$ ...(1)

Since (x_1, y_1) lies on (1),

$$\therefore \quad 16x_1^2 + 9y_1^2 = 145$$

$$\Rightarrow \quad 16(2)^2 + 9y_1^2 = 145$$

$$\Rightarrow \quad 9y_1^2 = 145 - 64$$

$$\Rightarrow \quad 9y_1^2 = 81$$

$$\Rightarrow \quad y_1^2 = 9$$

$$\Rightarrow \quad y_1 = 3 \quad [\because y_1 > 0]$$

Thus, the point is (2, 3).

Diff. (1) w.r.t. x,

$$32x+18y\frac{dy}{dx}=0$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{32x}{18y}$$

$$\Rightarrow \quad \frac{dy}{dx}=-\frac{16x}{9y}.$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{(2,3)}=-\frac{16}{9}\left(\frac{2}{3}\right)=-\left(\frac{32}{27}\right).$$

(*i*) The equation of tangent is :

$$y-3=\frac{-32}{27}(x-2)$$

$$\Rightarrow \quad 27y-81=-32x+64$$

$$\Rightarrow \quad 32x+27y=145.$$

(*ii*) The equation of normal is :

$$y-3=\frac{27}{32}(x-2)$$

$$\Rightarrow \quad 32y-96=27x+54$$

$$\Rightarrow \quad 27x-32y+42=0.$$

Example 6. Find the equations of the tangent and normal to the curve given by :

$x = a\sin^3\theta$, $y = a\cos^3\theta$ at a point, where $\theta = \frac{\pi}{4}$.

(C.B.S.E. 2014)

Solution. The given curve is :

$x=a\sin^3\theta,\ y=a\cos^3\theta$ ...(1)

$$\therefore \quad \frac{dx}{d\theta}=3a\sin^2\theta\cos\theta,\ \frac{dy}{dt}=-3a\cos^2\theta\sin\theta.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{-3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta}=\frac{-\cos\theta}{\sin\theta}.$$

$\therefore$ Slope of the tangent at $\theta=\frac{\pi}{4}$ is :

$$\left.\frac{dy}{dx}\right]_{\theta=\frac{\pi}{4}}=\frac{-\cos\frac{\pi}{4}}{\sin\frac{\pi}{4}}=-\cot\frac{\pi}{4}=-1$$

and slope of the normal = 1.

When $\theta=\frac{\pi}{4}$, then from (1),

$$x=a\left(\frac{1}{\sqrt{2}}\right)^3=\frac{a}{2\sqrt{2}} \text{ and } y=a\left(\frac{1}{\sqrt{2}}\right)^3=\frac{a}{2\sqrt{2}}.$$

$\therefore$ The equations of the tangent and normal to (1) at $\theta=\frac{\pi}{4}$ are :

$$y-\frac{a}{2\sqrt{2}}=(-1)\left(x-\frac{a}{2\sqrt{2}}\right) \Rightarrow x+y=\frac{2a}{2\sqrt{2}} \Rightarrow x+y=\frac{a}{\sqrt{2}}$$

$$\text{and } y-\frac{a}{2\sqrt{2}}=(1)\left(x-\frac{a}{2\sqrt{2}}\right)$$

$$\Rightarrow y-\frac{a}{2\sqrt{2}}=x-\frac{a}{2\sqrt{2}} \Rightarrow x-y=0.$$

Example 7. Find the equations of the tangent and the normal to the curve $x = 1 - \cos\theta$, $y = \theta - \sin\theta$ at $\theta = \frac{\pi}{4}$.

(A.I.C.B.S.E. 2010; J. & K.B. 2010)

Solution. The given curve is $x = 1 - \cos\theta$, $y = \theta - \sin\theta$.

$$\therefore \quad \frac{dx}{d\theta}=\sin\theta,\quad \frac{dy}{d\theta}=1-\cos\theta.$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{1-\cos\theta}{\sin\theta}=\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\tan\frac{\theta}{2}.$$

At $\theta=\frac{\pi}{4}$, $x=1-\cos\frac{\pi}{4}=1-\frac{1}{\sqrt{2}}$,

$$y=\frac{\pi}{4}-\sin\frac{\pi}{4}=\frac{\pi}{4}-\frac{1}{\sqrt{2}} \text{ and } \frac{dy}{dx}=\tan\frac{\pi}{8}.$$

(I) The equation of the tangent is :

$$y-\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right)=\tan\frac{\pi}{8}\left(x-1+\frac{1}{\sqrt{2}}\right).$$

(II) Now slope of the normal $=-\cot\frac{\pi}{8}$.

$\therefore$ The equation of the normal is :

$$y-\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right)=-\cot\frac{\pi}{8}\left(x-1+\frac{1}{\sqrt{2}}\right).$$

Example 8. Find the equation of the tangent to the curve $x^2 + 3y = 3$, which is parallel to the line $y - 4x + 5 = 0$.

(C.B.S.E. 2009 C)

Solution. The given curve is $x^2+3y=3$

$$\Rightarrow \quad y=\frac{1}{3}(3-x^2) \quad ...(1)$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{3}(0-2x)=-\frac{2}{3}x,$$

which is the slope of the tangent.

But the tangent is parallel to the line $y-4x+5=0$, whose slope is 4.

Thus $-\frac{2}{3}x=4$ $[\because m_1=m_2]$

$\Rightarrow x=-6.$

From (1), $y=\frac{1}{3}(3-36)=-11.$

Thus the point of contact is $(-6, -11)$.

$\therefore$ The equation of the tangent is :

$$y+11=4(x+6)$$

$$\Rightarrow \quad y+11=4x+24 \Rightarrow 4x-y+13=0.$$

Example 9. Determine the points on the curve $x^2 + y^2 = 13$, where the tangents are perpendicular to the line $3x - 2y = 0$.

Solution. The given curve is $x^2+y^2=13$ ...(1)

Let (x_1, y_1) be a point on (1).

Then $\quad x_1^2 + y_1^2 = 13 \quad$ ...(2)

Diff. (1) w.r.t. x, $2x + 2y\frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \Rightarrow \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = -\frac{x_1}{y_1}.$$

But the tangents are perp. to $3x - 2y = 0$.

$$\therefore \left(-\frac{x_1}{y_1}\right)\left(-\frac{3}{-2}\right) = -1 \qquad [\because m_1 m_2 = -1]$$

$$\Rightarrow \quad \frac{3x_1}{2y_1} = 1 \Rightarrow 3x_1 = 2y_1 \quad ...(3)$$

From (3), $\quad y_1 = \frac{3x_1}{2} \quad$ ...(4)

Putting in (2), $x_1^2 + \frac{9x_1^2}{4} = 13$

$$\Rightarrow \quad \frac{13x_1^2}{4} = 13 \Rightarrow x_1^2 = 4 \Rightarrow x_1 = \pm 2.$$

When $x_1 = 2$, then from (4), $y_1 = \frac{3}{2}(2) = 3$.

When $x_1 = -2$, then from (4), $y_2 = \frac{3}{2}(-2) = -3$.

Hence, the reqd. points are (2, 3) and (– 2, – 3).

Example 10. Show that the equation of normal at any point '*t*' on the curve :
$x = 3\cos t - \cos^3 t$ and $y = 3\sin t - \sin^3 t$ is :
$4(y\cos^3 t - x\sin^3 t) = 3\sin 4t$. *(C.B.S.E. 2016)*

Solution. The given curve is :
$x = 3\cos t - \cos^3 t$ and $y = 3\sin t - \sin^3 t$.

$$\therefore \frac{dx}{dt} = -3\sin t - 3\cos^2 t(-\sin t)$$
$$= -3\sin t(1 - \cos^2 t) = -3\sin t\sin^2 t = -3\sin^3 t$$

and $\frac{dy}{dt} = 3\cos t - 3\sin^2 t\cos t = 3\cos t(1 - \sin^2 t)$
$$= 3\cos t\cos^2 t = 3\cos^3 t.$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3\cos^3 t}{-3\sin^3 t} = -\frac{\cos^3 t}{\sin^3 t}.$$

$\therefore$ Slope of normal $= \frac{\sin^3 t}{\cos^3 t}$.

$\therefore$ The equation of the normal at 't' is:

$$y - (3\sin t - \sin^3 t) = \frac{\sin^3 t}{\cos^3 t}(x - 3\cos t + \cos^3 t)$$
$$\Rightarrow \quad y\cos^3 t - 3\sin t\cos^3 t + \sin^3 t\cos^3 t$$
$$= x\sin^3 t - 3\cos t\sin^3 t + \sin^3 t\cos^3 t$$
$$\Rightarrow y\cos^3 t - x\sin^3 t = 3\sin t\cos t(\cos^2 t - \sin^2 t)$$
$$\Rightarrow y\cos^3 t - x\sin^3 t = \frac{3\sin 2t\cos 2t}{2}$$
$$\Rightarrow y\cos^3 t - x\sin^3 t = \frac{3}{4}\sin 4t$$

$\Rightarrow 4(y\cos^3 t - x\sin^3 t) = 3\sin 4t$, which is true.

Example 11. Prove that :

$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$ for all $n \in N$ at the point (a, b).

(Meghalaya B. 2017)

Solution. The given curve is $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$.

Diff. w.r.t. x,

$$n\left(\frac{x}{a}\right)^{n-1}.\frac{1}{a} + n\left(\frac{y}{b}\right)^{n-1}.\frac{1}{b}.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b^n x^{n-1}}{a^n y^{n-1}}.$$

At (a, b), $\frac{dy}{dx} = -\frac{b^n . a^{n-1}}{a^n . b^{n-1}} = -\frac{b}{a}$.

$\therefore$ The equation of the tangent at (a, b) is :

$$y - b = -\frac{b}{a}(x - a)$$
$$\Rightarrow \quad ay - ab = -bx + ab$$
$$\Rightarrow \quad bx + ay = 2ab$$

$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = 2$, which is independent of n.

Hence, $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ touches the st. line $\frac{x}{a} + \frac{y}{b} = 2$, for all $n \in$ **N**.

Example 12. At what points will the tangent to the curve :
$y = 2x^3 - 15x^2 + 36x - 21$
be parallel to x-axis ?
Also, find the equations of tangents to the curves at those points.

Solution. (*i*) The given curve is :

$$y = 2x^3 - 15x^2 + 36x - 21 \quad ....(1)$$

Diff. w.r.t x, $\quad \frac{dy}{dx} = 6x^2 - 30x + 36$.

Since the tangents are parallel to x-axis,

$$\therefore \quad \frac{dy}{dx} = 0 \Rightarrow 6x^2 - 30x + 36 = 0$$
$$\Rightarrow \quad x^2 - 5x + 6 = 0 \Rightarrow (x - 2)(x - 3) = 0$$
$$\Rightarrow \quad x = 2, 3.$$

When $x = 2$, then from (1),
$$y = 2(8) - 15(4) + 36(2) - 21$$
$$= 16 - 60 + 72 - 21 = 88 - 81 = 7.$$

When $x = 3$, then from (1),
$$y = 2(27) - 15(9) + 36(3) - 21$$
$$= 54 - 135 + 108 - 21$$
$$= 162 - 156 = 6.$$

Hence, the reqd. points are (2, 7) and (3, 6).

(*ii*) The equations of the tangents are :
$$y - 7 = 0 \text{ and } y - 6 = 0$$
i.e. $\quad y = 7$ and $y = 6$.

Example 13. Show that the curves :

$xy = a^2$ and $x^2 + y^2 = 2a^2$ touch each other.

Solution. The given curves are $xy = a^2$ ...(1)

and $x^2 + y^2 = 2a^2$ ...(2)

From (1), $y = \frac{a^2}{x}$ ...(3)

Putting in (2), $x^2 + \frac{a^4}{x^2} = 2a^2$

$\Rightarrow x^4 - 2a^2x^2 + a^4 = 0$

$\Rightarrow (x^2 - a^2)^2 = 0$

$\Rightarrow x^2 - a^2 = 0$

$\Rightarrow x = \pm a.$

When $x = a, y = a.$

When $x = -a, y = -a.$

Thus (1) and (2) meet at A (a, a) and B $(-a, -a)$.

Now from (3), $\frac{dy}{dx} = \frac{-a^2}{x^2}.$

$\therefore$ Tangent at A (a, a) to (1) is $y - a = \frac{-a^2}{a^2}(x - a)$

$\Rightarrow y - a = -x + a$

$\Rightarrow x + y = 2a$ ...(4)

And from (2), $2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}.$

$\therefore$ Tangent at A (a, a) to (2) is $y - a = \frac{-a}{a}(x - a)$

$\Rightarrow y - a = -x + a$

$\Rightarrow x + y = 2a$ ...(5)

From (4) and (5), given curves touch each other at A (a, a).

Similarly, given curves touch each other at B $(-a, -a)$.

(Do it)

Example 14. Find the angle of intersection of the following curves :

$xy = 6$ and $x^2y = 12$.

Solution. The given curves are $xy = 6$ ...(1)

and $x^2y = 12$ ...(2)

From (1), $y = \frac{6}{x}$ ...(3)

Putting in (2), $x^2\left(\frac{6}{x}\right) = 12 \Rightarrow 6x = 12 \Rightarrow x = 2.$

Putting in (3), $y = \frac{6}{2} = 3.$

Thus the given curves intersect at P (2, 3).

Diff. (1) w.r.t. x, $x\frac{dy}{dx} + y(1) = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}.$

$\therefore m_1 = \left.\frac{dy}{dx}\right]_{(2,3)} = -\frac{3}{2}.$

Diff. (2), w.r.t. x, $x^2\frac{dy}{dx} + y(2x) = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{2y}{x}.$

$\therefore m_2 = \left.\frac{dy}{dx}\right]_{(2,3)} = -\frac{2(3)}{2} = -3.$

If 'θ' be the angle between the given curves,

then $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| = \left|\frac{-\frac{3}{2} + 3}{1 + \left(\frac{-3}{2}\right)(-3)}\right|$

$= \left|\frac{3}{11}\right| = \frac{3}{11}.$

Hence, $\theta = \tan^{-1}\left(\frac{3}{11}\right).$

Example 15. If the curves $ax^2 + by^2 = 1$ and $a'x^2 + b'y^2 = 1$ intersect orthogonally, prove that :

$$\frac{1}{a} - \frac{1}{a'} = \frac{1}{b} - \frac{1}{b'}.$$

Solution. If (x_1, y_1) be the point of intersection of the curves :

$ax^2 + by^2 = 1$ ...(1) and $a'x^2 + b'y^2 = 1$ ...(2),

then $ax_1^2 + by_1^2 = 1$ ...(3)

and $a'x_1^2 + b'y_1^2 = 1$ ...(4)

Diff. (1) w.r.t. x, $2ax + 2by\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{ax}{by}.$

$\therefore m_1 = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = -\frac{ax_1}{by_1}$ ...(5)

Similarly $m_2 = \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = -\frac{a'x_1}{b'y_1}$ ...(6)

The two curves intersect orthogonally

$\Rightarrow m_1m_2 = -1$

$\Rightarrow \left(-\frac{ax_1}{by_1}\right)\left(-\frac{a'x_1}{b'y_1}\right) = -1$ *[Using (5) & (6)]*

$\Rightarrow aa'x_1^2 = -bb'y_1^2$ ...(7)

Subtracting (4) from (3),

$$(a - a')\,x_1^2 + (b - b')\,y_1^2 = 0$$

$$\Rightarrow \qquad (a - a')\,x_1^2 = -(b - b')\,y_1^2 \qquad ...(8)$$

Dividing (8) by (7), $\dfrac{a - a'}{aa'} = \dfrac{b - b'}{bb'}$

$$\Rightarrow \frac{1}{a'} - \frac{1}{a} = \frac{1}{b'} - \frac{1}{b} \qquad \Rightarrow \qquad \frac{1}{a} - \frac{1}{a'} = \frac{1}{b} - \frac{1}{b'},$$

which is true.

Example 16. Find the values of 'x' for which f (x) = [x (x – 2)]^2 is an increasing function. Also, find the points on the curve, where the tangent is parallel to x-axis. *(C.B.S.E. 2010)*

Solution. (*i*) The given curve is :

$$y = f(x) = [x(x-2)]^2$$

$$= (x^2 - 2x)^2 = x^4 - 4x^3 + 4x^2 \qquad ...(1)$$

$$\therefore \qquad f'(x) = 4x^3 - 12x^2 + 8x \qquad ...(2)$$

$$\Rightarrow \qquad f'(x) = 4x(x^2 - 3x + 2)$$

$$\Rightarrow \qquad f'(x) = 4x(x-1)(x-2).$$

When $x < 0$, then :

$$f'(x) = 4\,(-\text{ ve})\,(-\text{ ve})\,(-\text{ ve}) < 0$$

When $0 < x < 1$, then :

$$f'(x) = 4\,(+\text{ ve})\,(-\text{ ve})\,(-\text{ ve}) > 0$$

When $1 < x < 2$, then :

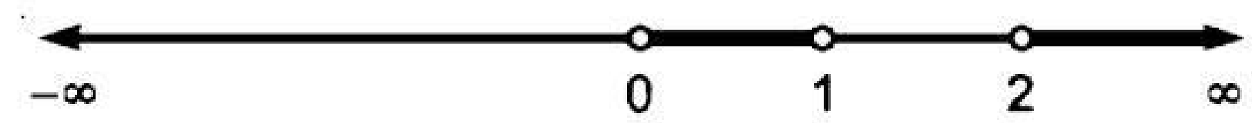
$$f'(x) = 4\,(+\text{ ve})\,(+\text{ ve})\,(-\text{ ve}) < 0$$

When $x > 2$, then :

$$f'(x) = 4\,(+\text{ ve})\,(+\text{ ve})\,(+\text{ ve}) > 0.$$

Hence, $f(x)$ is increasing when $0 < x < 1$ or $x > 2$

i.e. when $x \in (0, 1) \cup (2, \infty)$.



(*ii*) The tangent is parallel to x-axis $\Rightarrow f'(x) = 0$

$\Rightarrow 4x(x-1)(x-2) = 0 \Rightarrow x = 0, 1, 2.$

When $x = 0$, then from (1), $y = 0$.

When $x = 1$, then from (1), $y = 1 - 4 + 4 = 1$.

When $x = 2$, then from (1), $y = 16 - 32 + 16 = 0$.

Hence, the required points are :

(0, 0), (1, 1) and (2, 0).

EXERCISE 6 (c)

Fast Track Answer Type Questions

FTATQ

1. (*a*) Find the slopes of the tangents to the following curves :

(*i*) $y = (x-2)^2$ at $x = 1$ *(Kerala B. 2018)*

(*ii*) $y = x^3 - x$ at $x = 2$ *(N.C.E.R.T.)*

(*iii*) $y = 3x^4 - 4x$ at $x = 4$ *(N.C.E.R.T.)*

(*iv*) $y = 3x^4 - 4x$ at $x = 1$ *(Kashmir B. 2012)*

(*v*) $y = \dfrac{x-1}{x-2}, x \neq 2$ at $x = 10$. *(N.C.E.R.T.)*

(*b*) Find the slope of the tangent to the curve :

(*i*) $y = x^3 - 3x + 2$ at the point whose x-co-ordinate is 3 *(N.C.E.R.T.)*

(*ii*) $y = x^3 - x + 1$ at the point whose x – co-ordinate is 2 *(N.C.E.R.T.)*

(*iii*) $y = 3x^2 - 6x$ at the point on it. whose x-co-ordinate is 2. *(A.I.C.B.S.E. 2009 C)*

2. Find the slope of the tangent to the curve :

$y = x^3 - 2x + 8$ at the point (1, 7). *(Bihar B. 2014)*

3. Find the slope of the normal to the curve :

(*a*) (*i*) $y = 2x^2 - 1$ at (1, 1) *(Tripura B. 2016)*

(*ii*) $y = x^3 - x + 1$ at $x = 2$ *(H.B. 2011)*

(*b*) (*i*) $y = \tan^2 x + \sec x$ at $x = \dfrac{\pi}{4}$ *(H.B. 2011)*

(*ii*) $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ at $\theta = \dfrac{\pi}{2}$ *(H.B. 2011)*

(*iii*) $x = 1 - a\sin\theta, y = b\cos^2\theta$ at $\theta = \dfrac{\pi}{2}$ *(N.C.E.R.T. ; Jammu B. 2018, 14)*

(*iv*) $x = a\cos^3\theta, y = a\sin^3\theta$ at $\theta = \dfrac{\pi}{4}$. *(N.C.E.R.T.; Kashmir B. 2011)*

4. Find the equation of the tangent line to the curve :

(*i*) $y = 2x^2 + 3y^2 = 5$ at the point (1, 1) *(Kerala B. 2015)*

(*ii*) $y = x^3 - 3x + 5$ at the point (2, 7). *(H.B. 2013)*

5. Find the equation of tangent line to the curve :

(*i*) $y = \sin x$ at $x = \dfrac{\pi}{4}$

(*ii*) $y = \cot^2 x - 2\cot x + 2$ at $x = \dfrac{\pi}{4}$

(*iii*) $y = \sec^4 x - \tan^4 x$ at $x = \dfrac{\pi}{3}$.

Very Short Answer Type Questions

VSATQ

Find the equations of the tangent and normal lines to the following curves (6 – 9) :

6. (*i*) $y = x^2$ at (0, 0) *(N.C.E.R.T.)*

(*ii*) $y = x^3$ at (1, 1). *(N.C.E.R.T.)*

7. (*i*) $y = 2x^2 - 3x - 1$ at (1, – 2)

(*ii*) $x^{2/3} + y^{2/3} = 2$ at (1, 1).

(N.C.E.R.T. ; H.P.B. 2017; Kashmir B. 2012)

8. (*i*) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (1, 3)

(N.C.E.R.T. ; Jammu B. 2015, 13)

(*ii*) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at (0, 5).

(N.C.E.R.T. ; Jammu B. 2013)

9. $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{4}$.

(N.C.E.R.T. ; Jammu B. 2016, 13; Uttarakhand B. 2015 ; Jammu B. 2013)

10. Find the equation of the tangent to the curve :

$x = a \cos^3 \theta$, $y = a \sin^3 \theta$ at $\theta = \frac{\pi}{4}$. *(H.B. 2014)*

11. Find the equation of the normal line to the curve :

(*i*) $y = 2x^2 + 3 \sin x$ at $x = 0$

(*ii*) $y (x - 2) (x - 3) - x + 7 = 0$ at the point, where it meets x-axis.

Short Answer Type Questions

SATQ

Find the equations of the tangent and normal lines to the following curves (12 – 13) :

12. $y = \sin^2 x$ at $x = \frac{\pi}{2}$.

13. $y = \frac{1 + \sin x}{\cos x}$ at $x = \frac{\pi}{4}$.

14. Find the equations of the tangent and normal to the parabola :

$y^2 = 4ax$ at $(at^2 , 2at)$.

(N.C.E.R.T.; H.P.B. 2017, 16, 15; H.B. 2012)

15. (*i*) Find the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1).

(*ii*) Find the equations of the tangent and the normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(\sqrt{2}a, b)$.

(A.I.C.B.S.E. 2014)

16. (*i*) Find the equation of the normal to the curve $ay^2 = x^3$ at (am^2, am^3). *(N.C.E.R.T.; Kashmir B. 2013)*

(*ii*) Find the equation of tangent to the curve $2x^2 - y = -7$, which is parallel to the line $4x - y + 3 = 0$.

(Meghalaya B. 2014)

17. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$, which is parallel to the line $4x - 2y + 5 = 0$.

(N.C.E.R.T. ; H.P.B. 2016; P.B. 2010; C.B.S.E. 2009)

18. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$, which is :

(*i*) parallel to the line $2x - y + 9 = 0$

(Jammu B. 2017; H.P.B. 2014; P.B. 2012, 10)

(*ii*) perpendicular to the line $5y - 15x = 13$.

(N.C.E.R.T. Jammu B. 2017)

19. Find the equations of the tangents to the curve :

$y = x^3 + 2x - 4$,

which are perpendicular to line $x + 14y + 3 = 0$.

(A. I.C.B.S.E.B. 2016)

20. (*a*) (*i*) Find the equations of the normal lines to the curve : $y = x^3 + 2x + 6$, which are parallel to the line $x + 14y + 4 = 0$.

(N.C.E.R.T.; H.P.B. 2016, 11; Jammu B. 2015; C.B.S.E. 2010)

(*ii*) Find the equations of the normals to the curve : $3x^2 - y^2 = 8$ parallel to the line $x + 3y = 4$.

(J. & K.B. 2011)

(*iii*) Find the equations of the normals to the curve : $2x^2 - y^2 = 14$, which are parallel to the line $x + 3y = 6$.

(Rajasthan B. 2013)

(*b*) If the normal at any point on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the x-axis, then prove that the equation of the normal is :

$y \cos \phi - x \sin \phi = a \cos 2\phi$.

(W. Bengal B. 2016)

21. (*i*) Find the equation of the normal to the curve : $x^2 = 4y$, which passes through the point (1, 2). *(N.C.E.R.T.)*

(*ii*) Find the equation of the normal to the curve : $y^2 = 4x$, which passes through the point (1, 2). *(N.C.E.R.T.)*

22. (*a*) Find the equation of the tangent to the curve : $y = x^2 - 2x + 9$, which is parallel to the line : $2x - y + 7 = 0$ *(P.B. 2011)*

(*b*) Find the equation of the tangent to the curve : $y = x^3 - 2x^2 - 2x$ at which the tangent is parallel to the line $y = 2x - 3$. *(P. B. 2013)*

(*c*) Find the equation of the tangent to the curve $y = \sqrt{5x - 3}$, which is parallel to the line $4x - 2y + 3 = 0$.

(Meghalaya B. 2015)

23. Find the equations of tangent lines to the curve :

$y = 4x^3 - 3x + 5$,

which are perpendicular to the line $9y + x + 3 = 0$.

24. (*a*) (*i*) Find the equations of the normals to : the curve $y = x^3 + 2x + 6$, which are parallel to the line $x + 14y + 4 = 0$. *(N.C.E.R.T.; Kashmir B. 2017)* (*ii*) Find the equation of the normal to the curve : $y = x^3 + 5x^2 - 10x + 10$, where the normal is parallel to the line $x - 2y + 10 = 0$. **(P.B. 2016)**

(*b*) Find the equations of the normal to the curve $y = 4x^3 - 3x + 5$, which are perpendicular to the line : $9x - y + 5 = 0$. **(C.B.S.E. Sampe Paper 2019)**

25. (*i*) Find the equation of the tangent to the curve : $x = a\sin^3 t,\ y = b\cos^3 t$ at $t = \frac{\pi}{2}$. **(N.C.E.R.T.; H.B. 2018)**

(*ii*) Find the equation of the tangent at $t = \frac{\pi}{4}$ to the curve : $x = \sin 3t,\ y = \cos 2t$. **(Bihar B. 2014)**

26. Find the point(s) on the curve :
(*i*) $y = 3x^2 - 12x + 6$ **(H.B. 2011)**
(*ii*) $x^2 + y^2 - 2x - 3 = 0$ **(N.C.E.R.T.; H.B. 2011)**
at which the tangent is parallel to *x*-axis.

27. Find the point(s) on the curve :
(*i*) $y = \frac{1}{4}x^2$, where the slope of the tangent is $\frac{16}{3}$
(*ii*) $y = x^2 + 1$, at which the slope of the tangent is equal to :
(I) *x*-coordinate (II) *y*-coordinate.

28. Find the points on the curve $y = x^3 - 11x + 5$ at which tangent is parallel to the line $y = x - 11$. **(N.C.E.R.T.; H.P.B. 2018, 13 S, 13 ; C.B.S.E. 2012)**

29. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin. **(N.C.E.R.T.)**

30. Find the points on the following curve at which the tangents are parallel to *x*-axis : $y = x^3 - 3x^2 - 9x + 7$. **(N.C.E.R.T. ; H.P.B. 2018, 13 S, 13)**

31. At what point on the curve $x^2 + y^2 - 2x - 4y + 1 = 0$, is the tangent parallel to *x*-axis ?

32. Find the points on the curve :
$\frac{x^2}{4} + \frac{y^2}{25} = 1$ at which the tangents are :
(*i*) parallel to *x*-axis (*ii*) parallel to *y*-axis. **(N.C.E.R.T.)**

33. Show that the tangents to the curve $y = 7x^3 + 11$ at the points $x = 2$ and $x = -2$ are parallel. **(N.C.E.R.T.)**

34. Find the equations of all lines :
(*i*) having slope 0 and that are tangents to the curve : $y = \frac{1}{x^2 - 2x + 3}$ **(N.C.E.R.T.)**
(*ii*) having slope –1 and that are tangents to the curve : $y = \frac{1}{x-1},\ x \neq 1$ **(N.C.E.R.T.)**
(*iii*) having slope 2 and that are tangents to the curve : $y = \frac{1}{x-3},\ x \neq 3$. **(N.C.E.R.T.)**

35. Find the point of intersection of the tangent lines to the curve $y = 2x^2$ at the points (1, 2) and (– 1, 2).

36. Prove that the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) are at right-angles.

37. Find the angle of intersection of the curves :
(*i*) $y^2 = 4x$ and $x^2 = 4y$
(*ii*) $x^2 + y^2 - 4x - 1 = 0$ and $x^2 + y^2 - 2y - 9 = 0$.

38. Show that the following curves cut each other orthogonally :
(*i*) $y = x^3$ and $6y = 7 - x^2$ (*ii*) $x^2 + 4y^2 = 8$ and $x^2 - 2y^2 = 4$.

39. If the curves : $\alpha x^2 + \beta y^2 = 1$ and $\alpha' x^2 + \beta' y^2 = 1$ intersect orthogonally, prove that $(\alpha - \alpha')\beta\beta' = (\beta - \beta')\alpha\alpha'$.

40. (*i*) Prove that the curves $4x = y^2$ and $4xy = k$ cut at right angles if $k^2 = 512$.
(*ii*) Show that the curves $2x = y^2$ and $2xy = k$ cut at right angles if $k^2 = 8$.
(*iii*) Prove that the curves $y^2 = 4ax$ and $xy = c^2$ cut at right angles if $c^4 = 32a^4$.

41. Show that the normal to the curve :
$x = a\cos\theta + a\theta\sin\theta$; $y = a\sin\theta - a\theta\cos\theta$
at any point 'θ' is at a constant distance from the origin. **(N.C.E.R.T.)**

42. Find the equation(*s*) of the tangent(*s*) to the curve :
$y = (x^3 - 1)(x - 2)$
at the points, where the curve intersects the *x*-axis. **(C.B.S.E. Sample Paper 2018)**

43. (*i*) Find a point on the graph of $y = x^3$, where the tangent is parallel to the chord joining (1, 1) and (3, 27).
(*ii*) Find a point on the parabola $y = (x - 2)^2$, where the tangent is parallel to the line joining (2, 0) and (4, 4). **(H.P.B. 2018; Kerala B. 2018)**

44. Show that the area of the triangle formed by the tangent and the normal at the point (*a*, *a*) on the curve $y^2(2a - x) = x^3$ and the line $x = 2a$ is $\frac{5a^2}{4}$ sq. units.

Long Answer Type Questions

LATQ

45. Find the equation of the normal at a point on the curve $x^2 = 4y$, which passes through the point (1, 2). Also find the equation of the corresponding tangent. **(C.B.S.E. 2013)**

46. Find the equations of the tangents to the curve $3x^2 - y^2 = 8$, which pass through the point $\left(\frac{4}{3}, 0\right)$. **(A.I.C.B.S.E. 2013)**

47. Determine the values of '*x*' for which the function $f(x) = x^2 + 2x - 3$ is an increasing.
Also, find the co-ordinates of the point on the curve $y = x^2 + 2x - 3$, where the normal is parallel to the line $x - 4y + 7 = 0$.

48. Determine the intervals in which the function $f(x) = (x - 1)(x + 1)^2$ is increasing or decreasing. Find also the points at which the tangents to the curve are parallel to *x*-axis. **(J & K.B. 2010)**

Answers

1. (*a*) (*i*) – 2 (*ii*) 11 (*iii*) 764 (*iv*) 8 (*v*) $-\frac{1}{64}$

(*b*) (*i*) 24 (*ii*) 11 (*iii*) 6.

2. 1.

3. (*a*) (*i*) $-\frac{1}{4}$ (*ii*) $-\frac{1}{11}$

(*b*) (*i*) $-\frac{1}{4+\sqrt{2}}$ (*ii*) –1 (*iii*) $-\frac{a}{2b}$ (*iv*) 1.

4. (*i*) $2x + 3y - 5 = 0$

(*ii*) $9x - y - 11 = 0$.

5. (*i*) $4\sqrt{2}\,y - 4x + \pi - 4 = 0$

(*ii*) $y = 1$

(*iii*) $48x - \sqrt{3}\,y + 7\sqrt{3} - 16\pi = 0$.

6. (*i*) $y = 0, x = 0$

(*ii*) $3x - y - 2 = 0, x + 3y - 4 = 0$.

7. (*i*) $x - y - 3 = 0, x + y + 1 = 0$

(*ii*) $x + y - 2 = 0, x - y = 0$.

8. (*i*) $2x - y + 1 = 0, x + 2y - 7 = 0$

(*ii*) $10x + y - 5 = 0, x - 10y + 50 = 0$.

9. $x + y - \sqrt{2} = 0, x - y = 0$.

10. $x + y = \frac{a}{\sqrt{2}}$.

11. (*i*) $x + 3y = 0$ (*ii*) $20x + y - 140 = 0$.

12. $y - 1 = 0, 2x - \pi = 0$.

13. $y - (2+\sqrt{2})x + (2+\sqrt{2})\frac{\pi}{4} - \sqrt{2} - 1 = 0$,

$(2+\sqrt{2})y + x - (4+3\sqrt{2}) - \frac{\pi}{4} = 0$.

14. $x = ty - at^2,\ y = -tx + 2at + at^3$.

15. (*i*) $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

(*ii*) $\frac{\sqrt{2}}{a}x - \frac{y}{b} = 1,\ ax + \sqrt{2}\,by - \sqrt{2}(a^2 + b^2) = 0$.

16. (*i*) $2x + 3my = 3am^4 + 2am^2$ (*ii*) $4x - y + 5 = 0$.

17. $48x - 24y = 23$.

18. (*i*) $y - 2x - 3 = 0$ (*ii*) $36y + 12x - 227 = 0$.

19. $14x - y - 20 = 0, 14x - y + 12 = 0$.

20. (*a*) (*i*) $x + 14y - 254 = 0; x + 14y + 86 = 0$

(*ii*) $x + 3y - 8 = 0$; $x + 3y + 8 = 0$

(*iii*) $x + 3y - 9 = 0$; $x + 3y + 9 = 0$.

21. (*i*) – (*ii*) $x + y - 3 = 0$.

22. (*a*) $2x - y + 5 = 0$

(*b*) $2x - y - 8 = 0$ and $2x - y + \frac{40}{27} = 0$

(*c*) $80x - 40y = 23$.

23. $9x - y - 3 = 0, 9x - y + 13 = 0$.

24. (*a*) (*i*) $x + 14y - 254 = 0, x + 14y + 86 = 0$

(*ii*) $y - \frac{185}{27} = \frac{1}{2}\left(x - \frac{2}{3}\right)$; $y - 67 = \frac{1}{2}(x+4)$

(*b*) $x + 9y = 55, x + 9y = 35$.

25. (*i*) $y = 0$ (*ii*) $3y = 2\sqrt{2}\,x - 2$.

26. (*i*) (2, – 6) (*ii*) (1, ± 2).

27. (*i*) $\left(\frac{32}{3}, \frac{256}{9}\right)$

(*ii*) (I) (0, 1) (II) (1, 2)

28. (2, – 9), (– 2, 19).

29. (0, 0), (1, 2), (– 1, – 2).

30. (3, – 20), (– 1, 12).

31. (1, 0), (1, 4).

32. (*i*) (0, ± 5)

(*ii*) (± 2, 0).

34. (*i*) $y = \frac{1}{2}$

(*ii*) $x + y + 1 = 0, x + y - 3 = 0$

(*iii*) No tangent.

35. (0, – 2).

37. (*i*) 90°, $\tan^{-1}\left(\frac{3}{4}\right)$ (*ii*) $\frac{\pi}{4}$.

42. $3x + y - 3 = 0, 7x - y - 14 = 0$.

43. (*i*) $\left(\sqrt{\frac{13}{3}}, \left(\frac{13}{3}\right)^{3/2}\right)$ (*ii*) (3, 1).

45. $x + y - 3 = 0; x - 2y - 4 = 0$.

46. $3x \pm y = 4$.

47. Increasing on $(-1, \infty)$; $\left(2, \frac{1}{4}\right)$ and $\left(-2, -\frac{1}{4}\right)$

Decreasing on $(-\infty, -1)$; (3, 0).

48. Increasing on $(-\infty, -1) \cup \left(\frac{1}{3}, \infty\right)$

Decreasing on $\left(-1, \frac{1}{3}\right)$;

(– 1, 0) and $\left(\frac{1}{3}, -\frac{32}{27}\right)$.

Hints to Selected Questions

5. (*i*) When $x = \frac{\pi}{4}$, $y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

$\therefore$ Point of contact is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

And $\frac{dy}{dx} = \cos x\Big]_{x=\pi/4} = \frac{1}{\sqrt{2}}$.

9. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$.

$\therefore$ $\frac{dy}{dx}\Big]_{t=\frac{\pi}{4}} = -\cot\frac{\pi}{4} = -1$.

14. Here $y^2 = 4ax$.

$\therefore$ $2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$.

At $(at^2, 2at)$, $\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$.

18. **(*ii*)** Slope of tangent $= 2(x-1)$.

Slope of line $= 3$.

$\therefore$ $2(x-1)\,(3) = -1 \Rightarrow x = \frac{5}{6}$; etc.

24. (*i*) Here $\frac{dy}{dx} = 3x^2 + 2$.

Since the normal is parallel to $x + 14y + 4 = 0$,

$\therefore$ the tangent is perpendicular to $x + 14y + 4 = 0$.

$\therefore$ $(3x^2+2)\left(-\frac{1}{14}\right) = -1 \Rightarrow x = \pm\ 2$.

25. (*i*) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3b\sin t\cos^2 t}{3a\sin^2 t\cos t} = -\frac{b}{a}\cot t$.

At $t = \frac{\pi}{2}, \frac{dy}{dx} = -\frac{b}{a}\cot\frac{\pi}{2} = 0$.

26. (*i*) $\frac{dy}{dx} = 0 \Rightarrow 6x - 12 = 0 \Rightarrow x = 2$.

28. $\frac{dy}{dx} = 1 \Rightarrow 3x^2 - 11 = 1 \Rightarrow x = \pm\ 2$.

30. $\frac{dy}{dx} = 0$.

36. $\frac{dy}{dx} = 2x - 5$.

$\therefore$ $m_1 = 2\,(2) - 5 = -1$ and $m_2 = 2\,(3) - 5 = 1$

so that $m_1 m_2 = -1$.

40. (*i*) $x = \frac{y^2}{4} \Rightarrow 1 = \frac{2y}{4}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2}{y}$.

For $4xy = k$, $4\left(x\frac{dy}{dx} + y(1)\right) = 0$

$\Rightarrow$ $x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$.

$\therefore$ $\left(\frac{2}{y}\right)\left(\frac{-y}{x}\right) = -1 \Rightarrow x = 2$.

Also $4x = \frac{k^2}{16x^2}$ $\quad$ [$\because$ *each* $= y^2$]

$\Rightarrow$ $k^2 = 64x^3 \Rightarrow k^2 = 64(8) = 512$.

44. The given curve is $y^2(2a - x) = x^3$.

Diff. w.r.t. x, $2y\frac{dy}{dx}(2a - x) + y^2(-1) = 3x^2$

$\Rightarrow$ $\frac{dy}{dx} = \frac{3x^2 + y^2}{2y(2a - x)}$.

At (a, a), slope of tangent $= \frac{3a^2 + a^2}{2a(2a - a)} = \frac{4a^2}{2a^2} = 2$

and slope of normal $= -\frac{1}{2}$.

$\therefore$ The equation of tangent at (a, a) is :

$y - a = 2(x - a) \Rightarrow 2x - y = a$

and the equation of normal at (a, a) is :

$y - a = -\frac{1}{2}(x - a) \Rightarrow 2y - 2a = -x + a$

$\Rightarrow$ $x + 2y = 3a$.

Now we are to find the area of the triangle formed by the lines :

$2x - y = a$ ...(1)

$x + 2y = 3a$ ...(2)

and $x = 2a$ ...(3)

SUB CHAPTER
6.4

Differentials, Errors and Approximations

6.6. DIFFERENTIALS AND APPROXIMATIONS

Here we shall give the meaning of dx and dy in such a way that the meaning of the symbol $\frac{dy}{dx}$ coincides with the quotient when dy is divided by dx.

Let $y = f(x)$ be the given function.

Then Δx denotes the increment in x, which is also denoted by dx.

We also take $dy = \left(\frac{dy}{dx}\right).dx = f'(x)dx$.

Also $\quad \Delta y = f(x+\Delta x) - f(x)$.

We regard dy as an approximate value of Δy.

Geometrically, we have the meaning of Δx, Δy, dx and dy.

Practically, it is easy to calculate dy but not Δy.

dx is called the **differential** of x and dy, the **differential** of y.

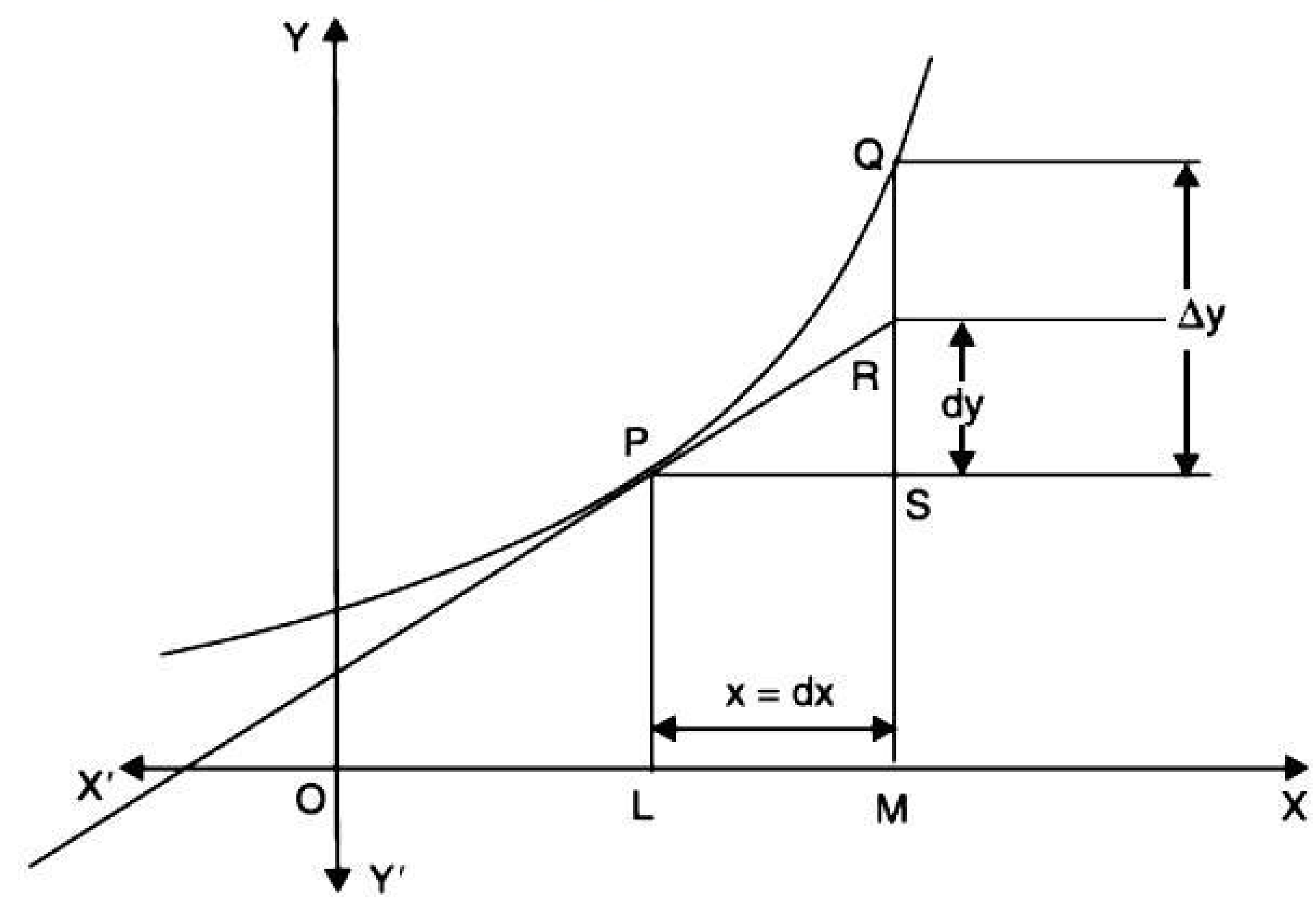


Fig.

