Complex Numbers Exercise 1 :
Single Option Correct Type Questions

- This section contains **30 multiple choice questions.** Each question has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct

1. If $\cos(1-i) = a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$, then

(a) $a = \frac{1}{2}\left(e - \frac{1}{e}\right)\cos 1,\ b = \frac{1}{2}\left(e + \frac{1}{e}\right)\sin 1$

(b) $a = \frac{1}{2}\left(e + \frac{1}{e}\right)\cos 1,\ b = \frac{1}{2}\left(e - \frac{1}{e}\right)\sin 1$

(c) $a = \frac{1}{2}\left(e + \frac{1}{e}\right)\cos 1,\ b = \frac{1}{2}\left(e + \frac{1}{e}\right)\sin 1$

(d) $a = \frac{1}{2}\left(e - \frac{1}{e}\right)\cos 1,\ b = \frac{1}{2}\left(e - \frac{1}{e}\right)\sin 1$

2. Number of roots of the equation $z^{10} - z^5 - 992 = 0$, where real parts are negative, is

(a) 3 (b) 4 (c) 5 (d) 6

3. If z and $\bar{z}$ represent adjacent vertices of a regular polygon of n sides with centre at origin and if $\frac{\text{Im}(z)}{\text{Re}(z)} = \sqrt{2} - 1$, the value of n is equal to

(a) 2 (b) 4 (c) 6 (d) 8

4. If $\prod_{p=1}^{r} e^{ip\theta} = 1$, where $\prod$ denotes the continued product and $i = \sqrt{-1}$, the most general value of θ is

(a) $\frac{2n\pi}{r(r-1)}, n \in I$ (b) $\frac{2n\pi}{r(r+1)}, n \in I$

(c) $\frac{4n\pi}{r(r-1)}, n \in I$ (d) $\frac{4n\pi}{r(r+1)}, n \in I$

(where, n is an integer)

5. If $(3 + i)(z + \bar{z}) - (2 + i)(z - \bar{z}) + 14i = 0$, where $i = \sqrt{-1}$, then $z\bar{z}$ is equal to

(a) 10 (b) 8
(c) -9 (d) -10

6. The centre of a square $ABCD$ is at $z = 0$, A is z_1. Then, the centroid of ΔABC is

(a) $z_1(\cos\pi \pm i\sin\pi)$ (b) $\frac{z_1}{3}(\cos\pi \pm i\sin\pi)$

(c) $z_1\left(\cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2}\right)$ (d) $\frac{z_1}{3}\left(\cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2}\right)$

(where, $i = \sqrt{-1}$)

7. If $z = \frac{\sqrt{3} - i}{2}$, where $i = \sqrt{-1}$, then $(i^{101} + z^{101})^{103}$ equals to

(a) iz (b) z
(c) $\bar{z}$ (d) None of these

8. Let a and b be two fixed non-zero complex numbers and z is a variable complex number. If the lines $a\bar{z} + \bar{a}z + 1 = 0$ and $b\bar{z} + \bar{b}z - 1 = 0$ are mutually perpendicular, then

(a) $ab + \bar{a}\bar{b} = 0$ (b) $ab - \bar{a}\bar{b} = 0$
(c) $\bar{a}b - a\bar{b} = 0$ (d) $a\bar{b} + \bar{a}b = 0$

9. If $\alpha = \cos\left(\frac{8\pi}{11}\right) + i\sin\left(\frac{8\pi}{11}\right)$, where $i = \sqrt{-1}$, then $\text{Re}(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$ is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 0 (d) None of these

10. The set of points in an Argand diagram which satisfy both $|z| \le 4$ and $0 \le \arg(z) \le \frac{\pi}{3}$, is

(a) a circle and a line (b) a radius of a circle
(c) a sector of a circle (d) an infinite part line

11. If $f(x) = g(x^3) + xh(x^3)$ is divisible by $x^2 + x + 1$, then

(a) $g(x)$ is divisible by $(x - 1)$ but not $h(x)$
(b) $h(x)$ is divisible by $(x - 1)$ but not $g(x)$
(c) both $g(x)$ and $h(x)$ are divisible by $(x - 1)$
(d) None of the above

12. If the points represented by complex numbers $z_1 = a + ib$, $z_2 = c + id$ and $z_1 - z_2$ are collinear, where $i = \sqrt{-1}$, then

(a) $ad + bc = 0$ (b) $ad - bc = 0$
(c) $ab + cd = 0$ (d) $ab - cd = 0$

13. Let C denotes the set of complex numbers and R is the set of real numbers. If the function $f: C \to R$ is defined by $f(z) = |z|