GUIDE-LINES

Step (i) Let $y = f(x)$.

Step (ii) Let Δx be a small change in x and Δy, the corresponding change in y.

Step (iii) $\quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} = f'(x)$ ...(1)

$\Rightarrow \quad \frac{\Delta y}{\Delta x} = f'(x) + \varepsilon$, where $\varepsilon \to 0$ when $\Delta x \to 0$

$\Rightarrow \quad \Delta y = f'(x)\,\Delta x + \varepsilon\,\Delta x \Rightarrow \Delta y = f'(x)\,\Delta x$ approximately.

Hence, $\quad \Delta y = \frac{dy}{dx}\Delta x$, approximately. [*Using (1)*]

6.7. IMPORTANT DEFINITIONS

(i) Absolute Error. *The error Δx in x is called the absolute error in x.*

(ii) Relative Error. $\frac{\Delta x}{x}$ *is called the relative error in x.*

(iii) Percentage Error. $\frac{\Delta x}{x} \times 100$ *is called the percentage error in x.*

ILLUSTRATIVE EXAMPLES

Example 1. Using differentials, find the approximate value of $\sqrt{26}$.

Solution. Let $y = f(x) = \sqrt{x}$.

Take $\quad x = 25,\ x + \Delta x = 26$ so that $\Delta x = 26 - 25 = 1$.

When $x = 25$, then $y = \sqrt{25} = 5$.

Let $\quad dx = \Delta x = 1$.

Now $\ y = \sqrt{x}\ $ so that $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$.

$\therefore \quad \left.\frac{dy}{dx}\right]_{x=25} = \frac{1}{2(5)} = \frac{1}{10}.$

Now $\quad dy = \frac{dy}{dx}dx \Rightarrow dy = \frac{1}{10}(1) = 0{\cdot}1$

$\Rightarrow \quad \Delta y = 0{\cdot}1 \quad$ [$\because$ *Δy is approximately = dy*]

Hence, $\sqrt{26} = y + \Delta y = 5 + 0{\cdot}1 = 5{\cdot}1$ app.

Example 2. Using differentials, find the approximate value of $\sqrt[3]{0\cdot 026}$, upto three places of decimals. *(P.B. 2010)*

Solution. Let $y = f(x) = x^{1/3}$.

Take $x = 0{\cdot}027$, $x + \Delta x = 0{\cdot}026$

so that $\Delta x = 0{\cdot}026 - 0{\cdot}027 = -0{\cdot}001$.

When $x = 0{\cdot}027$, then $y = \sqrt[3]{0\cdot 027} = 0{\cdot}3$.

Let $\quad dx = \Delta x = -0{\cdot}001$.

Now $y = x^{1/3}$ so that $\frac{dy}{dx} = \frac{1}{3}x^{-2/3}$.

$\therefore \quad \left.\frac{dy}{dx}\right]_{x=0\cdot027} = \frac{1}{3}(0\cdot 027)^{-2/3} = \frac{1}{3}(0\cdot 3)^{-2}$

$= \frac{1}{3}\frac{1}{\cdot 09} = \frac{1}{0\cdot 27}.$

Now $\quad dy = \frac{dy}{dx}dx \Rightarrow dy = \frac{1}{0{\cdot}27}(-0{\cdot}001) = -\frac{1}{270}$

$\Rightarrow \quad \Delta y = -\frac{1}{270}. \quad$ [$\because$ *Δy is approximately = dy*]

Hence, $\sqrt[3]{0\cdot 026} = y + \Delta y = 0{\cdot}3 - \frac{1}{270} = 0{\cdot}296.$

Example 3. Find the approximate change in the volume 'V' of a cube of side 'x' metres caused by increasing the side by 1%. *(N.C.E.R.T.)*

Solution. V, the volume of the cube = x^3.

Now $\quad dV = \left(\frac{dV}{dx}\right)\Delta x$

$= (3x^2)\ \Delta x$

$= (3x^2)(0{\cdot}01x)$ [$\because$ *1 % of $x = 0{\cdot}01\,x$*]

$= 0{\cdot}03\ x^3\ \text{m}^3.$

Hence, the approximate change in volume = $0{\cdot}03\ x^3\ \text{m}^3$.

Example 4. If the radius of a sphere is measured as 9 m with an error of 0·03 m, then find the approximate error in calculating its surface area.

(N.C.E.R.T.; A.I.C.B.S.E. 2011)

Solution. Let 'r' be the radius of the sphere and Δr be the error in measuring the radius.

Then $\quad r = 9\ m$ and $\Delta r = 0{\cdot}03$ m.

Now S, the surface area of the sphere is given by :

$S = 4\pi r^2.$

$\therefore \quad \frac{dS}{dr} = 8\pi r.$

Now $dS = \left(\frac{dS}{dr}\right)\Delta r = (8\pi r)\ \Delta r$

$= (8\pi(9))(0{\cdot}03)$

$= 2{\cdot}16\ \pi\ \text{m}^2.$

Hence, the approximate error in calculating the surface area is $2{\cdot}16\ \pi\ \text{m}^2$.

Example 5. If $y = x^4 + 10$ and x changes from 2 to 1·99, find the approximate change in y.

Solution. We have : $y = x^4 + 10$.

$\therefore \quad \frac{dy}{dx} = 4x^3.$

Since x changes from 1 to 1·99,

$\therefore \quad x = 2$ and $x + \Delta x = 1{\cdot}99$

so that $\Delta x = 1{\cdot}99 - 2 = -0{\cdot}01$.

Let $\quad dx = \Delta x = -0{\cdot}01$.

Now Δy is approximately equal to dy and

$dy = \left(\frac{dy}{dx}\right)dx = 4x^3(-0{\cdot}01)$

$= 4(2)^3(-0{\cdot}01)$ at $x = 2$

$= -0{\cdot}32.$

Hence, the approximate decrease in the value of $y = 0{\cdot}32$.

Example 6. Use differentials to calculate approximate value of $\log_e(9\cdot01)$. **(Given $\log_e 3 = 1{\cdot}0986$)**

Solution. Let $y = f(x) = \log_e x$.

Take $x = 9$ and $x + \Delta x = 9{\cdot}01$ so that $\Delta x = 0{\cdot}01$.

When $x = 9$, $f(9) = \log_e 9 = \log_e 3^2 = 2\log_e 3$

$= 2(1{\cdot}0986) = 2{\cdot}1972.$

Let $\quad dx = \Delta x = 0{\cdot}01$.

Now $y = \log_e x$ so that $\frac{dy}{dx} = \frac{1}{x}$.

$\therefore \quad \left.\frac{dy}{dx}\right]_{x=9} = \frac{1}{9}.$

Now $\quad dy = \frac{dy}{dx}dx \Rightarrow dy = \frac{1}{9}\times 0{\cdot}01 = 0{\cdot}0011$

$\Rightarrow \quad \Delta y = 0{\cdot}0011. \quad$ [$\because$ *Δy is approximately = dy*]

Hence, $\log_e(9{\cdot}01) = y + \Delta y$

$= 2{\cdot}1972 + 0{\cdot}0011 = 2{\cdot}1983.$

Example 7. Use differentials to find the approximate value of tan 46°, if it is being given that 1° = 0·01745 radian.

Solution. Let $y = f(x) = \tan x$.

Take $x = 45° = \left(\frac{\pi}{4}\right)^c$ and $x + \Delta x = 46°$ so that :

$\Delta x = 1° = 0{\cdot}01745$ radian.

When $x = \frac{\pi}{4}$, $y = f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} = 1$.

Let $\quad dx = \Delta x = 0{\cdot}01745$.

Now $y = \tan x$ so that $\frac{dy}{dx} = \sec^2 x$.

$$\therefore \quad \left.\frac{dy}{dx}\right]_{x=\frac{\pi}{4}} = \sec^2\frac{\pi}{4} = 2.$$

Now $\quad dy = \frac{dy}{dx}\,dx$

$\Rightarrow \quad dy = 2\,(0{\cdot}01745) = 0{\cdot}03490$

$\Rightarrow \quad \Delta y = 0{\cdot}03490$. [$\because$ Δy *is approximately* $= dy$]

Hence, $\tan 46° = y + \Delta y = 1 + 0{\cdot}03490 = 1{\cdot}0349$.

Example 8. If in a triangle ABC, the side c and the angle C remain constant, while the remaining elements are changed slightly, using differentials, show that :

$$\frac{\mathbf{da}}{\mathbf{cos\ A}} + \frac{\mathbf{db}}{\mathbf{cos\ B}} = \mathbf{0}.$$

Solution. Since side 'c' and angle 'C' are given,

therefore, $\frac{c}{\sin C}$ = constant (= k say).

By *Sine Formula*, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\Rightarrow \quad \frac{a}{\sin A} = \frac{b}{\sin B} = k$ (say)

$\Rightarrow \quad a = k\sin A$ and $b = k\sin B$

so that $\quad \frac{da}{dA} = k\cos A$ and $\frac{db}{dB} = k\cos B$.

Now $\quad da = \frac{da}{dA}\cdot dA$

$\Rightarrow \quad da = k\cos A\cdot dA$

$\Rightarrow \quad \frac{da}{\cos A} = k\,dA \quad$ ...(1)

And $\quad db = \frac{db}{dB}\cdot dB$

$\Rightarrow \quad db = k\cos B\,.\,dB$

$\Rightarrow \quad \frac{db}{\cos B} = k\,dB \quad$ ...(2)

$\therefore \quad \frac{da}{\cos A} + \frac{db}{\cos B} = k\,dA + k\,dB \quad$ [*Using (1) & (2)*]

$= k\,(dA + dB) = kd\,(A + B)$

$= kd\,(\pi - C)$

[$\because A + B + C = \pi \Rightarrow A + B = \pi - C$]

$= 0$.

Hence, $\frac{da}{\cos A} + \frac{db}{\cos B} = 0$.

Example 9. The time T of a complete oscillation of simple pendulum of length l is given by the equation $T = 2\pi\sqrt{\frac{l}{g}}$, where 'g' is constant. What is the percentage error in T when l is increased by 1% ?

Solution. Let Δl be the change in l and ΔT, the corresponding error in T.

By the question, $\frac{\Delta l}{l}\times 100 = 1 \quad$ ...(1)

Let $\quad \Delta l = dl$.

$\therefore$ From (1), $\frac{dl}{l}\times 100 = 1 \quad$ ...(2)

Now $\quad T = 2\pi\sqrt{\frac{l}{g}}$.

Taking logs.,

$$\log T = \log 2 + \log \pi + \frac{1}{2}\log l - \frac{1}{2}\log g$$

so that $\quad \frac{1}{T}\frac{dT}{dl} = 0 + 0 + \frac{1}{2}\cdot\frac{1}{l} - 0 \;\Rightarrow\; \frac{dT}{dl} = \frac{T}{2l} \quad$ ...(3)

Now $\quad dT = \frac{dT}{dl}\,dl \Rightarrow dT = \frac{T}{2l}\,dl \quad$ [*Using (3)*]

$\Rightarrow \quad \frac{dT}{T} = \frac{1}{2}\frac{dl}{l}$

$\Rightarrow \quad \frac{dT}{T}\times 100 = \frac{1}{2}\left(\frac{dl}{l}\times 100\right)$

$\Rightarrow \quad \frac{dT}{T}\times 100 = \frac{1}{2}(1) \quad$ [*Using (2)*]

Since ΔT is approximately equal to dT,

$\therefore \quad \frac{dT}{T}\times 100 = \frac{1}{2} \;\Rightarrow\; \frac{\Delta T}{T}\times 100 = \frac{1}{2}$.

Hence, there is $\left(\frac{1}{2}\right)$% error in calculating the time period T.

EXERCISE 6 (d)

Short Answer Type Questions

SATQ

In the following (1 – 13), find the approximate values, using differentials :

1. (*i*) $\sqrt{37}$ (*ii*) $\sqrt{50}$.

2. (*i*) $\sqrt{401}$ **(*N.C.E.R.T.; Nagaland B. 2018*)**

   (*ii*) $\sqrt{360}$ **(*P.B. 2017*)**

3. $\sqrt{0 \cdot 037}$.

4. $\sqrt{0 \cdot 0037}$. **(*N.C.E.R.T.*)**

5. (*i*) $\sqrt{25 \cdot 2}$

   (*ii*) $\sqrt{25 \cdot 3}$ **(*P.B. 2015, 13*)**

   (*iii*) $\sqrt{49 \cdot 5}$ **(*P.B. 2014; C.B.S.E. 2012*)**

   (*iv*) $\sqrt{36 \cdot 6}$ **(*Kerala. B. 2016*)**

   (*v*) $\sqrt{16.3}$ **(*P.B. 2015*)**

   (*vi*) $\sqrt{0 \cdot 6}$. **(*N.C.E.R.T.*)**

6. (*i*) $\sqrt{0 \cdot 17}$

   (*ii*) $\sqrt{0 \cdot 26}$ **(*P.B. 2010 S*)**

   (*iii*) $\sqrt{0 \cdot 82}$.

7. (*i*) $\sqrt{0 \cdot 24}$

   (*ii*) $\sqrt{0 \cdot 50}$. **(*P.B. 2010 S*)**

8. (*i*) $(26)^{1/3}$ **(*N.C.E.R.T.; H.B. 2015 ; H.P.B. 2010 S*)**

   (*ii*) $(28)^{1/3}$ **(*P.B. 2014 S; H.P.B. 2012*)**

   (*iii*) $(25)^{1/3}$ **(*N.C.E.R.T.*)**

   (*iv*) $(127)^{1/3}$ **(*Meghalaya B. 2015 ; H.B. 2012*)**

   (*v*) $(26 \cdot 57)^{1/3}$ **(*N.C.E.R.T.*)**

   (*vi*) $(0 \cdot 731)^{1/3}$. **(*P.B. 2016*)**

9. (*i*) $\sqrt[3]{0 \cdot 009}$ **(*N.C.E.R.T. ; H.P.B. 2011; P.B. 2010*)**

   (*ii*) $\sqrt[3]{0 \cdot 007}$. **(*P.B. 2010*)**

10. (*i*) $(15)^{1/4}$ **(*N.C.E.R.T.; H.B. 2015 ; H.P.B. 2013 S, 10 S*)**

    (*ii*) $(82)^{1/4}$ **(*N.C.E.R.T.; H.P.B. 2013 S*)**

    (*iii*) $(255)^{1/4}$ **(*N.C.E.R.T.; H.P.B. 2013 S, 10 S*)**

    (*iv*) $(81 \cdot 5)^{1/4}$ **(*N.C.E.R.T.*)**

    (*v*) $\left(\frac{17}{81}\right)^{1/4}$. **(*N.C.E.R.T.*)**

11. $(32 \cdot 15)^{1/5}$. **(*N.C.E.R.T.*)**

12. (*i*) $(0 \cdot 999)^{1/10}$ (*ii*) $(3 \cdot 968)^{3/2}$. **(*N.C.E.R.T.*)**

13. $(33)^{-1/5}$.

14. Find the approximate value of :

(*i*) $f(3 \cdot 02)$, where $f(x) = 3x^2 + 15x + 3$

(*ii*) $f(5 \cdot 001)$, where $f(x) = x^3 - 7x^2 + 15$. **(*N.C.E.R.T.*)**

15. Find the approximate change in the volume V of a cube of side 'x' metres caused by increasing the side by 2%. **(*N.C.E.R.T.; Karanataka B. 2014*)**

16. Find the approximate change in the surface area of a cube of side 'x' metres caused by decreasing the side by 1%. **(*N.C.E.R.T.*)**

17. If the radius of a sphere is measured as 9 cm with an error of 0·03 cm, then find the approximate error in calculating its volume. **(*N.C.E.R.T.*)**

18. Find $\cos 61°$, it being given that $\sin 60° = 0 \cdot 86603$ and $1° = 0 \cdot 01745$ radian.

19. Find the approximate change in the value of $\frac{1}{x^2}$, when x changes from $x = 2$ to $x = 2.002$. **(*C.B.S.E. Sample Paper 2018*)**

20. Using differentiation, find the approximate value of $f(3.01)$, where $f(x) = 4x^2 + 5x + 2$. **(*H. B. 2013*)**

21. Use differentials, find the approximate value of the following :

(*i*) $\sin \frac{22}{4}$ (*ii*) $\cos \frac{11\pi}{36}$.

22. If $y = \sin x$ and x changes from $\frac{\pi}{2}$ to $\frac{22}{14}$, what is the approximate change in y ?

23. A circular metal plate expands under heating so that its radius increases by 2%. Find the approximate increase in the area of the plate if the radius of the plate before heating is 10 cm.

24. Find the percentage error in calculating the surface area of a cubical box if an error of 1% is made in increasing the lengths of edges of the cube.

25. The radius of a spherical diamond is measured as 6 cm with an error of 0·04 cm. Obtain the approximate error in calculating its volume. If the cost of 1 cm^3 diamond is ₹ 1600, what is the loss to the buyer of the diamond ?

Answers

1. (*i*) 6·16 (*ii*) 7·07.
2. (*i*) 20·025 (*ii*) 18.974.
3. 0·1924.
4. 0·0608.
5. (*i*) 5·02 (*ii*) 5·03 (*iii*) 7·036 (*iv*) 6·05 (*v*) 4·037 (*vi*) 0·78.
6. (*i*) 0·412 (*ii*) 0·51 (*iii*) 0·905.
7. (*i*) 0·49 (*ii*) 0·71.
8. (*i*) $\frac{80}{27}$ (*ii*) $\frac{82}{27}$ (*iii*) 2·926 (*iv*) 5·03 (*v*) 2·984 (*vi*) 0·9008.
9. (*i*) 0·208 (*ii*) 0·1917.
10. (*i*) 1·96875 (*ii*) 3·009 (*iii*) $\frac{1023}{256}$ (*iv*) 3·0046 (*v*) 0·677.
11. 2·00187.
12. (*i*) 0·9999 (*ii*) 7·904.
13. 0·497.
14. (*i*) 75·66 (*ii*) – 34·995.
15. 0·06 x^3 m^3.
16. 0·12$x^2$$m^2$.
17. 9·72 π cm^3.
18. 0·4997.
19. Decrease of 0·0005.
20. 53·29.
21. (*i*) 1 (*ii*) 0·575575.
22. No change.
23. 4π.
24. 2%.
25. Loss ₹ 28,476.

Hints to Selected Questions

2. (*i*) Take $y=\sqrt{x}$, $x = 400$, $x + \Delta x = 401$.

10. (*i*) Take $y=x^{1/4}$, $x = 16$, $x + \Delta x = 15$.

15. $V = x^3$ so that $\frac{dV}{dx}=3x^2$.

Thus $\Delta V = 3x^2\,\Delta x = 3x^2\left(\frac{2x}{100}\right)=\frac{6x^3}{100}$.

Change in volume $= \frac{6}{100}x^3 = 0\cdot06x^3\text{m}^3$.

16. $S = 6x^2$ so that $\frac{dS}{dx}=12x$.

Thus $\Delta S = 12x\ \Delta x = 12x\left(-\frac{x}{100}\right)=-0\cdot12x^2\text{ m}^2$.

17. $V = \frac{4}{3}\pi r^3$ so that $\frac{dV}{dr}=4\pi r^2$.

Thus $\Delta V = 4\pi r^2\ \Delta r = (4 \times 9^2)(\pm 0\cdot03) = \pm 9\cdot72\ \pi\text{cm}^3$.

18. Take $y = \cos x$, $x = 60°=\frac{\pi}{3}, x+\Delta x=61°$.

19. $y=x^4-10 \Rightarrow \frac{dy}{dx}=4x^3$.

Since x changes from 2 to 1·99,

$\therefore\ x = 2$ and $x + dx = 1\cdot99$

so that $dx = 1\cdot99 - 2 = -0\cdot01$.

$\therefore\ \Delta y = \left(\frac{dy}{dx}\right)dx=4x^3(-0\cdot01)$

$= 4\,(2)^3\,(-0\cdot01)$ at $x = 2$

$= -0\cdot32$.

Hence, approximate decrease in the value of $y = 0\cdot32$.

22. We have : $y = \sin x\ \therefore\ \frac{dy}{dx} = \cos x$.

Since x changes from $\frac{\pi}{2}$ to $\frac{22}{14}$,

$\therefore\ x = \frac{\pi}{2}$ and $x + dx = \frac{22}{14}$

so that $dx = \frac{22}{14} - \frac{\pi}{2} = \frac{22}{14} - \frac{355}{226}=\frac{1}{1582}$. $\left[\because \pi=\frac{355}{113}\right]$

$\therefore\ dy = \left(\frac{dy}{dx}\right)dx = \cos x\,dx = \cos x \cdot \frac{1}{1582}$

$= \cos\frac{\pi}{2}\cdot\frac{1}{1582}$ at $x = \frac{\pi}{2}$

$= 0.\ \frac{1}{1582} = 0 \Rightarrow$ No change in y.

23. Let 'A' be the area and 'r' be the radius of the circular plate.

$\therefore\ A = \pi r^2$ so that $\frac{dA}{dr}=2\pi r$.

Let A = 10 and dr = 2% of 10 = $\frac{2}{10}$; etc.

SUB CHAPTER
6.5 Maxima and Minima

In this section, we shall use differentiation to calculate the maximum (or minimum) values of various functions. In fact we shall determine '*turning points*' of the graph of a function and consequently the points at which the graph reaches its highest (or lowest) *locally.* Also we shall find the absolute maximum (or absolute minimum) of a function, which is necessary for the solution of many applied problems.

6.8. DEFINITIONS

(*a*) Maxima and Minima.

Definition

Let a real valued function 'f' be defined on a set D_f, the domain of f.

(*i*) The function 'f' is said to have **absolute maxima** at $c \in D_f$ if $f(x) \le f(c) \, \forall \, x \in D_f$.

Here 'c' is called the **point of absolute maxima** and $f(c)$ is called the **absolute maximum value** of 'f' on D_f.

(*ii*) The function 'f' is said to have **absolute minima** at $d \in D_f$ if $f(x) \ge f(d) \, \forall \, x \in D_f$.

Here 'd' is called the **point of absolute minima** and $f(d)$ is called **absolute minimum value** of 'f' on D_f.

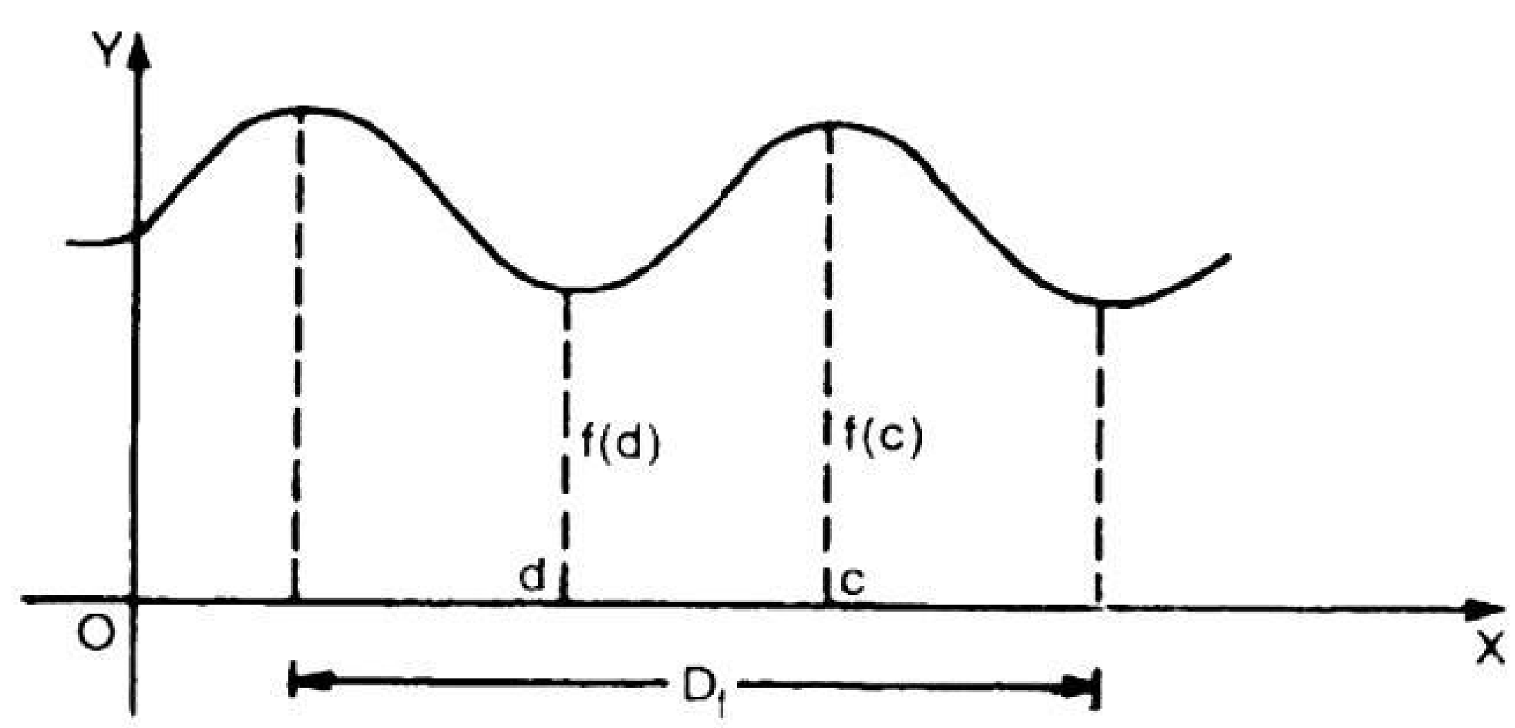


Fig.

Clearly the absolute maximum and absolute minimum values of a function (if they exist) are unique. However, these values may exist at more than one point of D_f.

GUIDE-LINES TO FIND MAX. AND MIN. VALUES IN A GIVEN INTERVAL

Step (i) *Find all points, where* $\frac{dy}{dx} = 0$.

Step (ii) *Take end points of the interval.*

Step (iii) *At all these points, calculate the values of y.*

Step (iv) *Take the maximum and minimum values out of the values calculated in step (iii).*

This is illustrated in Exs. 3–6.

(*b*) Relative Maxima and Minima.

(*i*) 'f' is said to have a **local (or relative) maxima** at $c \in D_f$ if there exists a real number $\delta > 0$ such that $f(x) \le f(c)$ $\forall \, x \in (c - \delta, c + \delta)$.

Here 'c' is called the **point of local maxima** and $f(c)$ is called the **local maximum value.**

(*ii*) 'f' is said to have a **local (or relative) minima** at $d \in D_f$ if there exists a real number $\delta > 0$ such that $f(x) \geq f(d)$ $\forall\ x \in (d - \delta, d + \delta)$.

Here 'd' is called the **point of local minima** and $f(d)$ is called the **local minimum value**.

(*iii*) A point of D_f, which is either a point of **local maxima** or **local minima**, is called an **extreme point** and the value of the function at this point is called an **extreme value.**

Geometrically. The graph of a function has a **peak** (or **trough**) at a point according as the point is of a local maxima or local minima.

(*iv*) **Critical point.** A point $c \in D_f$ is said to be **critical** (or **turning** or **saddle**) point of a continuous function 'f' if either $f'(c) = 0$, $f'(c)$ is infinite or $f'(c)$ does not exist.

KEY POINT

1. A local maxima (or minima) may not be absolute maxima (or minima).

2. A local maximum value at some point may be less than a local minimum value of the function at another point.

3. It is not essential that an extreme point is a critical point.

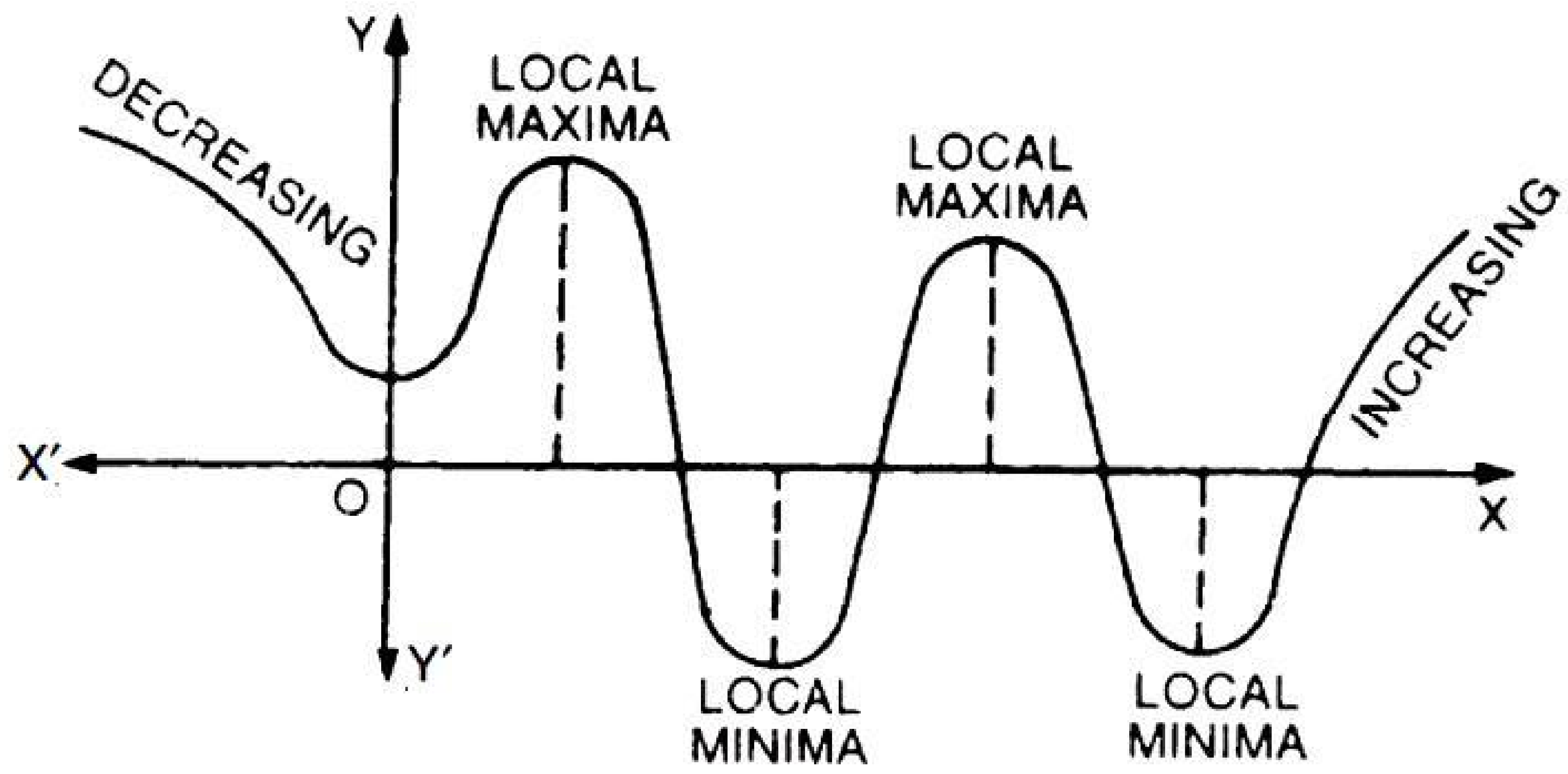


Fig.

6.9. NECESSARY CONDITIONS FOR EXTREME VALUES

If f(a) is an extreme value of f(x) and f(x) is derivable at a, then $f'(a) = 0$.

Case I. *When $f(a)$ is max. value of $f(x)$.*

Here $f(a)$ is the greatest value of $f(x)$ in the interval $(a - h, a + h)$.

Thus $f(x)$ is an increasing function in the sub-interval $(a - h, a)$ and a decreasing function in the sub-interval $(a, a + h)$

$\Rightarrow f'(x) > 0$ in $(a - h, a)$ and $f'(x) < 0$ in $(a, a + h)$.

If $f'(x)$ is continuous at 'a', [*Assumed*]

then $f'(x)$ must be zero at 'a' while changing from +ve to –ve values. Thus $f'(a) = 0$.

Case II. *When $f(a)$ is min. value of $f(x)$.*

Here as above, $f'(x)$ must be zero at 'a' while changing from –ve to +ve values.

Thus $f'(a) = 0$.

Hence, the result.

The converse of the above theorem is not true *i.e.* if $f'(a) = 0$, then $f(a)$ may not be the extreme value.

For Example : *Consider* $f(x) = x^3$ at $x = 0$.

Here $f'(x) = 3x^2$, $\therefore\ f'(0) = 0$. But 0 is neither max. nor min. value of $f(x)$.

Definition

Stationary values *are the values of $f(x)$ for those points at which $f'(x) = 0$.*

6.10. NECESSARY AND SUFFICIENT CONDITIONS FOR MAXIMUM AND MINIMUM VALUES (FIRST DERIVATIVE TEST)

We know that $f(x)$ has an extreme value at $x = a$ if $f'(a) = 0$ ...(1)

Three cases arise :

Case I. *When $f'(x)$ changes from +ve to -ve while passing through 'a'.*

Here $f(x)$ increases in the left-neighbourhood of 'a' and decreases in the right-neighbourhood of 'a'.

Hence, $f(x)$ has a maximum value at $x = a$.

Case II. *When $f'(x)$ changes from -ve to +ve while passing through 'a'.*

Here $f(x)$ decreases in the left-neighbourhood of 'a' and increases in the right-neighbourhood of 'a'.

Hence, $f(x)$ has a minimum value at $x = a$.

GUIDE-LINES TO FIND THE MAX. OR MIN. VALUE OF y = f (x)

Step (i) Put $\frac{dy}{dx} = 0$. Solve it for 'x' giving $x = a, b, c,$ ....

Step (ii) Select $x = a$ (say).

Study the sign of $\frac{dy}{dx}$ when (*i*) $x < a$ slightly (*ii*) $x > a$ slightly.

(*a*) If the former is +ve and latter is –ve, then $f(x)$ is max. at $x = a$.

(*b*) If the former is – ve and latter is +ve, then $f(x)$ is min. at $x = a$.

Step (iii) Putting these values of 'x' for which $f(x)$ is max. or min. and get the corresponding max. or min. values of $f(x)$.

6.11. USE OF SECOND DERIVATIVE (SECOND DERIVATIVE TEST)

Theorem. (I) A function f (x) is maximum at x = a if f′(a) = 0 and f″(a) < 0.

(II) A function f (x) is minimum at x = a if f′(a) = 0 and f″(a) > 0.

Proof. (I) Since $f'(a) = 0$, $\therefore$ $f(x)$ may be either max. or min. at $x = a$.

Again $f'(x)$ is a function of x, $\therefore$ $f''(a)$ (derivative of $f'(x)$ at $x = a$) is –ve [*Given*]

$\Rightarrow$ $f'(x)$ is a decreasing function at $x = a$.

But $f'(a) = 0$ and $f'(x)$ exists in the immediate nhd. of 'a'

$\Rightarrow$ $f'(x)$ changes from +ve to – ve in this nhd. while passing through 'a'.

Hence, $f(x)$ is maximum at $x = a$.

(II) Since $f'(a) = 0$, $\therefore$ $f(x)$ may be either max. or min. at $x = a$.

Again $f'(x)$ is a function of x, $\therefore$ $f''(a)$ (derivative $f'(x)$ at $x = a$) is +ve [*Given*]

$\Rightarrow$ $f'(x)$ is an increasing function at $x = a$.

But $f'(a) = 0$ and $f'(x)$ exists in the immediate nhd. of 'a'

$\Rightarrow$ $f'(x)$ changes from -ve to +ve in this nhd. while passing through 'a'.

Hence, $f(x)$ is minimum at $x = a$.

Generalisation : *When* $f''(a) = 0$, the **students are urged to remember the following facts without proofs.**

(*i*) If $f'''(a) \neq 0, x = a$ is a **point of inflexion.**

(*ii*) If $f'''(a) = 0$ and $f^{iv}(a) \neq 0, f(a)$ is maximum at $x = a$ if $f^{iv}(a) < 0$ and is minimum at $x = a$ if $f^{iv}(a) > 0$.

(*iii*) If $f^{iv}(a) = 0$ and $f^{v}(a) \neq 0, x = a$ is a point of inflexion.

(*iv*) If $f^{v}(a) = 0$ and $f^{vi}(a) \neq 0$, which is same as in (*ii*).

And so on.

Thus if n is odd and $f^{n}(a) \neq 0$, *result is as in (i) and if n is even and* $f^{n}(a) \neq 0$, *result is as in (ii).*

GUIDE-LINES TO FIND THE MAX. OR MIN. VALUES

Step (i) Put y = given function $f(x)$ and find $\frac{dy}{dx}$ *i.e.* $f'(x)$.

Step (ii) Put $\frac{dy}{dx} = 0$ *i.e.* $f'(x) = 0$ and solve it for 'x' giving $x = a, b, c$, ...

Step (iii) Select $x = a$. Find $\frac{d^2y}{dx^2}$ *i.e.* $f''(x)$ at $x = a$.

(*a*) If $\left(\frac{d^2y}{dx^2}\right)_{x=a}$ *i.e.* $f''(a)$ is – ve, $x = a$ gives the max. value of the function.

(*b*) If $\left(\frac{d^2y}{dx^2}\right)_{x=a}$ *i.e.* $f''(a)$ is +ve, $x = a$ gives the min. value of the function.

Similarly for $x = b, c$, ...

ILLUSTRATIVE EXAMPLES

Example 1. Find the maximum and minimum values, if any, of the functions given by :

(i) $f(x) = x^2, x \in \mathbf{R}$ (ii) $f(x) = |x|, x \in \mathbf{R}$.

(N.C.E.R.T.)

Solution. (*i*) The given function is :

$$f(x) = x^2, x \in \mathbf{R} \qquad ...(1)$$

Its graph is as shown in the adjoining figure :

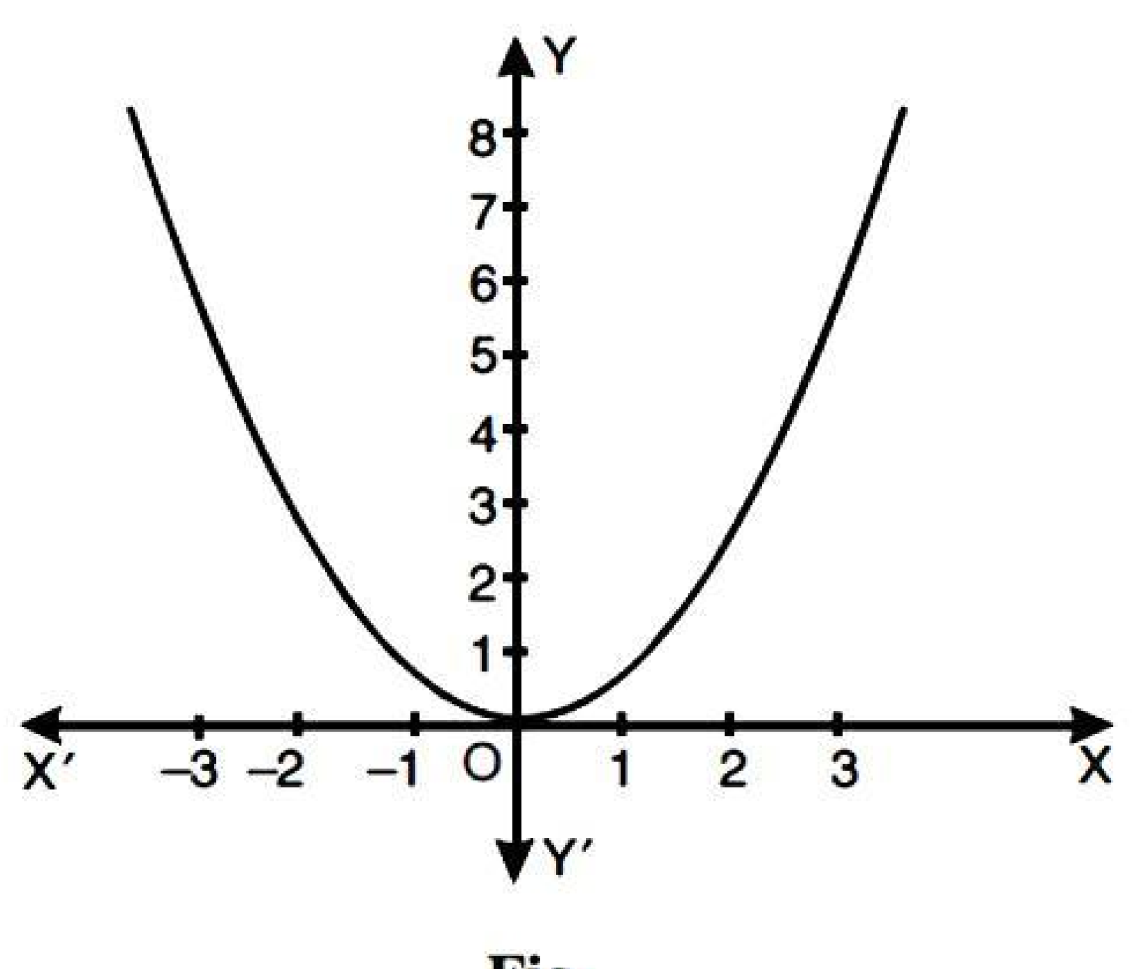


Fig.

From the graph, we have : $f(x) = 0$ when $x = 0$.

Also $f(x) \geq 0 \ \forall \ x \in \mathbf{R}$.

Thus minimum value of 'f' is 0 and point of minimum value of 'f' is $x = 0$.

And 'f' has no maximum value and hence no point of maximum value of 'f' in $\mathbf{R}$.

(*ii*) The given function is $f(x) = |x|, \forall \ x \in \mathbf{R}$ ...(1)

Its graph is as shown in the following figure :

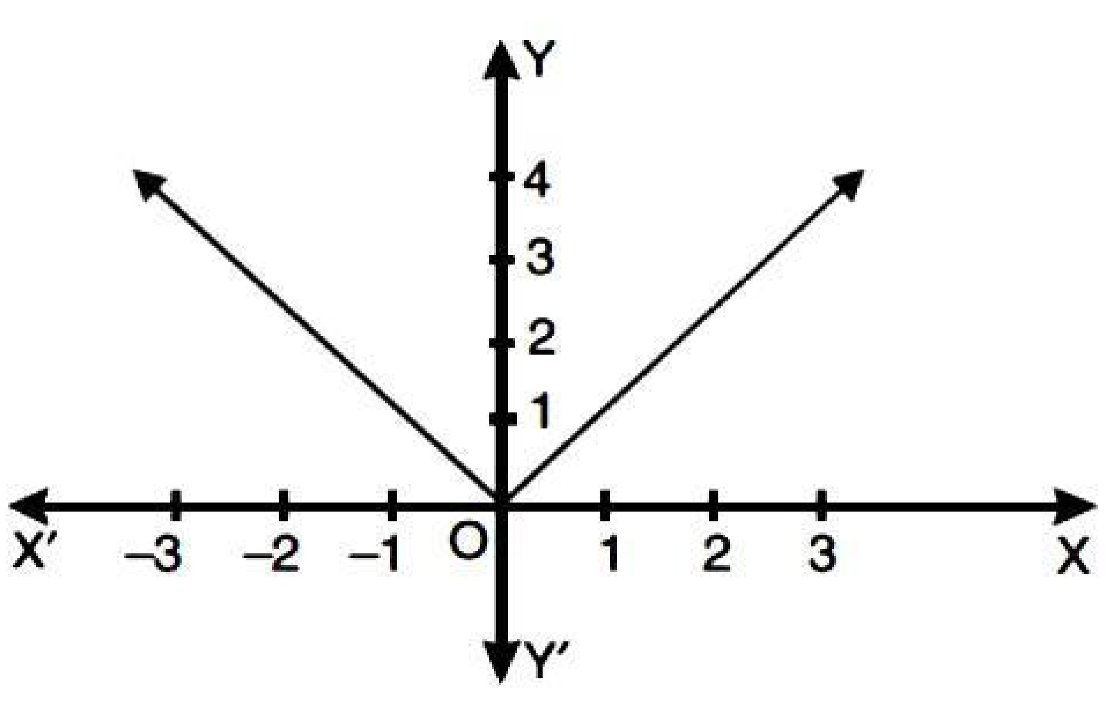


Fig.

From the graph, we have :

$f(x) \geq 0 \ \forall x \in \mathbf{R}$ and $f(x) = 0$ when $x = 0$.

Thus minimum value of 'f' is 0 and point of minimum value of 'f' is $x = 0$.

And 'f' has no maximum value and has no point of maximum value of 'f' in **R.**

Example 2. Find the maximum and minimum values, if any, of the following functions without using derivatives :

(i) $f(x) = (2x-1)^2 + 3$ *(Kerala B. 2016)*

(ii) $f(x) = 16x^2 - 16x + 28$

(iii) f(x) = – □x + 1□+ 3 (iv) f(x) = sin 2x + 5

(v) f(x) = sin (sin x).

Solution. (*i*) We have : $f(x)=(2x-1)^2+3$.

Here $D_f = \mathbf{R}$.

Now $f(x) \geq 3$. [$\because (2x-1)^2 \geq 0$ *for all* $x \in \mathbf{R}$]

Hence, the minimum value = 3.

However, maximum value does not exist.

[$\because$ *f (x) can be made as large as we please*]

(*ii*) We have : $f(x) = 16x^2 - 16x + 28$.

Here $D_f = \mathbf{R}$.

Now $$f(x) = 16\left(x^2 - x + \frac{1}{4}\right) + 24$$

$$= 16\left(x - \frac{1}{2}\right)^2 + 24$$

$\Rightarrow$ $f(x) \geq 24$.

$$\left[\because 16\left(x - \frac{1}{2}\right)^2 \geq 0 \text{ for all } x \in \mathbf{R}\right]$$

Hence, the minimum value is 24.

However, maximum value does not exist.

[$\because$ *f (x) can be made as large as we please*]

(*iii*) We have : $f(x) = -|x+1| + 3$

$\Rightarrow f(x) \leq 3$. [$\because -|x+1| \leq 0$]

Hence, the maximum value = 3.

However, the minimum value does not exist.

[$\because$ *f (x) can be made as small as we please*]

(*iv*) We have : $f(x) = \sin 2x + 5$.

Since $-1 \leq \sin 2x \leq 1$ for all $x \in \mathbf{R}$,

$\therefore -1 + 5 \leq \sin 2x + 5 \leq 1 + 5$ for all $x \in \mathbf{R}$

$\Rightarrow 4 \leq \sin 2x + 5 \leq 6$ for all $x \in \mathbf{R}$

$\Rightarrow 4 \leq f(x) \leq 6$ for all $x \in \mathbf{R}$.

Hence, the maximum value = 6 and minimum value = 4.

(*v*) We have : $f(x) = \sin(\sin x)$.

We know that $-1 \leq \sin x \leq 1$ for all $x \in \mathbf{R}$

$\Rightarrow \sin(-1) \leq \sin(\sin x) \leq \sin 1$ for all $x \in \mathbf{R}$

$\Rightarrow -\sin 1 \leq f(x) \leq \sin 1$.

Hence, max. value = sin 1 and min. value = – sin 1.

Example 3. Determine the absolute maximum and absolute minimum values of each of the following in the stated domains :

(i) $y = \frac{1}{2}x^2 + 5x + \frac{3}{2}; -6 \leq x \leq -2$

(ii) $f(x) = (x+1)^{2/3}$; $0 \leq x \leq 8$.

Solution. (*i*) We have : $y = \frac{1}{2}x^2 + 5x + \frac{3}{2}$ ...(1)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(2x) + 5(1) + 0$$

$$\Rightarrow \quad \frac{dy}{dx} = x + 5 \quad ...(2)$$

Now $\frac{dy}{dx} = 0$ gives : $x + 5 = 0 \Rightarrow x = -5 \in [-6, -2]$.

Now $$y\Big]_{x=-6} = \frac{1}{2}(-6)^2 + 5(-6) + \frac{3}{2}$$

$$= 18 - 30 + \frac{3}{2} = -\frac{21}{2}$$

$$y\Big]_{x=-5} = \frac{1}{2}(-5)^2 + 5(-5) + \frac{3}{2}$$

$$= \frac{25}{2} - 25 + \frac{3}{2} = -11$$

and $$y\Big]_{x=-2} = \frac{1}{2}(-2)^2 + 5(-2) + \frac{3}{2}$$

$$= 2 - 10 + \frac{3}{2} = -\frac{13}{2}.$$

Hence, the absolute max. value of $y = -\frac{13}{2}$ and absolute min. value of $y = -11$.

(*ii*) We have : $f(x) = (x+1)^{2/3}$ ...(1)

$$\therefore \quad f'(x) = \frac{2}{3}(x+1)^{-1/3}.$$

But $f'(x) = 0$ gives no value in $0 \le x \le 8$.

So we are to examine the max./min. at the end points only.

Now $f(0) = (0 + 1)^{2/3} = 1$

and $f(8) = (8 + 1)^{2/3} = 9^{2/3} = \sqrt[3]{81} = 3\sqrt[3]{3}$.

Hence, the absolute maximum value of $f(x) = 3\sqrt[3]{3}$ and absolute minimum value of $f(x) = 1$.

Example 4. Calculate the absolute maximum and absolute minimum value of the function $f(x) = \frac{x+1}{\sqrt{x^2+1}}$, $0 \le x \le 2$. (*J. & K.B. 2011*)

Solution. We have : $f(x) = \frac{x+1}{\sqrt{x^2+1}}$.

$$\therefore \quad f'(x)$$

$$= \frac{\sqrt{x^2+1}\,(1+0) - (x+1)\frac{1}{2\sqrt{x^2+1}}(2x+0)}{(x^2+1)}$$

$$= \frac{\sqrt{x^2+1} - \frac{x^2+x}{\sqrt{x^2+1}}}{x^2+1} = \frac{x^2+1-x^2-x}{(x^2+1)^{3/2}}$$

$$= \frac{1-x}{(x^2+1)^{3/2}}.$$

Now $f'(x) = 0 \Rightarrow 1 - x = 0 \Rightarrow x = 1 \in [0, 2]$.

Now $f(0) = \frac{0+1}{\sqrt{0+1}} = \frac{1}{1} = 1,$

$$f(1) = \frac{1+1}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

and $f(2) = \frac{2+1}{\sqrt{4+1}} = \frac{3}{\sqrt{5}}$.

Hence, absolute maximum value is $\frac{3}{\sqrt{5}}$ and absolute min. value is 1.

Example 5. Find the absolute maximum and the absolute minimum value of the function given by :

$f(x) = \sin^2 x - \cos x,\ x \in [0, \pi]$.

(*A.I.C.B.S.E. 2015*)

Solution : We have :

$$f(x) = \sin^2 x - \cos x.$$

$$\therefore \quad f'(x) = 2\sin x \cos x + \sin x = \sin x(2\cos x + 1).$$

Now $f'(x) = 0 \Rightarrow \sin x(2\cos x + 1) = 0$

$$\Rightarrow \quad \sin x = 0 \quad \text{or} \cos x = -\frac{1}{2}.$$

$$\Rightarrow \quad x = 0, \pi \text{ or } x = \frac{2\pi}{3}.$$

Now $f(0) = 0 - 1 = -1$

$$f\left(\frac{2\pi}{3}\right) = \sin^2\left(\frac{2\pi}{3}\right) - \cos\frac{2\pi}{3}$$

$$= \left(\frac{\sqrt{3}}{2}\right)^2 - \left(-\frac{1}{2}\right) = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

and $f(\pi) = \sin^2 \pi - \cos \pi = (0)^2 - (-1) = 1$.

Hence, absolute maximum value is $\frac{5}{4}$ and absolute minimum values is -1.

Example 6. Find the points of local maxima and local minima, if any, of the function :

$f(x) = (x - 1)(x + 2)^2$.

Find also the local maximum and local minimum values.

Solution. We have : $f(x) = (x - 1)(x + 2)^2$.

$$\therefore \quad f'(x) = (x-1)\,.\,\frac{d}{dx}(x+2)^2 + (x+2)^2\,.\,\frac{d}{dx}(x-1)$$

$$= (x-1) \times 2(x+2) + (x^2+4x+4) \times 1$$

$$= 2(x^2+x-2) + x^2+4x+4$$

$$= 3x^2 + 6x = 3x(x+2).$$

Now $f'(x) = 0 \Rightarrow 3x(x+2) = 0 \Rightarrow x = 0, -2$

and $f''(x) = 6x + 6 = 6(x + 1)$ ...(1)

Now $f''(x)\Big]_{x=0} = 6\,(0 + 1) = 6$ (+ ve)

and $f''(x)\Big]_{x=-2} = 6\,(-2 + 1) = -6$ (– ve)

Hence, $f(x)$ has local max. at $x = -2$ and local min. at $x = 0$.

Also $f(-2) = (-2 - 1)(-2 + 2)^2 = (-3)(0) = 0$, local max. value.

and $f(0) = (0 - 1)(0 + 2)^2 = (-1)(4) = -4$, local min. value.

Aliter. Upto (1), repeat as above.

At $x = 0$. When x is slightly less than 0, then $f'(x)$ is – ve.

When x is slightly greater than 0, then $f'(x)$ is + ve.

Thus $f'(x)$ changes sign from –ve to +ve.

So $x = 0$ is a point of local minima and local minimum value $= f(0) = (0 - 1)(0 + 2)^2 = -4$.

At $x = -2$. When x is slightly less than –2, then $f'(x)$ is +ve.

When x is slightly greater than –2, then $f'(x)$ is –ve.

Thus $f'(x)$ changes sign from +ve to –ve.

So $x = -2$ is a point of local maxima and local maximum value

$= f(-2) = (-2 - 1)(-2 + 2)^2$

$= (-3)(0)^2 = (-3)(0) = 0$.

Example 7. Find the points of local maxima and local minima, if any, of the following function :

$$\mathbf{f(x) = \sin x + \frac{1}{2}\cos 2x;\ 0 < x < \frac{\pi}{2}.}$$

Solution. We have : $f(x) = \sin x + \dfrac{1}{2}\cos 2x$.

$\therefore \quad f'(x) = \cos x + \dfrac{1}{2}(-\sin 2x)(2)$

$= \cos x - \sin 2x$.

For local max./min., $f'(x) = 0$

$\Rightarrow \quad \cos x - \sin 2x = 0$

$\Rightarrow \quad \cos x - 2\sin x \cos x = 0$

$\Rightarrow \quad \cos x\,(1 - 2\sin x) = 0$

$\Rightarrow \quad \cos x = 0$ or $1 - 2\sin x = 0$

$\Rightarrow \cos x = 0$ or $\sin x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{2}$ or $x = \dfrac{\pi}{6}$.

Thus $\quad x = \dfrac{\pi}{6}. \qquad \left[\because \dfrac{\pi}{6} \in \left(0, \dfrac{\pi}{2}\right)\right]$

Now $\quad f''(x) = -\sin x - 2\cos 2x$.

$\therefore \quad f''\left(\dfrac{\pi}{6}\right) = -\sin\dfrac{\pi}{6} - 2\cos\dfrac{2\pi}{6}$

$= -\sin\dfrac{\pi}{6} - 2\cos\dfrac{\pi}{3} = -\dfrac{1}{2} - 2.\dfrac{1}{2}$

$= -\dfrac{1}{2} - 1 = -\dfrac{3}{2} < 0$.

Hence, $f(x)$ is local max. at $x = \dfrac{\pi}{6}$.

Example 8. Find all the points of local maxima and local minima of the function 'f' given by :

$f(x) = 2x^3 - 6x^2 + 6x + 5$. *(N.C.E.R.T.)*

Solution. We have : $\quad f(x) = 2x^3 - 6x^2 + 6x + 5$.

$\therefore \quad f'(x) = 6x^2 - 12x + 6$

$= 6(x - 1)^2$.

$\therefore \quad f''(x) = 12(x - 1)$.

Now $\quad f'(x) = 0$ gives $x = 1$.

Also $\quad f''(1) = 0$.

Thus $x = 1$ is neither a point of maxima nor of minima.

Now $\quad f'''(x) = 12$.

And $\quad f'''(x)\Big]_{x=1} = 12 \neq 0$.

Hence, $x = 1$ is a point of inflexion.

EXERCISE 6 (e)

Fast Track Answer Type Questions

FTATQ

Find the maximum or minimum values, if any, of the following functions (1– 4) without using the derivatives :

1. (*a*) (*i*) $f(x) = -(x - 1)^2 + 2$

(*ii*) $f(x) = -(x - 1)^2 + 10$ *(N.C.E.R.T.)*

(*iii*) $f(x) = (2x - 1)^2 + 3$ *(N.C.E.R.T.)*

(*iv*) $f(x) = 9x^2 + 12x + 2$

(*v*) $f(x) = x + 1,\ x \in (-1, 1)$. *(N.C.E.R.T.)*

(*b*) $g(x) = x^3 + 1$. *(N.C.E.R.T.)*

2. (*i*) $f(x) = \square x + 2\square - 1$

(*ii*) $g(x) = -\square x - 1\square + 3.$ **(N.C.E.R.T.)**

3. (*i*) $f(x) = \sin 2x + 5$ **(N.C.E.R.T.)**

(*ii*) $f(x) = \square \sin 4x + 3\square.$ **(N.C.E.R.T.)**

4. The function $f(x) = x^2$, $x \in \mathbf{R}$ has no maximum value. **(True/False) (Kashmir B. 2012)**

5. If $x > 0$, $y > 0$ and $xy = 25$, then find the minimum value of $x + y$. **(W. Bengal B. 2018)**

Very Short Answer Type Questions

VSATQ

6. Find the points of absolute maximum and minimum of each of the following :

(*i*) $y = x(1 + 10x - x^2)$; $3 \le x \le 9$

(*ii*) $y = \frac{1}{3}x^{3/2} - 4x$; $0 \le x \le 64$

(*iii*) $y = \sqrt{5}\left(\sin x + \frac{1}{2}\cos 2x\right)$; $0 \le x \le \frac{\pi}{2}$.

7. (*a*) Find the maximum and minimum values, if any, of the function given by :

$f(x) = x, x \in (0, 1).$ **(N.C.E.R.T.)**

(*b*) Prove that the maximum value of the function $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$. **(W. Bengal B. 2017)**

8. Find the absolute minimum value of $y = x^2 - 3x$ in $0 \le x \le 2$.

9. At what points in the interval $[0, 2\pi]$ does the function $\sin 2x$ attain its maximum value ?

(N.C.E.R.T.; Jammu B. 2017)

10. Find the maximum and minimum values of the function :

$f(x) = 2x^3 - 15x^2 + 36x + 11.$ **(Jharkhand B. 2013)**

11. Find the local minimum value of the function 'f' given by $f(x) = 3 + |x|$, $x \in \mathbf{R}$. **(N.C.E.R.T.)**

Short Answer Type Questions

SATQ

Find the absolute maximum and minimum values of each of the following (12 – 19) in the given intervals :

12. $f(x) = x^{50} - x^{20}$; $[0, 1]$.

13. (*i*) $f(x) = 4x - \frac{1}{2}x^2$; $x \in \left[-2, \frac{9}{2}\right]$ **(N.C.E.R.T.)**

(*ii*) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$. **(N.C.E.R.T.)**

14. (*i*) $f(x) = x^3 - 3x$; $-3 \le x \le 3$ **(J.&K.B. 2010)**

(*ii*) $f(x) = x^3 - \frac{5}{2}x^2 - 2x + 1$; $0 \le x \le 3$.

15. $f(x) = x^3$ in $[-2, 2]$. **(N.C.E.R.T.; Kashmir B. 2016)**

16. (*i*) $f(x) = (x - 1)^2 + 3$ in $[-3, 1]$

(N.C.E.R.T.; Kashmir B. 2016)

(*ii*) $f(x) = 2x^3 - 15x^2 + 36x + 1$ in $[1, 5]$.

(N.C.E.R.T.; Jharkhand B. 2016)

17. (*i*) $f(x) = \sin x + \cos x$ in $[0, \pi]$

(N.C.E.R.T. ; Kashmir B. 2012 ; H.B. 2012)

(*ii*) $f(x) = \cos^2 x + \sin x$ in $[0, \pi]$. **(N.C.E.R.T.)**

18. $y = x + \sin 2x$; in $[0, 2\pi]$. **(N.C.E.R.T.)**

19. $y = 2\cos 2x - \cos 4x$; $0 \le x \le \pi$. **(J & K B. 2011)**

20. Find both maximum and minimum values of :

$$3x^4 - 8x^3 + 12x^2 - 48x + 25$$

in the interval $[0, 3]$.

(N.C.E.R.T.; Jammu B. 2018 ; H.P.B. 2010)

21. Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$.

Find the maximum value of the same function in $[-3, -1]$.

(N.C.E.R.T.; H.P.B. 2010)

Find the points of local maxima and local minima, if any, of the following (22 – 30) functions. Find also the local maximum and local minimum values :

22. (*i*) The constant function α

(*ii*) $f(x) = x^2$ **(N.C.E.R.T.)**

(*iii*) $f(x) = x^3 - 3x.$ **(N.C.E.R.T.).**

23. (*i*) $f(x) = \cos x$, $0 < x < \pi$

(*ii*) $f(x) = \sin x + \cos x$, $0 < x < \frac{\pi}{2}$

(N.C.E.R.T. ; Assam B. 2017; Jammu B. 2013)

(*iii*) $f(x) = \sin x - \cos x$, $0 < x < 2\pi$.

(N.C.E.R.T. ; Assam B. 2018; C.B.S.E. 2015 ; Jammu B. 2015, 13, H.P.B. 2013)

24. $g(x) = \frac{x}{2} + \frac{2}{x}, x \neq 0$.
(N.C.E.R.T.; Assam B. 2017; Kashmir B. 2015)

25. $g(x) = \frac{1}{x^2 + 2}$. *(N.C.E.R.T.)*

26. $f(x) = x\sqrt{1-x}, x > 0$. *(N.C.E.R.T.)*

27. (*i*) $f(x) = x^3 - 12x^2 + 36x - 4$

(*ii*) $f(x) = x^3 - 6x^2 + 9x + 15 ; 0 \le x \le 6$
(N.C.E.R.T.; Jammu B. 2014)

(*iii*) $f(x) = x^3 - 3x + 3$. *(N.C.E.R.T.)*

28. (*i*) $f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$

(*ii*) $f(x) = 3x^4 + 4x^3 - 12x^2 + 12$.
(N.C.E.R.T.; Assam B. 2016)

29. $f(x) = x\sqrt{1-x} ; x > 0$. *(H.P.B. 2013)*

30. (*i*) $f(x) = -x + 2\sin x ; 0 \le x \le 2\pi$

(*ii*) $f(x) = \sin^4 x + \cos^4 x ; 0 < x < \frac{\pi}{2}$.

31. Prove that x^x has minimum value at $x = \frac{1}{e}$.

Long Answer Type Questions

LATQ

32. The curve $y = \frac{x^2 + ax + b}{x - 10}$ has a turning point at (4, 1). Find the values of '*a*' and '*b*' and also show that *y* is maximum at this point. Point out the maximum value without any calculation.

33. $y = \frac{ax - b}{(x-1)(x-4)}$ has a turning point P (2, –1). Find the values of '*a*' and '*b*' and show that *y* is maximum at P.

Answers

1. (*a*) (*i*) Max. = 2 (*ii*) Max. = 10

(*iii*) Min. = 3 (*iv*) Min. = – 2

(*v*) Max. = 2; Min. $x = 0$.

(*b*) No max. or no min.

2. (*i*) Min. = – 1 (*ii*) Max. = 3.

3. (*i*) Min. = 4, Max. = 6

(*ii*) Min. = 2, Max. = 4.

4. True. **5.** 10.

6. (*i*) Max. at $x = \frac{10 + \sqrt{103}}{3}$; Min. at $x = 3$

(*ii*) Max. at $x = 0$; Min. at $x = 64$

(*iii*) Max. at $x = \frac{\pi}{6}$; Min. at $x = \frac{\pi}{2}$.

7. (*a*) Neither maximum nor minimum.

8. $-\frac{9}{4}$.

9. $\left(\frac{\pi}{4}, 1\right)$, $\left(\frac{5\pi}{4}, 1\right)$.

10. Max. = 39, Min. = 38.

11. 3.

12. $\left(\frac{2}{5}\right)^{5/3} - \left(\frac{2}{5}\right)^{2/3}$, 0.

13. (*i*) 8, – 10 (*ii*) $18, -\frac{9}{4}$.

14. (*i*) 18, – 18 (*ii*) 1, – 5.

15. 8, – 8.

16. (*i*) 19, 3 (*ii*) 56, 24.

17. (*i*) $\sqrt{2}, -1$ (*ii*) $\frac{5}{4}$, – 1.

18. 2π, 0. **19.** $\frac{3}{2}$, – 3.

20. Max. = 25, Min = – 39.

21. Max. = 89 at $x = 3$,

Max. = 139 at $x = -2$.

22. (*i*) None (*ii*) Min. at $x = 0$ is 0

(*iii*) Min. at $x = 1$ is – 2 ; Max. at $x = -1$ is 2.

23. (*i*) None

(*ii*) Max. at $x = \frac{\pi}{4}$ is $\sqrt{2}$

(*iii*) Max. at $x = \frac{3\pi}{4}$ is $\sqrt{2}$;

Min. at $x = \frac{7\pi}{4}$ is $-\sqrt{2}$.

24. Min. at $x = 2$ is 2.

25. Max. at $x = 0$ is $\frac{1}{2}$.

26. Max. at $x = \frac{2}{3}$ is $\frac{2\sqrt{3}}{9}$.

27. (*i*) Max. at $x = 2$ is 28 ; Min. at $x = 6$ is – 4

(*ii*) Max. at $x = 1$ is 19 ; Min. at $x = 3$ is 15

(*iii*) Max. at $x = -1$ is 5 ; Min. at $x = 1$ is 1.

28. (*i*) Max. at $x = 0$ is 105 ;

Min. at $x = -3$ is 57·75

Max. at $x = -5$ is 73·75

(*ii*) Max. at $x = 0$, is 12 ; Min at $x = -1$ is 7

Min. at $x = -2$ is – 20.

29. Max. at $x = \frac{2}{3}$ is $\frac{2\sqrt{3}}{9}$.

30. (*i*) Max. at $x = \frac{\pi}{3}$ is $\sqrt{3} - \frac{\pi}{3}$;

Min. at $x = \frac{5\pi}{3}$ is $-\sqrt{3} - \frac{5\pi}{3}$

(*ii*) Min. at $x = \frac{\pi}{4}$ is $\frac{1}{2}$.

32. $a = -7$, $b = 6$; Max. value is 1.

33. $a = 1$, $b = 0$.

Hints to Selected Questions

9. Here $f(x) = \sin 2x,\ 0 \le x \le 2\pi$.

$\therefore \quad f'(x) = 2\cos 2x,\ 0 < x < 2\pi$.

Now $f'(x) = 0 \Rightarrow 2\cos 2x = 0 \Rightarrow \cos 2x = 0$

$\Rightarrow \quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

11. Since $|x| \ge 0,\ x \in \mathbf{R}$,

$\therefore$ local minimum value of 'f' = 3 + 0 = 3.

12. Here $f'(x) = 50x^{49} - 20x^{19}$.

Now $f'(x) = 0 \Rightarrow 50x^{49} - 20x^{19} = 0$

$\Rightarrow 10x^{19}(5x^{30} - 2) = 0$

$\Rightarrow \quad x = 0, \left(\frac{2}{5}\right)^{1/30}$; etc.

20. Here $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$.

$\therefore \quad f'(x) = 12x^3 - 24x^2 + 24x - 48$.

Now $\quad f'(x) = 0 \Rightarrow x = 2$; etc.

31. Let $y = x^x$.

$\therefore \quad \frac{dy}{dx} = x^x(1 + \log x)$.

Now $\frac{dy}{dx} = 0 \Rightarrow \log x = -1 \Rightarrow x = \frac{1}{e}$.

32. (I) $\frac{dy}{dx} = 0$ at $(4, 1) \Rightarrow 10a + b = -64$ ...(1)

Also (4, 1) lies on the given curve $\Rightarrow b = -4a - 22$ ...(2)

Solving (1) and (2), $a = -7$, $b = 6$; etc.

(II) Since $x = 4$ makes y maximum,

$\therefore$ max. value = ordinate of the point = 1.

GENERAL PROBLEMS

SOME USEFUL RESULTS

In the problems to follow, we shall use the following results :

(*i*) **Rectangle.** If 'x' and 'y' be its sides, then Area = xy, Perimeter = $2(x + y)$.

(*ii*) **Square.** If 'x' be its side, then Area = x^2, Perimeter = $4x$.

(*iii*) **Equilateral triangle.** If 'x' be its side, then Area = $\frac{\sqrt{3}x^2}{4}$, Perimeter = $3x$.

(*iv*) **Cube.** If 'x' be its edge, then volume = x^3, Surface area = $6x^2$.

(*v*) **Cuboid.** If x, y, z be its sides, then Volume = xyz ; Surface area = $2(xy + yz + zx)$.

(*vi*) **Trapezium.** Area = $\frac{1}{2}$ (Sum of parallel sides) × (distance between them).

(*vii*) **Circle.** If 'r' be the radius, then Area = πr^2 ; Circumference = $2\pi r$.

(*viii*) **Sphere.** If 'r' be the radius, then Volume = $\frac{4}{3}\pi r^3$; Surface area = $4\pi r^2$.

(*ix*) **Right-circular Cone.** If 'r' be the radius, 'h' the height and 'l' the slant height, then Volume = $\frac{1}{3}\pi r^2 h$; Curved Surface Area = πrl and Total Surface Area = $\pi r^2 + \pi rl$.

(*x*) **Right-Circular Cylinder.** If 'r' be the radius, 'h' the height, then volume = $\pi r^2 h$; Curved Surface = $2\pi rh$ and Total Surface Area = $2\pi rh + 2\pi r^2$.

Frequently Asked Questions

FAQs

Example 1. Find two positive numbers whose sum is 24 and their sum of squares is minimum. *(H.B. 2014)*

Solution. Let 'x' and 'y' be two positive numbers.

By the question, $x + y = 24$ ...(1)

Let $S = x^2 + y^2$

$\Rightarrow$ $S = x^2 + (24 - x)^2$ [*Using (1)*]

$= x^2 + 576 + x^2 - 48x$

$= 2x^2 - 48x + 576.$

$\therefore$ $\frac{dS}{dx} = 4x - 48.$

For S to be minimum, $\frac{dS}{dx} = 0$, which gives :

$4x - 48 = 0 \Rightarrow 4x = 48 \Rightarrow x = 12.$

Now $\frac{d^2S}{dx^2} = 4$, which is +ve for $x = 12$.

Hence, S is minimum when the two positive numbers are 12 and (24 – 12) *i.e.* 12 and 12.

Example 2. Show that, of all rectangles with a given perimeter, the square has the largest area.

(P.B. 2015 ; Kerala B. 2013; C.B.S.E. 2011)

Solution. Let 'x' and 'y' be the sides of the rectangle.

By the question, perimeter = p (say) [*Given*]

$\Rightarrow 2x + 2y = p \Rightarrow y = \frac{1}{2}(p - 2x)$ ...(1)

Now area, $A = xy = x \cdot \frac{1}{2}(p - 2x)$ [*Using (1)*]

$= \frac{1}{2}(px - 2x^2).$

For A to be largest, $\frac{dA}{dx} = 0$ and $\frac{d^2A}{dx^2}$ is – ve.

Here $\frac{dA}{dx} = \frac{1}{2}(p - 4x)$ and $\frac{d^2A}{dx^2} = -2.$

Now $\frac{dA}{dx} = 0$ gives $\frac{1}{2}(p - 4x) = 0$

$\Rightarrow p - 4x = 0 \Rightarrow p = 4x.$

And $\frac{d^2A}{dx^2}$ is –ve .

Thus A is largest when $p = 4x$.

From (1), $y = \frac{1}{2}(4x - 2x) \Rightarrow y = \frac{1}{2}(2x) \Rightarrow y = x.$

Hence, area is largest when $x = y$ *i.e.* when rectangle becomes a square.

Example 3. Show that rectangle of maximum perimeter, which can be inscribed in a circle of radius 'a' is a square of side $\sqrt{2}$ a .

Solution. Let 'x' and 'y' be the length and breadth respectively of the rectangle inscribed in a circle of radius 'a'.

$\therefore$ $x^2 + y^2 = (2a)^2$ [*Pythagoras' Theorem*]

$\Rightarrow$ $x^2 + y^2 = 4a^2$ ...(1)

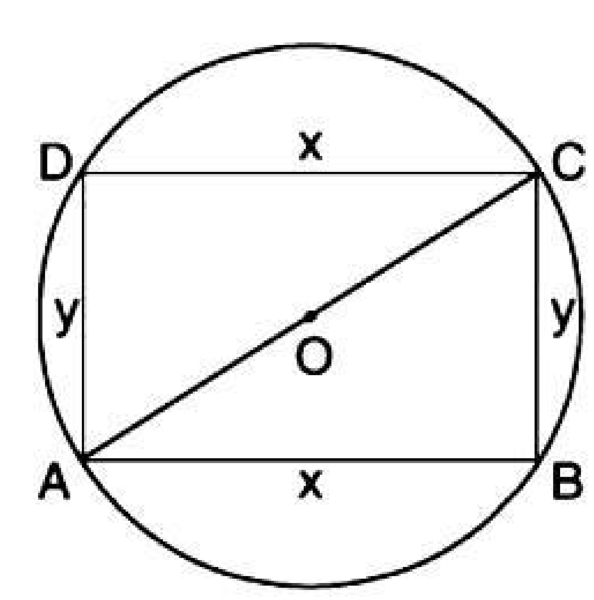


Fig.

$\therefore$ Perimeter = 2 (x + y)

$$\Rightarrow \quad P(x) = 2\left[x + \sqrt{4a^2 - x^2}\right]. \qquad [Using\ (1)]$$

$$\therefore \quad P'(x) = 2\left[1 + \frac{1}{2\sqrt{4a^2 - x^2}}(0 - 2x)\right]$$

$$= 2\left[1 - \frac{x}{\sqrt{4a^2 - x^2}}\right] \qquad ...(2)$$

$$\text{and } P''(x) = 2\left[0 - \frac{\sqrt{4a^2 - x^2}\,(1) - x \cdot \frac{1}{2}(4a^2 - x^2)^{-1/2}(-2x)}{(4a^2 - x^2)}\right]$$

$$= 2\left[\frac{-\sqrt{4a^2 - x^2} - \frac{x^2}{\sqrt{4a^2 - x^2}}}{(4a^2 - x^2)}\right]$$

$$= 2\left[\frac{-(4a^2 - x^2) - x^2}{(4a^2 - x^2)^{3/2}}\right]$$

$$= 2\left[\frac{-4a^2}{(4a^2 - x^2)^{3/2}}\right]$$

$$= \frac{-8a^2}{(4a^2 - x^2)^{3/2}} \qquad ...(3)$$

For P (x) to be maximum, $P'(x) = 0$ and $P''(x) < 0$.

Now, $P'(x) = 0$

$$\Rightarrow \quad 2\left[1 - \frac{x}{\sqrt{4a^2 - x^2}}\right] = 0 \qquad [Using\ (2)]$$

$$\Rightarrow \quad 1 - \frac{x}{\sqrt{4a^2 - x^2}} = 0$$

$$\Rightarrow \quad 1 = \frac{x}{\sqrt{4a^2 - x^2}}$$

$$\Rightarrow \quad \sqrt{4a^2 - x^2} = x$$

$$\Rightarrow \quad 4a^2 - x^2 = x^2 \qquad [Squaring]$$

$$\Rightarrow \quad 2x^2 = 4a^2$$

$$\Rightarrow \quad x^2 = 2a^2$$

$$\Rightarrow \quad x = \sqrt{2}a. \quad [Taking\ +ve\ value]$$

From (3),

$$P''(\sqrt{2}a) = \frac{-8a^2}{(4a^2 - 2a^2)^{3/2}} = \frac{-8a^2}{(2a^2)^{3/2}}, \text{ which is } - \text{ve.}$$

Hence, P (x) is maximum when $x = \sqrt{2}\,a$.

$\therefore$ From (1), $y^2 = 4a^2 - x^2 = 4a^2 - 2a^2 = 2a^2$

$\Rightarrow \quad y = \sqrt{2}\,a$.

Thus $\quad x = y$.

Hence, the rectangle is a square of each side $\sqrt{2}\,a$.

Example 4. Show that the triangle of maximum area that can be inscribed in a given circle is an equilateral triangle.

Solution. Let O be the centre and '*r*', the radius of the circle in which ΔABC is inscribed.

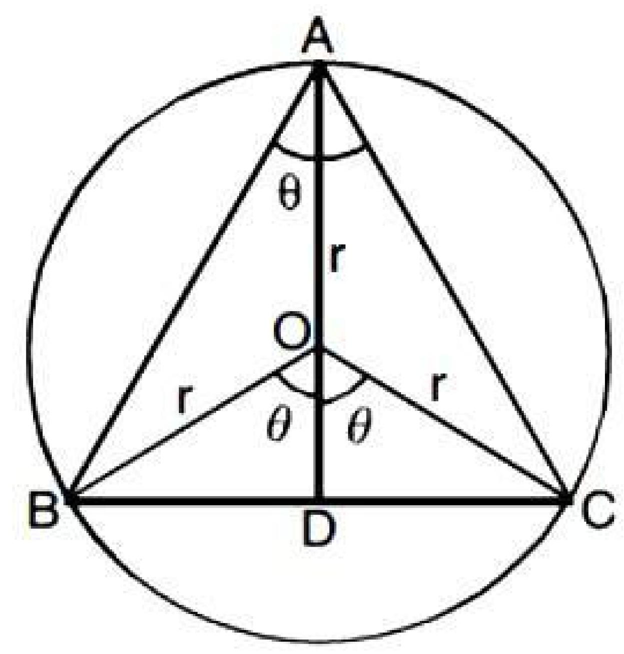


Fig.

For maximum area, the vertex A should be at a maximum distance from the base BC

⇒ A must lie on the diameter, which is perp. to BC

⇒ ΔABC is isosceles.

Let ∠BAC = θ. Then ∠BOC = 2θ ⇒ ∠DOC = θ.

Now BC = 2DC = 2 OC sin θ = 2*r* sin θ ...(1)

and AD = AO + OD = AO + OC cos θ

= *r* + *r* cos θ = *r* (1 + cos θ) ...(2)

If 'A' be the area of the triangle, then :

$$A = \frac{1}{2}\,(BC)\,(AD) = \frac{1}{2}\,(2r\sin\theta)\,r\,(1+\cos\theta)$$

[Using (1) & (2)]

$$\Rightarrow \quad A = r^2(\sin\theta + \sin\theta\,\cos\theta).$$

$$\therefore \quad \frac{dA}{d\theta} = r^2(\cos\theta + \cos^2\theta - \sin^2\theta)$$

$$= r^2(\cos\theta + \cos 2\theta)$$

$$= r^2(\cos 2\theta + \cos\theta)$$

$$= r^2(2\cos^2\theta - 1 + \cos\theta)$$

$$= r^2(2\cos^2\theta + \cos\theta - 1) \qquad \text{...(3)}$$

and $$\frac{d^2A}{d\theta^2} = r^2(-4\cos\theta\,\sin\theta - \sin\theta) \qquad \text{...(4)}$$

For 'A' to be maximum, $\frac{dA}{d\theta} = 0$ and $\frac{d^2A}{d\theta^2} < 0$.

Now $\frac{dA}{d\theta} = 0$

$\Rightarrow r^2(2\cos^2\theta + \cos\theta - 1) = 0$ *[Using (3)]*

$\Rightarrow \quad 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2}, -1$

$\Rightarrow \quad \theta = \frac{\pi}{3}, \pi \quad \Rightarrow \quad \theta = \frac{\pi}{3}.$

[$\because$ *$\theta = \pi \Rightarrow \angle BOC = 2\theta = 2\pi$, which is not possible*]

From (4), $$\left.\frac{d^2A}{d\theta^2}\right]_{\theta=\frac{\pi}{3}} = r^2\left(-4\cos\frac{\pi}{3}\sin\frac{\pi}{3} - \sin\frac{\pi}{3}\right)$$

$$= r^2\left(-4.\frac{1}{2}.\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}r^2 < 0.$$

Thus A is maximum when $\theta = \frac{\pi}{3}$.

But ΔABC is isosceles.

Hence, for maximum area, triangle is equilateral.

Example 5. An open box with a square base is to be made out of a given iron sheet of area 27 sq. m. Show that the maximum volume of the box is 13·5 cu. cm.

(P.B. 2012)

Solution. Let 'x' be the side of the square base and 'y' the height of the box.

$\therefore$ Area of the square base $= x^2$

and area of four walls $= 4xy$.

By the question, $x^2 + 4xy = 27$ ...(1)

Now V, volume of the box = (area of the base) × height

$$= x^2y = x^2 . \frac{27 - x^2}{4x} \qquad [Using\ (1)]$$

$$= \frac{1}{4}(27x - x^3) \qquad \text{...(2)}$$

$$\therefore \quad \frac{dV}{dx} = \frac{1}{4}(27 - 3x^2) \qquad \text{...(3)}$$

and $$\frac{d^2V}{dx^2} = -\frac{3}{2}x \qquad \text{...(4)}$$

Now $\frac{dV}{dx} = 0$ gives :

$$\frac{1}{4}(27 - 3x^2) = 0 \Rightarrow x^2 = \frac{27}{3} = 9$$

$\Rightarrow \quad x = \pm 3$

$\Rightarrow \quad x = 3.$ [$\because$ *x can't be – ve*]

$\therefore$ V can be max. only when $x = 3$ m.

When $x = 3$, from (4), $\frac{d^2V}{dx^2} = -\frac{3}{2}(3) = -\frac{9}{2}$, which is – ve.

Thus $x = 3$ gives the max. value of V.

Hence, max. V $= \frac{1}{4}[(27)(3) - 27]$ *[Putting $x = 3$ in (2)]*

$$= \frac{27}{4}(3 - 1) = \frac{27}{2}$$

$$= 13{\cdot}5 \text{ cm}^3.$$

Example 6. A given quantity of metal is to be cast into a solid half circular cylinder (i.e. with rectangular base and semi-circular ends). Show that in order that the total surface area may be minimum, the ratio of the length of the cylinder to the diameter of its circular ends is $\pi : (\pi + 2)$.

Solution. Let 'r' be the radius and 'h', the height of the cylinder.

$\therefore$ V = Volume of half cylinder

$$= \frac{1}{2}\pi r^2 h \qquad ...(1)$$

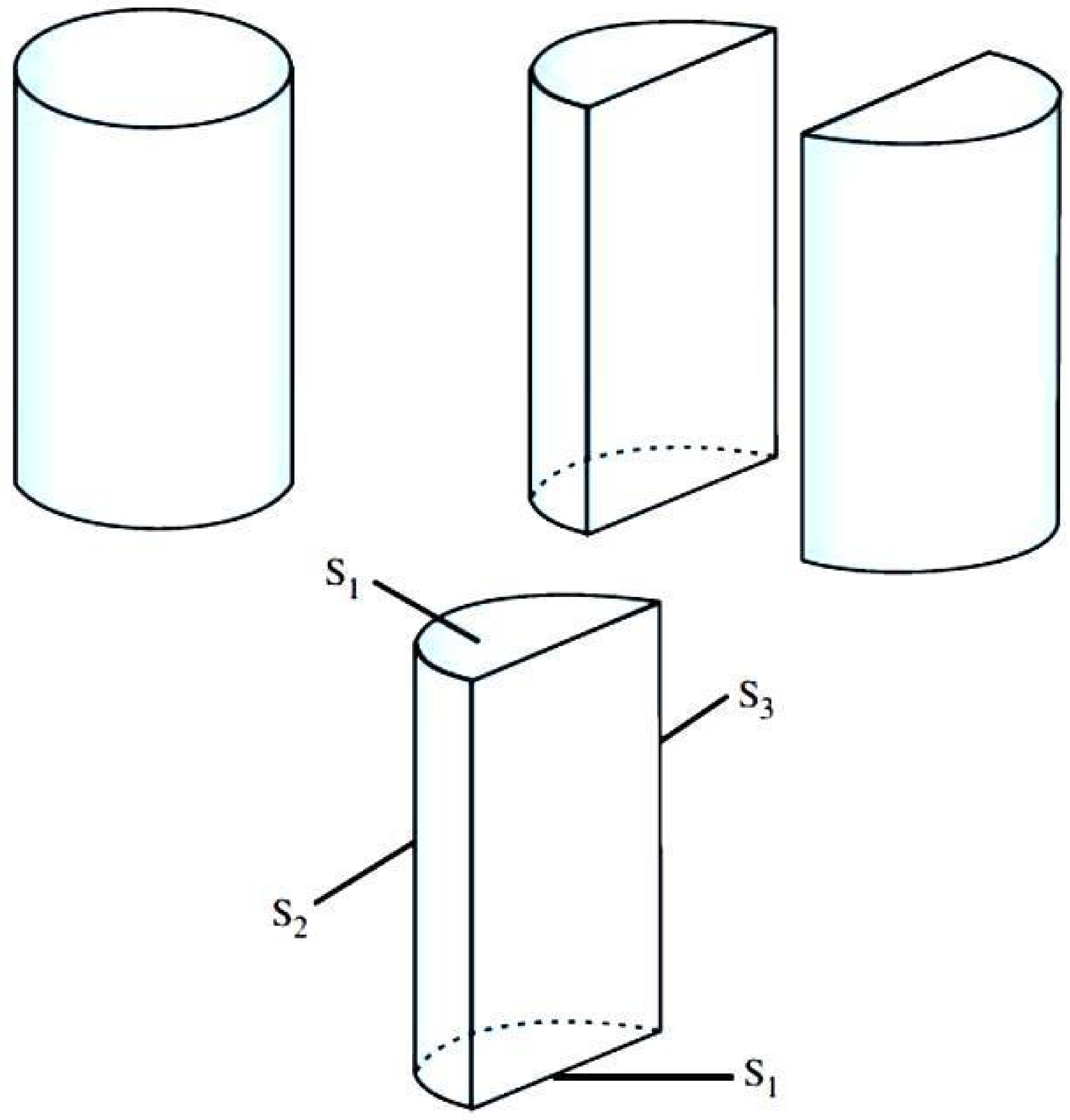


Fig.

Total surface area, S = S_1 (Surface area of semi-circular ends) + S_2 (Curved surface of half cylinder) + S_3 (Surface area of rectangular base having height h and width $2r$)

$$= \left(\frac{1}{2}\pi r^2 + \frac{1}{2}\pi r^2\right) + \frac{1}{2}(2\pi rh) + 2rh$$

$$= \pi r^2 + \pi rh + 2rh$$

$$= \pi r^2 + (\pi+2)\, r\frac{2V}{\pi r^2} \qquad [Using\ (1)]$$

$$\therefore \quad S = \pi r^2 + \frac{2V}{\pi r}(\pi+2) \qquad ...(2)$$

$$\therefore \quad \frac{dS}{dr} = 2\pi r - \frac{2V}{\pi r^2}(\pi+2).$$

For min. S, $\frac{dS}{dr} = 0$

$$\Rightarrow 2\pi r - \frac{2V}{\pi r^2}(\pi+2) = 0$$

$$\Rightarrow \quad \pi^2 r^3 = V(\pi+2) \qquad ...(3)$$

Also $$\frac{d^2S}{dr^2} = 2\pi + \frac{4V(\pi+2)}{\pi r^3}$$

$$= 2\pi + \frac{4\pi^2 r^3}{\pi r^3} \qquad [Using\ (3)]$$

$= 2\pi + 4\pi = 6\pi$, which is + ve.

$\therefore$ S is minimum when $\pi^2 r^3 = V(\pi + 2)$

$$\Rightarrow \quad \pi^2 r^3 = \frac{1}{2}\pi r^2 h(\pi+2) \qquad [Using\ (1)]$$

$$\Rightarrow \quad \frac{h}{2r} = \frac{\pi}{\pi+2}.$$

Hence, $h : 2r = \pi : (\pi + 2)$, which is true.

Example 7. A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window.

(A.I.C.B.S.E. 2011)

Solution. Let 'x' and 'y' be the length and breadth of the window.

By the question,

$$x + 2y + 2x = 12$$

$$\Rightarrow \quad 2y = 12 - 3x$$

$$\Rightarrow \quad y = 6 - \frac{3}{2}x \qquad ...(1)$$

Now area of the window

$$= xy + \frac{\sqrt{3}x^2}{4}$$

$$= x\left(6 - \frac{3}{2}x\right) + \frac{\sqrt{3}}{4}x^2 \qquad [Using\ (1)]$$

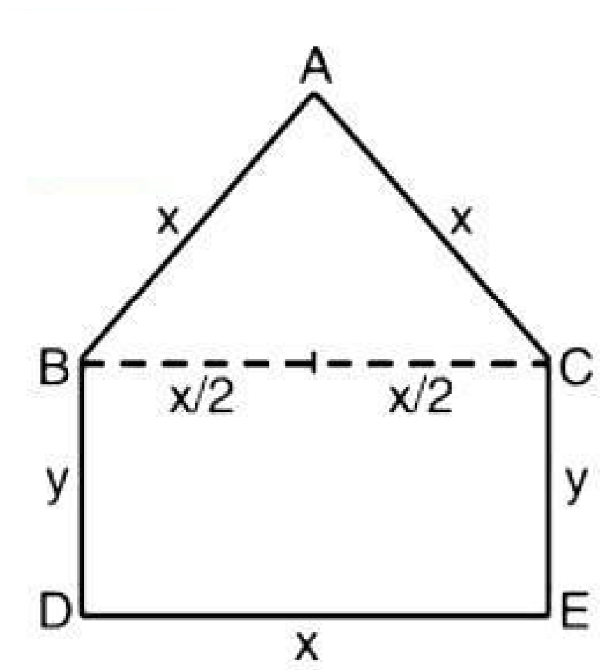


Fig.

$$= 6x - \frac{3}{2}x^2 + \frac{\sqrt{3}}{4}x^2$$

$$= 6x + \left(\frac{\sqrt{3}}{4} - \frac{3}{2}\right)x^2.$$

Thus $\quad A(x) = 6x + \frac{\sqrt{3}-6}{4}x^2.$

$\therefore \quad A'(x) = 6 + \frac{\sqrt{3}-6}{4}(2x) = 6 + \frac{\sqrt{3}-6}{2}x.$

And $\quad A''(x) = \frac{\sqrt{3}-6}{2}.$

For Max./Min. of A (x), A′ (x) = 0

$\Rightarrow 6 + \frac{\sqrt{3}-6}{2}x = 0$

$\Rightarrow \quad x = \frac{12}{6-\sqrt{3}}$

and A″ (x) is – ve for all x.

Thus area is largest when

$$x = \frac{12}{6-\sqrt{3}}$$

and consequently, $y = 6 - \frac{3}{2}\left(\frac{12}{6-\sqrt{3}}\right)$ [*Using (1)*]

$$= 6 - \frac{18}{6-\sqrt{3}} = \frac{18-6\sqrt{3}}{6-\sqrt{3}}.$$

Hence, the dimensions of rectangle are $\frac{12}{6-\sqrt{3}}$ and $\frac{18-6\sqrt{3}}{6-\sqrt{3}}$ units.

Example 8. Show that the height of the circular cylinder of maximum volume that can be inscribed in a given right-circular cone of height h is $\frac{1}{3}$h.

(Mizoram B. 2018; H.P.B. 2015)

Solution. Let ‘x’ be the radius, ‘z’, the height of the cylinder.

Then OC = OE – CE = h – z and CD = x.

If ‘θ’ be the semi-vertical angle of right-circular cone,

then $\tan\theta = \frac{CD}{OC} = \frac{x}{h-z}$

$\Rightarrow \quad x = (h-z)\tan\theta \quad ...(1)$

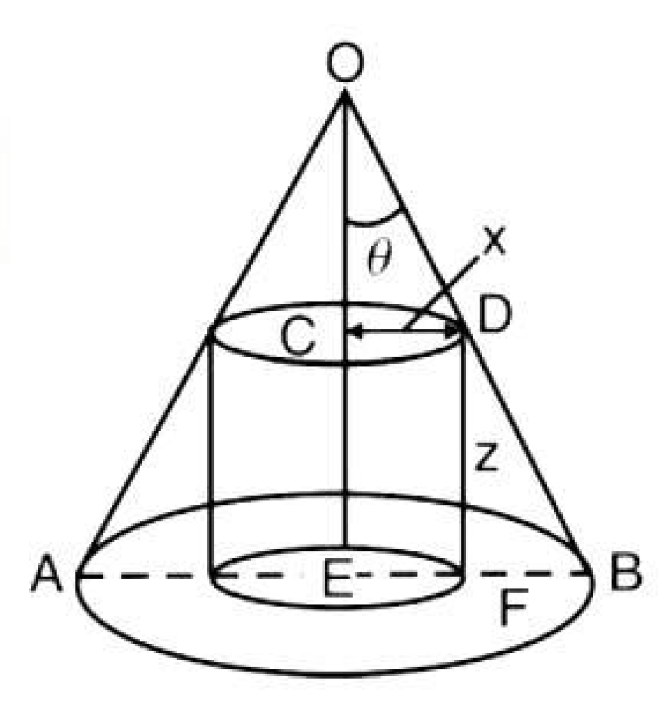


Fig.

Clearly ‘θ’ is constant. [∵ *Cone is given*]

Now V, the volume of the cylinder

$= \pi x^2 z = \pi\,[(h-z)^2\tan^2\theta\,]z$ [*Using (1)*]

$= \pi z\,.\,(h-z)^2\,.\,\tan^2\theta \quad ...(2)$

V is max. when $\frac{dV}{dz} = 0$ and $\frac{d^2V}{dz^2} < 0.$

Diff. V w.r.t z, we get :

$$\frac{dV}{dz} = \pi\,[(h-z)^2 \times 1 + z\,.\,2(h-z)(-1)]\tan^2\theta$$

$$= \pi\tan^2\theta\,[(h-z)^2 - 2z(h-z)]$$

$$= \pi\tan^2\theta\,[h^2 - 2hz + z^2 - 2hz + 2z^2]$$

$$= \pi\tan^2\theta\,[h^2 - 4hz + 3z^2] \quad ...(3)$$

$$= \pi\tan^2\theta\,(h-z)(h-3z).$$

Now $\frac{dV}{dz} = 0 \quad \Rightarrow \pi\tan^2\theta\,(h-z)(h-3z) = 0$

$\Rightarrow \quad h = z, 3z \quad \Rightarrow \quad z = h$ or $\frac{h}{3}$.

Clearly $z \neq h$ [∵ *Cylinder is inscribed in the cone*]

Thus $\quad z = \frac{h}{3} \quad ...(4)$

Diff. (3) w.r.t. z, we get : $\frac{d^2V}{dz^2} = \pi\tan^2\theta\,(0 - 4h + 6z)$

$$= \pi\tan^2\theta\left(-4h + 6\frac{h}{3}\right) \text{ [Using (4)]}$$

$$= \pi\tan^2\theta\,(-2h) < 0.$$

Hence, the volume of the inscribed cylinder is maximum when its height is $\frac{h}{3}$.

Example 9. Let AP and BQ be two vertical poles at points A and B respectively. If AP = 16 m, BQ = 22 m and AB = 20 m, then find the distance of a point R on AB from the point A such that $RP^2 + RQ^2$ is minimum.

(N.C.E.R.T.)

Solution. Let R be a point an AB such that :

AR = x m so that RB = 20 – x [∵ *AB = 20 m*]

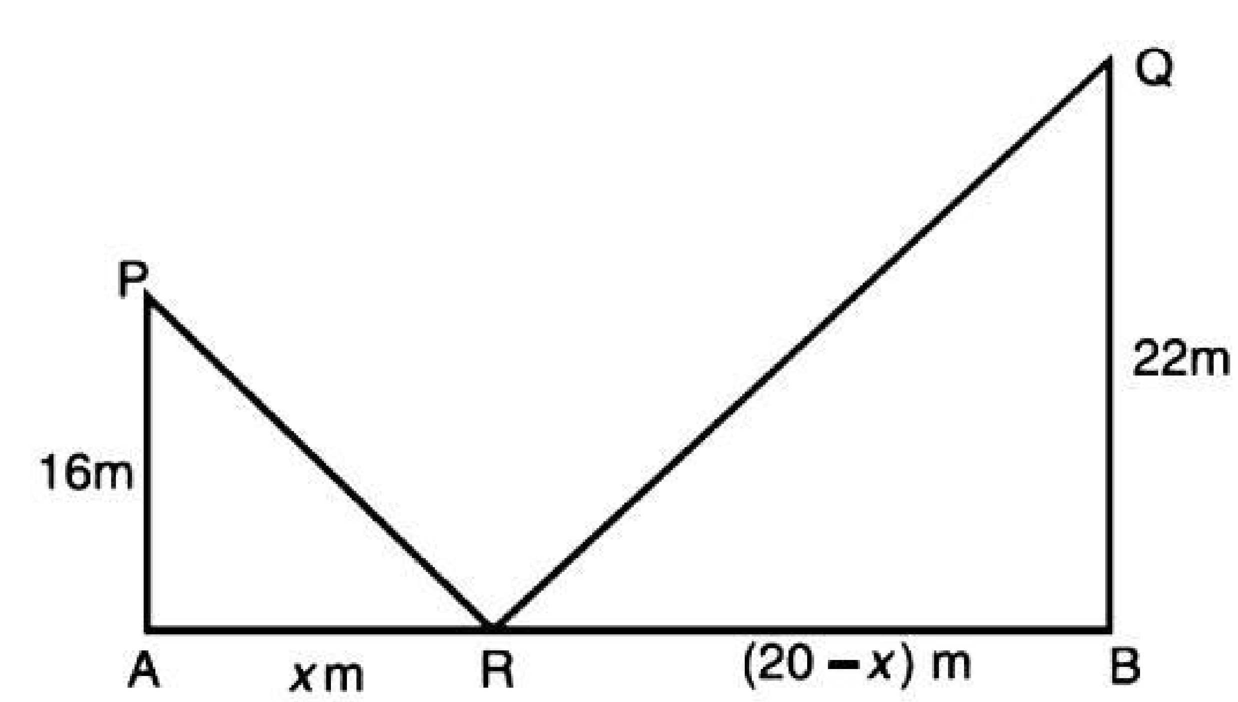


Fig.

Also AP = 16 m and BQ = 22 m.

In *rt.* ∠*d* Δ APR, $\quad RP^2 = AR^2 + AP^2 \quad ...(1)$

In rt. $\angle d.\ \Delta$ BQR, $RQ^2 = RB^2 + BQ^2$ ...(2)

Adding (1) and (2),

$$RP^2 + RQ^2 = AR^2 + AP^2 + RB^2 + BQ^2$$
$$= x^2 + (16)^2 + (20 - x)^2 + (22)^2$$
$$= x^2 + 256 + 400 + x^2 - 40x + 484$$
$$= 2x^2 - 40x + 1140.$$

Let $S(x) = RP^2 + RQ^2$ (*Say*)

$$= 2x^2 - 40x + 1140.$$

$\therefore$ $S'(x) = 4x - 40$

and $S''(x) = 4.$

Now $S'(x) = 0 \Rightarrow 4x - 40 = 0 \Rightarrow x = 10.$

Also $S''(x)\big]_{x=10} = 4 > 0.$

$\therefore$ Point of local minima of S is $x = 10$.

Hence, distance of R from A on AB = x = 10 m.

Example 10. If the lengths of three sides of a trapezium other than base are equal to 10 cm, then find the area of the trapezium when it is maximum. *(N.C.E.R.T.; A.I.C.B.S.E. 2010)*

Solution.

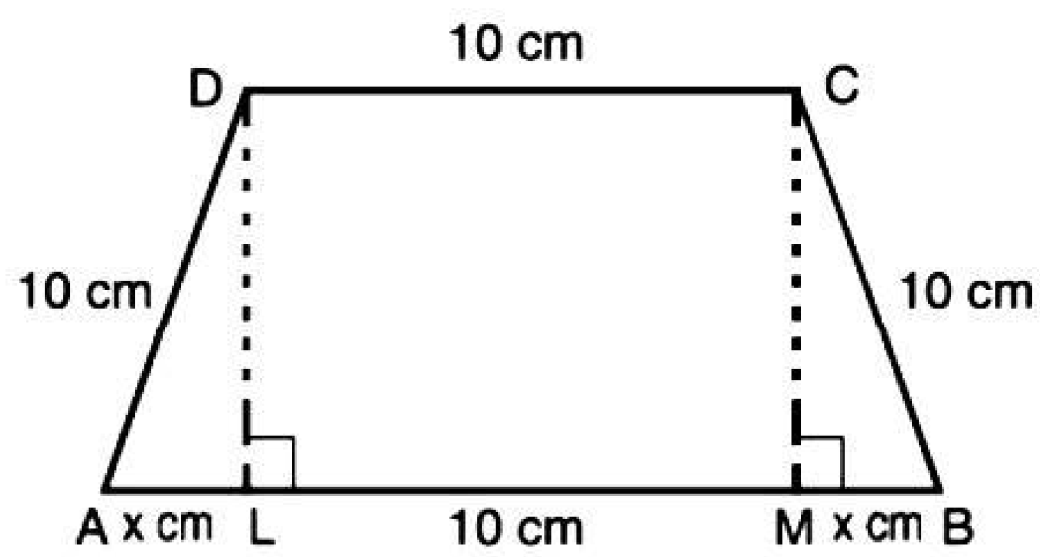


Fig.

Let ABCD be the trapezium such that :

AD = BC = CD = 10 cm.

Draw DL and CM perpendiculars on AB.

Let AL = x cm.

Now Δ ALD $\cong$ Δ BMC.

$\therefore$ MB = AL = x cm. Also LM = 10 cm.

And $DL = CM = \sqrt{100 - x^2}$.

Let 'A' be the area of the trapezium.

Then $A(x) = \frac{1}{2}(10 + (10 + 2x))\sqrt{100 - x^2}$.

[*Area of Trap.* = $\frac{1}{2}$ (*sum of* ∥ *sides*) (*height*)]

$$= \frac{1}{2}(2x + 20)\sqrt{100 - x^2}$$
$$= (x + 10)\sqrt{100 - x^2} \quad ...(1)$$

$$\therefore A'(x) = (x + 10)\frac{1}{2\sqrt{100 - x^2}}(-2x) + \sqrt{100 - x^2}\,(1)$$

$$= \frac{-x^2 - 10x + 100 - x^2}{\sqrt{100 - x^2}} = \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}}.$$

And
$$A''(x) = \frac{\sqrt{100 - x^2}\,(-4x - 10) - (-2x^2 - 10x + 100)\frac{1}{2\sqrt{100 - x^2}}(-2x)}{100 - x^2}$$

$$= \frac{(100 - x^2)(-4x - 10) + x(-2x^2 - 10x + 100)}{(100 - x^2)^{3/2}}$$

$$= \frac{2x^3 - 300x - 1000}{(100 - x^2)^{3/2}}.$$

Now $A'(x) = 0 \Rightarrow -2x^2 - 10x + 100 = 0$

$\Rightarrow x^2 + 5x - 50 = 0 \Rightarrow (x + 10)(x - 5) = 0$

$\Rightarrow x = -10, 5.$

But $x \neq -10$. [$\because$ *x, being the distance, can't be – ve*]

Thus x = 5 cm.

And
$$A''(5) = \frac{2(5)^3 - 300(5) - 1000}{(100 - 25)^{3/2}} = \frac{-2250}{75\sqrt{75}} < 0.$$

Thus A (x) is maximum when $x = 5$.

Hence, $A(5) = (5 + 10)\sqrt{100 - 25}$ [*Using (1)*]

$$= 15\sqrt{75} = 75\sqrt{3}\ \text{cm}^2.$$

Example 11. A square piece of tin of side 24 cm is to be made into a box without top by cutting a square from each corner and folding up the flaps to form a box. What should be the side of the square to be cut off so that the volume of the box is maximum ? Also, find this maximum volume. *(P.B. 2014 S ; H.B. 2014)*

Solution. Let 'x' cm. be the side of the square cut off from each corner of the square of side 24 cm.

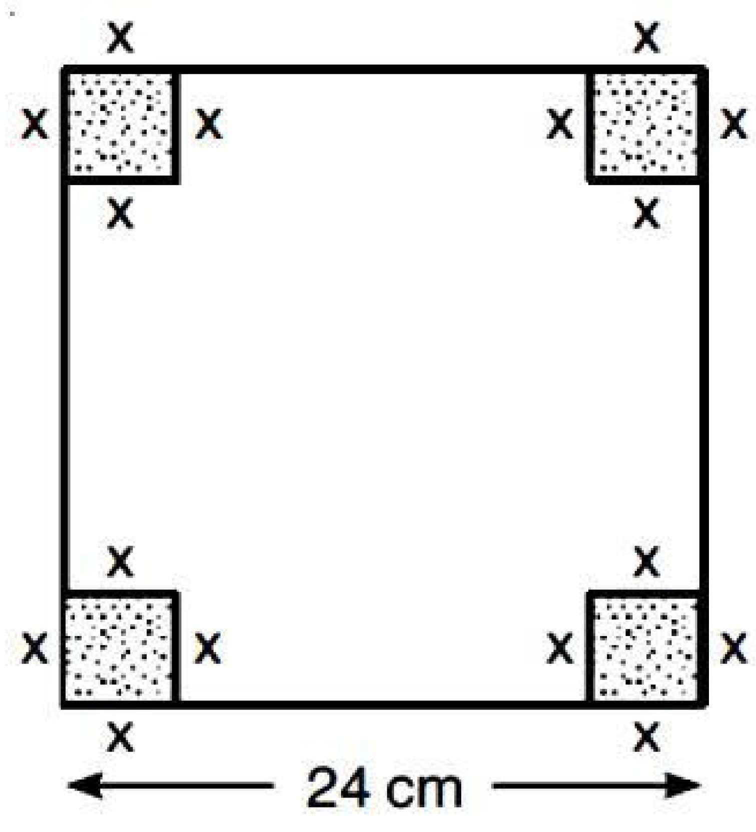


Fig.

Length of resulting box = $(24 - 2x)$ cm.

Breadth of resulting box = $(24 - 2x)$ cm.

Height of resulting box = x cm.

$\therefore$ Volume = $(24 - 2x)^2 x$ cm^3.

Let $V(x) = x(24 - 2x)^2$ ...(1)

$$\therefore V'(x) = x \cdot 2(24 - 2x)(-2) + (24 - 2x)^2(1)$$
$$= (24 - 2x)(-4x + 24 - 2x)$$
$$= (24 - 2x)(24 - 6x) \quad ...(2)$$

and $\quad V''(x) = (24-2x)(-6) + (24-6x)(-2)$

$= -144 + 12x - 48 + 12x$

$= 24x - 192 \qquad ...(3)$

For max./min., $\quad V'(x) = 0$

$\Rightarrow \quad (24-2x)(24-6x) = 0$

$\Rightarrow \quad x = 12, 4.$

But $x = 12$ is not possible.

[$\because$ *In this case, the whole tin sheet will be cut off*]

Thus $x = 4$.

Now $V''(4) = 24(4) - 192 = -96$ (– ve). [*Using (3)*]

Thus $V(x)$ is max. when $x = 4$.

Hence, the length of the side of the square cut off from each corner is 4 cm. and maximum volume,

$V = (4)(24-8)^2 = 1024 \text{ cm}^3$. [*Using (1)*]

Example 12. A metal box with a square base and vertical sides is to contain 1024 cm³. The material for the top and bottom costs ₹ 5 per cm² and the material for the sides costs ₹ 2.50 per cm². Find the least cost of the box. *(C.B.S.E. 2017)*

Solution. Let 'x' cm be the side of square base and 'y' cm the height.

$\therefore$ volume of the box $= x \times x \times y = x^2y$.

By the question, $\quad x^2y = 1024 \qquad ...(1)$

Now the cost, $\quad C = 5(x^2) + \frac{5}{2}(4xy)$

$= 5x^2 + 10xy$

$= 5x^2 + 10x\left(\frac{1024}{x^2}\right) \qquad ...(2)$

[*Using (1)*]

$$\therefore \quad \frac{dC}{dx} = 10x - \frac{10240}{x^2}.$$

$$\text{and} \quad \frac{d^2C}{dx^2} = 10 + \frac{20480}{x^3}.$$

Now for least cost, $\frac{dC}{dx} = 0$

$$\Rightarrow \quad 10x - \frac{10240}{x^2} = 0 \Rightarrow x^3 = 1024$$

$$\Rightarrow \quad x = (1024)^{1/3}.$$

$$\text{And} \quad \frac{d^2C}{dx^2} > 0 \text{ for } x = (1024)^{1/3}.$$

Hence, $\quad$ least cost $= 5(1024)^{2/3} + \frac{10240}{(1024)^{1/3}}$

$= 5(1024)^{2/3} + 10(1024)^{2/3}$

$= ₹\ 15(1024)^{2/3}$.

Example 13. Show that the height of the cylinder, open at the top, of given surface area and greatest volume is equal to the radius of its base. *(C.B.S.E. 2010)*

Solution. Let 'r' and 'h' be the radius and height respectively of the cylinder.

$\therefore \quad$ S, the surface area $= \pi r^2 + 2\pi rh$

$$\Rightarrow \quad h = \frac{S - \pi r^2}{2\pi r} \qquad ...(1)$$

And V, the volume $= \pi r^2 h$

$$i.e. \quad V = \pi r^2 \left(\frac{S - \pi r^2}{2\pi r}\right) \qquad [Using\ (1)]$$

$$\Rightarrow \quad V = \frac{r}{2}(S - \pi r^2) \Rightarrow V = \frac{1}{2}(Sr - \pi r^3).$$

$$\therefore \quad \frac{dV}{dr} = \frac{1}{2}(S - 3\pi r^2) \qquad ...(2)$$

$$\text{and} \quad \frac{d^2V}{dr^2} = -\frac{3\pi}{2}(2r) = -3\pi r \qquad ...(3)$$

For greatest volume, $\frac{dV}{dr} = 0$ and $\frac{d^2V}{dr^2}$ is – ve.

$$\text{Now} \quad \frac{dV}{dr} = 0 \quad \Rightarrow \quad \frac{1}{2}(S - 3\pi r^2) = 0$$

$$\Rightarrow \quad S = 3\pi r^2 \quad \Rightarrow \quad r = \sqrt{\frac{S}{3\pi}}.$$

Putting in (3),

$$\frac{d^2V}{dr^2} = -3\pi\sqrt{\frac{S}{3\pi}} = -\sqrt{3\pi S}, \text{ which is – ve.}$$

Thus V is greatest when $r = \sqrt{\frac{S}{3\pi}}$.

And from (1),

$$h = \frac{S - \pi\left(\frac{S}{3\pi}\right)}{2\pi\sqrt{\frac{S}{3\pi}}} = \frac{\frac{2S}{3}}{\frac{2}{\sqrt{3}}\sqrt{\pi S}} = \sqrt{\frac{S}{3\pi}}.$$

Hence, height = radius of the base.

Example 14. Find the point on the curve $y^2 = 4x$, which is nearest to the point (2, 1).

Solution. Let P (x, y) be any point on the curve $y^2 = 4x \qquad ...(1)$

Let A (2, 1) be the given point.

$\therefore \quad AP^2 = (x-2)^2 + (y-1)^2$

$$= \left(\frac{y^2}{4} - 2\right)^2 + (y-1)^2 \qquad [Using\ (1)]$$

Let $\quad D = AP^2$.

Then D is maximum or minimum according as AP is maximum or minimum.

$$\text{Now} \quad D = \left(\frac{y^2}{4} - 2\right)^2 + (y-1)^2.$$

$$\therefore \quad \frac{dD}{dy} = 2\left(\frac{y^2}{4} - 2\right)\frac{2y}{4} + 2(y-1)$$

$$= 2\left(\frac{y^2}{4} - 2\right)\frac{y}{2} + 2y - 2 = \frac{y^3}{4} - 2$$

and $\frac{d^2 D}{dy^2} = \frac{3y^2}{4}$.

For D to be maximum/minimum, $\frac{dD}{dy} = 0$

$\Rightarrow \quad \frac{y^3}{4} - 2 = 0 \quad \Rightarrow y^3 = 8 \quad \Rightarrow \quad y = 2.$

And $\left.\frac{d^2 D}{dy^2}\right]_{y=2} = \frac{3(4)}{4} = 3$, which is + ve.

Thus D is minimum when $y = 2$

and putting in (1), $4 = 4x \Rightarrow x = 1$.

Hence, the point (1, 2) on $y^2 = 4x$ is nearest to the given point (2, 1).

Example 15. An Apache helicopter of enemy is flying along the curve given by :

$$\mathbf{y = x^2 + 7.}$$

A soldier, placed at (3, 7), wants to shoot down the helicopter when it is nearest to him. Find the nearest distance. *(N.C.E.R.T.)*

Solution. Let P $(x, x^2 + 7)$ be the position of the helicopter at any instant.

Also A is (3, 7).

Now $|AP| = \sqrt{(x-3)^2 + (x^2 + 7 - 7)^2}$

$= \sqrt{(x-3)^2 + x^4}$.

Let $\quad f(x) = (x-3)^2 + x^4$.

$\therefore \quad f'(x) = 2(x-3) + 4x^3 = 4x^3 + 2x - 6$

$= 2(2x^3 + x - 3) = 2(x-1)(2x^2 + 2x + 3)$.

Now $f'(x) = 0 \quad \Rightarrow \quad x - 1 = 0$ or $2x^2 + 2x + 3 = 0$

$\Rightarrow \quad x = 1.$

[$\because$ *Roots of* $2x^2 + 2x + 3 = 0$ *are not real*]

Thus there is only one point $x = 1$.

And $\quad f(1) = (1-3)^2 + 1^4 = 4 + 1 = 5$.

$\therefore$ Distance between soldier and helicopter $= \sqrt{5}$.

Now $\sqrt{5}$ is either max. value or min. value.

Since $\sqrt{f(0)} = \sqrt{(0-3)^2 + (0)^4} = 3 > \sqrt{5}$

$\Rightarrow \sqrt{5}$ is min. value of $\sqrt{f(x)}$.

Hence, $\sqrt{5}$ is the reqd. minimum distance between the soldier and the helicopter.

Example 16. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. *(C.B.S.E. 2018)*

Solution. Let x, x and y units be the length, breadth and height respectively of the tank.

$\therefore$ Volume, $\quad V = (x)(x)(y) = x^2y \quad$ ...(1)

Total surface area $= x^2 + 4xy$.

Thus, $\quad S = x^2 + 4x\left(\frac{V}{x^2}\right) \quad$ [*Using (1)*]

$\Rightarrow \quad S = x^2 + \frac{4V}{x}$.

$\therefore \quad \frac{dS}{dx} = 2x - \frac{4V}{x^2} \quad$ ...(2)

and $\quad \frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3} \quad$ ...(3)

For least cost, $\frac{dS}{dx} = 0$

$\Rightarrow \quad 2x - \frac{4V}{x^2} = 0$

$\Rightarrow \quad 2x^3 = 4V$

$\Rightarrow \quad 2x^3 = 4x^2y \quad$ [*Using (1)*]

$\Rightarrow \quad x = 2y. \quad$ [$\because$ *x* $\neq$ *0*]

Also, $\quad \frac{d^2S}{dx^2} = 2 + \frac{8V}{8y^3} = 2 + \frac{V}{y^3} > 0.$

Hence, S is minimum when $x = 2y$

i.e. when depth (height) of the tank is half of the width.

Example 17. A cuboidal shaped godown with square base is to be constructed. Three times as much cost per square metre is incurred for constructing the roof as compared to the walls. Find the dimensions of the godown if it is to enclose a given volume and minimize the cost of constructing the roof and the walls.

(C.B.S.E. Sampe Paper 2019)

Solution. Let 'x' be the length and breadth of the base and 'y', the height of the godown.

Let 'V' be the given volume and 'C' the cost of constructing the godown.

Since cost is proportional to the area,

$\therefore \quad C = k(3x^2 + 4xy) \quad$ ...(1),

where $k > 0$ is the constant of proportionality

and $\quad x^2y = V$ (constant) $\quad$ ...(2)

From (2), $y = \frac{V}{x^2} \quad$ ...(3)

Putting in (1),

$$C = k\left(3x^2 + 4x.\frac{V}{x^2}\right)$$

$$\Rightarrow \quad C = k\left(3x^2 + \frac{4V}{x}\right)$$

$\therefore \quad \frac{dC}{dx} = k\left(6x - \frac{4V}{x^2}\right) \quad$ ...(4)

$\frac{d^2C}{dx^2} = k\left(6 + \frac{8V}{x^3}\right) \quad$ ...(5)

For minimum cost, $\frac{dC}{dx}=0$

$$\Rightarrow 6x-\frac{4V}{x^2} = 0 \Rightarrow 6x^3 = 4V \Rightarrow x = \left(\frac{2V}{3}\right)^{1/3}.$$

Putting in (5), $\frac{d^2C}{dx^2} = k\left(6+8V.\frac{3}{2V}\right) = k(18) > 0.$

Hence, minimum cost when

$$x = \left(\frac{2V}{3}\right)^{1/3} \text{ and } y=\frac{(18V)^{1/3}}{2}.$$

Example 18. A manufacturer can sell 'x' items at a price of ₹ $\left(5-\frac{x}{100}\right)$ each. The cost price of 'x' items is ₹ $\left(\frac{x}{5}+500\right)$. Find the number of items he should sell to earn maximum profit.

(N.C.E.R.T.; A.I.C.B.S.E. 2009)

Solution. Here C (x), cost price = ₹ $\left(\frac{x}{5}+500\right)$

and S(x), selling price = $\left(5-\frac{x}{100}\right)x$ = ₹ $\left(5x-\frac{x^2}{100}\right)$.

$\therefore$ P(x), profit = S(x) – C (x)

$$= \left(5x-\frac{x^2}{100}\right)-\left(\frac{x}{5}+500\right)$$

$$= \frac{24}{5}x-\frac{x^2}{100}-500.$$

$\therefore$ $P'(x) = \frac{24}{5}-\frac{x}{50}$ and $P''(x) = -\frac{1}{50}$.

Now $P'(x) = 0 \Rightarrow \frac{24}{5}-\frac{x}{50}=0 \Rightarrow x = 240.$

And $P''(240) = -\frac{1}{50}<0$.

Thus $x = 240$ is the point of maxima.

Hence, the manufacturer should sell 240 items in order to earn maximum profit.

Example 19. A person wants to plant some trees in his community park. The local nursery has to perform this task. It charges the cost of planting trees by the formula :

$$C(x) = x^3 - 45x^2 + 600x,$$

where 'x' is the number of trees and C(x) is cost of planting 'x' trees in rupees. The local authority has imposed a restriction that it can plant 10 to 20 trees in one community park for a fair-distribution. For how many trees should the person place the order so that he has to spend the least amount ? How much is the least amount ? Use Calculus to answer these questions. *(C.B.S.E. Sample Paper 2018)*

Solution. We have : $C(x) = x^3 - 45x^2 + 600x$ ...(1),

where $10 \le x \le 20$.

Diff. w.r.t. x, $C'(x) = 3x^2 - 90x + 600$ ...(2)

Again diff. w.r.t. x, $C''(x) = 6x - 90$ ...(3)

Now, $C'(x) = 0 \Rightarrow 3x^2 - 90x + 600 = 0$

$\Rightarrow x^2 - 30x + 200 = 0$

$\Rightarrow (x-10)(x-20) = 0$

$\Rightarrow x = 10, 20.$

And, $C''(10)= 6(10) - 90 = -30$ (–ve)

$C''(20)= 6(20) - 90 = 30$ (+ve).

Thus the amount is least when **20** plants are placed on order.

The least amount to be spent

$= (20)^3 - 45(20)^2 + 600(20)$

$= 8000 - 18000 + 12000$

$= 20000 - 18000 =$ ₹ 2000.

EXERCISE 6 (f)

Short Answer Type Questions

SATQ

1. Find two positive numbers whose sum is (*i*) 14 (*ii*) 16 and product is maximum. *(H.B. 2016)*

2. (*a*) Amongst all pairs of positive numbers with product (*i*) 256 (*ii*) 64, find those whose sum is least.

(*b*) Find two numbers whose sum is 15 and the square of one multiplied by the cube of the other is maximum.

3. Find two positive numbers whose sum is 15 and the sum of whose squares is minimum. *(N.C.E.R.T.)*

4. Find two positive numbers 'x' and 'y' such that their sum is 35 and the product x^2y^5 is maximum.

(N.C.E.R.T.; H.P.B. 2017, 13 S)

5. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

(N.C.E.R.T. ; Kerala B. 2017; H.P.B. 2017 S, 13 ; Assam B. 2013)

6. Find two positive numbers 'x' and 'y' such that : $x + y = 60$ and xy^3 is maximum.

(N.C.E.R.T.; H.P.B. 2017, 13, 10S ; Karnataka B. 2017)

7. How should we choose two numbers, each greater than or equal to – 2 whose sum is $\frac{1}{2}$ so that the sum of square of the first and cube of the second is minimum ?

8. Find the maximum slope of the curve : $y=-x^3+3x^2+2x-27$.

9. Two sides of a triangle are a and b. Find the angle between them such that area shall be maximum. (P.B. 2017)

10. A wire of 36 m length is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What could be the lengths of the two pieces so that the combined area of the square and the circle is minimum ? (P.B. 2017, 10)

11. A wire of length 36 cm is cut into two pieces. One of the pieces is to be made into a square and the other into an equilateral triangle. Find the length of each piece so that the sum of the areas of the square and the triangle is minimum. (Meghalaya B. 2018)

12. (*a*) Prove that the perimeter of a right-angled triangle of given hypotenuse equal to 5 cm is maximum when the triangle is isosceles.

(*b*) Prove that the area of right-angled triangle of given hypotenuse is maximum when the triangle is isosceles.

13. Prove that the least perimeter of an isosceles triangle in which a circle of radius 'r' can be inscribed is $6r\sqrt{3}$.

14. Show that, of all the rectangles with a given area, the square has the smallest perimeter.(P.B. 2015 ; C.B.S.E. 2011)

15. (*i*) Show that the rectangle of maximum area that can be inscribed in a circle of radius 'r' is a square of side $\sqrt{2}r$. (H.B. 2016, 10; Uttarakhand B. 2015)

(*ii*) Show that the rectangle of maximum area that can be inscribed in a circle is a square.

16. Show that, of all rectangles inscribed in a given fixed circle, the square has the maximum area. (N.C.E.R.T. ; P.B. 2015 ; Nagaland B. 2015 ; H.P.B. 2012 ; C.B.S.E. 2011)

17. A rectangle is inscribed in a semi-circle of radius 'r' with one of its sides on diameter of semi-circle. Find the dimensions of the rectangle so that its area is maximum. Find also the area. (Nagaland B. 2016)

18. Of all rectangles, each of which has perimeter :
(*i*) 40 cm (*ii*) 60 cm (P.B. 2010)
Find the one having maximum area. Also find that area.

19. An open box with a square base is to be made out of a given quantity of card board of area c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units. (A.I.C.B.S.E. 2012)

20. Show that the semi-vertical angle of the right-circular cone of maximum volume and of given slant height is

(*i*) $\tan^{-1}\sqrt{2}$ (N.C.E.R.T. ; Mizoram B. 2015 ; Nagaland B. 2015 ; Uttarakhand B. 2013)

(*ii*) $\cos^{-1}\frac{1}{\sqrt{3}}$. (A.I.C.B.S.E. 2016; C.B.S.E. 2014)

21. Prove that the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. (C.B.S.E. 2014)

Long Answer Type Questions

LATQ

22. Show that the volume of the greatest cylinder, which can be inscribed in a cone of height 'h' and semi-vertical angle 'θ' is $\frac{4}{27}\pi h^3 \tan^2\theta$. (N.C.E.R.T. ; H.P.B. 2015)

23. Show that the volume of the greatest cylinder, which can be inscribed in a cone of height 'h' and semi-vertical angle 30° is $\frac{4}{81}\pi h^3$.

24. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius 'r' is $4r/3$. Also show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.
(N.C.E.R.T. ; P.B. 2018; Meghalaya B. 2017, 14; C.B.S.E. 2016, 10 C; H.P.B. 2016, 15; A.I.C.B.S.E. 2014, 10)

25. (*i*) Find the volume of the largest cone that can be inscribed in a sphere of radius 'r'.

(*ii*) Find the volume of the largest cylinder that can be inscribed in a sphere of radius 'r'. (W. Bengal B. 2016; C.B.S.E. 2009)

26. Show that the right-circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base. (N.C.E.R.T.; H.P.B. 2016; A.I.C.B.S.E. 2011)

27. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius 'R' is $\frac{2R}{\sqrt{3}}$. Also find maximum volume.
(N.C.E.R.T. ; Mizoram B. 2017; Meghalaya B. 2016; Nagaland B. 2016; H.P.B. 2015 ; A.I.C.B.S.E. 2014; C.B.S.E. 2013)

28. (*a*) Show that the radius of right-circular cylinder of maximum volume, that can be inscribed in a sphere of radius 18 cm, is $6\sqrt{6}$ cm. (Type : P.B. 2017; P.B. 2016)

(*b*) Prove that the radius of the right-circular cylinder of greatest curved surface, which can be inscribed in a given cone, is half of that of the cone.
(N.C.E.R.T.; A.I.C.B.S.E. 2012 ; C.B.S.E. 2010 C)

29. Of all the closed cylindrical cans (right-circular), which enclose a given volume of :
(*i*) 100 cubic centimeters (N.C.E.R.T.; H.P.B. 2012)
(*ii*) 128 π cubic centimeters, (C.B.S.E. 2014)
find the dimensions of the can, which has the minimum surface area. (N.C.E.R.T ; H.P.B. 2012, 09)

30. Show that surface area of a closed cuboid with square base and given volume is minimum, when it is a cube. (A.I.C.B.S.E. 2017)

31. (*a*) A figure consists of a semi-circle with a rectangle on its diameter. Given perimeter of the figure, find the dimensions in order that the area may be maximum.

(*b*) (*i*) A window is in the form of a rectangle surrounded by a semi-circular opening. The total perimeter of the window is 10 metres. Find the dimensions of the window so as to admit maximum light through the whole opening.

(*Mizoram B. 2018; A.I.C.B.S.E. 2017*)

(*ii*) A window is in the form of a rectangle, surmounted by a semi-circle. If the perimeter be 30 metres, find the dimensions so that the greatest possible amount of light may be admitted in order that its area may be maximum.

(*P.B. 2018*)

32. Show that a cylinder of given volume, open at the top, has minimum total surface area, if its height is equal to radius of its base. (*C.B.S.E. 2009 C*)

33. Show that the height of a cylinder, which is open at the top, having a given surface and the greatest volume, is equal to the radius of its base.

34. Show that the height of a closed right-circular cylinder of given volume and least surface area is equal to its diameter.

35. Given the sum of the perimeters of a square and a circle, show that the sum of their areas is least when the side of the square is equal to radius of the circle.

36. (*i*) An expensive square piece of golden color board of side 24 centimetres is to be made into a box without top by cutting a square from each corner and folding the flaps to form a box. What should be the side of the square piece to be cut from each corner of the board to hold maximum volume and minimize the wastage ?

(*ii*) A square tank of capacity 250 cubic metres has to be dug out. The cost of the land is ₹ 50 per square metre. The cost of digging increases with the depth and for the whole tank is ₹ $400 \times h^2$, where '*h*' metres is the depth of the tank. What should be the dimensions of the tank so that the cost be minimum ?

37. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per sq. metre for the base and ₹ 45 per sq. metre for sides, what is the cost of least expensive tank ?

(*N.C.E.R.T.; C.B.S.E. 2009*)

38. A rectangular sheet of tin 45 cm. by 24 cms. is to be made into a box without top by cutting off squares from the corners and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

(*N.C.E.R.T. ; H.P.B. 2015*)

39. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of the material will be least when the depth of the tank is half of its width.

40. A canon is fired at an angle $\theta\left(0 \le \theta \le \frac{\pi}{2}\right)$ with the horizontal. If '*v*' is the initial velocity of the canon ball, the height '*h*' of the ball at time '*t*', ignoring wind resistance, is given by $h = (v \sin \theta)\, t - 4{\cdot}9\, t^2$.

(*a*) When will the ball return to the ground ?

(*b*) How far will the ball have travelled horizontally at the time it hits the ground, assuming there are no forces in the horizontal direction ?

(*c*) Determine 'θ' so that the horizontal range of the ball is maximum.

41. Find the maximum profit that a company can make, if the profit function is given by :

(*i*) $P(x) = 41 - 24x - 18x^2$

(*N.C.E.R.T.; H.P.B. 2013, 10*)

(*ii*) $P(x) = 41 - 72x - 18x^2$

(*Kashmir B. 2016; Jammu B. 2015*)

(*iii*) $P(x) = 41 - 24x - 6x^2$. (*Kerala B. 2014*)

42. Find the point on the curve $y^2 = 4x$, which is nearest to the point (2, – 8). (*P.B. 2013*)

43. (*i*) Find a point on the curve $y^2 = 2x$, which is at a minimum distance from the point (1, 4).

(*A.I.C.B.S.E. 2009 C*)

(*ii*) Find the point on the curve $y^2 = 2x$, which is nearest to the point (1, – 4). c

(*iii*) Find the point on the parabola $x^2 = 8y$, which is nearest to the point (2, 4).

44. A helicopter is flying along the curve $y = x^2 + 2$. A soldier is placed at the point (3, 2). Find the nearest distance between the soldier and the helicopter.

45. A manufacturer can sell '*x*' items at a price of ₹ $(250 - x)$ each. The cost of producing '*x*' items is ₹ $(x^2 - 50x + 12)$. Determine the number of items to be sold so that he can make maximum profit.

46. (*i*) A factory can sell '*x*' items per week at price of ₹ $\left(20 - \frac{x}{1000}\right)$ each. If the cost price of one item is ₹ $\left(5 + \frac{2000}{x}\right)$, find the number of items, the factory should produce every week for maximum profit. If price is reduced, how it will effect the sale ? Give reasons.

(*ii*) Profit function of a company is given as :

$$P(x) = \frac{24x}{5} - \frac{x^2}{100} - 500,$$

where 'x' is the number of units produced.

What is the maximum profit of the company ?

47. Let 'p' be the price per unit of a certain product, when there is a sale of 'x' units. The total revenue function is :

$$R(x) = \frac{100x}{3x+1} - 4x.$$

(*i*) Find the marginal revenue function, rate of change of total revenue function with respect to x.

(*ii*) When $x = 10$, find the relative change of revenue R, *i.e.*, Rate of change of R with respect to x and also the percentage rate of change of R at $x = 10$.

48. If performance of the students 'y' depends on the number of hours 'x' given by the relation :

$$y = 4x - \frac{x^2}{2}.$$

Find the number of hours, the students work to have the best performence.

Answers

1. (*i*) 7, 7 (*ii*) 8, 8.

2. (*a*) (*i*) 16, 16 (*ii*) 8, 8 (*b*) 6, 9.

3. 7·5, 7·5.

4. 25, 10. **5.** 8, 8. **6.** 15, 45.

7. $\frac{1}{6}, \frac{1}{3}$. **8.** 5. **9.** 90°

10. $\frac{144}{4+\pi}$ m and $\frac{36\pi}{4+\pi}$ m.

11. $\frac{144\sqrt{3}}{9+4\sqrt{3}}$ cm, $\frac{324}{9+4\sqrt{3}}$ cm.

17. $\frac{r}{\sqrt{2}}, \sqrt{2}\,r$; Area $= r^2$.

18. (*i*) 100 sq. cm. (*ii*) 225 sq. cm.

25. (*i*) $\frac{32\pi r^3}{81}$ (*ii*) $\frac{4\pi r^3}{3\sqrt{3}}$. **27.** $\frac{4\pi R^3}{3\sqrt{3}}$ cube units.

29. (*i*) Radius$=\left(\frac{50}{\pi}\right)^{1/3}$ cm. ; Height $= \frac{100}{\pi}\left(\frac{\pi}{50}\right)^{2/3}$ cm

(*ii*) Radius = 4 cm ; Height = 8 cm.

31. (*a*) Length$=\frac{2p}{4+\pi}$; Breadth$=\frac{p}{4+\pi}$; Radius $= \frac{p}{4+\pi}$,

where 'p' is the perimeter.

(*b*) (*i*) Length $= \frac{20}{4+\pi}$; Breadth $= \frac{10}{4+\pi}$; Radius $= \frac{10}{4+\pi}$

(*ii*) Length $= \frac{60}{4+\pi}$; Breadth $= \frac{30}{4+\pi}$; Radius $= \frac{30}{4+\pi}$.

36. (*i*) 4 cm ;

(*ii*) Square base of side 10 m and depth 2·5 m.

37. ₹ 1,000.

38. 5 cm.

40. (*a*) $\frac{v\sin\theta}{4\cdot 9}$ units of time

(*b*) $\frac{v^2\sin\theta\cos\theta}{4\cdot 9}$ (*c*) $\theta = \frac{\pi}{4}$.

41. (*i*) 49 (*ii*) 113 (*iii*) 65. **42.** (4, – 4).

43. (*i*) (2, 2) (*ii*) (2, – 2) (*iii*) (4, 2).

44. $\sqrt{5}$ units.

45. 75 items.

46. (*i*) $x = 7500$. On reduction of price, the sale will increase but the profit will decrease.

(*ii*) Max. profit = ₹ 76 when $x = 240$

47. (*i*) $R(x) = \frac{100x}{3x+1} - 4x$; $MR = \frac{100}{(3x+1)^2} - 4$.

(*ii*) Relative change of R $= \left.\frac{dR/dx}{R}\right]_{x=10} = 0\cdot 503$.

Percentage rate of change at $x = 10$

$$= \frac{dR/dx}{R} \times 100 = 50\cdot 3\%.$$

48. 4 hours per day.

Hints to Selected Questions

3. Let 'x' and 'y' be two numbers so that $x + y + 15$ *i.e.*, $y = 15 - x$ ...(1)

Let $S = x^2 + y^2 \Rightarrow S = x^2 + (15 - x)^2$ ...(2) [*Using (1)*]

$\therefore \quad \frac{dS}{dx} = 2x + 2(15 - x)(-1) = 2x - 30 + 2x = 4x - 30.$

For S to be minimum, $\frac{dS}{dx} = 0$, which gives :

$4x - 30 = 0 \Rightarrow x = \frac{30}{4} = \frac{15}{4} = 7.5.$

Now $\frac{d^2S}{dx^2} = 4$, which is +ve.

Thus S is minimum when $x = 7{\cdot}5$

Hence, the numbers are 7·5 and 15 – 7·5

i.e., 7·5 and 7·5.

12. (*a*) Here 5 cm is the hypotenuse and let x, y the other two sides of the right-angled triangle.

$\therefore \quad \sqrt{x^2 + y^2} = 5 \Rightarrow x^2 + y^2 = 25 \Rightarrow y = \sqrt{25^2 - x^2}.$

Now, perimeter $P = 5 + x + y = 5 + x + \sqrt{25 - x^2}$.

$\therefore \frac{dP}{dx} = 0 \Rightarrow 1 - \frac{x}{\sqrt{25 - x^2}} = 0 \Rightarrow 1 = \frac{x}{\sqrt{25 - x^2}}$

$\Rightarrow \quad x = \frac{5}{\sqrt{2}}.$

Also show that $\left.\frac{d^2P}{dx^2}\right]_{x = \frac{5}{\sqrt{2}}}$ is – ve.

Hence, the perimeter is maximum when $x = \frac{5}{\sqrt{2}}$

and then $y = \sqrt{25 - \frac{25}{2}} = \sqrt{\frac{25}{2}} = \frac{5}{\sqrt{2}}$

i.e. when $x = y$ *i.e.* when the triangle is isosceles.

13. Let 'θ' be the semi-vertical angle of the triangle.

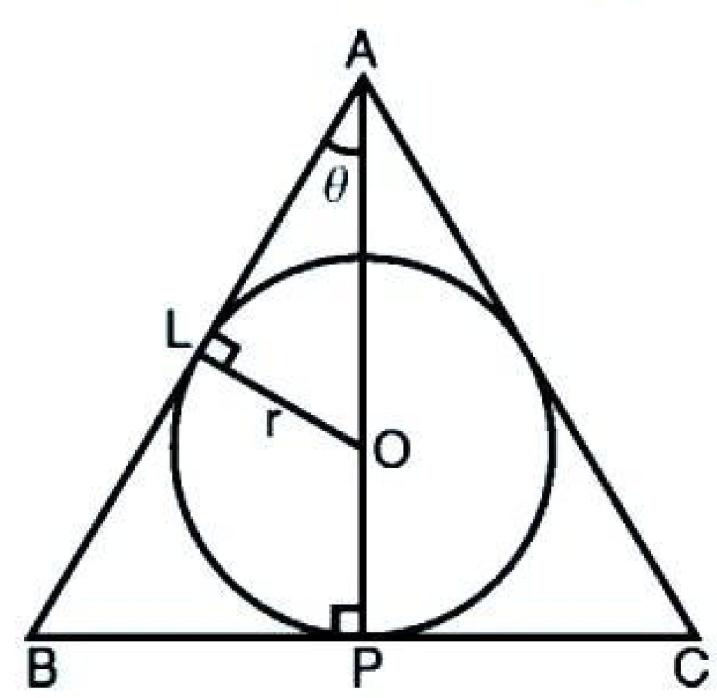


Fig.

Then $AL = r \cot \theta$, $OA = r \operatorname{cosec} \theta$.

$\therefore \quad AP = AO + OP = r \operatorname{cosec} \theta + r = r + r \operatorname{cosec} \theta.$

Now $\quad BP = AP \tan \theta = (r + r \operatorname{cosec} \theta) \tan \theta$

$= r(\tan \theta + \sec \theta).$

The perimeter, $2s$, of the triangle

$= 2AL + 4PB = 2r \cot \theta + 4r(\tan \theta + \sec \theta)$

$\therefore \quad s = r(\cot \theta + 2 \tan \theta + 2 \sec \theta)$ ...(1)

$\therefore \quad \frac{ds}{d\theta} = r(-\operatorname{cosec}^2\theta + 2\sec^2\theta + 2\sec\theta\tan\theta)$ ...(2)

and $\frac{d^2s}{d\theta^2} = r(2\operatorname{cosec}^2\theta\cot\theta + 4\sec^2\theta\tan\theta + 2\sec^3\theta + 2\sec\theta\tan^2\theta)$ ...(3)

Now $\frac{ds}{d\theta} = 0$ gives : $-\operatorname{cosec}^2\theta + 2\sec^2\theta + 2\sec\theta\tan\theta = 0$

$\Rightarrow \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}.$ **[Do it]**

Putting in (3),

$\frac{d^2s}{d\theta^2} = r\,[2\operatorname{cosec}^2 30° \cot 30° + 4\sec^2 30° \tan 30° + 2\sec^3 30° + 2\sec 30° \tan^2 30°]$

$= r\left[2(4)(\sqrt{3}) + 4\left(\frac{4}{3}\right)\left(\frac{1}{\sqrt{3}}\right) + 2\left(\frac{8}{3\sqrt{3}}\right) + 2\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{3}\right)\right]$, which is +ve.

$\therefore \; s$ is least when $\theta = \frac{\pi}{6}$.

Hence, the least perimeter

$= 2r\left[\cot\frac{\pi}{6} + 2\tan\frac{\pi}{6} + 2\sec\frac{\pi}{6}\right] \quad \left[\textit{Putting } \theta = \frac{\pi}{6}\right]$

$= 2r\left[\sqrt{3} + 2.\frac{1}{\sqrt{3}} + 2.\frac{2}{\sqrt{3}}\right]$

$= 2r\left[\frac{9}{\sqrt{3}}\right] = \frac{18r}{\sqrt{3}} = 6r\sqrt{3}.$

14. Let 'x' and 'y' be the sides of the rectangle.

By the question, $xy = A \Rightarrow y = \frac{A}{x}$, where A is given.

Now the perimeter,

$$p = 2x + 2y = 2(x + y) = 2\left(x+\frac{A}{x}\right).$$

$$\therefore \frac{dp}{dx} = 2\left(1-\frac{A}{x^2}\right) \text{ and } \frac{d^2p}{dx^2}=\frac{4A}{x^3}.$$

For p to be smallest, $\frac{dp}{dx} = 0$ and $\frac{d^2p}{dx^2}$ is +ve.

Now $\frac{dp}{dx} = 0 \Rightarrow 2\left(1-\frac{A}{x^2}\right) = 0$

$$\Rightarrow 1-\frac{A}{x^2} = 0 \quad \Rightarrow x = \sqrt{A} \quad ...(1)$$

and $\left.\frac{d^2p}{dx^2}\right]_{x=\sqrt{A}} = \frac{4A}{A\sqrt{A}}=\frac{4}{\sqrt{A}}$, which is +ve.

Hence, P is least when $x = \sqrt{A}$

and then $y = \frac{A}{\sqrt{A}}=\sqrt{A}$

i.e., when the rectangle becomes a square. [$\because x = y$]

19. Let 'x' be the side of the square base and 'y' the height of the box.

$\therefore$ Area of the square base $= x^2$.

Area of four walls $= 4xy$.

By the question, $x^2 + 4xy = c^2$ ...(1)

Now V, volume of the box = (area of the box) × height

$$= x^2 y = x^2.\frac{c^2-x^2}{4x}=\frac{1}{4}(c^2x-x^3) \quad ...(2)$$

$$\therefore \frac{dV}{dx} = \frac{1}{4}(c^2-3x^2) \quad ...(3)$$

$$\text{and } \frac{d^2V}{dx^2} = -\frac{3}{2}x \quad ...(4)$$

Now $\frac{dV}{dx} = 0$ gives : $\frac{1}{4}(c^2-3x^2) = 0$

$$\Rightarrow x^2 = \frac{c^2}{3} \Rightarrow x = \pm\frac{c}{\sqrt{3}}.$$

$\therefore$ V can be max. only for these value of x.

When $x = \frac{c}{\sqrt{3}}$, from (4), $\frac{d^2V}{dx^2}=-\frac{3}{2}\left(\frac{c}{\sqrt{2}}\right) = -$ve

When $x = \frac{c}{\sqrt{3}}$, from (4), $\frac{d^2V}{dx^2}=-\frac{3}{2}\left(-\frac{c}{\sqrt{2}}\right) = +$ve.

Thus, $x = \frac{c}{\sqrt{3}}$ gives the max. value of V.

Hence, max. V $= \frac{1}{4}\left(c^2\frac{c}{\sqrt{3}}-\frac{c^3}{3\sqrt{3}}\right)$

$\left[\textit{Putting } x = \frac{c}{\sqrt{3}} \textit{ in}(2)\right]$

$$= \frac{1}{4}\left(\frac{2c^3}{3\sqrt{3}}\right) = \frac{c^3}{6\sqrt{3}}.$$

20. (*i*) Let 'θ' be the semi-vertical angle.

r, radius of the base $= l \sin\theta$

and h, height of the cone $= l\cos\theta$.

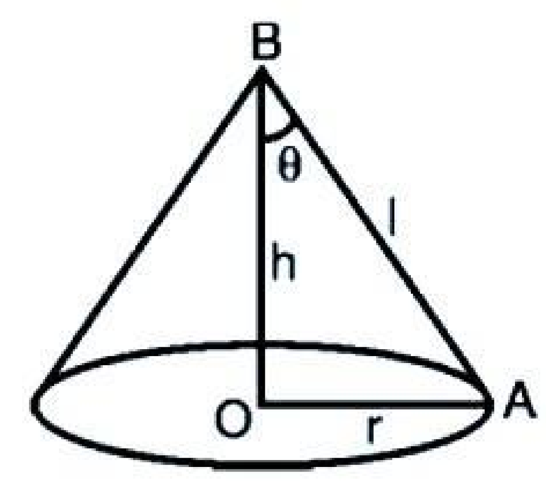


Fig.

Now V, volume of the cone

$$= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(l\sin\theta)^2(l\cos\theta)$$

$$= \frac{1}{3}\pi l^3 \sin^2\theta\cos\theta = k\sin^2\theta\cos\theta \quad ...(1),$$

where $k = \frac{1}{3}\pi l^3$.

$$\therefore \frac{dV}{d\theta} = k(2\sin\theta\cos^2\theta - \sin^3\theta)$$

$$= k\sin\theta\cos^2\theta(2-\tan^2\theta)$$

$$= k\sin\theta\cos^2\theta(\sqrt{2}+\tan\theta)(\sqrt{2}-\tan\theta) \quad ...(2)$$

Now $\frac{dV}{d\theta} = 0$ gives : $\tan\theta = \sqrt{2}$ as the only admissible value. [$\because$ *Other values of* θ *are 0,* $\frac{\pi}{2}$ *and obtuse*]

$\therefore \theta = \tan^{-1}\sqrt{2}$; etc.

Hence, the semi-vertical angle $= \tan^{-1}\sqrt{2}$.

22. First of all do **Example 9.**

Radius of the cylinder = CD

$$= OC\tan\theta = \left(h-\frac{h}{3}\right)\tan\theta = \frac{2}{3}h\tan\theta.$$

$\therefore$ Volume of the cylinder

$$= \pi\left(\frac{2}{3}h\tan\theta\right)^2\left(\frac{h}{3}\right) = \frac{4}{27}\pi h^3\tan^2\theta.$$

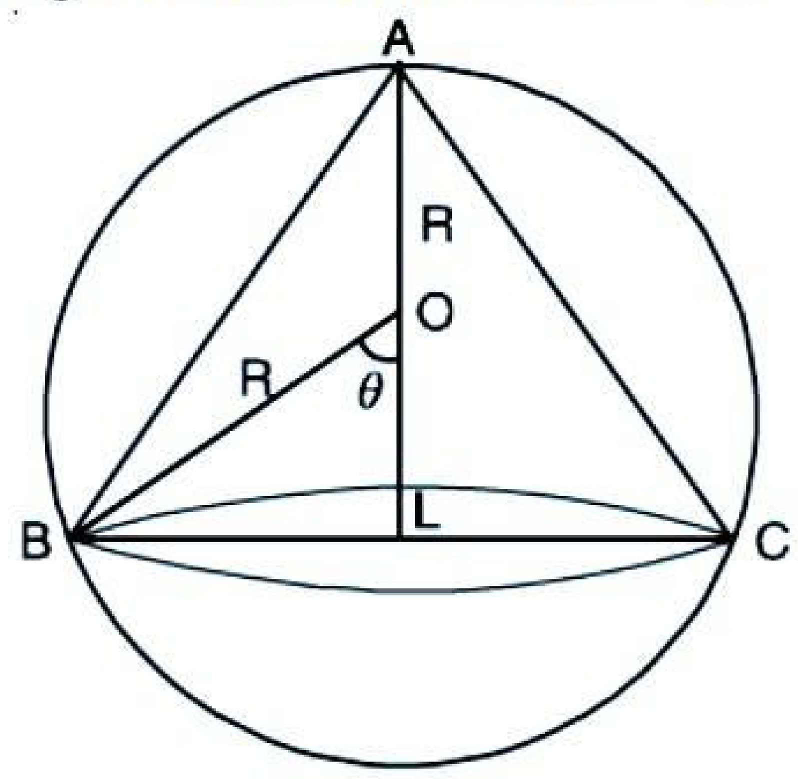
23. As in Ex. 8.

24. Let the cone ABC be inscribed in a sphere of radius R.

Let $\angle BOL = \theta$.

$\therefore$ BL, radius of the base of the cone = R sin θ

and AL height of the cone = AO + OL = R + R cos θ.



Now volume of the cone,

$$V = \frac{1}{3}\pi\,(BL)^2\,(AL)$$

$$= \frac{1}{3}\pi R^2\sin^2\theta\,(R + R\cos\theta)$$

$$= \frac{1}{3}\pi R^3\sin^2\theta\,(1+\cos\theta).$$

$$\therefore \frac{dV}{d\theta} = \frac{1}{3}\pi.R^3\,[\sin^2\theta\,(-\sin\theta) + (1+\cos\theta)\,2\sin\theta\cos\theta]$$

$$= \frac{1}{3}\pi R^3\,[-\sin^3\theta + 2\sin\theta\,(1-\sin^2\theta) + 2\sin\theta\cos\theta]$$

$$= \frac{1}{3}\pi R^3\,[-3\sin^3\theta + 2\sin\theta + 2\sin\theta\cos\theta].$$

For max./min. $\frac{dV}{d\theta} = 0$

$\Rightarrow \frac{1}{3}\pi R^3\,[-3\sin^3\theta + 2\sin\theta + 2\sin\theta\cos\theta] = 0$

$\Rightarrow -3\sin^3\theta + 2\sin\theta + 2\sin\theta\cos\theta = 0$

$\Rightarrow -3\sin^2\theta + 2 + 2\cos\theta = 0$ $\quad$ [$\because$ *$\sin\theta \neq 0$*]

$\Rightarrow -3\,(1-\cos^2\theta) + 2 + 2\cos\theta = 0$

$\Rightarrow 3\cos^2\theta + 2\cos\theta - 1 = 0$

$\Rightarrow (3\cos\theta - 1)(\cos\theta + 1) = 0$

$\Rightarrow \cos\theta = \frac{1}{3}, -1$

But $\cos\theta \neq -1$ [$\because$ *$\cos\theta = -1 \Rightarrow \theta = \pi$, which is not possible*]

Thus $\cos\theta = \frac{1}{3}$.

When $\cos\theta = \frac{1}{3}$, then $\sin\theta = \sqrt{1-\frac{1}{9}} = \sqrt{\frac{8}{9}}$

$$= \frac{2\sqrt{2}}{3}.$$

Now $\frac{dV}{d\theta} = \frac{1}{3}\pi R^3\,(-3\sin^3\theta + 2\sin\theta + 2\sin\theta\cos\theta)$

$$= \frac{1}{3}\pi R^3\sin\theta\,(-3\sin^2\theta + 2 + 2\cos\theta)$$

$$= \frac{1}{3}\pi R^3\sin\theta\,(3\cos\theta - 1)(\cos\theta + 1).$$

When $\theta < \cos^{-1}\left(\frac{1}{3}\right)$ slightly,

$\sin\theta$ = + ve, $3\cos\theta - 1$ = + ve.

[$\because$ *As θ decreases, $\cos\theta$ increases*]

$\cos\theta + 1$ = + ve.

$\therefore \frac{dV}{d\theta}$ = (+) (+) (+) = + ve.

When $\theta > \cos^{-1}\left(\frac{1}{3}\right)$ slightly,

$\sin\theta$ = + ve, $3\cos\theta - 1$ = + ve.

[$\because$ *As θ increases, $\cos\theta$ decreases*]

$\cos\theta + 1$ is + ve.

$\therefore \frac{dV}{d\theta}$ = (+) (–) (+) = – ve

Thus $\frac{dV}{d\theta}$ changes from + ve to – ve.

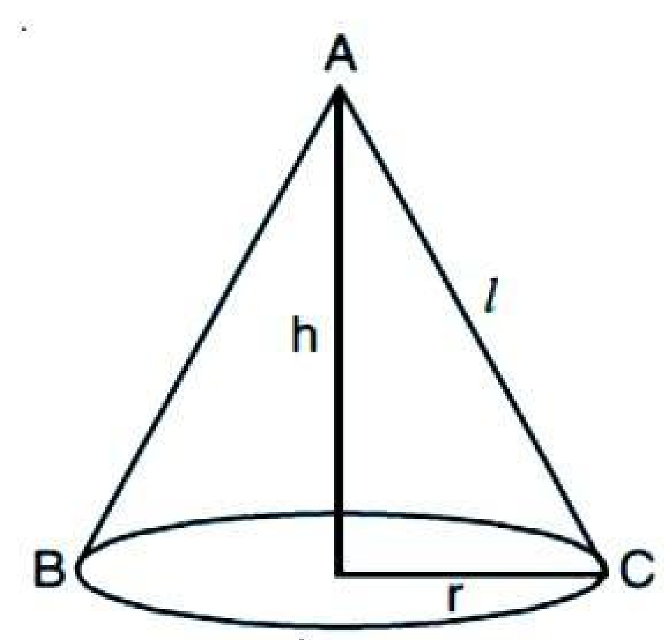
Hence, V is maximum when $\theta = \cos^{-1}\left(\frac{1}{3}\right)$

and max. volume of cone $= \frac{1}{3}\pi R^3\left(\frac{2\sqrt{2}}{3}\right)^2\left(1+\frac{1}{3}\right)$

$$= \frac{1}{3}\pi R^3\left(\frac{8}{9}\right)\left(\frac{4}{3}\right) = \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{8}{27} \times \text{Volume of the sphere.}$$

26. Let '*r*' and '*h*' be the radius and height respectively of the cone ABC.



Volume, $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}k$ (constant)

$\therefore \quad r^2 h = k$

$\Rightarrow \quad h = \dfrac{k}{r^2}$ ....(1)

Now, surface, $S = \pi r l = \pi r\sqrt{h^2 + r^2}$

$= \pi r\sqrt{\dfrac{k^2}{r^4} + r^2}$ [*Using (1)*]

$= \dfrac{\pi\sqrt{k^2 + r^6}}{r}.$

$$\therefore \frac{dS}{dr} = \pi\left[\frac{r\dfrac{1}{2\sqrt{k^2+r^6}}(6r^5) - \sqrt{k^2+r^6}\,.1}{r^2}\right]$$

$$= \pi\frac{3r^6 - (r^6 + k^2)}{r^2\sqrt{k^2+r^6}} = \frac{2r^6 - k^2}{r^2\sqrt{k^2+r^6}}\pi.$$

For least surface, $\dfrac{dS}{dr} = 0$

$\Rightarrow 2r^6 - k^2 = 0$

$\Rightarrow k^2 = 2r^6$ ...(2)

Also $\dfrac{dS}{dr}$ changes from – ve to + ve as r increases through the point $h^2 = 2r$.

From (1), $k^2 = h^2 r^4$

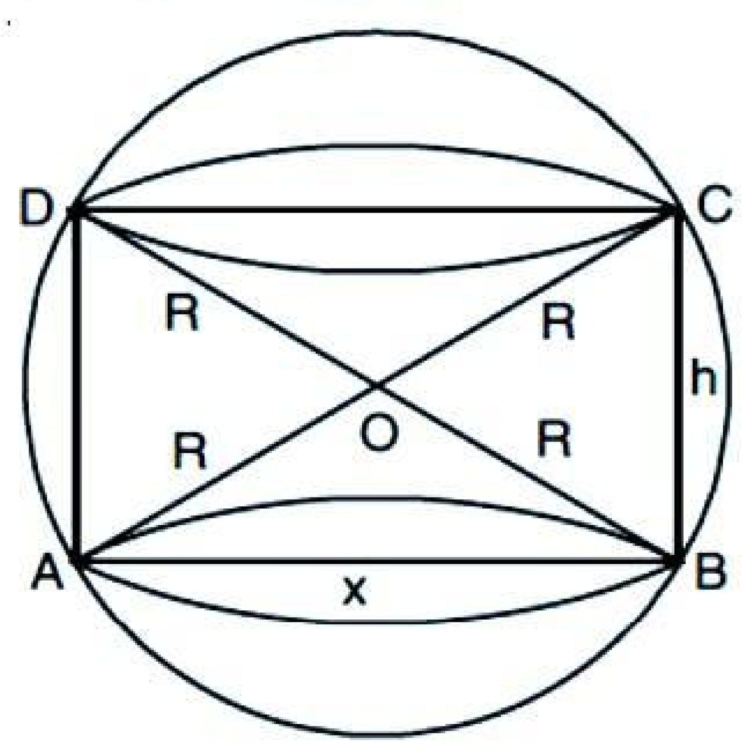
$\Rightarrow \quad h^2 r^4 = 2r^6$ [*Using (2)*]

$\Rightarrow \quad h^2 = 2r^2.$

Hence, $h = \sqrt{2}r$.

27. Let 'h' be the height and 'x' the diameter of the bars of the inscribed cylinder.

Then $h^2 + x^2 = 4R^2$ ....(1)



Fig

Now V, the volume of the cylinder $= \pi\left(\dfrac{x}{2}\right)^2 h$

$= \dfrac{1}{4}\pi x^2 h = \dfrac{1}{4}\pi(4R^2 - h^2)\,h$ [*Using (1)*]

$= \pi R^2 h - \dfrac{1}{4}\pi h^3$ ...(2)

$\therefore \dfrac{dV}{dh} = \pi R^2 - \dfrac{3}{4}\pi h^2 = \pi\left(R^2 - \dfrac{3}{4}h^2\right).$

Now $\dfrac{dV}{dh} = 0 \Rightarrow \pi\left(R^2 - \dfrac{3}{4}h^2\right) = 0 \Rightarrow R^2 = \dfrac{3}{4}h^2$

$\Rightarrow h = \dfrac{2R}{\sqrt{3}}$ ...(3)

And $\dfrac{d^2V}{dh^2} = -\dfrac{3}{4}\pi\,(2h) = -\dfrac{3}{2}\pi h.$

At $h = \dfrac{2R}{\sqrt{3}}$, $\dfrac{d^2V}{dh^2} = -\dfrac{3}{2}\pi\left(\dfrac{2R}{\sqrt{3}}\right)$

$= -\sqrt{3}\pi R$, which is – ve.

Hence, V is maximum when $h = \dfrac{2R}{\sqrt{3}}$

and maximum value $= \pi R^2 . \dfrac{2R}{\sqrt{3}} - \dfrac{1}{4}\pi\left(\dfrac{8R^3}{3\sqrt{3}}\right)$

[*Putting in (2)*]

$= \dfrac{2\pi R^3}{\sqrt{3}} - \dfrac{2\pi R^3}{3\sqrt{3}} = \dfrac{2\pi R^3}{\sqrt{3}}\left(1 - \dfrac{1}{3}\right)$

$= \dfrac{4\pi R^3}{3\sqrt{3}}.$

28. (*b*) Let OC = r be the radius of the cone and 6A = h be its height.

Let a cylinder of radius OE = x be inscribed in the given cone.

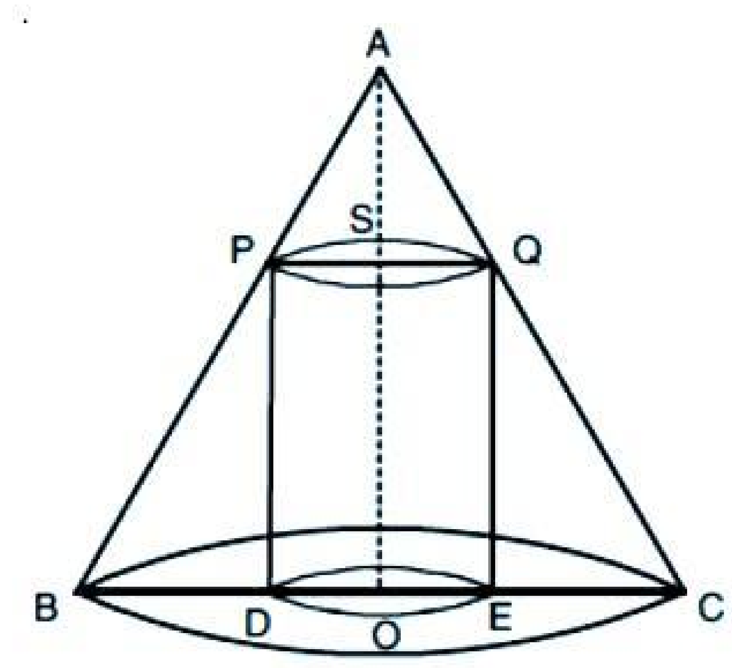


Fig.

The length QE of the cylinder is given by :

$\dfrac{QE}{OA} = \dfrac{EC}{OC}$ [$\because \Delta QEC \sim \Delta AOC$]

$\Rightarrow \dfrac{QE}{h} = \dfrac{r - x}{r}$

$\Rightarrow QE = \dfrac{h\,(r-x)}{r}.$

If S be the curved surface of the given cylinder, then :

$$S(x) = 2\pi\left(\frac{h(r-x)}{r}\right)x = \frac{2\pi h}{r}(rx - x^2).$$

$$\therefore \quad S'(x) = \frac{2\pi h}{r}(r-2x)$$

and $S''(x) = -\frac{4\pi h}{r}(-ve).$

Now $S'(x) = 0 \Rightarrow r = 2x$ *i.e.* $x = \frac{r}{2}$ and $S''(x)$ is – ve.

Thus S is maximum when $x = \frac{r}{2}$.

Hence the radius of the cylinder of greatest curved surface area, which can be inscribed in a given cone, is half that of the cone.

37. Let 'x' and 'y' be the length and breadth respectively of the tank.

And depth of the tank = 2 m.

$\therefore$ Volume of the tank $= 2xy$.

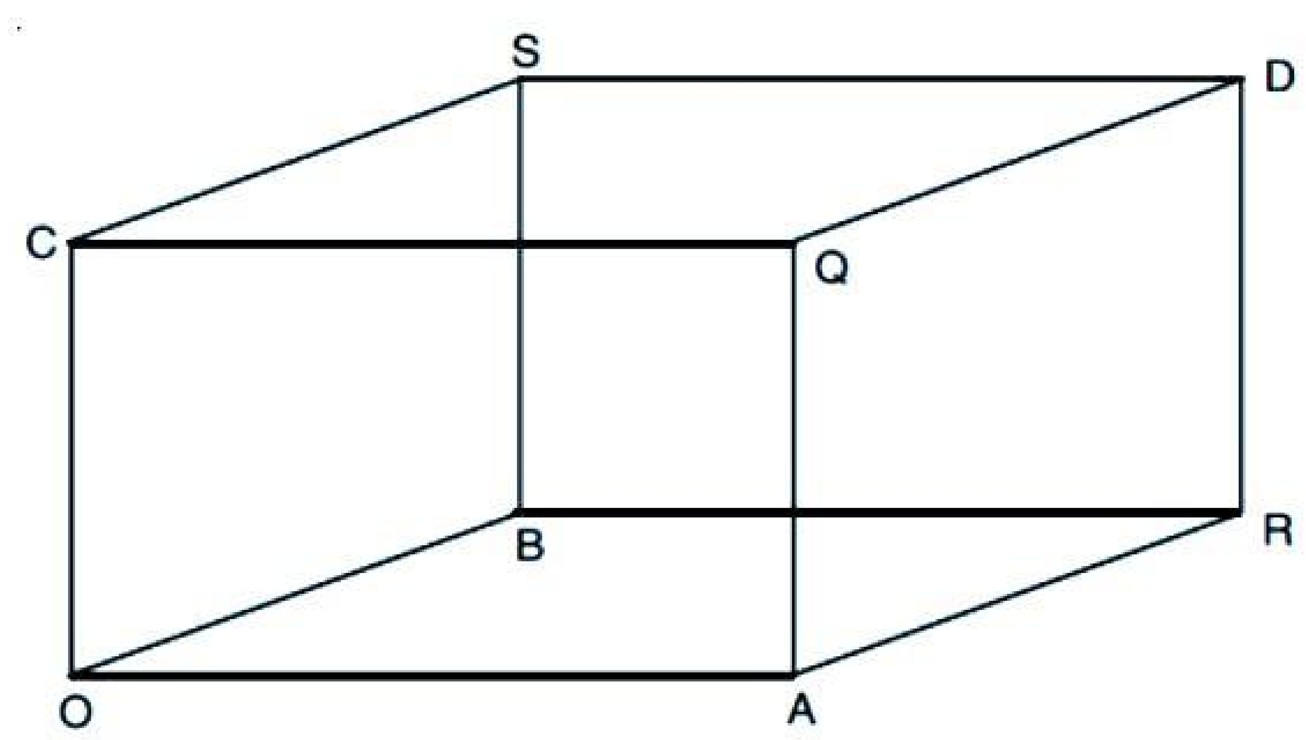


Fig.

By the question, $2xy = 8 \Rightarrow xy = 4$ ....(1)

Now area of the base $= xy$

and area of the sides $= 2(x + y)(2) = 4(x + y)$.

$\therefore$ Cost of construction = ₹ $(70xy + 45(4(x + y))$

= ₹ $(70xy + 180(x + y))$ ....(2)

$\therefore$ C, the cost of construction

$$= 70(4) + 180\left(x + \frac{4}{x}\right) \text{ [Using (1)]}$$

$$= 280 + 180\left(x + \frac{4}{x}\right) \quad ...(3)$$

$$\therefore \frac{dC}{dx} = 180\left(1 - \frac{4}{x^2}\right) = 180\left(\frac{x^2-4}{x^2}\right).$$

For max./min., $\frac{dC}{dx} = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2.$

Thus $x = 2$. [$\because$ *Length can't be – ve*]

$\therefore$ From (1), $y = \frac{4}{2} = 2.$

Thus the tank is a cube of side 2 m.

$\therefore$ Least cost of constitution

$$= ₹\left[280 + 180\left(2 + \frac{4}{2}\right)\right] \quad \text{[From (3)]}$$

= ₹ (280 + 720) = ₹ 1000.

42. Let P (x, y) be any point on $y^2 = 4x$ ...(1)

Also A (2, – 8) is the given point.

Now S = (Distance of P from A = | PA |)^2

$$= (x-2)^2 + (y+8)^2$$

$$= \left(\frac{y^2}{4} - 2\right)^2 + (y+8)^2$$

$$= \frac{y^4}{16} + 4 - y^2 + y^2 + 16y + 64$$

$$= \frac{1}{16}y^4 + 16y + 68.$$

$$\therefore \frac{dS}{dy} = \frac{1}{4}y^3 + 16 \text{ and } \frac{d^2S}{dy^2} = \frac{3}{4}y^2.$$

Now $\frac{dS}{dy} = 0 \Rightarrow \frac{1}{4}y^3 + 16 = 0$

$\Rightarrow y^3 = -64 \Rightarrow y = -4.$

From (1), $16 = 4x \Rightarrow x = 4.$

Thus the point is (4, – 4) and

$$\frac{d^2S}{dy^2} = \frac{3}{4}(-4)^2 = 12 \text{ (+ve)}.$$

Hence, (4, – 4) is nearest to the given point (2, – 8).

44. Find a point $(x, x^2 + 2)$, which is nearest to (3, 2).

NCERT-FILE

Questions from NCERT Book

(For each unsolved question, refer : "Solution of Modern's abc of Mathematics")

Exercise 6.1

1. Find the rate of change of the area of a circle with respect to its radius r when :

(a) $r = 3$ cm (b) $r = 4$ cm.

Solution : Here 'r' is the radius of the circle.

Then A, the area of the circle $= \pi r^2$.

$$\therefore \quad \frac{dA}{dr} = 2\pi r.$$

Hence, the rate of change of the area of the circle $= 2\pi r$.

(a) When $r = 3$ cm, then the rate of change of the area of the circle $= 2\pi(3) = 6\pi \text{ cm}^2/\text{s}$.

(b) When $r = 4$ cm, then the rate of change of the area of the circle $= 2\pi(4) = 8\pi \text{ cm}^2/\text{s}$.

2. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of an edge is 12 cm ?

[Solution : Refer Q. 12(b) ; Ex. 6(a)]

3. The radius of a circle is increasing uniformly at the rate of 3 cm/s. Find the rate at which the area of the circle is increasing when the radius is 10 cm.

[Solution : Refer Q. 6 ; Ex. 6(a)]

4. An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long ?

[Solution : Refer Q. 22 ; Ex. 6(a)]

5. A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At the instant when the radius of the circular wave is 8 cm, how fast is the enclosed area increasing ?

Solution : Let 'r' be the radius of the circular wave.

Then $\quad A = \pi r^2$,

where A is the enclosed area at time t.

Differentiating w.r.t. t, we have :

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r(5) \qquad \left[\because \frac{dr}{dt} = 5 \text{ cm/s}\right]$$

$$= 10\pi r.$$

When $r = 8$ cm, $\dfrac{dA}{dt} = 10\pi(8)$

$$= 80\pi \text{ cm}^2/\text{sec}.$$

Hence, the enclosed area is increasing at the rate of $80\pi \text{ cm}^2/\text{s}$, where $r = 8$ cm.

6. The radius of a circle is increasing at the rate of 0·7 cm/s. What is the rate of increase of its circumference ?

[Solution : Refer Q. 5(i) ; Ex. 6(a)]

7. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rates of change of (a) the perimeter, and (b) the area of the rectangle.

Solution : We have : $\dfrac{dx}{dt} = -5 \quad$ ...(1)

and $\dfrac{dy}{dt} = 4 \quad$ ...(2)

(a) Perimeter, $p = 2x + 2y$.

$$\therefore \quad \frac{dp}{dt} = 2\frac{dx}{dt} + 2\frac{dy}{dt}$$

$$= 2(-5) + 2(4) \qquad [Using\ (1)\ and\ (2)]$$

$$= -10 + 8 = -2.$$

Hence, $\left.\dfrac{dp}{dt}\right]_{\substack{x=8 \\ y=6}} = -2$ cm/m.

(b) Area, $A = xy$.

$$\therefore \quad \frac{dA}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

Hence, $\left.\dfrac{dA}{dt}\right]_{\substack{x=8 \\ y=6}} = 8(-5) + 6(4)$

$$= -40 + 24 = -16 \text{ cm}^2/\text{m}.$$

8. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon increases when the radius is 15 cm.

[Solution : Refer Q. 11 (i) ; Ex. 6(a)]

9. A balloon, which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius when the later is 10 cm.

Solution : Let 'r' be the variable radius of the balloon.

$$\therefore \quad V = \frac{4}{3}\pi r^3.$$

$$\therefore \quad \frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2.$$

Hence, $$\left.\frac{dV}{dr}\right]_{r=10} = 4\pi(10)^2 = 400\pi \text{ cm}^3/\text{cm}.$$

10. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ?

[Solution : Refer Q. 24; Ex. 6(*a*)]

11. A particle moves along the curve $6y = x^3 + 2$. Find the points on the curve at which the *y*-coordinate is changing 8 times as fast as the *x*-coordinate.

[Solution : Refer Q. 13(*ii*) ; Ex. 6(*a*)]

12. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. At what rate is the volume of the bubble increasing when the radius is 1 cm ?

[Solution : Refer Q. 7 ; Ex. 6(*a*)]

13. A balloon, which always remains spherical, has a variable diameter $\frac{3}{2}(2x+1)$. Find the rate of change of its volume with respect to *x*.

[Solution : Refer Q. 10 ; Ex. 6(*a*)]

14. Sand is pouring from a pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm ?

Solution : Let '*r*' and '*h*' be the radius and height of the sand cone respectively at time *t*.

By the question, $h = \frac{r}{6}$ ...(1)

Now V, the volume of the cone

$$= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(6h)^2 h \quad [\textit{Using (1)}]$$

$$= 12\pi h^3.$$

$$\therefore \quad \frac{dV}{dt} = 36\pi h^2 \frac{dh}{dt} \quad ...(2)$$

Given : $\frac{dV}{dt} = 12$ cm³/s. ...(3)

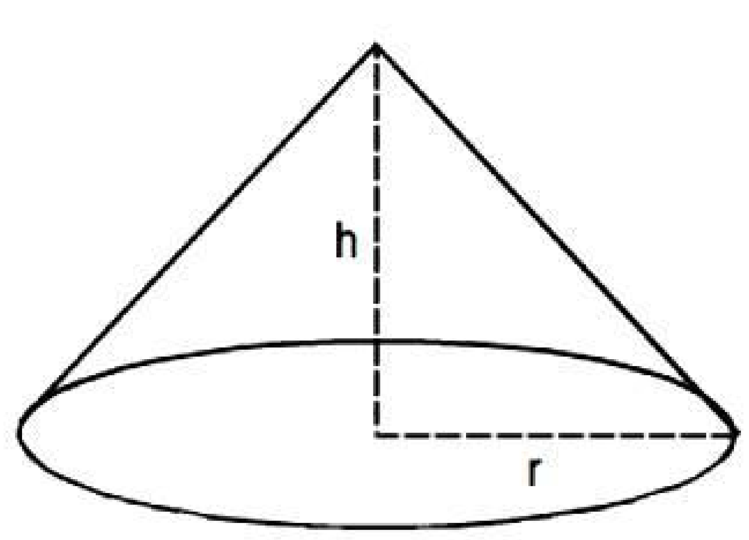


Fig.

From (2) and (3), $36\pi h^2 \frac{dh}{dt} = 12$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{12}{36\pi h^2} = \frac{1}{3\pi h^2}.$$

When, $h = 4$ cm, $$\frac{dh}{dt} = \frac{1}{3\pi(4)^2} = \frac{1}{48\pi}.$$

Hence, the rate of increase of height of the sand cone, when $h = 4$ cm $= \frac{1}{48\pi}$ cm./s.

15. The total cost C(*x*) in ₹ associated with the production of *x* units of an item is given by :

$$C(x) = 0{\cdot}007x^3 - 0{\cdot}003x^2 + 15x + 4000.$$

Find the marginal cost when 17 units are produced.

Solution : Marginal cost is the rate of change of total cost with respect to output.

Now $C(x) = 0{\cdot}007x^3 - 0{\cdot}003x^2 + 15x + 4000.$

$\therefore$ Marginal cost (MC) $= \frac{dC}{dx}$

$= (0{\cdot}007)(3x^2) - 0{\cdot}003(2x) + 15.$

When $x = 17$, MC $= (0{\cdot}007)(3(17)^2) - 0{\cdot}003(2(17)) + 15$

$= 6{\cdot}069 - 0{\cdot}102 + 15 = 20{\cdot}967.$

Hence, the reqd. marginal cost is ₹ 21 (nearly).

16. The total revenue in ₹ received from the sale of *x* units of a product is given by :

$$R(x) = 13x^2 + 26x + 15.$$

Find the marginal revenue when $x = 7$.

Solution : Marginal revenue is the rate of change of total revenue with respect to number of units sold.

Now $R(x) = 13x^2 + 26x + 15.$

$\therefore$ Marginal Revenue (MR) $= \frac{dR}{dx}$

$= 26x + 26.$

When $x = 7$, MR $= 26(7) + 26$

$= 182 + 26 = 208.$

Hence, the reqd. marginal revenue is ₹ 208.

Choose the correct answer in the Exercises 17 and 18.

17. The rate of change of the area of a circle with respect to its radius *r* at *r* = 6 cm is :

(A) 10π (B) 12π (C) 8π (D) 11π.

[Ans. (B)]

18. The total revenue in ₹ received from the sale of *x* units of a product is given by :

$R(x) = 3x^2 + 36x + 5$. The marginal revenue, when $x = 15$ is :

(A) 116 (B) 96 (C) 90 (D) 126.

[Ans. (D)]

Exercise 6.2

1. Show that the function given by $f(x) = 3x + 17$ is strictly increasing on **R**.

2. Show that the function given by $f(x) = e^{2x}$ is strictly increasing on **R**.

[Solutions : (1–2) : Refer Q. 1 ; Ex. 6(*b*)]

3. Show that the function given by $f(x) = \sin x$ is :

(*a*) strictly increasing in $\left(0, \frac{\pi}{2}\right)$

(*b*) strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(*c*) neither increasing nor decreasing in $(0, \pi)$.

[Solution : Refer Q. 7(*i*) ; Ex. 6(*b*)]

4. Find the intervals in which the function 'f' given by $f(x) = 2x^2 - 3x$ is :

(*a*) strictly increasing (*b*) strictly decreasing.

Solution : We have :

$$f(x) = 2x^2 - 3x.$$

$$\therefore f'(x) = 4x - 3.$$

(*a*) For $f(x)$ to be strictly increasing function of x,

$$f'(x) > 0$$

$$\Rightarrow 4x - 3 > 0$$

$$\Rightarrow x > \frac{3}{4}.$$

Hence, 'f' is strictly increasing in $\left(\frac{3}{4}, \infty\right)$.

(*b*) For $f(x)$ to be strictly decreasing function of x,

$$f'(x) < 0 \Rightarrow 4x - 3 < 0$$

$$\Rightarrow x < \frac{3}{4}.$$

Hence, 'f' is strictly decreasing in $\left(-\infty, \frac{3}{4}\right)$.

5. Find the intervals in which the function 'f' given by $f(x) = 2x^3 - 3x^2 - 36x + 7$ is :

(*a*) strictly increasing (*b*) strictly decreasing.

[Solution : Refer Q. 22 (*ii*) ; Ex. 6(*b*)]

6. Find the intervals in which the following functions are strictly increasing or decreasing :

(*a*) $x^2 + 2x - 5$ (*b*) $10 - 6x - 2x^2$

(*c*) $-2x^3 - 9x^2 - 12x + 1$ (*d*) $6 - 9x - x^2$

(*e*) $(x + 1)^3 (x - 3)^3$.

[(*a*) Solution : Refer Q. 20(*ii*) ; Ex. 6(*b*)]

[(*b*) Solution : Refer Q. 20(*iii*) ; Ex. 6(*b*)]

[(*c*) Solution : Refer Q. 25(*i*) ; Ex. 6(*b*)]

[(*d*) Solution : Refer Q. 20(*iv*) ; Ex. 6(*b*)]

[(*e*) Solution : Refer Q. 28 ; Ex. 6(*b*)]

7. Show that $y = \log(1+x) - \frac{2x}{2+x}$, $x > -1$, is an increasing function of 'x' throughout its domain.

Solution : Let $f(x) = \log(1+x) - \frac{2x}{2+x}$.

$$\therefore f'(x) = \frac{1}{1+x} - 2\left[\frac{(2+x)(1) - x(0+1)}{(2+x)^2}\right]$$

$$= \frac{1}{1+x} - 2\left[\frac{2}{(2+x)^2}\right]$$

$$= \frac{(x+2)^2 - 4(x+1)}{(x+1)(x+2)^2} = \frac{x^2}{(x+1)(x+2)^2}$$

$$> 0$$

$$\left[\because \frac{x^2}{(x+2)^2}, \text{ being a perfect sq., is } +ve \text{ and } x > -1 \Rightarrow x + 1 > 0\right]$$

Hence, 'f' is an increasing function of x throughout its domain.

8. Find the values of 'x' for which $y = [x(x - 2)]^2$ is an increasing function.

[Solution : Refer Q. 15 ; Ex. 6(*b*)]

9. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$. ***(A.I.C.B.S.E. 2016)***

Solution :

We have : $f(\theta) = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$.

$$\therefore f'(\theta) = 4\frac{(2+\cos\theta)(\cos\theta) - \sin\theta(0 - \sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= 4\frac{2\cos\theta + \cos^2\theta + \sin^2\theta}{(2+\cos\theta)^2} - 1 = 4\frac{2\cos\theta + 1}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$$

$$= \frac{\cos\theta(4 - \cos\theta)}{(2+\cos\theta)^2} > 0 \text{ for all } \theta \in \left[0, \frac{\pi}{2}\right].$$

Hence, 'f' is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

10. Prove that the logarithmic function is strictly increasing on $(0, \infty)$.

[Solution : Refer Q. 9 ; Ex. 6(*b*)]

11. Prove that the function f given by $f(x) = x^2 - x + 1$ is neither strictly increasing nor strictly decreasing on $(-1, 1)$.

[Solution : Refer Q. 13 ; Ex. 6(*b*)]

12. Which of the following functions are strictly decreasing on $\left(0, \frac{\pi}{2}\right)$?

(A) $\cos x$ (B) $\cos 2x$ (C) $\cos 3x$ (D) $\tan x$.

[Solution : Refer Q. 35 ; Ex. 6(*b*)]

13. On which of the following intervals is the function '*f*' given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing ?

(A) $(0, 1)$ (B) $\left(\frac{\pi}{2}, \pi\right)$

(C) $\left(0, \frac{\pi}{2}\right)$ (D) None of these

[Solution : Refer Q. 30 ; Ex. 6(*b*)]

14. Find the least value of '*a*' such that the function '*f*' given by $f(x) = x^2 + ax + 1$ is strictly increasing on $(1, 2)$.

Solution : We have : $f(x) = x^2 + ax + 1,\ 1 < x < 2.$

$\therefore \quad f'(x) = 2x + a.$

Now $1 < x < 2 \Rightarrow 2 < 2x < 4$

$\Rightarrow \quad 2 + a < 2x + a < 4 + a$

$\Rightarrow \quad 2 + a < f'(x) < 4 + a.$

For f(x) to be strictly increasing, $f'(x) \geq 0$

$\Rightarrow 4 + a \geq 0 \quad \Rightarrow \quad a \geq -4.$

Hence, the least value of '*a*' is -4.

15. Let I be any interval disjoint from $(-1, 1)$. Prove that the function '*f* given by $f(x) = x + \frac{1}{x}$ is strictly increasing on I.

Solution : We have : $f(x) = x + \frac{1}{x}$.

$\therefore \quad f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}.$

Now $f'(x) > 0 \Rightarrow \frac{x^2 - 1}{x^2} > 0$

$\Rightarrow x^2 - 1 > 0 \Rightarrow |x| > 1$

$\Rightarrow \quad x < -1$ or $x > 1$

$\Rightarrow \quad x \in (-\infty, -1)$ or $x \in (1, \infty)$

$\Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty) = \mathbf{R} - (-1, 1).$

Hence, '*f*' is strictly increasing on I, where I is any interval disjoint from $(-1, 1)$.

16. Prove that the function '*f*' given by $f(x) = \log \sin x$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

17. Prove that the function *f* given by $f(x) = \log \cos x$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.

[Solution : (16–17) Refer Q. 10 ; Ex. 6(*b*)]

18. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in **R**

[Solution : Refer Q. 5(*i*) ; Ex. 6(*b*)]

19. The interval in which $y = x^2 e^{-x}$ is increasing is :

(A) $(-\infty, \infty)$ (B) $(-2, 0)$ (C) $(2, \infty)$ (D) $(0, 2)$.

[Ans. (D)]

Exercise 6.3

1. Find the slope of the tangent to the curve $y = 3x^4 - 4x$ at $x = 4$.

2. Find the slope of the tangent to the curve $y = \frac{x-1}{x-2}$, $x \neq 2$ at $x = 10$.

3. Find the slope of the tangent to curve $y = x^3 - x + 1$ at the point whose *x*-coordinate is 2.

4. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose *x*-coordinate is 3.

[Solutions : (1 – 4) : Refer Q. 1 ; Ex. 6(*c*)]

5. Find the slope of the normal to the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

6. Find the slope of the normal to the curve $x = 1 - a\sin\theta$, $y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.

[Solutions : (5 – 6) : Refer Q. 3(*b*) ; Ex. 6(*c*)]

7. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the *x*-axis.

[Solution : Refer Q. 30 ; Ex. 6(*c*)]

8. Find a point on the curve $y = (x - 2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

Solution : The given curve is $f(x) = y = (x - 2)^2$ ...(1)

$\therefore \quad f'(x) = 2(x - 2).$

Since $2(x - 2) = \frac{4 - 0}{4 - 2}$ $[\because m_1 = m_2]$

$\Rightarrow \quad 2x - 4 = 2$

$\Rightarrow \quad 2x = 6 \quad \Rightarrow \quad x = 3.$

When $x = 3$, then from (1), $y = (3 - 2)^2 = 1$.

Hence, the reqd. point is (3, 1).

9. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$.

[Solution : Refer Q. 28 ; Ex. 6(*c*)]

10. Find the equation of all lines having slope -1 that are tangents to the curve $y = \frac{1}{x-1}, x \neq 1$.

11. Find the equations of all lines having slope 2 which are tangents to the curve $y = \frac{1}{x-3}, x \neq 3$.

12. Find the equations of all lines having slope 0 which are tangent to the curve $y = \frac{1}{x^2 - 2x + 3}$.

[Solutions : (10 – 12) Refer Q. 34 ; Ex. 6(*c*)]

13. Find points on the curve $\frac{x^2}{9}+\frac{y^2}{16}=1$ at which the tangents are :

(i) parallel to x-axis (ii) parallel to y-axis.

Solution : The given curve is :

$$\frac{x^2}{9}+\frac{y^2}{16} = 1 \quad ...(1)$$

Diff. w.r.t. x, $\frac{2x}{9}+\frac{2y}{16}.\frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{-2x}{9}}{\frac{y}{8}} = -\frac{16x}{9y}.$$

(i) The tangent is parallel to x-axis

if slope of the tangent $= 0$

if $-\frac{16x}{9y} = 0$ if $x = 0$.

Putting in (1), $0+\frac{y^2}{16} = 1$

$\Rightarrow$ $y^2 = 16$ $\Rightarrow$ $y = \pm 4$.

Hence, the tangents are parallel to x-axis at $(0, \pm 4)$.

(ii) The tangent is parallel to y-axis

if the slope of the tangent $\to \pm\infty$

if the slope of the normal $= 0$

if $\frac{9y}{16x} = 0 \Rightarrow y = 0$.

Putting is (1) $\frac{x^2}{9}+0 = 1 \Rightarrow x^2 = 9$

$\Rightarrow$ $x = \pm 3$.

Hence, the tangents are parallel to y-axis at $(\pm 3, 0)$.

14. Find the equations of the tangent and normal to the given curves at the indicated points :

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(0, 5)$

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ at $(1, 3)$.

[Solutions : (i) – (ii) Refer Q. 8 ; Ex. 6(c)]

(iii) $y = x^3$ at $(1, 1)$

(iv) $y = x^2$ at $(0, 0)$.

[Solutions : (iii) – (iv) Refer Q. 6 ; Ex. 6(c)]

(v) $x = \cos t$, $y = \sin t$ at $t = \frac{\pi}{4}$.

Solution : We have : $x = \cos t$, $y = \sin t$.

$$\therefore \quad \frac{dx}{dt} = -\sin t,\ \frac{dy}{dt} = \cos t.$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t.$$

At $t = \frac{\pi}{4}$, $x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $y = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$,

$$\frac{dy}{dx} = -\cot\frac{\pi}{4} = -1,$$

which is the slope of tangent.

$\therefore$ The equation of tangent is :

$$y-\frac{1}{\sqrt{2}} = -1.\left(x-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad y-\frac{1}{\sqrt{2}} = -x+\frac{1}{\sqrt{2}} \Rightarrow x+y-\sqrt{2}$$

Now slope of normal $= -1$.

$\therefore$ The equation of normal is :

$$y-\frac{1}{\sqrt{2}} = 1.\left(x-\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \quad y = x.$$

15. Find the equation of the tangent line to the curve $y = x^2 - 2x + 7$ which is :

(a) parallel to the line $2x - y + 9 = 0$

(b) perpendicular to the line $5y - 15x = 13$.

[Solution : Refer Q. 18 ; Ex. 6(c)]

16. Show that the tangents to the curve $y = 7x^3 + 11$ at the points where $x = 2$ and $x = -2$ are parallel.

[Solution : Refer Q. 33 ; Ex. 6(c)]

17. Find the points on the curve $y = x^3$ at which the slope of the tangent is equal to the y-coordinate of the point.

Solution : Let P (x_1, y_1) be any point on the curve :

$$y = x^3 \quad ...(1)$$

$$\therefore \quad \frac{dy}{dx} = 3x^2 \Rightarrow \left.\frac{dy}{dx}\right]_{(x_1,\, y_1)} = 3x_1^2.$$

By the question, slope of the tangent at $(x_1, y_1) = y_1$.

$$\therefore \quad 3x_1^2 = y_1 \quad ...(2)$$

Since (x_1, y_1) lies on (1), $\therefore$ $y_1 = x_1^3$ ...(3)

From (2) and (3), $3x_1^2 = x_1^3$

$\Rightarrow$ $x_1^2(3 - x_1) = 0$

$\Rightarrow$ $x_1 = 0$ or 3.

When $x_1 = 0$, then $y_1 = (0)^3 = 0$.

When $x_1 = 3$, then $y_1 = 3^3 = 27$.

Hence, the reqd. points are $(0, 0)$ and $(3, 27)$.

18. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.

[Solution : Refer Q. 29 ; Ex. 6(c)]

19. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to the x-axis.

Solution :

The given curve is $x^2 + y^2 - 2x - 3 = 0$ ...(1)

Diff. w.r.t. x, $2x+2y\frac{dy}{dx}-2 = 0$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2-2x}{2y}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1-x}{y}.$$

Now tangent is parallel to x-axis

$$\Rightarrow \quad \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{1-x}{y} = 0$$

$\Rightarrow$ $1 - x = 0$ $\Rightarrow$ $x = 1$.

Putting in (1), $1 + y^2 - 2(1) - 3 = 0$

$\Rightarrow$ $y^2 = 4$ $\Rightarrow$ $y = \pm 2$.

Hence, the reqd. points are $(1, \pm 2)$.

20. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

[Solution : Refer Q. 16(*i*) ; Ex. 6(*c*)]

21. Find the equation of the normals to the curve $y = x^3 + 2x + 6$, which are parallel to the line $x + 14y + 4 = 0$.

[Solution : Refer Q. 24 (*i*) ; Ex. 6(*c*)]

22. Find the equations of the tangent and normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$.

[Solution : Refer Q. 14 ; Ex. 6(*c*)]

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$. ***(H.B. 2018)***

Solution : The given curves are :

$$x = y^2 \quad ...(1) \quad \text{and } xy = k \quad ...(2)$$

From (2), using (1), we get :

$$y^2 y = k \Rightarrow y^3 = k \Rightarrow y = k^{1/3}.$$

Putting in (1), $x = k^{2/3}$.

Thus the given curves intersect at P $(k^{2/3}, k^{1/3})$.

Diff. (1) w.r.t. x,

$$1 = 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{2y}.$$

$$\therefore \quad m_1 = \left.\frac{dy}{dx}\right]_P = \frac{1}{2k^{1/3}} \quad ...(3)$$

Diff. (2) w.r.t. x, $x\dfrac{dy}{dx} + y(1) = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}.$$

$$\therefore \quad m_2 = \left.\frac{dy}{dx}\right]_P = -\frac{k^{1/3}}{k^{2/3}} = -\frac{1}{k^{1/3}} \quad ...(4)$$

Since the given curves cut at right angles at P,

$$\therefore \quad m_1 m_2 = -1$$

$$\Rightarrow \left(\frac{1}{2k^{1/3}}\right)\left(-\frac{1}{k^{1/3}}\right) = -1 \quad \text{[Using (3) \& (4)]}$$

$$\Rightarrow 2k^{2/3} = 1 \Rightarrow 8k^2 = 1, \text{ which is true.}$$

24. Find the equations of the tangent and normal to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at the point (x_0, y_0).

(H.B. 2018; H.P.B. 2017; Kashmir B. 2016)

Solution : The given hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

(I) Diff. w.r.t. x, $\dfrac{2x}{a^2} - \dfrac{2y}{b^2}\cdot\dfrac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{2x}{a^2}\cdot\frac{b^2}{2y} = \frac{b^2 x}{a^2 y}.$$

$\therefore$ Slope of the tangent at $(x_0, y_0) = \dfrac{b^2 x_0}{a^2 y_0}$.

$\therefore$ The equation of the tangent at (x_0, y_0) is :

$$y - y_0 = \frac{b^2 x_0}{a^2 y_0}(x - x_0)$$

$$\Rightarrow a^2 y_0 (y - y_0) = b^2 x_0 (x - x_0)$$

$$\Rightarrow b^2 x x_0 - a^2 y y_0 = b^2 x_0^2 - a^2 y_0^2$$

$$\Rightarrow \frac{xx_0}{a^2} - \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2}. \quad \textit{[Dividing by } a^2b^2\textit{]}$$

But (x_0, y_0) lies on (1), $\therefore \dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$.

Hence, the equation of the tangent is $\dfrac{xx_0}{a^2} - \dfrac{yy_0}{b^2} = 1$.

(II) Slope of the normal at $(x_0, y_0) = -\dfrac{a^2 y_0}{b^2 x_0}$.

$\therefore$ The equation of the normal at (x_0, y_0) is :

$$y - y_0 = -\frac{a^2 y_0}{b^2 x_0}(x - x_0)$$

$$\Rightarrow \frac{y - y_0}{a^2 y_0} = -\frac{x - x_0}{b^2 x_0}$$

$$\Rightarrow \frac{y - y_0}{a^2 y_0} + \frac{x - x_0}{b^2 x_0} = 0 \Rightarrow \frac{x - x_0}{b^2 x_0} + \frac{y - y_0}{a^2 y_0} = 0.$$

25. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$, which is parallel to the line $4x - 2y + 5 = 0$.

[Solution : Refer Q. 17 ; Ex. 6(*c*)]

Choose the correct answer in Exercises 26 and 27.

26. The slope of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$ is :

(A) 3 (B) $\dfrac{1}{3}$ (C) -3 (D) $-\dfrac{1}{3}$.

[Ans. (D)]

27. The line $y = x + 1$ is a tangent to the curve $y^2 = 4x$ at the point :

(A) (1, 2) (B) (2, 1) (C) (1, –2,) (D) (–1, 2).

[Ans. (A)]

Exercise 6.4

1. Using differentials, find the approximate value of each of the following up to 3 places of decimal.

(*i*) $\sqrt{25{\cdot}3}$ (*ii*) $\sqrt{49{\cdot}5}$ (*iii*) $\sqrt{0{\cdot}6}$

(*iv*) $(0{\cdot}009)^{1/3}$ (*v*) $(0{\cdot}999)^{1/10}$ (*vi*) $(15)^{1/4}$

(*vii*) $(26)^{1/3}$ (*viii*) $(255)^{1/4}$ (*ix*) $(82)^{1/4}$

(*x*) $(401)^{1/2}$ (*xi*) $(0{\cdot}0037)^{1/2}$ (*xii*) $(26{\cdot}57)^{1/3}$

(*xiii*) $(81\cdot5)^{1/4}$ (*xiv*) $(3\cdot968)^{3/2}$ (*xv*) $(32\cdot15)^{1/5}$.

[Solutions : Refer Q. 2, 4-5, 8-12 ; Ex. 6(*d*)]

2. Find the approximate value of $f(2.01)$, where $f(x) = 4x^2 + 5x + 2$.

Solution : Here $f(x) = 4x^2 + 5x + 2$ ... (1)

Take $x = 2$ and $\Delta x = 0\cdot01$ ... (2)

Then $f(2\cdot01) = f(x + \Delta x)$

$= 4(x + \Delta x)^2 + 5(x + \Delta x) + 2$ [*Putting x = 2·01 in (1)*]

Now $\Delta y = f(x + \Delta x) - f(x)$

$\Rightarrow f(x + \Delta x) = f(x) + \Delta y$

$= f(x) + f'(x)\Delta x$ [$\because dx = \Delta x$]

$\therefore f(2\cdot01) = (4x^2 + 5x + 2) + (8x + 5)\Delta x$

$= (4(2)^2 + 5(2) + 2) + (8(2) + 5)(0\cdot01)$ [*Using* (2)]

$= (16 + 10 + 2) + (16 + 5)(0\cdot01)$

$= 28 + 0\cdot21 = 28\cdot21$.

Hence, approximate value of $f(2\cdot01) = 28\cdot21$.

3. Find the approximate value of $f(5\cdot001)$, where $f(x) = x^3 - 7x^2 + 15$.

[Solution : Refer Q. 14(*ii*) ; Ex. 6(*d*)]

4. Find the approximate change in the volume V of a cube of side x metres caused by increasing the side by 1%.

Solution : V, the volume of the cube $= x^3$.

Now $dV = \left(\frac{dV}{dx}\right)\Delta x$

$= (3x^2)\ \Delta x$

$= (3x^2)(0\cdot01x)$ [$\because$ *1 % of x = 0·01 x*]

$= 0\cdot03\, x^3\ m^3$.

Hence, the approximate change in volume $= 0\cdot03\, x^3\ m^3$.

5. Find the approximate change in the surface area of a cube of side 'x' metres caused by decreasing the side by 1%.

[Solution : Refer Q. 16 ; Ex. 6(*d*)]

6. If the radius of a sphere is measured as 7 m with an error of 0·02 m, then find the approximate error in calculating its volume.

Solution : If 'r' be the radius of the sphere,

then $V = \frac{4}{3}\pi r^3$

so that $\frac{dV}{dr} = \frac{4}{3}\pi\left(3r^2\right) = 4\pi r^2$.

Thus $dV \simeq (4\pi r^2)\,\Delta r$

$= (4\pi 7^2)(\pm 0\cdot02) = \pm 3\cdot92\,\pi\ m^3$.

Hence, approx. error in calculating volume

$= 3\cdot92\,\pi\ m^3$.

7. If the radius of a sphere is measured as 9 m with an error of 0·03 m, then find the approximate error in calculating its surface area.

[Solution : Refer Q. 17 ; Ex. 6(*d*)]

8. If $f(x) = 3x^2 + 15x + 5$, then the approximate value of $f(3.02)$ is

(A) 47·66 (B) 57·66

(C) 67·66 (D) 77·66.

[Ans. (D)]

9. The approximate change in the volume of a cube of side x metres caused by increasing the side by 3% is :

(A) $0.06\,x^3\ m^3$ (B) $0.6\,x^3\ m^3$

(C) $0.09\,x^3\ m^3$ (D) $0.9\,x^3\ m^3$

[Ans. (C)]

Exercise 6.5

1. Find the maximum and minimum values, if any, of the following functions given by :

(*i*) $f(x) = (2x - 1)^2 + 3$ (*ii*) $f(x) = 9x^2 + 12x + 2$

(*iii*) $f(x) = -(x - 1)^2 + 10$ (*iv*) $g(x) = x^3 + 1$.

[Solutions : Refer Q. 1–2 ; Ex. 6(*e*)]

2. Find the maximum and minimum values, if any, of the following functions given by :

(*i*) $f(x) = |x + 2| - 1$ (*ii*) $g(x) = -|x + 1| + 3$

(*iii*) $h(x) = \sin(2x) + 5$ (*iv*) $f(x) = |\sin 4x + 3|$

(*v*) $h(x) = x + 1, x \in (-1, 1)$.

[Solutions : Refer Q. 2 – 3 ; Ex. 6(*e*)]

3. Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be :

(*i*) $f(x) = x^2$ (*ii*) $g(x) = x^3 - 3x$

(*iii*) $h(x) = \sin x + \cos x, 0 < x < \frac{\pi}{2}$

(*iv*) $f(x) = \sin x - \cos x, 0 < x < 2\pi$

(*v*) $f(x) = x^3 - 6x^2 + 9x + 15$

(*vi*) $g(x) = \frac{x}{2} + \frac{2}{x}, x > 0$

(*vii*) $g(x) = \frac{1}{x^2 + 2}$

(*viii*) $f(x) = x\sqrt{1 - x}, x > 0$.

[Solutions : Refer Q. 22 – 27 ; Ex. 6(*e*)]

4. Prove that the following functions do not have maxima or minima :

(*i*) $f(x) = e^x$ (*ii*) $g(x) = \log x$

(*iii*) $h(x) = x^3 + x^2 + x + 1$.

Solution : (*i*) We have : $f(x) = e^x$.

$\therefore f'(x) = e^x$.

Now $f'(x) = 0$

$\Rightarrow \quad e^x = 0,$

which gives no real value of x.

Hence, $f(x)$ does not have maximum or minimum values.

(*ii*) We have $f(x) = \log x$.

$\therefore \quad f'(x) = \frac{1}{x}$.

Now $f'(x) = 0$

$\Rightarrow \quad \frac{1}{x} = 0,$

which gives no real value of x.

Hence, $f(x)$ does not have maximum or minimum values.

(*iii*) We have : $f(x) = x^3 + x^2 + x + 1$.

$\therefore \quad f'(x) = 3x^2 + 2x + 1$.

Now $f'(x) = 0$

$\Rightarrow 3x^2 + 2x + 1 = 0$

$$\Rightarrow \quad x = \frac{-2 \pm \sqrt{4-12}}{6} = \frac{-2 \pm \sqrt{-8}}{6},$$

which are non-real.

Hence, $f(x)$ does not have maximum and minimum values.

5. Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:

(*i*) $f(x) = x^3, x \in [-2, 2]$

(*ii*) $f(x) = \sin x + \cos x, x \in [0, \pi]$

(*iii*) $f(x) = 4x - \frac{1}{2}x^2, x \in \left[-2, \frac{9}{2}\right]$

(*iv*) $f(x) = (x-1)^2 + 3, x \in [-3, 1]$.

[Solution : Refer Q. 15-17 ; Ex. 6(*e*)]

6. Find the maximum profit that a company can make, if the profit function is given by :

$$p(x) = 41 - 72x - 18x^2.$$

[Solution : Refer Q. 41(*ii*) ; Ex. 6(*f*)]

7. Find both the maximum value and the minimum value of

$3x^4 - 8x^3 + 12x^2 - 48x + 25$ on the interval $[0, 3]$.

[Solution : Refer Q. 20 ; Ex. 6(*e*)]

8. At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value ?

[Solution : Refer Q. 9 ; Ex. 6(*e*)]

9. What is the maximum value of the function :

$\sin x + \cos x$?

Solution. We have :

$f(x) = \sin x + \cos x ; x \in [0, 2\pi]$ (say)

$\therefore \quad f'(x) = \cos x - \sin x$.

For extreme values, $f'(x) = 0$

$\Rightarrow \quad \cos x - \sin x = 0$

$\Rightarrow \quad \sin x = \cos x \quad \Rightarrow \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$.

Now $f(0) = \sin 0 + \cos 0 = 0 + 1 = 1$

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$f\left(\frac{5\pi}{4}\right) = \sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

and $f(2\pi) = \sin 2\pi + \cos 2\pi = 0 + 1 = 1$.

Hence, max. value of $f(x) = \sqrt{2}$.

10. Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$.

Solution : We have :

$f(x) = 2x^3 - 24x + 107, x \in [1, 3]$.

$\therefore \quad f'(x) = 6x^2 - 24$.

(*i*) For extreme values, $f'(x) = 0$

$\Rightarrow 6x^2 - 24 = 0$

$\Rightarrow \quad x^2 = 4 \Rightarrow x = \pm 2$

$\Rightarrow \quad x = 2 \in [1, 3]$.

Now $f(1) = 2(1) - 24(1) + 107$

$= 2 - 24 + 107 = 85$

$f(2) = 2(8) - 24(2) + 107$

$= 16 - 48 + 107 = 75$

and $f(3) = 2(27) - 24(3) + 107$

$= 54 - 72 + 107 = 89$.

Hence, max. value $= 89$, which is attained at $x = 3$.

(*ii*) We have :

$f(x) = 2x^3 - 24x + 107, x \in [-3, -1]$.

$\therefore \quad f'(x) = 6x^2 - 24$.

For extreme values,

$f'(x) = 0$

$\Rightarrow 6x^2 - 24 = 0 \quad \Rightarrow x^2 - 4 = 0 \Rightarrow x^2 = 4$

$\Rightarrow \quad x = \pm 2 \Rightarrow x = -2 \in [-3, -1]$.

Now $f(-3) = 2(-27) - 24(-3) + 107$

$= -54 + 72 + 107 = 125$

$f(-2) = 2(-8) - 24(-2) + 107$

$= -16 + 48 + 107 = 139$

$f(-1) = 2(-1) - 24(-1) + 107$

$= -2 + 24 + 107 = 129$.

Hence, max. value $= 139$,

which is attained at $x = -2$.

11. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0, 2]$. Find the value of 'a'.

Solution : We have : $f(x) = x^4 - 62x^2 + ax + 9$.

$\therefore \quad f'(x) = 4x^3 - 124x + a$

and $\quad f''(x) = 12x^2 - 124$.

Now $f'(1) = 0$ [*Given*]

$\Rightarrow 4(1) - 124(1) + a = 0$

$\Rightarrow \quad a = 120.$

Now $f''(1) = 12(1) - 124 = -112 < 0.$

Thus $f(x)$ attains its maximum value at $x = 1$, where $a = 120$.

Hence, $a = 120$.

12. Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.

[Solution : Refer Q. 18 ; Ex. 6(*e*)]

13. Find two numbers whose sum is 24 and whose product is as large as possible. **(*Meghalaya B. 2016*)**

Solution : Let 'x' and 'y' be two numbers.

By the question, $x + y = 24$ ...(1)

Let the product, $P = xy \Rightarrow P = x(24 - x)$ [*Using (1)*]

$\Rightarrow P = 24x - x^2.$

$\therefore \frac{dP}{dx} = 24 - 2x.$

For P to be largest, $\frac{dP}{dx} = 0$, which gives :

$24 - 2x = 0 \Rightarrow 2x = 24 \Rightarrow x = 12.$

Now $\frac{d^2P}{dx^2} = -2$, which is – ve for $x = 12$.

Hence, product is maximum when the numbers are 12 and $(24 - 12)$ *i.e.* 12 and 12.

14. Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

[Solution : Refer Q. 6 ; Ex. 6(*f*)]

15. Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

[Solution : Refer Q. 4 ; Ex. 6(*f*)]

16. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.

[Solution : Refer Q. 5 ; Ex. 6(*f*)]

17. A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible. **(*Meghalaya B. 2018; Nagaland B. 2018*)**

Solution : Let x cm, be the side of the square cut from each corner.

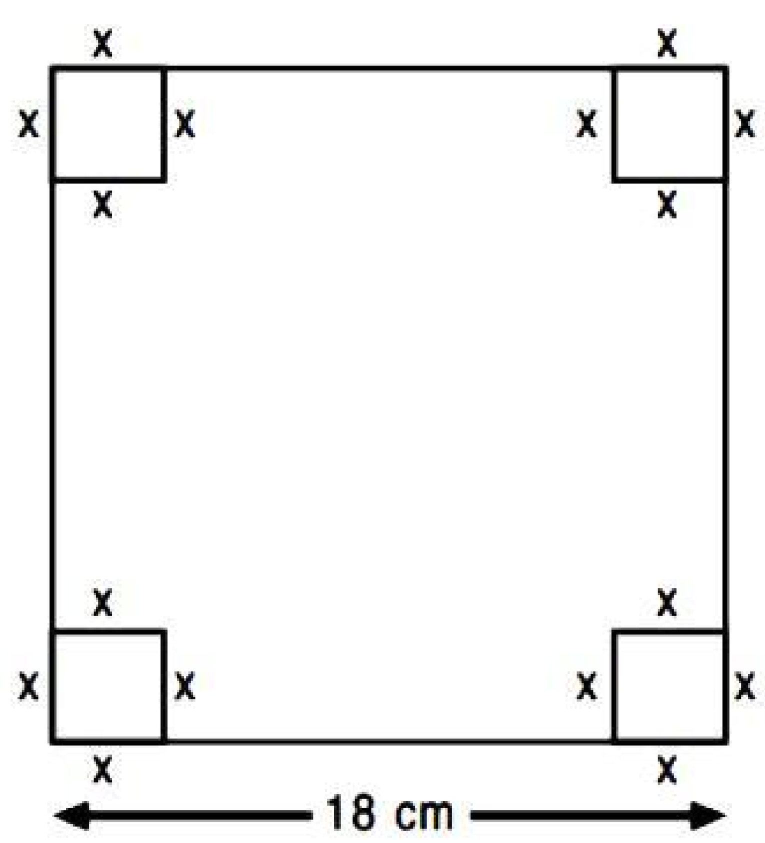


Fig.

Length of the resulting box $= 18 - 2x$;

Breadth of the resulting box $= 18 - 2x$;

Height of the resulting box $= x$.

$\therefore$ Volume $= (18 - 2x)^2 . x$

Let $V(x) = x(18 - 2x)^2.$

$\therefore V'(x) = x.2(18 - 2x)(-2) + (18 - 2x)^2(1)$

$= (18 - 2x)[-4x + 18 - 2x]$

$= (18 - 2x)(18 - 6x).$

For max./min. volume, $V'(x) = 0$

$\Rightarrow (18 - 2x)(18 - 6x) = 0$

$\Rightarrow x = 9$ or $x = 3$.

But $x = 9$ is impossible.

[$\because$ *The whole of thin sheet will be cut off*]

Thus $x = 3$.

Now $V''(x) = (18 - 2x)(-6) + (18 - 6x)(-2)$

$= -108 + 12x - 36 + 12x$

$= 24x - 144.$

$\therefore V''(x = 3) = 24(3) - 144$

$= 72 - 144 = -72,$

which is – ve.

$\therefore V(x)$ is max. when $x = 3$.

Hence, the length of the side of the square cut off from each corner = 3 cm.

18. A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum ?

[Solution : Refer Q. 38 (*i*) ; Ex. 6(*f*)]

19. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

[Solution : Refer Q. 16 ; Ex. 6(*f*)]

20. Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.

Solution : Let 'S' and 'V' be the surface area and volume of the cylinder, whose radius is 'r' and height 'h'.

$\therefore$ Surface area, $S = 2\pi r^2 + 2\pi rh$

$$\Rightarrow \quad h = \frac{S - 2\pi r^2}{2\pi r} \quad ...(1)$$

Now the volume, $V = \pi r^2 h$

$$= \pi r^2 \left[\frac{S - 2\pi r^2}{2\pi r}\right] \quad [\textit{Using (1)}]$$

$$= \frac{1}{2} r\left(S - 2\pi r^2\right)$$

$$= \frac{1}{2}\left(Sr - 2\pi r^3\right).$$

$$\therefore \frac{dV}{dr} = \frac{1}{2}\left(S - 6\pi r^2\right).$$

For max./min,, $\frac{dV}{dr} = 0$

$\Rightarrow \frac{1}{2}\left(S - 6\pi r^2\right) = 0$

$\Rightarrow \quad S = 6\pi r^2.$

From (1), $h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = \frac{4\pi r^2}{2\pi r} = 2r.$

Now $\frac{d^2 V}{dr^2} = \frac{1}{2}(-12\pi r) = -6\pi r < 0.$

$\therefore$ V is max. when $h = 2r$

i.e. when height equals the diameter of the base.

21. Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimeters, find the dimensions of the can which has the minimum surface area ?

[Solution : Refer Q. 29(*i*) ; Ex. 6(*f*)]

22. A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum ?

(P.B. 2017; H.B. 2016)

Solution : Let one piece be '*x*' m.

$\therefore$ Other piece = $(28 - x)$ m.

Let '*x*' metres be made into a circle and $(28 - x)$ metres be made into a square.

If '*r*' to the radius of the circle,

then $2\pi r = x \Rightarrow r = \frac{x}{2\pi}$.

Thus area $= \pi\left(\frac{x}{2\pi}\right)^2$ ...(1)

And side of the square $= \frac{28 - x}{4} = \left(7 - \frac{x}{4}\right)$ metres.

Its area $= \left(7 - \frac{x}{4}\right)^2$ ...(2)

Let 'A' represent their combined area.

$\therefore \quad A = \pi\left(\frac{x}{2\pi}\right)^2 + \left(7 - \frac{x}{4}\right)^2$. [*Using (1) and (2)*]

$\therefore \quad \frac{dA}{dx} = \pi \times 2\left(\frac{x}{2\pi}\right) \times \frac{1}{2\pi} + 2\left(7 - \frac{x}{4}\right)\left(\frac{-1}{4}\right)$

$= \frac{x}{2\pi} - \frac{1}{2}\left(7 - \frac{x}{4}\right)$

$= \frac{x}{2\pi} - \frac{7}{2} + \frac{x}{8}.$

For maxima/minima, $\frac{dA}{dx} = 0$

$\Rightarrow \quad \frac{x}{2\pi} - \frac{7}{2} + \frac{x}{8} = 0 \Rightarrow x\left(\frac{1}{2\pi} + \frac{1}{8}\right) = \frac{7}{2}$

$\Rightarrow \quad x\left(\frac{4 + \pi}{8\pi}\right) = \frac{7}{2} \Rightarrow x = \frac{7 \times 8\pi}{2(4 + \pi)} = \frac{28\pi}{4 + \pi}.$

Also $\quad \frac{d^2 A}{dx^2} = \frac{1}{2\pi} + \frac{1}{8}.$

$\therefore \quad \left.\frac{d^2 A}{dx^2}\right]_{x = \frac{28\pi}{4 + \pi}} = \frac{1}{2\pi} + \frac{1}{8}$, which is + ve.

Thus area A is minimum.

$\therefore$ The wire is to be cut at a distance of $\frac{28\pi}{4 + \pi}$ m from one end.

Hence, the lengths of the two pieces are :

$\frac{28\pi}{4 + \pi}$ m and $28 - \frac{28\pi}{4 + \pi}$ *i.e.* $\frac{112}{4 + \pi}$ m.

23. Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

[Solution : Refer Q. 24 ; Ex. 6(*f*)]

24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

[Solution : Refer Q. 26 ; Ex. 6(*f*)]

25. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

[Solution : Refer Q. 20(*i*) ; Ex. 6(*f*).]

26. Show that the semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.

(H.P.B. 2018; Nagaland B. 2018; Mizoram B. 2017)

Solution. Let 'S' be the given surface of the cone whose radius is '*r*', '*h*' the height and '*l*', the slant height.

$\therefore \quad S = \pi r^2 + \pi rl$ ...(1)

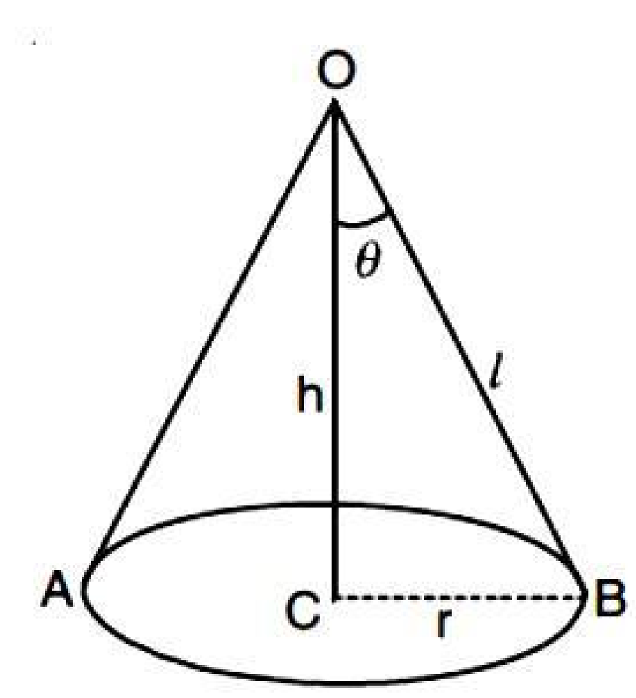


Fig.

If 'V' be the volume of the cone,

then $V = \frac{1}{3}\pi r^2 h$

$$\Rightarrow \quad V^2 = \frac{1}{9}\pi^2 r^4 h^2 = \frac{1}{9}\pi^2 r^4 (l^2 - r^2)$$

$[\because l^2 = r^2 + h^2]$

$$= \frac{1}{9}\pi^2 r^4 \left[\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right]$$

$\left[\because \text{From } (1), l = \frac{S - \pi r^2}{\pi r}\right]$

$$= \frac{1}{9}\pi^2 r^4 \left[\frac{(S - \pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2}\right]$$

$$= \frac{1}{9} r^2 (S^2 - 2\pi r^2 S)$$

$$= \frac{1}{9} S (S r^2 - 2\pi r^4).$$

Putting $V^2 = Z$ so that V is max./min. according as Z is max./min.

$$\therefore \quad Z = \frac{1}{9} S (Sr^2 - 2\pi r^4)$$

so that $\frac{dZ}{dr} = \frac{1}{9} S (2Sr - 8\pi r^3)$ ...(2)

and $\frac{d^2 Z}{dr^2} = \frac{1}{9} S (2S - 24\pi r^2)$ ...(3)

For Z to be maximum, $\frac{dZ}{dr} = 0$

$$\Rightarrow \quad \frac{1}{9} S (2Sr - 8\pi r^3) = 0$$

$$\Rightarrow \quad 2Sr - 8\pi r^3 = 0$$

$$\Rightarrow \quad S = 4\pi r^2.$$

Putting in (3), $\frac{d^2 Z}{dr^2} = \frac{1}{9} S(2S - 6S) = -\frac{4S^2}{9} < 0.$

Thus Z is max. when $S = 4\pi r^2$

$\Rightarrow$ V is max. when $S = 4\pi r^2$.

Now $S = 4\pi r^2 \Rightarrow \pi r^2 + \pi r l = 4\pi r^2$

[Using (1)]

$\Rightarrow 3\pi r^2 = \pi r l \quad \Rightarrow 3r = l \Rightarrow l = 3r$ ...(4)

If 'θ' be the semi-vertical angle of the cone,

then $\sin\theta = \frac{r}{l} \Rightarrow \sin\theta = \frac{r}{3r} = \frac{1}{3}$. *[Using (4)]*

Hence, $\theta = \sin^{-1}\left(\frac{1}{3}\right)$, which is true.

Choose the correct answer in the Exercises 27 and 29.

27. The point on the curve $x^2 = 2y$, which is nearest to the point (0, 5) is :

(A) $(2\sqrt{2}, 4)$ (B) $(2\sqrt{2}, 0)$
(C) (0, 0) (D) (2, 2).

[Ans. (A)**]**

28. For all real values of x, the minimum value of $\frac{1 - x + x^2}{1 + x + x^2}$ is :

(A) 0 (B) 1 (C) 3 (D) $\frac{1}{3}$.

[Ans. (D)**]**

29. The maximum value of $[x(x-1)+1]^{1/3}$, $0 \le x \le 1$ is :

(A) $\left(\frac{1}{3}\right)^{1/3}$ (B) $\frac{1}{2}$ (C) 1 (D) 0.

[Ans. (C)**]**

Miscellaneous Exercise on Chapter 6

1. Using differentials, find the approximate value of each of the following :

(*a*) $\left(\frac{17}{81}\right)^{1/4}$

[Solution : Refer Q. 10(*v*) ; Ex. 6(*d*)]

(*b*) $(33)^{-1/5}$

[Solution : Refer Q. 13 ; Ex. 6(*d*)]

2. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$.

[Solution : Refer Q. 24 ; Rev. Ex.]

3. The two equal sides of an isosceles triangle with fixed base 'b' are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ?

[Solution : Refer Q. 3 ; Rev. Ex.]

4. Find the equation of the normal to curve $y^2 = 4x$ at the point (1, 2).

Solution : The given curve is $y^2 = 4x$.

Diff. w.r.t. x, $2y\dfrac{dy}{dx} = 4 \Rightarrow \dfrac{dy}{dx} = \dfrac{2}{y}$.

At (1, 2), $\left.\dfrac{dy}{dx}\right]_{(1,2)} = \dfrac{2}{2} = 1$.

Slope of the normal = – 1.

∴ The equation of the normal is :

$y - 2 = (-1)(x - 1)$

$\Rightarrow x + y - 3 = 0$.

5. Show that the normal at any point 'θ' to the curve $x = a\cos\theta + a\theta\sin\theta$, $y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin.

[Solution : Refer Q. 41 ; Ex. 6(c)]

6. Find the intervals in which the function 'f' given by :

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$$

is (i) increasing (ii) decreasing.

Solution : We have :

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x} = \frac{4\sin x}{2+\cos x} - x.$$

$$\therefore \quad f'(x) = \frac{(2+\cos x).4\cos x - 4\sin x(-\sin x)}{(2+\cos x)^2} - 1$$

$$= \frac{8\cos x + 4\cos^2 x + 4\sin^2 x}{(2+\cos x)^2} - 1$$

$$= \frac{8\cos x + 4}{(2+\cos x)^2} - 1$$

$$= \frac{8\cos x + 4 - (2+\cos x)^2}{(2+\cos x)^2}$$

$$= \frac{8\cos x + 4 - 4 - \cos^2 x - 4\cos x}{(2+\cos x)^2}$$

$$= \frac{4\cos x - \cos^2 x}{(2+\cos x)^2} = \frac{\cos x(4-\cos x)}{(2+\cos x)^2}.$$

Now $(2 + \cos x)^2$, being a perfect square, is always non-negative

but $(2 + \cos x)^2 \neq 0$ as $2 + \cos x \neq 0$ [$\because$ *cos x* $\neq -2$]

∴ $(2 + \cos x)^2$ is always +ve and

$4 - \cos x$ is also always +ve.

[$\because$ *cos x is always numerically* ≤ 1]

∴ $f'(x)$ is +ve or –ve according as $\cos x$ is +ve or – ve

according as $0 < x < \dfrac{\pi}{2}$ and $\dfrac{3\pi}{2} < x < 2\pi$

or $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$.

Hence, $f(x)$ is an increasing function in $\left(0, \dfrac{\pi}{2}\right) \cup \left(\dfrac{3\pi}{2}, 2\pi\right)$

and a decreasing function in $\left(\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.

7. Find the intervals in which the function 'f' given by

$f(x) = x^3 + \dfrac{1}{x^3}$, $x \neq 0$ is :

(A) increasing (B) decreasing

[Solution : Refer Q. 18(ii) ; Ex. 6(b)]

8. Find the maximum area of an isosceles triangle inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

[Solution : Refer Q. 35 ; Rev. Ex.]

9. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per sq metres for the base and ₹ 45 per square metre for sides. What is the cost of least expensive tank ?

[Solution : Refer Q. 37 ; Ex. 6(f)]

10. The sum of the perimeter of a circle and square is k, where 'k' is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

[Solution : Refer Q. 25 ; Rev. Ex.]

11. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Solution : Let 'x' and 'y' be the length and breadth of the rectangle ABCD.

Radius of the semi-circle = $\dfrac{x}{2}$.

Circumference of the semi-circle = $\dfrac{\pi x}{2}$.

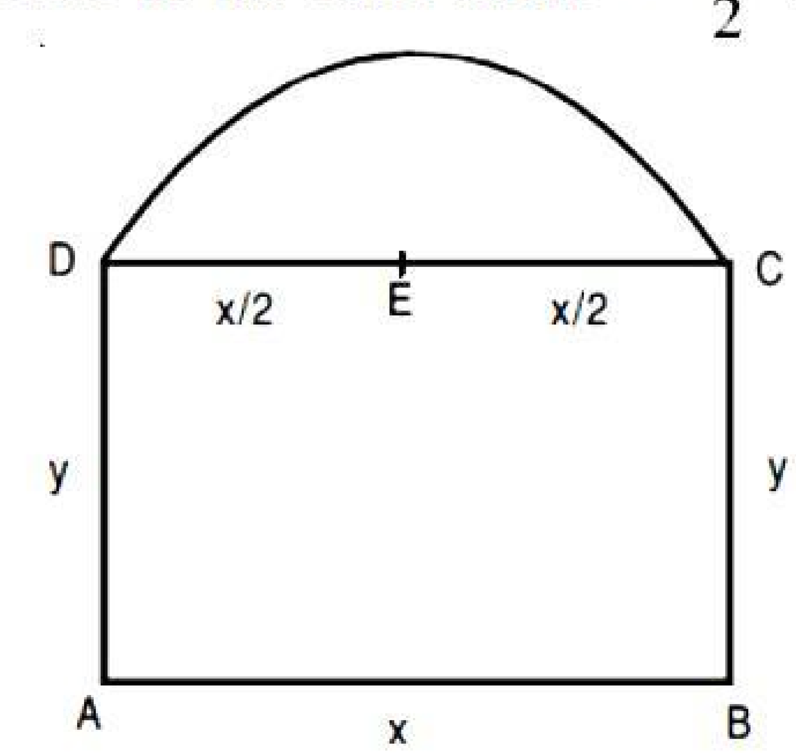


Fig.

By the question, $x + 2y + \dfrac{\pi x}{2} = 10$

$\Rightarrow \quad 2x + 4y + \pi x = 20 \Rightarrow y = \dfrac{20 - (2+\pi)x}{4}$ ....(1)

∴ Area of the figure $= xy + \dfrac{1}{2}\pi\left(\dfrac{x}{2}\right)^2$

$= x\dfrac{20-(2+\pi)x}{4} + \pi\dfrac{x^2}{8}$.

[*Using (1)*]

Thus $\quad A(x) = \frac{20x-(2+\pi)x^2}{4}+\frac{\pi x^2}{8}.$

$\therefore \quad A'(x) = \frac{20-(2+\pi)(2x)}{4}+\frac{2\pi x}{8}$

and $\quad A''(x) = \frac{-(2+\pi)2}{4}+\frac{2\pi}{8}$

$$= \frac{-4-2\pi+\pi}{4} = \frac{-4-\pi}{4}.$$

For Max./Min. of A (x), A$'(x) = 0$

$\Rightarrow \frac{20-(2+\pi)(2x)}{4}+\frac{2\pi x}{8} = 0$

$\Rightarrow 20 - (2+\pi)\,2x + \pi x = 0 \Rightarrow 20 + x(\pi - 4 - 2\pi) = 0$

$\Rightarrow 20 - x(4+\pi) = 0 \qquad \Rightarrow x = \frac{20}{4+\pi}.$

And A$''(x)$ is – ve for all x.
Hence, the area is max. *i.e.* A (x) is max.

when length $= x = \frac{20}{4+\pi}$

and $\quad$ breadth $= y = \frac{20-(2+\pi)\frac{20}{4+\pi}}{4}$

$$= \frac{80+20\pi-40-20\pi}{4(4+\pi)}$$

$$= \frac{40}{4(4+\pi)} = \frac{10}{4+\pi}.$$

And radius of semi-circle $= \frac{10}{4+\pi}.$

12. A point on the hypotenuse of a triangle is at distance 'a' and 'b' from the sides of the triangle.

Show that the minimum length of the hypotenuse is $(a^{2/3}+b^{2/3})^{3/2}$.

Solution : Let ABC be a right-triangle with hypotenuse AB.

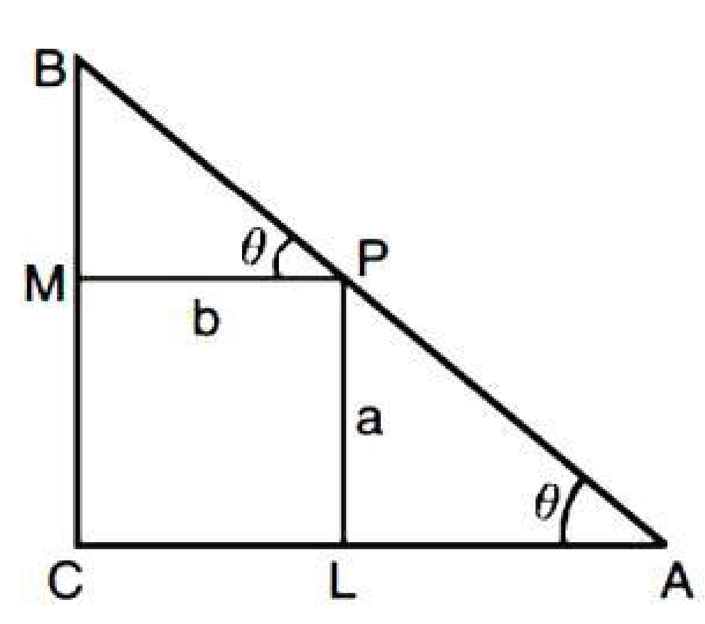


Fig.

Let P be on AB such that :

$$PL = a \text{ and } PM = b.$$

Let $\quad \angle CAB = \theta.$

$\therefore \quad AP = a \operatorname{cosec}\theta$ and $PB = b\sec\theta.$

If 'l' be the length of the hypotenuse AB,

then $\quad l = AP + PB$

$$= a\operatorname{cosec}\theta + b\sec\theta.$$

$\therefore \quad \frac{dl}{d\theta} = -a\operatorname{cosec}\theta\cot\theta + b\sec\theta\tan\theta$

and $\quad \frac{d^2l}{d\theta^2} = a\operatorname{cosec}^3\theta + a\operatorname{cosec}\theta\cot^2\theta + b\sec^3\theta + b\sec\theta\tan^2\theta.$

For maximum/minimum, $\frac{dl}{d\theta} = 0$

$\Rightarrow -a\operatorname{cosec}\theta\cot\theta + b\sec\theta\tan\theta = 0$

$\Rightarrow -\frac{a\cos\theta}{\sin^2\theta}+\frac{b\sin\theta}{\cos^2\theta} = 0$

$\Rightarrow \quad \frac{a\cos\theta}{\sin^2\theta} = \frac{b\sin\theta}{\cos^2\theta}$

$\Rightarrow \quad \tan^3\theta = \frac{a}{b}$

$\Rightarrow \quad \tan\theta = \left(\frac{a}{b}\right)^{1/3}.$

$\therefore \sin\theta = \frac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}$ and $\cos\theta = \frac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}.$

Also $\quad \frac{d^2l}{d\theta^2} > 0$ when $\tan\theta = \left(\frac{a}{b}\right)^{1/3}.$

Thus l is minimum when $\tan\theta = \left(\frac{a}{b}\right)^{1/3}.$

$\therefore$ Minimum value of l

$$= a\sqrt{1+\cot^2\theta}+b\sqrt{1+\tan^2\theta}$$

$$= a\sqrt{1+\left(\frac{b}{a}\right)^{2/3}}+b\sqrt{1+\left(\frac{a}{b}\right)^{2/3}}$$

$$= a^{2/3}\sqrt{a^{2/3}+b^{2/3}}+b^{2/3}\sqrt{b^{2/3}+a^{2/3}}$$

$$= (a^{2/3}+b^{2/3})\sqrt{a^{2/3}+b^{2/3}} = (a^{2/3}+b^{2/3})^{3/2},$$

which is true.

13. Find the points at which the function f given by : $f(x) = (x-2)^4(x+1)^3$ has :

(A) local maxima (B) local minima
(C) point of inflexion

[Solution : Refer Q. 26 ; Rev. Ex.]

14. Find the absolute maximum and minimum values of the function f given by :

$$f(x) = \cos^2 x + \sin x, x \in [0, \pi].$$

[Solution : Refer Q. 17(*ii*) ; Ex. 6(*e*)]

15. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

[Solution : Refer Q. 34 ; Rev. Ex.]

16. Let 'f' be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that 'f' is an increasing function on (a, b).

Solution : Take $x_1, x_2 \in (a, b)$ so that $x_1 < x_2$.

Let the sub–interval be $[x_1, x_2]$.

Since $f(x)$ is differentiable on (a, b) and $[x_1, x_2] \subset (a, b)$,

$\therefore f(x)$ is continuous on $[x_1, x_2]$ and differentiable in (x_1, x_2).

$\therefore$ By LMV Theorem, there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Now $f'(x) > 0$ for all $x \in (a, b) \Rightarrow f'(c) > 0$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

$\Rightarrow f(x_2) - f(x_1) > 0$ $\quad$ [$\because$ $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$]

$\Rightarrow$ $f(x_1) < f(x_2)$ if $x_1 < x_2$

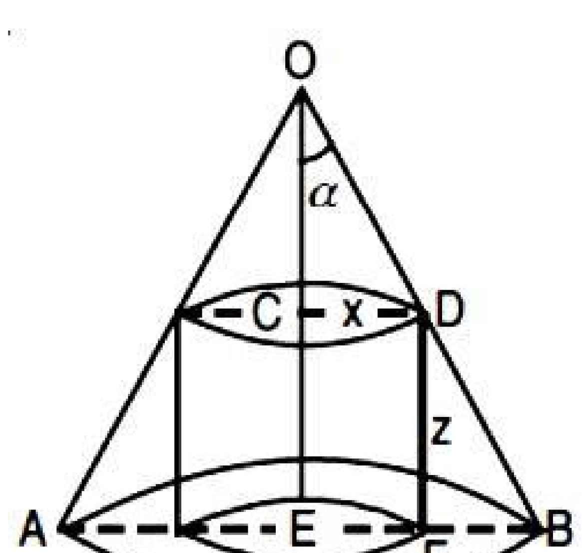
Hence, 'f' is increasing in (a, b). $\quad$ [$\because$ *x_1, x_2 are arbitrary*]

17. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

[Solution : Refer Q. 27 ; Ex. 6(*f*)]

18. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height 'h' and semi vertical angle 'α' is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

Solution : Let 'x' be the radius, 'z', the height of the cylinder.



Then OC = OE – CE = $h - z$ and CD = x.

Now $\tan \alpha = \frac{CD}{OC} = \frac{x}{h - z}$

$\Rightarrow x = (h - z) \tan \theta$ $\quad$ ...(1),

where 'θ' is the semi-vertical angle of the cone.

Clearly θ is constant. $\quad$ [$\because$ *Cone is given*]

Now V, the volume of the cylinder

$= \pi x^2 z = \pi [(h - z)^2 \tan^2 \alpha]z$ $\quad$ [*Using (1)*]

$= \pi z. (h - z)^2. \tan^2 \alpha$ $\quad$ ...(2)

V is max. when $\frac{dV}{dz} = 0$ and $\frac{d^2V}{dz^2} < 0$.

Diff. V w.r.t. z, we get :

$$\frac{dV}{dz} = \pi [(h - z)^2 \times 1 + z . 2 (h - z) (- 1)] \tan^2 \theta$$

$= \pi \tan^2 \alpha (h - z)^2 - 2z (h - z)$

$= \pi \tan^2 \alpha [h^2 - 2hz + z^2 - 2hz + z^2]$

$= \pi \tan^2 \alpha (h^2 - 4hz + 3z^2$

$= \pi \tan^2 \alpha (h - z) (h - 3z)$ $\quad$ ...(3)

Now $\frac{dV}{dz} = 0 \Rightarrow (h - z) (h - 3z) = 0$

$\Rightarrow h = z, 3z \Rightarrow z = h$ or $\frac{h}{3}$.

Clearly $z \neq h$ $\quad$ [$\because$ *Cylinder is inscribed in the cone*]

Hence $z = \frac{h}{3}$.

Diff. (3) w.r.t. z, we get :

$$\frac{d^2V}{dz^2} = \pi \tan^2 \theta (0 - 4h + 6z)$$

$$= \pi \tan^2 \theta \left(-4h + 6\frac{h}{3}\right)$$

$$= \pi \tan^2 \theta (-2h) < 0.$$

Hence, the volume of the inscribed cylinder is maximum when its height is $\frac{h}{3}$.

Radius of the cylinder = CD = OC $\tan \theta$

$$= \left(h - \frac{h}{3}\right)\tan\theta = \frac{2}{3}h \tan\theta.$$

$\therefore$ Volume of the cylinder

$$= \pi\left(\frac{2}{3}h \tan\theta\right)^2\left(\frac{h}{3}\right) = \frac{4}{27}\pi h^3 \tan^2 \theta.$$

Choose the correct answer in the Exercises from 19 to 24.

19. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of :

(A) 1 m^3/h $\quad$ (B) 0.1 m^3/h

(C) 1.1 m^3/h $\quad$ (D) 0.5 m^3/h.

[Ans. (A)]

20. The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2, –1) is :

(A) $\frac{22}{7}$ $\quad$ (B) $\frac{6}{7}$ $\quad$ (C) $\frac{7}{6}$ $\quad$ (D) $\frac{-6}{7}$

[Ans. (B)]

21. The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of m is :

(A) 1 $\quad$ (B) 2 $\quad$ (C) 3 $\quad$ (D) $\frac{1}{2}$.

[Ans. (A)]

22. The normal at the point (1, 1) on the curve $2y + x^2 = 3$ is :

(A) $x + y = 0$ (B) $x - y = 0$
(C) $x + y + 1 = 0$ (D) $x - y = 0$.

[Ans. (B)]

23. The normal to the curve $x^2 = 4y$ passing (1, 2) is :

(A) $x + y = 3$ (B) $x - y = 3$
(C) $x + y = 1$ (D) $x - y = 1$.

[Ans. (A)]

24. The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are :

(A) $\left(4, \pm\frac{8}{3}\right)$ (B) $\left(4, \frac{-8}{3}\right)$

(C) $\left(4, \pm\frac{3}{8}\right)$ (D) $\left(\pm 4, \frac{3}{8}\right)$ **[Ans.** (A)]

Questions From NCERT Exemplar

Example 1. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/s , then how fast is the slope of curve changing when x = 3 ?

Solution. We have : $y = 5x - 2x^3$.

$\therefore$ Slope of the curve, $\frac{dy}{dx} = 5 - 6x^2$.

$$\therefore \quad \frac{d}{dt}\left(\frac{dy}{dx}\right) = -12x.\frac{dx}{dt}$$

$$= -12\,(3)\,(2) = -72.$$

Hence, the slope of the curve is decreasing at the rate of 72 units/s when x is increasing at the rate of 2 units/s.

Example 2. Prove that the function f (x) = tan x – 4x is strictly decreasing on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Solution. We have : $f(x) = \tan x - 4x$.

$\therefore \quad f'(x) = \sec^2 x - 4$.

When $-\frac{\pi}{3} < x < \frac{\pi}{3}, 1 < \sec x < 2$

$\Rightarrow 1 < \sec^2 x < 4 \Rightarrow -3 < \sec^2 x - 4 < 0$.

Thus for $-\frac{\pi}{3} < x < \frac{\pi}{3}$, $f'(x) < 0$.

Hence, 'f' is strictly decreasing on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$.

Example 3. Show that the function :
$f(x) = 4x^3 - 18x^2 + 27x - 7$
has neither maximum nor minimum.

Solution. We have : $f(x) = 4x^3 - 18x^2 + 27x - 7$.

$\therefore f'(x) = 12x^2 - 36x + 27$

$= 3\,(4x^2 - 12x + 9) = 3\,(2x - 3)^2$.

Now $f'(x) = 0 \Rightarrow 3\,(2x - 3)^2 = 0 \Rightarrow (2x - 3)^2 = 0$

$\Rightarrow x = \frac{3}{2}$. **(Critical Point)**

Since $f'(x) > 0$ for all $x < \frac{3}{2}$ and for all $x > \frac{3}{2}$.

$\therefore x = \frac{3}{2}$ is a point of inflexion.

Hence, 'f' has neither maximum nor minimum.

Example 4. Find the condition for the curves : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$; $xy = c^2$ to intersect orthogonally.

Solution. The given curves are $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ...(1)

and $xy = c^2$ ...(2)

Let these curves intersect at (x_1, y_1).

Diff. (1), $\frac{2x}{a^2} - \frac{2y}{b^2}.\frac{dy}{dx} = 0 = \frac{dy}{dx} \Rightarrow \frac{b^2 x}{a^2 y}$.

At (x_1, y_1), $\frac{dy}{dx}(= m_1) = \frac{b^2 x_1}{a^2 y_1}$.

Again diff. (2), $x\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$.

At (x_1, y_1), $\frac{dy}{dx}(= m_2) = -\frac{y_1}{x_1}$.

For orthogonality, $m_1 m_2 = -1$

$\Rightarrow \left(\frac{b^2 x_1}{a^2 y_1}\right)\left(-\frac{y_1}{x_1}\right) = -1 \Rightarrow \frac{b^2}{a^2} = 1$

$\Rightarrow a^2 - b^2 = 0$, which is the reqd. condition.

Example 5. Find the difference between the greatest and least values of the function f (x) = sin 2x – x, on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Solution. We have : $f(x) = \sin 2x - x$.

$\therefore \quad f'(x) = 2\cos 2x - 1$.

Now $f'(x) = 0 \Rightarrow 2\cos 2x - 1 = 0 \Rightarrow \cos 2x = \frac{1}{2}$

$\Rightarrow \quad 2x = -\frac{\pi}{3}$ or $\frac{\pi}{3} \Rightarrow x = -\frac{\pi}{6}$ or $\frac{\pi}{6}$.

Now $f\left(-\frac{\pi}{2}\right) = \sin(-\pi) + \frac{\pi}{2} = -0 + \frac{\pi}{2} = \frac{\pi}{2}$

$f\left(-\frac{\pi}{6}\right) = \sin\left(-\frac{2\pi}{6}\right) + \frac{\pi}{6} = -\sin\frac{\pi}{3} + \frac{\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{\pi}{6}$

$f\left(\frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{6}\right) - \frac{\pi}{6} = \sin\frac{\pi}{3} - \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{\pi}{6}$

and $f\left(\frac{\pi}{2}\right) = \sin\pi - \frac{\pi}{2} = 0 - \frac{\pi}{2} = -\frac{\pi}{2}$.

Thus $\frac{\pi}{2}$ is the greatest value and $-\frac{\pi}{2}$ is the least value.

Hence, the reqd. difference $= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$.

Example 6. Find the area of greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Solution. Let ABCD be the rectangle of maximum area with sides AB = $2x$ and BC = $2y$, where C(x, y) is a point on the

ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ...(1)

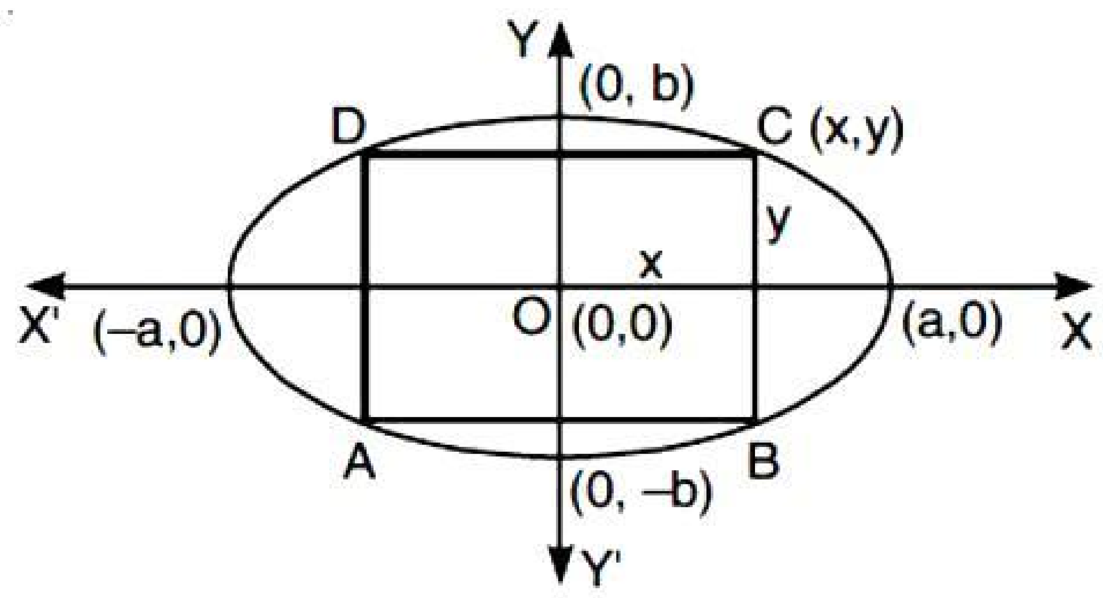


Fig.

$\therefore$ A, the area of the rectangle = $4xy$

$\Rightarrow$ $A^2 = 16x^2y^2 = s$ (say)

$\therefore\ s = 16x^2\left(1-\frac{x^2}{a^2}\right)b^2 \Rightarrow s = \frac{16b^2}{a^2}(a^2x^2 - x^4)$.

$\therefore\ \frac{ds}{dx} = \frac{16b^2}{a^2}(2a^2x - 4x^3)$.

Now $\frac{ds}{dx} = 0 \Rightarrow 2a^2x - 4x^3 = 0$

$\Rightarrow x = \frac{a}{\sqrt{2}}$ and consequently $y = \frac{b}{\sqrt{2}}$.

Now $\frac{d^2s}{dx^2} = \frac{16b^2}{a^2}(2a^2 - 12x^2)$.

At $x = \frac{a}{\sqrt{2}}, \frac{d^2s}{dx^2} = \frac{16b^2}{a^2}(2a^2 - 6a^2) = \frac{16b^2}{a^2}(-4a^2) < 0$.

Thus at $x = \frac{a}{\sqrt{2}}, y = \frac{b}{\sqrt{2}}$, s is maximum and hence, the area A is maximum.

$\therefore$ Maximum area $= 4xy = 4.\frac{a}{\sqrt{2}}.\frac{b}{\sqrt{2}} = 2ab$ sq. units.

Example 7. An isosceles triangle of vertical angle 2θ is inscribed in a circle of radius 'a'. Show that the area of triangle is maximum when $\theta = \frac{\pi}{6}$.

Solution. Let ABC be the isosceles triangle, which is inscribed in a circle of radius 'a' such that AB = AC.

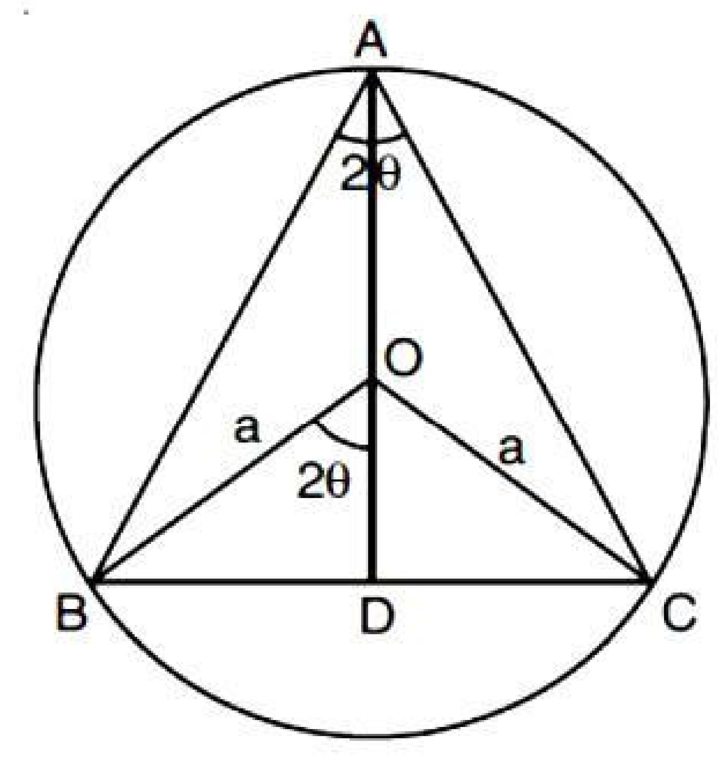


Fig.

$$AD = AO + OD$$
$$= a + a\cos 2\theta$$

and $BC = 2BD = 2a\sin 2\theta$.

$\therefore\ \Delta$, area of triangle ABC $= \frac{1}{2}$BC.AD

$\Rightarrow \Delta = \frac{1}{2}2a\sin 2\theta(a + a\cos 2\theta)$

$\Rightarrow \Delta = a^2\sin 2\theta\,(1 + \cos 2\theta)$

$\Rightarrow \Delta = a^2\sin 2\theta + \frac{1}{2}a^2\sin 4\theta$.

$\therefore\ \frac{d\Delta}{d\theta} = 2a^2\cos 2\theta + \frac{1}{2}a^2\cos 4\theta.4$

$= 2a^2(\cos 2\theta + \cos 4\theta)$.

$\therefore\ \frac{d\Delta}{d\theta} = 0 \Rightarrow 2a^2(\cos 2\theta + \cos 4\theta) = 0$

$\Rightarrow \cos 2\theta + \cos 4\theta = 0$

$\Rightarrow \cos 2\theta = -\cos 4\theta = \cos(\pi - 4\theta)$

$\Rightarrow 2\theta = \pi - 4\theta \Rightarrow 6\theta = \pi \Rightarrow \theta = \frac{\pi}{6}$.

And $\frac{d^2\Delta}{d\theta^2} = 2a^2(-2\sin 2\theta - 4\sin 4\theta)$

< 0 at $\theta = \frac{\pi}{6}$.

Hence, the area of triangle is maximum when $\theta = \frac{\pi}{6}$.

Exercise

1. A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.

2. A kite is moving horizontally at the height of 151·5 metres. If the speed of the kite is 10 m/s, how fast is the string being let out, when the kite is 250 m away from the key who is flying the kite ? The height of the boy is 1·5 m.

3. The volume of a cube is increasing at a constant rate. Prove that the increase in its surface area varies inversely as the length of the side.

4. x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of second square with respect to the area of the first square.

5. Show that for $a \geq 1, f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing in **R**.

6. Show that the function 'f' given by :

$f(x) = \tan^{-1}(\sin x + \cos x), x > 0$

is always increasing function in $\left(0, \frac{\pi}{4}\right)$.

7. Find for which values of 'x', the function :

$$y = x^4 - \frac{4x^3}{3}$$

is increasing and for which values, it is decreasing.

8. Show that $f(x) = 2x + \cot^{-1} x + \log(\sqrt{1+x^2} - x)$ is increasing in **R.**

9. Find the angle of intersection of the curves $y = 4 - x^2$ and $y = x^2$.

10. Prove that the curves $xy = 4$ and $x^2 + y^2 = 8$ touch each other.

11. What is the slope of the tangent to the curve :

$x = t^2 + 3t - 8, y = 2t^2 - 2t - 5$ at the point $(2, -1)$?

12. Using differentials, find the approximate value of $\sqrt{0 \cdot 082}$.

13. Find the approximate value of $(1{\cdot}999)^5$.

14. Find the approximate volume of metal in a hollow spherical shell, whose internal and external radii are 3 cm and 3·0005 cm respectively.

15. At what point, the slope of the curve :

$y = -x^3 + 3x^2 + 9x - 27$

is maximum ? Also, find the maximum slope.

Answers

2. 8 m/sec (approx.). **4.** $2x^2 - 3x + 1$.

7. Increasing : $(1, \infty)$ Decreasing : : $(-\infty, 1)$.

9. $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$. **11.** $\frac{6}{7}$.

12. 0·2867.

13. 31·92.

14. $0{\cdot}18\,\pi$ cm^3.

15. $(1, -16)$; Maximum slope = 12.

Revision Exercise

1. A car starts from a point P at time $t = 0$ seconds and stops at point Q. The distance x, in metres, covered by it, in t seconds is given by :

$$\mathbf{x = t^2\left(2 - \frac{t}{3}\right).}$$

Find the time taken by it to reach Q. Also find the distance between P and Q. *(N.C.E.R.T.)*

Solution : We have : $x = t^2\left(2 - \frac{t}{3}\right)$.

$\therefore \qquad v = \frac{dx}{dt} = 4t - t^2 = t(4 - t)$.

Now $v = 0 \Rightarrow t = 0$ or 4.

Now $v = 0$ at P as well as at Q.

Thus at P, $t = 0$ and at Q, $t = 4$.

Thus the car will reach the point Q after 4 seconds.

Also distance travelled in 4 seconds

$$= 4^2\left(2 - \frac{4}{3}\right) = 16\left(\frac{2}{3}\right) = \frac{32}{3} \text{ m.}$$

2. A water tank has the shape of an inverted right-circular cone with its axis vertical and vertex lower most. Its semi-vertical angle is $\tan^{-1}(0{\cdot}5)$. Water is poured into it at a constant rate of 5 cubic metre per hour. Find the rate at which the level of the water is rising at the instant when the depth of water in the tank is 4 m. *(N.C.E.R.T.)*

3. The two equal sides of an isosceles triangle with fixed base 'b' are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base ? *(N.C.E.R.T.)*

4. The bottom of a rectangular swimming tank is 25 m by 40 m. Water is pumped into the tank at the rate of 500 cubic metres per minute. Find the rate at which the level of the water in the tank is rising.

5. A ladder 13 m long leans against a wall. The foot of the ladder is pulled along the ground away from the wall at the rate of 1·5 m/sec. How fast is the angle 'θ' between the ladder and the ground changing when the foot of the ladder is 12 m away from the wall ?

6. The radius of a cylinder is increasing at the rate of 2 cm/sec and its altitude is decreasing at the rate of 3 cm/sec. Find the rate of change of volume when radius is 3 cm and altitude is 5 cm.

7. A kite is 120 m high and 130 m of string is out. If the kite is moving away horizontally at the rate of 5.2 m/sec., find the rate at which the string is being pulled out. How a festival enhance national integration ? *(C.B.S.E.)*

8. Show that the function 'f' given by :

$f(x) = \tan^{-1}(\sin x + \cos x),\ x > 0$

is always an increasing function in $\left(0, \frac{\pi}{4}\right)$.

(N.C.E.R.T.)

Solution : We have : $f(x) = \tan^{-1}(\sin x + \cos x),\ x > 0.$

$$\therefore\ f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \cdot (\cos x - \sin x)$$

$$= \frac{\cos x - \sin x}{1 + (\sin^2 x + \cos^2 x) + 2\sin x \cos x}$$

$$= \frac{\cos x - \sin x}{2 + \sin 2x}.$$

Now $2 + \sin 2x > 0$ for all x in $\left(0, \frac{\pi}{4}\right)$.

Thus $f'(x) > 0$ when $\cos x - \sin x > 0$

i.e., when $\cot x > 1$.

Now $\cot x > 1 \Rightarrow \tan x < 1 \Rightarrow 0 < x < \frac{\pi}{4}$.

Thus $f'(x) > 0$ in $\left(0, \frac{\pi}{4}\right)$.

Hence, 'f' is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

9. Let 'f' be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that 'f' is an increasing function on (a, b). *(N.C.E.R.T.)*

10. Find the equations of the tangents to the curve $y = \cos(x + y),\ -2\pi \le x \le 2\pi$ that are parallel to the line $x + 2y = 0$. *(N.C.E.R.T.)*

Solution : We have : $y = \cos(x + y)$ ...(1)

Diff. w.r.t. x, $\frac{dy}{dx} = -\sin(x + y)\left(1 + \frac{dy}{dx}\right)$

$$\Rightarrow (1 + \sin(x + y))\frac{dy}{dx} = -\sin(x + y)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sin(x + y)}{1 + \sin(x + y)}$$

$\Rightarrow$ Slope of the tangent at $(x, y) = -\frac{\sin(x + y)}{1 + \sin(x + y)}$.

Since the tangents to the curve are parallel to the line $x + 2y = 0$, [*Given*]

$$\therefore \frac{-\sin(x + y)}{1 + \sin(x + y)} = -\frac{1}{2} \qquad [\because m_1 = m_2]$$

$$\Rightarrow 2\sin(x + y) = 1 + \sin(x + y)$$

$$\Rightarrow \sin(x + y) = 1 = \sin\frac{\pi}{2}$$

$$\Rightarrow x + y = 2n\pi + \frac{\pi}{2},\ n \in \mathbf{I}.$$

Then $y = \cos(x + y) = \cos\left(2n\pi + \frac{\pi}{2}\right),\ n \in \mathbf{I}$

$= 0.$

Since $-2\pi \le x \le 2\pi$, $\therefore x = -\frac{3\pi}{2}$ and $\frac{\pi}{2}$.

Thus the tangents are parallel to the $x + 2y = 0$ at the points $\left(-\frac{3\pi}{2}, 0\right)$ and $\left(\frac{\pi}{2}, 0\right)$.

And the tangents are $y - 0 = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right)$

i.e. $2x + 4y + 3\pi = 0$

and $y - 0 = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)$

i.e. $2x + 4y - \pi = 0.$

11. Find the angle between the parabolas :

$y^2 = 4ax$ and $x^2 = 4by$

at their point of intersection other than the orgin.

(N.C.E.R.T.)

12. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on the curve $x^2y^2 = (x^2 - y^2)$.

13. Show that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point, where the curve crosses the y-axis.

14. If the straight line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that :

$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha.$$

15. If $x\cos\alpha + y\sin\alpha = p$ touches the curve $x^m y^n = a^{m+n}$, prove that :

$$p^{m+n}.m^m.n^n = (m+n)^{m+n}a^{m+n}\cos^m\alpha\sin^n\alpha.$$

16. Find the points on the curve $y = 3x^2 - 9x + 8$ at which the tangents are equally inclined to the axes.

17. The equation of the tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the values of 'a' and 'b'.

18. A circular disc of radius 3 cm is being heated. Due to expansion its radius increases at the rate of 0·05 cm/s. Find the rate at which area is increasing when radius is 3·2 cm.

(N.C.E.R.T.)

19. The radius of a sphere shrinks from 10 to 9·8 cm. Find approximately the decrease in its volume.

20. If the error committed in measuring the radius of a circle is 0·01%, find the corresponding error in calculating the area.

21. If a triangle ABC, inscribed in a fixed circle, be slightly varied in such a way as to have its vertices always on the circle, then show that :

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$

22. The area S of a triangle is calculated by measuring b, c and A. If there be an error ΔA in the measurement of A, show that the relative error in area is given by $\frac{\Delta S}{S} = \cos A . \Delta A$.

23. The pressure 'p' and volume 'V' of a gas are connected by the relation $pV^{1·4}$ = constant.

Find the percentage error in 'p' corresponding to decrease of $\frac{1}{2}$% in V.

24. Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at $x = e$. *(N.C.E.R.T.)*

25. The sum of the perimeter of a circle and square is k, where 'k' is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle. *(N.C.E.R.T.)*

26. Find the points at which the function 'f' given by :

$$f(x) = (x-2)^4 (x+1)^3$$

has (*i*) local maxima (*ii*) local minima (*iii*) point of inflexion. *(N.C.E.R.T.)*

27. Show that $\sin^p \theta \cos^q \theta$ attains a maximum when $\theta = \tan^{-1}\sqrt{\frac{p}{q}}$.

28. Which fraction exceeds its pth power by the greatest possible number ?

29. If the sum of the lengths of the hypotenuse and a side of a right-angled triangle is given, show that the area of the triangle is maximum when the angle between them is $\frac{\pi}{3}$.

30. Divide 4 into two positive numbers such that the sum of the square of one and cube of the other is a minimum.

31. A can is to be made to hold 1 litre of oil. Find the dimensions which will minimize the cost of the metal to make the can.

32. Find the shortest distance of the point (0, c) from the curve $y = x^2$, where $0 \le c \le 5$. *(N.C.E.R.T.)*

33. A beam of length 'l' is supported at one end. If W is the uniform load per unit length, the bending moment M at a distance 'x' from the end is given by $M = \frac{1}{2}lx - \frac{1}{2}Wx^2$.

Find the point on the beam at which the bending moment has maximum value.

34. Show that the altitude of the right-circular cone of maximum volume, that can be inscribed in a sphere of radius 'r', is $\frac{4}{3}r$. *(N.C.E.R.T. ; C.B.S.E. (F) 2012)*

35. Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with vertex at one end of the major axis.

(N.C.E.R.T.; Assam B. 2015 ; C.B.S.E. 2010 C)

36. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *(A.I.C.B.S.E. 2013)*

37. A window of fixed perimeter (including the base of the arc) is in the form of a rectangle surrounded by a semi-circle. The semi-circular portion is filled with coloured glass while the rectangular part is filled with clear glass. The clear glass transmits three times as much light per square metre as the coloured glass does. What is the ratio of the sides of the rectangle so that the window transmits the maximum light ?

Solution : Let 'x' and 'y' be the length and breadth of rectangular portion. Then the perimeter of the rectangular portion = $2x + 2y$

and the perimeter of the semi-circle = $\pi . \frac{x}{2}$

$\therefore$ Total perimeter = $2x + 2y + \frac{1}{2}\pi x = k$ (say) ...(1),

where 'k' is a constant.

If the amount of light per sq. metre for the coloured glass = c, then the amount of light per sq. metre for clear glass = $3c$.

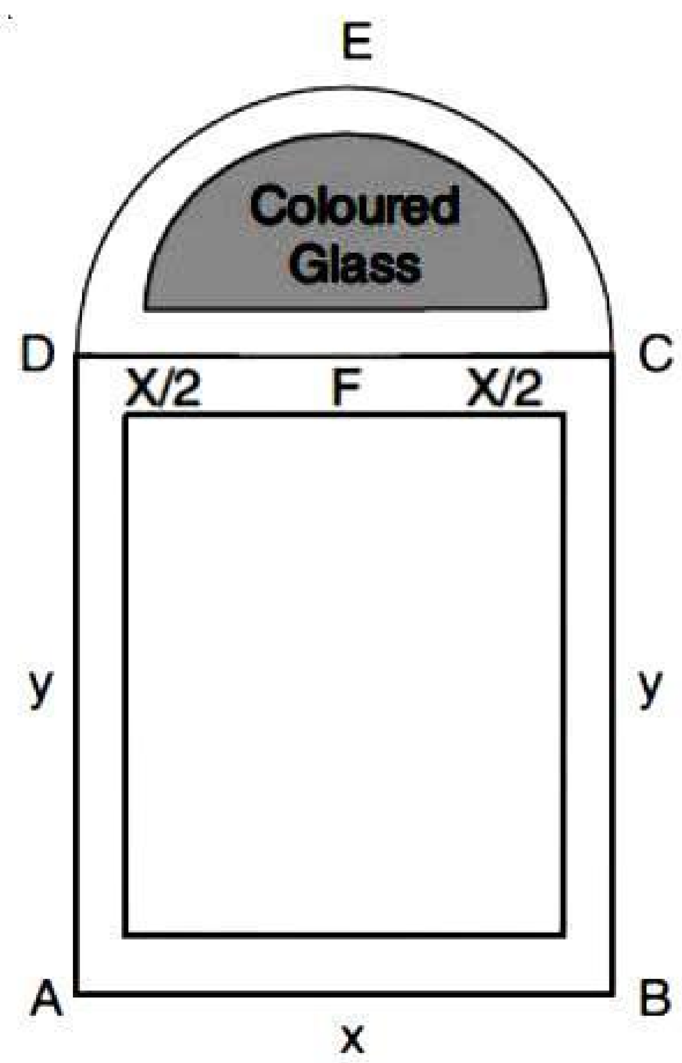


Fig.

Then S, the total amount of light, is given by :

$$S = (xy)(3c) + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 c$$

$$= \frac{c}{8}(24xy + \pi x^2)$$

$$= \frac{c}{8}\left[24x\left(\frac{k - \frac{1}{2}\pi x - 2x}{2}\right) + \pi x^2\right] \quad [Using\ (1)]$$

$$= \frac{c}{8}[12kx - 6\pi x^2 - 24x^2 + \pi x^2]$$

$$= \frac{c}{8}[12kx - 5\pi x^2 - 24x^2].$$

Then $\frac{dS}{dx} = \frac{c}{8}(12k - 10\pi x - 48x)$

and $\frac{d^2S}{dx^2} = \frac{c}{8}(-10\pi - 48).$

Now S is max. when $\frac{dS}{dx} = 0, \frac{d^2S}{dx^2} < 0.$

Now $\frac{dS}{dx} = 0 \Rightarrow \frac{c}{8}(12k - 10\pi x - 48x) = 0$

$\Rightarrow 12k - 10\pi x - 48x = 0$

$\Rightarrow x = \frac{12k}{10\pi + 48} = \frac{6k}{5\pi + 24}$ and $\frac{d^2S}{dx^2} < 0.$

Thus S is max. when $x = \frac{6k}{5\pi + 24}$

$\Rightarrow 5\pi x + 24x = 6\left[2x + 2y + \frac{1}{2}\pi x\right]$ *[Using (1)]*

$\Rightarrow x[5\pi + 24 - 12 - 3\pi] = 12y$

$\Rightarrow \frac{x}{y} = \frac{12}{2\pi + 12} = \frac{6}{\pi + 6}.$

Hence, $x : y = 6 : \pi + 6$.

Answers

2. $\frac{35}{88}$ m/h. **3.** $\sqrt{3}b\,\text{cm}^2/\text{s}$.

4. 0·5 m/m. **5.** 0·3 rad/s.

6. $33\pi\,\textbf{cm}^3/\text{s}$. **7.** 4.8 m/s. In a festival many people participate with full happiness and share their lives and enjoy it.

11. $\tan^{-1}\left(\frac{3a^{1/3}b^{1/3}}{2\left(a^{2/3} + b^{2/3}\right)}\right)$.

16. $\left(\frac{4}{3}, \frac{4}{3}\right); \left(\frac{5}{3}, \frac{4}{3}\right)$.

17. $a = 2, b = -7$. **18.** $0{\cdot}32\,\pi\ \text{cm}^2/\text{s}$.

19. $80\pi\,\text{cm}^3$. **20.** 0·02%.

23. 0·7%.

26. (*i*) Local min. at $x = 2$

(*ii*) Local max. at $x = \frac{2}{7}$

(*iii*) Point of inflexion at $x = -1$.

28. $\left(\frac{1}{p}\right)^{\frac{1}{p-1}}$. **30.** $\frac{8}{3}, \frac{4}{3}$.

31. $r = \left(\frac{500}{\pi}\right)^{1/3}$ cm, $h = \frac{1000}{\pi^{1/3}(500)^{2/3}}$ cm.

32. $\frac{\sqrt{4c-1}}{2}$. **33.** $x = \frac{l}{2W}$.

35. $\frac{3\sqrt{3}}{4}ab$ sq. units. **36.** $2ab$ sq units.

Hints to Selected Questions

4. $V = 25 \times 40 \times h$ so that $\frac{dV}{dt} = 1000.\frac{dh}{dt}$.

Obtain $\frac{dh}{dt}$ when $\frac{dV}{dt} = 500.$

5. Let the bottom of the ladder be at a distance 'x' from the wall.

Let the top be a height 'y' from the ground.

$\therefore x^2 + y^2 = 13^2$ and $\tan\theta = \frac{y}{x}$.

6. $V = \pi r^2 h, \frac{dr}{dt} = 2$ and $\frac{dh}{dt} = -3.$

$\therefore \frac{dV}{dt} = \pi\left(r^2 \frac{dh}{dt} + 2rh\frac{dr}{dt}\right) = \pi(9(-3) + 2(3)(5)(2))$

$= \pi(-27 + 60) = 33\pi.$

7. $y^2 = x^2 + (120)^2.$

$\therefore 2y\frac{dy}{dx} = 2x\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = 52\frac{x}{y}.$

14. $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ is identical with $x\cos\alpha + y\sin\alpha = p$

$\Rightarrow \frac{x_1/a^2}{\cos\alpha} = \frac{y_1/b^2}{\cos\alpha} = \frac{1}{p}$

$\Rightarrow \frac{x_1}{a} = \frac{a\cos\alpha}{p}, \frac{y_1}{b} = \frac{b\sin\alpha}{p}$. Square and add.

15. $x^m y^n = a^{m+n} \Rightarrow \left.\frac{dy}{dx}\right]_{(x_1, y_1)} = -\frac{my_1}{nx_1}.$

Tangent at (x_1, y_1) is $my_1 x + nx_1 y = (m+n)x_1 y_1.$

This is identical with $x\cos\alpha + y\sin\alpha = p.$

16. For equally inclined, $\frac{dy}{dx} = \pm 1.$

Here $\frac{dy}{dx} = \pm 1 \Rightarrow 6x - 9 = \pm 1 \Rightarrow x = \frac{5}{3}, \frac{4}{3}.$

17. (2, 3) lies on $y^2 = ax^3 + b$

$\Rightarrow 9 = 8a + b \Rightarrow 8a + b = 9$ ...(1)

Now $y = 4x - 5$ is a tangent at (2, 3)

$\therefore \left.\frac{dy}{dx}\right]_{(2,3)}$ = Slope of tangent *i.e.* $y = 4x - 5$

$\Rightarrow 2a = 4 \Rightarrow a = 2.$

19. Let 'x' be the radius and 'y' the volume.

Then $y = \frac{4}{3}\pi x^3.$

Let $x = 10, x + \Delta x = 9{\cdot}8$. Then $dx = \Delta x = -0{\cdot}2.$

Now $\frac{dy}{dx} = \frac{4}{3}\pi(3x^2) = 4\pi x^2$

$\therefore \left.\frac{dy}{dx}\right]_{x=10} = 4\pi(10)^2 = 400\pi$

$\therefore dy = \frac{dy}{dx}dx = 400\pi(-0{\cdot}2) = -80\pi.$

23. $pV^{1{\cdot}4} = k \Rightarrow \log p + 1{\cdot}4 \log V = \log k$

$\therefore \frac{1}{p}.\frac{dp}{dV} + \frac{1{\cdot}4}{V} = 0$; etc.

28. Let $y = x - x^p$. Now $\frac{dy}{dx} = 0$

$\Rightarrow x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$; etc.

31. $\pi r^2 h = 1000$

$\Rightarrow h = \frac{1000}{\pi r^2}.$

$S = (2\pi r^2 + 2\pi rh) = 2\pi r^2 + \frac{2000}{r}.$

33. $\frac{dM}{dt} = \frac{1}{2} - Wx = 0$

$\Rightarrow x = \frac{l}{2W}$ and $\frac{d^2M}{dx^2} = -W < 0.$

CHECK YOUR UNDERSTANDING

1. The radius of a soap bubble is increasing at the rate of 0·2 cm/s. Find the rate of increase of its surface area when radius = 4 cm.
Ans. 64π cm^2/s.
2. Is the function $f(x) = x^2, x \in \mathbf{R}$ increasing ?
Ans. No.
3. The function $f(x) = x^2 - 6x + 9$ is increasing for $x > 3$. ***(True/False) (Kashmir B. 2015)***
Ans. True.
4. Find the slope of the tangent to the curve $y = 3x^2 - 4x$ at the point, whose x-co-ordinate is 2.
Ans. 8.
5. Find the equation of the tangent to the curve $y = 3x^2$ at (1, 1). ***(Kerala B. 2016)***
Ans. $y = 6x - 5$.
6. The function $f(x) = x^2, x \in \mathbf{R}$ has no minimum value. *(True/False)*
Ans. True.
7. What is the absolute minimum value of $y = x^2 - 3x$ in [0, 2] ?
Ans. $-\frac{9}{4}$.
8. What are the maximum and minimum values, if any, of $f(x) = x, x \in (0, 1)$?
Ans. Neither maximum nor minimum.
9. Has the function $f(x) = x^n$ minimum value at $x = \frac{1}{e}$?
Ans. Yes.
10. Find two positive numbers whose product is 49 and their sum is minimum.
Ans. 7, 7.

SUMMARY

APPLICATION OF DERIVATIVES

DEFINITIONS AND IMPORTANT RESULTS

1. DERIVATIVE AS A RATE MEASURE

$f'(x)$ is the rate measure of $f(x)$ w.r.t. x.

2. INCREASING AND DECREASING FUNCTIONS

(*i*) A function $f(x)$ is said to be an **increasing function** of x

if $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2)$ or

$x_1 \ge x_2 \Rightarrow f(x_1) \ge f(x_2)$.

(*ii*) A function $f(x)$ is said to be a **strictly increasing function** of x

if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ or

$x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$.

(*iii*) A function $f(x)$ is said to be a **decreasing function** of x,

if $x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2)$ or

$x_1 \ge x_2 \Rightarrow f(x_1) \le f(x_2)$.

(*iv*) A function $f(x)$ is said to be a **strictly decreasing function** of x,

if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ or

$x_1 > x_2 \Rightarrow f(x_1) < f(x_2)$.

(*v*) **Monotone Function.** A function is said to be monotone if it is either increasing or decreasing.

3. TANGENTS AND NORMALS

(*i*) If $y = f(x)$, then the **slope of the tangent** at P ($x = c$) is

$$\left.\frac{dy}{dx}\right]_{x=c}.$$

(*ii*) **Equation of the tangent** at $P(x', y')$ to the curve $y = f(x)$ is $y - y' = \left(\frac{dy}{dx}\right)_P (x - x')$.

(*iii*) **Equation of the normal** at $P(x', y')$ to the curve $y = f(x)$ is $y - y' = -\frac{1}{\left(\frac{dy}{dx}\right)_P}(x - x')$.

4. MAXIMA AND MINIMA

(*i*) **Method to find absolute max. and min., values in a given interval.**

(I) Find all points when $\frac{dy}{dx} = 0$.

(II) Take end points of the interval.

(III) At all these points, calculate values of y.

(IV) Take the maximum and minimum values out of these values.

(*ii*) If $f(a)$ is an **extreme value** of $f(x)$, then $f'(a) = 0$.

(*iii*) **Local Maximum and Minimum values.**

(a) First Derivative Test. **GUIDE-LINES :** Let $y = f(x)$.

Step (i) Put $\frac{dy}{dx} = 0$. Solve it for getting $x = a, b, c, \ldots..$

Step (ii) Select $x = a$.

Study the sign of $\frac{dy}{dx}$ when (I) $x < a$ slightly

(II) $x > a$ slightly.

(*a*) If the former is +ve and latter is –ve, then $f(x)$ is max. at $x = a$.

(*b*) If the former is –ve and latter is +ve, then $f(x)$ is min. at $x = a$.

Step (iii) Putting those values of x for which $f(x)$ is max. or min. and get the corresponding max. or min. values of $f(x)$.

(b) Second Derivative Test. **GUIDE-LINES :**

Step (i) Put $y = f(x)$ and find $\frac{dy}{dx}$ *i.e.*, $f'(x)$.

Step (ii) Put $\frac{dy}{dx} = 0$ *i.e.* $f'(x) = 0$ and solve it for x giving : $x = a, b, c, \ldots..$

Step (iii) Select $x = a$. Find $\frac{d^2y}{dx^2}$ *i.e.* $f''(x)$ at $x = a$.

(I) If $\left.\frac{d^2y}{dx^2}\right]_{x=a}$ *i.e.* $f''(a)$ is –ve, then $x = a$ gives the max. value.

(II) If $\left.\frac{d^2y}{dx^2}\right]_{x=a}$ *i.e.* $f''(a)$ is +ve, then $x = a$ gives the min. value.

Similarly for $x = b, c, \ldots\ldots$

MULTIPLE CHOICE QUESTIONS

For Board Examinations

1. The interval, in which $y = x^2e^{-x}$ is increasing is :

(A) $(-\infty, \infty)$ (B) $(-2, 0)$
(C) $(2, \infty)$ (D) $(0, 2)$. *(H.P.B. 2018, 15)*

2. The slope of tangent to the curve :

$x = a\cos^3\theta, y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$ is :

(A) 1 (B) 2
(C) -1 (D) None of these.

(H.B. 2018)

3. The slope of normal to the curve :

(*i*) $x = a\cos^3\theta, y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$ is :

(A) 1 (B) -1
(C) 3 (D) -2. *(H.B. 2018)*

4. The maximum and minimum values of function

$f(x) = \sin 3x + 4$ are respectively :

(A) 5 and 3 (B) 6 and 4
(C) 4 and 3 (D) None of these.

(H.B. 2018)

5. The function $f(x) = \cos x - \sin x$ has maximum or minimum value at $x = ...$

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$
(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{3}$. *(H.B. 2018)*

6. The angle x, which increases twice as fast as it size is :

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$
(C) π (D) $\frac{3\pi}{2}$. *(Nagaland B. 2018)*

7. $f(x)$ is a strictly increasing function, if $f'(x)$ is :

(A) positive (B) negative
(C) 0 (D) None of these.

(Kerala B. 2018)

8. The approximate change in the volume V of a cube of side x metres caused by increasing the side by

(*i*) 2% is :

(A) $0.06\ x^3m^3$ (B) $0.02\ x^3m^3$
(C) $0.6\ x^3m^3$ (D) $0.006\ x^3m^3$.

(H.P.B. 2017)

9. The rate of change of the area of a circle with respect to its radius at :

$r = 5$, is :

(A) 10π (B) 8π
(C) 12π (D) 13π. *(H.B. 2017)*

10. The radius of a circle is increasing at the rate of 0.7 cm/sec. The rate of increase of its circumference is :

(A) 3.3π cm/sec (B) 1.4π cm/sec
(C) 2.2π cm/sec (D) 4.4π cm/sec.

(H.B. 2017)

11. (*i*) The point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$, is :

(A) $(-2, 0)$ (B) $(3, 7)$
(C) $(0, 2)$ (D) $(2, -9)$. *(H.B. 2017)*

12. If the rate of change of area of a circle is equal to the rate of change of its diameter, then its radius is :

(A) $\frac{1}{\pi}$ (B) $\frac{2}{\pi}$
(C) $\frac{\pi}{2}$ (D) π. *(Nagaland B. 2017)*

13. The volume of a cube is increasing at the rate of $9\ cm^3/s$. How fast is surface area increasing when the length of an edge is 10 cm ?

(A) $1.8\ cm^2/s$ (B) $2.7\ cm^2/s$
(C) $3.6\ cm^2/s$ (D) None of these.

(H.P.B. 2016)

14. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/s. The volume of the bubble increasing at the rate :

When radius 1 cm, is :

(A) $2\pi\ cm^3/s$ (B) $3\pi\ cm^3/s$
(C) $\frac{3}{2}\pi\ cm^3/s$ (D) None of these.

(H.B. 2016)

15. The rate of change of volume of a sphere with respect to its radius when radius is 1 unit is :

(A) 4π (B) 2π
(C) π (D) $\frac{\pi}{2}$. *(Kerala B. 2017)*

16. The slope of the tangent to the curve given by :

$x = 1 - \cos\theta, y = \theta - \sin\theta$ at $\theta = \frac{\pi}{2}$ is :

(A) 0 (B) -1
(C) 1 (D) Not defined.

(Kerala B. 2016)

17. Given $P(x) = x^4 + ax^3 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$.

If $P(-1) < P(1)$, then in the interval $[-1, 1]$.

(A) $P(-1)$ is the minimum and $P(1)$ is the maximum of P

(B) $P(-1)$ is not minimum but $P(1)$ is the maximum of P

(C) $P(-1)$ is the minimum but $P(1)$ is not the maximum of P

(D) Neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P. *(A.I.E.E.E. 2009)*

18. The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the x-axis is :

(A) $y = 0$ (B) $y = 1$
(C) $y = 2$ (D) $y = 3$. *(A.I.E.E.E. 2010)*

19. Let $f : \mathbf{R} \to \mathbf{R}$ be a positive increasing function with :

$\lim_{x\to\infty} \frac{f(3x)}{f(x)} = 1$. Then $\lim_{x\to\infty} \frac{f(2x)}{f(x)} =$

(A) 1 (B) $\frac{2}{3}$

(C) $\frac{3}{2}$ (D) 3. *(A.I.E.E.E. 2010)*

20. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by :

$$f(x) = \begin{cases} k - 2x, \text{if } x \le -1 \\ 2x + 3, \text{if } x > -1. \end{cases}$$

If 'f has a local maximum at $x = -1$, then a possible value of k is :

(A) 1 (B) 0

(C) $-\frac{1}{2}$ (D) -1. *(A.I.E.E.E. 2010)*

21. The shortest distance between $y - x = 1$ and curve $x = y^2$ is :

(A) $\frac{\sqrt{3}}{4}$ (B) $\frac{3\sqrt{2}}{8}$

(C) $\frac{8}{3\sqrt{2}}$ (D) $\frac{4}{\sqrt{3}}$. *(A.I.E.E.E. 2011)*

22. A spherical balloon is filled with $4500\,\pi$ cubic metres of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic metres per minute, then the rate (in metres per minute) at which the radius of the balloon decreases 49 minutes after the leakage begins is :

(A) $\frac{9}{7}$ (B) $\frac{7}{9}$

(C) $\frac{2}{9}$ (D) $\frac{9}{2}$. *(A.I.E.E.E. 2012)*

23. The real number k for which the equation $2x^3 + 3x + k = 0$ has two distinct real roots in $[0, 1]$:

(A) lies between 2 and 3

(B) lies between -1 and 0

(C) does not exist

(D) lies between 1 and 2. *(J. E. E. (Main) 2013)*

24. If f and g are differentiable functions on $[0, 1]$ satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$:

(A) $2f'(c) = 3g'(c)$ (B) $f'(c) = g'(c)$
(C) $f'(c) = 2g'(c)$ (D) $2f'(c) = g'(c)$.

(J.E.E. (Main) 2014)

25. If $x = -1$ and $x = 2$ are extreme points of :
$f(x) = \alpha \log|x| + \beta x^2 + x$, then :

(A) $\alpha = -6, \beta = \frac{-1}{2}$ (B) $\alpha = 2, \beta = \frac{-1}{2}$

(C) $\alpha = 2, \beta = \frac{1}{2}$ (D) $\alpha = -6, \beta = \frac{1}{2}$.

(J.E.E. (Main) 2014)

26. The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at $(1, 1)$:

(A) does not meet the curve again

(B) meets the curve again in the second quadrant

(C) meets the curve again in the third quadrant

(D) meets the curve again in the fourth quadrant.

(JEE (Main) 2015)

27. Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$. If $\lim_{x\to 0}\left[1 + \frac{f(x)}{x^2}\right] = 3$, then $f(2)$ is equal to :

(A) -8 (B) -4
(C) 0 (D) 4.

(JEE (Main) 2015)

28. Consider $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x \in \left(0, \frac{\pi}{2}\right)$.

A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point :

(A) $\left(0, \frac{2\pi}{3}\right)$ (B) $\left(\frac{\pi}{6}, 0\right)$

(C) $\left(\frac{\pi}{4}, 0\right)$ (D) $(0, 0)$.

(J.E. E. (Main)2016)

29. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. if the sum of the areas of the square and the circle so formed is minimum, then :

(A) $(4-\pi)x = \pi r$ (B) $x = 2r$
(C) $2x = r$ (D) $2x = (\pi + 4)\ r$.

(J.E. E. (Advances) 2016)

30. The least value of $a \in \mathbf{R}$ for which $4ax^2 + \frac{1}{x} \geq 1$ for all $x > 0$, is :

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$
(C) $\frac{1}{27}$ (D) $\frac{1}{25}$.

(J.E. E. (Advanced) 2016)

31. The normal to the curve $y(x-2)(x-3) = x+6$ at the point where the curve intersects the y-axis, passes through the point :

(A) $\left(\frac{1}{2}, -\frac{1}{3}\right)$ (B) $\left(\frac{1}{2}, \frac{1}{3}\right)$
(C) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (D) $\left(\frac{1}{2}, \frac{1}{2}\right)$.

(J.E.E. (Main) 2017)

32. Twenty metres of wave is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower bed is :

(A) 25 (B) 30
(C) 12.5 (D) 10.

(J.E.E. (Main) 2017)

33. If the curves :
$y^2 = 6x,\ 9x^2 + by^2 = 16$
intersect each other at right angles, then the value of 'b' is :

(A) 6 (B) $\frac{7}{2}$
(C) 4 (D) $\frac{9}{2}$.

(J.E.E. (Main) 2018)

34. Let $f(x) = x^2 - \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}, x \in \mathbf{R} - \{-1, 0, 1\}$.

If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is :

(A) 3 (B) -3
(C) $-2\sqrt{2}$ (D) $2\sqrt{2}$.

Answers

1. (D)	**2.** (C)	**3.** (A)	**4.** (D)	**5.** (B)	**6.** (A)	**7.** (A)	**8.** (A)	**9.** (A)	**10.** (B)
11. (A)	**12.** (A)	**13.** (C)	**14.** (A)	**15.** (C)	**16.** (B)	**17.** (B)	**18.** (D)	**19.** (A)	**20.** (D)
21. (B)	**22.** (C)	**23.** (C)	**24.** (C)	**25.** (B)	**26.** (D)	**27.** (C)	**28.** (A)	**29.** (B)	**30.** (C)
31. (D)	**32.** (B)	**33.** (D)	**34.** (D).						

Hints/Solutions

RCQ Pocket

17. (B) We have : $P(x) = x^4 + ax^3 + bx^2 + cx + d.$

$\therefore \quad P'(x) = 4x^3 + 3ax^3 + 2bx + c.$

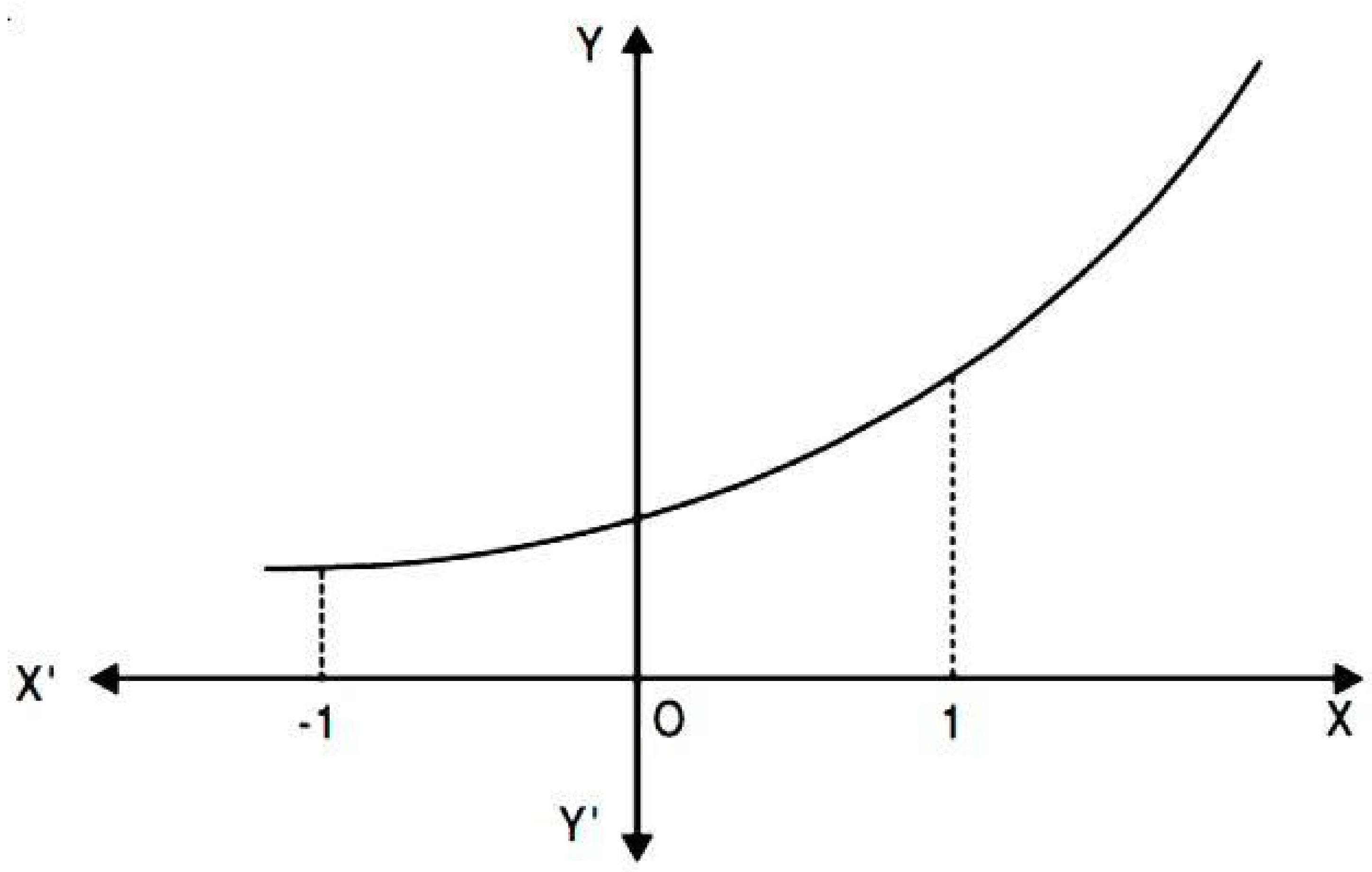
Now $\quad P'(0) = 0 \Rightarrow c = 0.$

$\therefore \quad P'(x) = 4x^3 + 3ax^2 + 2bx$
$= x(4x^2 + 3ax + 2b).$

Since $P'(x) = 0$ has no real root except $x = 0$,

$\therefore$ Disc. of $4x^2 + 3ax + 2b$ is < 0

i.e. $\quad 9a^2 - 32b < 0.$



18. (D) Here $\quad y = x + \frac{4}{x^2} \quad ...(1)$

$\therefore \quad \frac{dy}{dx} = 1 - \frac{8}{x^3}.$

Since the tangent is parallel to x-axis, $\therefore \frac{dy}{dx} = 0$

$\Rightarrow 1 - \frac{8}{x^3} = 0 \Rightarrow x^3 = 8 \Rightarrow x = 2.$

Putting in (1), $y = 2 + \frac{4}{4} = 2 + 1 = 3.$

$\therefore$ The equation of the tangent is $y - 3 = 0.\ (x-2)$

$\Rightarrow \quad y = 3.$

19. (A) Since $f(x)$ is a positive increasing function,

$\therefore \quad 0 < f(x) < f(2x) < f(3x)$

$\Rightarrow \quad 0 < 1 < \frac{f(2x)}{f(x)} < \frac{f(3x)}{f(x)}$

[*Dividing by $f(x)$*]

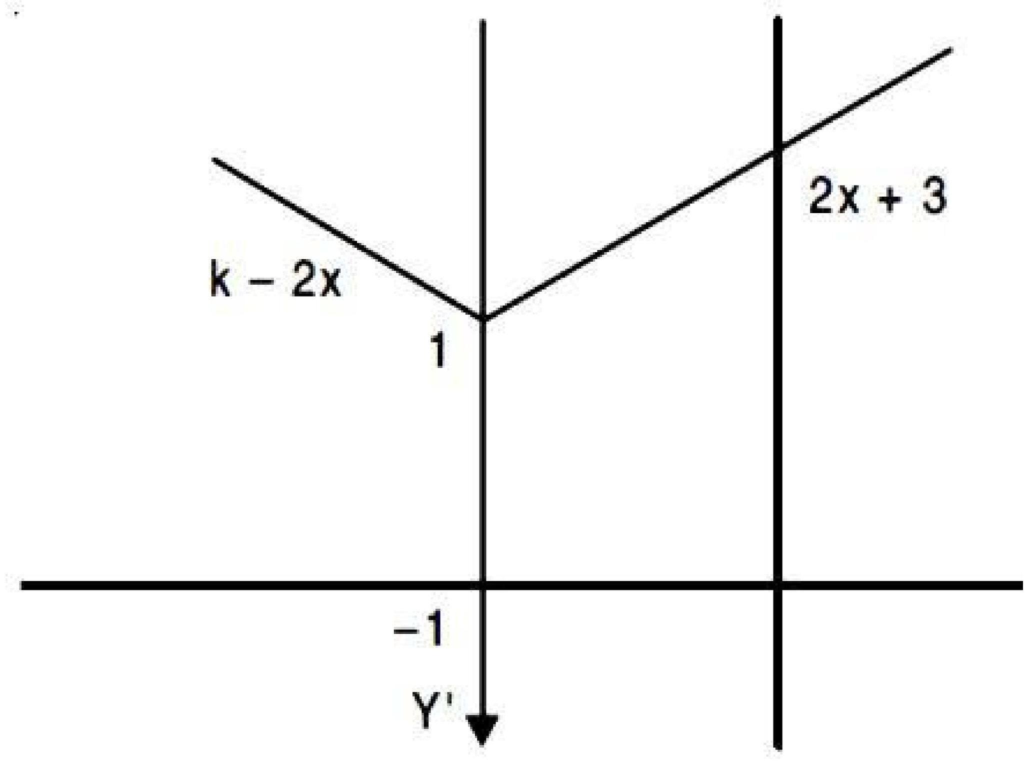
Taking limits,

$$\lim_{x \to \infty} 1 \le \lim_{x \to \infty} \frac{f(2x)}{f(x)} \le \lim_{x \to \infty} \frac{f(3x)}{f(x)}$$

$$\Rightarrow 1 \le \lim_{x \to \infty} \frac{f(2x)}{f(x)} \le 1.$$

By *Sequeeze Principle*, $\lim_{x \to \infty} \frac{f(2x)}{f(x)} = 1.$

20. (D) Here $f(x) = k - 2x$ if $x \le -1$
$= 2x + 3$ if $x > -1$.



Now $k - 2x > 1, k + 2 = 1$

$\Rightarrow k > 1 + 2x, k = -1$

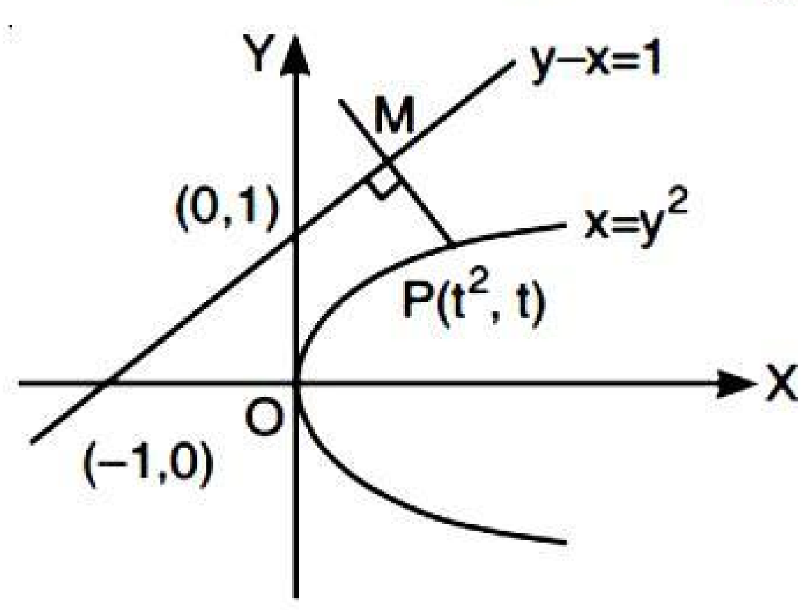
$\Rightarrow k > 1 + 2(-1)$

$\Rightarrow k > -1.$

21. (B) Shortest distance between line and the curve occurs along the common normal.

Now $\frac{dy}{dx} = 1 \Rightarrow -\frac{dx}{dy} = -1.$

And $1 = 2y\frac{dy}{dx} \Rightarrow -\frac{dx}{dy} = -\frac{1}{2y}.$



$\Rightarrow 1 = \frac{1}{2y} \Rightarrow 1 = \frac{1}{2t} \Rightarrow t = \frac{1}{2}.$

$\therefore$ Point P is $\left(\frac{1}{4}, \frac{1}{2}\right)$.

$\therefore$ Shortest distance $= |PM| = \frac{\left|\frac{1}{2} - \frac{1}{4} - 1\right|}{\sqrt{1+1}}$

$= \frac{\frac{3}{4}}{\sqrt{2}} = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$ units.

22. (C) $V = \frac{4}{3}\pi r^3$ ...(1)

After 49 minutes, volume $= 4500\pi - 49(72\pi) = 972\pi$.

$\therefore \frac{4}{3}\pi r^3 = 972\pi \Rightarrow r^3 = 729 \Rightarrow r = 9.$

Diff. (1) w.r.t. t, $\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$

$\Rightarrow 72\pi = 4\pi r^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{72\pi}{4\pi r^2} = \frac{18}{r^2}.$

Hence $\left.\frac{dr}{dt}\right]_{r=9} = \frac{18}{9 \times 9} = \frac{2}{9}.$

23. (C) If $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1], then $f'(x)$ will change sign.

But $f'(x) = cx^2 + 3 > 0.$

Hence, no value of k exists.

24. (C) Let $h(x) = f(x) - 2g(x)$.

$\therefore h'(x) = f'(x) - 2g'(x).$

Here $h(0) = h(1) = 2.$

$\therefore$ *By Rolle's Theorem*, $h'(c) = 0$

$\Rightarrow f'(c) = 2g'(c).$

25. (B) $f(x) = \alpha \log|x| + \beta x^2 + x.$

$\therefore f'(x) = \frac{\alpha}{x} + 2\beta x + 1.$

Since $x = -1$ and $x = 2$ are extreme points,

$\therefore f'(-1) = 0$ and $f'(2) = 0$

$\Rightarrow \frac{\alpha}{-1} + 2\beta(-1) + 1 = 0$ and $\frac{\alpha}{2} + 2\beta(2) + 1 = 0$

$\Rightarrow \alpha + 2\beta = 1$ ...(1)

and $\alpha + 8\beta + 2 = 0$ .....(2)

(2) – (1) gives : $6\beta = -3 \Rightarrow \beta = -\frac{1}{2}.$

Putting in (1), $\alpha - 1 = 1 \Rightarrow \alpha = 2.$

Hence, $\alpha = 2, \beta = -\frac{1}{2}.$

26. (D) The given curve is $x^2 + 2xy - 3y^2 = 0$ ...(1)

Diff w.r.t. x, $2x + 2\left(x\frac{dy}{dx} + y\right) - 6y\frac{dy}{dx} = 0$

$\Rightarrow 2x + 2x\frac{dy}{dx} + 2y - 6y\frac{dy}{dx} = 0$

$\Rightarrow (2x - 6y)\frac{dy}{dx} + (2x + 2y) = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x+y}{x-3y}.$

$\therefore \quad \left.\frac{dy}{dx}\right]_{(1,1)} = -\frac{1+1}{1-3} = 1.$

Slope of the normal = – 1.

∴ The equation of the normal at (1, 1) is :

$y - 1 = (-1)(x - 1)$

$\Rightarrow x + y = 2 \Rightarrow y = 2 - x$ ...(2)

Putting in (1), $x^2 + 2x(2-x) - 3(2-x)^2 = 0$

$\Rightarrow x^2 + 4x - 2x^2 - 3(4 - 4x + x^2) = 0$

$\Rightarrow x^2 + 4x - 2x^2 - 12 + 12x - 3x^2 = 0$

$\Rightarrow -4x^2 + 16x - 12 = 0 \Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1, 3.$

When $x = 1$, then from (2), $y = 2 - 1 = 1$.

When $x = 3$, then from (2), $y = 2 - 3 = -1$.

Hence, the curve meets at (3, – 1) *i.e.* in the fourth quadrant.

27. (C) $\lim_{x\to 0}\left(1+\frac{f(x)}{x^2}\right) = 3 \Rightarrow 1 + \lim_{x\to 0}\left(\frac{f(x)}{x^2}\right) = 3$

$\Rightarrow \lim_{x\to 0}\frac{f(x)}{x^2} = 2$

$\Rightarrow f(x) = ax^4 + bx^3 + 2x^2 + 0.x + 0.$

$\therefore f'(x) = 4ax^3 + 3bx^2 + 4x.$

$\therefore f'(1) = 0 \Rightarrow 4a + 3b + 4 = 0$ ...(1)

and $f'(2) = 0 \Rightarrow 32a + 12b + 8 = 0$

$\Rightarrow 8a + 3b + 2 = 0$ ...(2)

Solving (1) and (2), $a = \frac{1}{2}, b = -2.$

$\therefore \quad f(x) = \frac{1}{2}x^4 - 2x^3 + 2x^2.$

Hence, $\quad f(2) = 8 - 16 + 8 = 0.$

28. (A)

$f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$

$= \tan^{-1}\left(\frac{1+\sin x}{\sqrt{1-\sin^2 x}}\right) = \tan^{-1}\left(\frac{1+\sin x}{\sqrt{\cos^2 x}}\right)$

$= \tan^{-1}\left(\frac{1+\sin x}{\cos x}\right)$

$= \tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right) = \tan^{-1}\left(\frac{1+\tan x/2}{1-\tan x/2}\right)$

$[\because x \in (0, \frac{\pi}{2})]$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{2}+\frac{x}{2}\right)\right) = \frac{\pi}{4} + \frac{x}{2}.$

$\therefore \quad f'(x) = \frac{1}{2}.$

∴ Slope of normal = –2.

∴ Equation of normal is $y - \frac{\pi}{3} = -2\left(x - \frac{\pi}{6}\right)$

$\left[\because \textit{When } x = \frac{\pi}{6}, y = \frac{\pi}{3}\right]$

$\Rightarrow \quad 2x + y = \frac{2\pi}{3}.$

Hence, it passes through $\left(0, \frac{2\pi}{3}\right)$.

29. (B) Here $4x + 2\pi r = 2$

$\Rightarrow \quad r = \frac{2-4x}{2\pi} \Rightarrow r = \frac{1-2x}{\pi}$ ...(1)

Now $\quad A = x^2 + \pi r^2$

$= x^2 + \frac{\pi}{\pi^2}(1-2x)^2$ [*Using (1)*]

$= x^2 + \frac{1}{\pi}(2x-1)^2.$

For Max./Min., $\frac{dA}{dx} = 0 \Rightarrow 2x + \frac{4}{\pi}(2x-1) = 0$

$\Rightarrow x = \frac{2}{\pi+4}.$

Also $\frac{d^2A}{dx^2} > 0$ for $x = \frac{2}{\pi+4}.$

From (1), $r = \frac{1}{\pi+4}$. Comparing, $x = 2r$.

30. (C) Let $f(x) = 4ax^2 + \frac{1}{x} (x > 0).$

$\therefore \quad f'(x) = 8ax - \frac{1}{x^2}$

and $f''(x) = 8ax + \frac{2}{x^3}.$

Now $f'(x) = 0 \Rightarrow 8ax - \frac{1}{x^2} = 0 \Rightarrow x^3 = \frac{1}{8a}$

$\Rightarrow x = \left(\frac{1}{8a}\right)^{1/3}.$

Then 'f' attains its minimum at $x_0 = \left(\frac{1}{8a}\right)^{1/3}.$

Since $f(x) \geq 1 \;\forall\; x > 0,$

$\therefore \quad f(x) \geq 1$

$\Rightarrow \quad 4ax_0^2 + \frac{1}{x_0} \geq 1 \Rightarrow 4ax_0^3 + 1 \geq x_0$

$\Rightarrow 4a\left(\frac{1}{8a}\right) + 1 \geq \left(\frac{1}{8a}\right)^{\frac{1}{3}}$

$$\Rightarrow \quad \frac{3}{2} \geq \left(\frac{1}{8a}\right)^{\frac{1}{3}} \quad \geq \frac{27}{8} \geq \frac{1}{8a}$$

$$\Rightarrow \quad a \geq \frac{1}{27}.$$

Hence, the least value of $a \in \mathbf{R} = \frac{1}{27}$.

31. (D) The given curve is $y = \dfrac{x+6}{(x-2)(x-3)}$.

This meets y-axis at (0, 1).

Now, $$\frac{dy}{dx} = \frac{(x^2-5x+6)(1)-(x+6)(2x-5)}{(x^2-5x+6)^2}$$

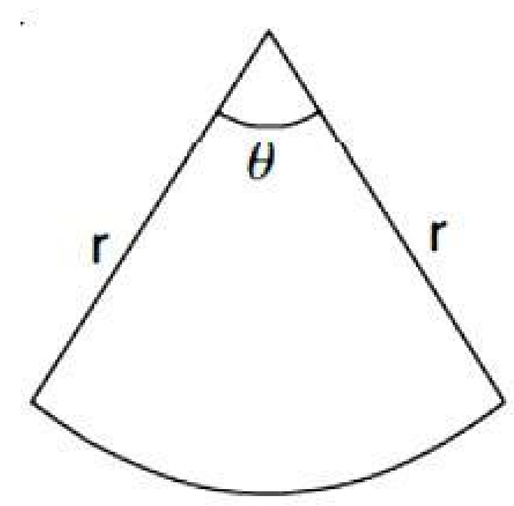
$$\therefore \left.\frac{dy}{dx}\right]_{(0,1)} = \frac{6+30}{36} = 1.$$

Slope of the normal = – 1.

The equation of the normal is $y - 1 = -1.(x - 0)$

$\Rightarrow x + y = 1$, which passes through $\left(\frac{1}{2}, \frac{1}{2}\right)$.

32. (A) Total length = $r + r + 2\theta = 20$



$$\Rightarrow \quad \theta = \frac{20-2r}{r} \quad ...(1)$$

$$\therefore \quad \text{Area, A} = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}r^2\,\frac{20-2r}{r}$$

$$= 10r - r^2 \quad ...(2)$$

$$\therefore \quad \frac{dA}{dr} = 10 - 2r = 0 \Rightarrow r = 5$$

and $$\frac{d^2A}{dr^2} = -2 \ (-\text{ ve})$$

$\therefore$ $r = 5$ gives max-area.

Hence, from (2) maximum area, A = 10(5) – 25 = 25.

33. (D) The given curves are : $y^2 = 6x$ ...(1)

and $9x^2 + by^2 = 16$ ...(2)

Diff. (1) w.r.t. x, $2y\dfrac{dy}{dx} = 6 \Rightarrow \dfrac{dy}{dx} = \dfrac{3}{y}$.

$\therefore$ Slope of the tangent at (x_1, y_1), $m_1 = \dfrac{3}{y_1}$.

Diff. (2) w.r.t. x, $18x + 2by\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = -\dfrac{9x}{by}$.

$\therefore$ Slope of the tangent at (x_1, y_1), $m_2 = -\dfrac{9x_1}{by_1}$.

By the question, $m_1 m_2 = -1$

$$\Rightarrow \left(\frac{3}{y_1}\right)\left(-\frac{9x_1}{by_1}\right) = -1 \Rightarrow \frac{-27x_1}{by_1^2} = -1$$

$$\Rightarrow \frac{-27x_1}{b(6x_1)} = -1 \quad [\because (x_1, y_1) \text{ lies on(1)} \Rightarrow y_1^2 = 6x_1]$$

Hence, $b = \dfrac{9}{2}$.

34. (D) Here, $$h(x) = \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} = \frac{\left(x - \frac{1}{x}\right)^2 + 2}{x - \frac{1}{x}}$$

$$= \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}}$$

$$= t + \frac{2}{t}; t > 2 \text{ putting } x - \frac{1}{x} = t.$$

Now AM $\geq$ GM

$$\Rightarrow \quad \frac{t + 2/t}{2} \geq \sqrt{t.\frac{2}{t}}.$$

Hence, $$t + \frac{2}{t} \geq 2\sqrt{2}.$$

CHAPTER TEST 6

Time Allowed : 1 Hour **Max. Marks : 34**

Notes : **1.** *All questions are compulsory.*
2. *Marks have been indicated against each question.*

1. The radius of a circle is increasing at the rate of 0·7 cm/s.
What is the rate of increasing of its circumference ? **(1)**

2. Show that the function $f(x) = x^3 - 3x^2 + 4x$, $x \in \mathbf{R}$ is strictly increasing on **R.** **(1)**

3. Find the slope of the tangent to the curve :
$y = x^3 - 3x + 2$ at the point whose x-co-ordinate is 3. **(2)**

4. A man 2 metres high walks at a uniform speed of 5 km/h away from a lamp-post 6 metres high. Find the rate at which the length of his shadow increases. **(2)**

5. Find the intervals in which the function 'f' given by :
$f(x) = \sin x + \cos x,\ 0 \leq x \leq 2\pi$
is strictly increasing or strictly decreasing. **(4)**

6. Show that the curves $x = y^2$ and $xy = k$ cut at right-angles if $8k^2 = 1$. **(4)**

7. Evaluate $\sqrt{401}$, using differentials. **(4)**

8. If is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value on the interval [0, 2]. Find the value of 'a'. **(4)**

9. Find the equations of the tangents to the curve $3x^2 - y^2 = 8$, which pass through the point $\left(\frac{4}{3}, 0\right)$. **(6)**

10. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius 'R' is $\frac{2R}{\sqrt{3}}$.
Also find maximum volume. **(6)**

Answers

1. $1{\cdot}4\,\pi$ cm/s. **3.** 24. **4.** $\frac{5}{2}$ km/h.

5. Strictly increasing in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$, Strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

7. 20·025. **8.** $a = 120$. **9.** $3x \pm y = 4$.

10. $\frac{4\pi R^3}{3\sqrt{3}}$ cubic units.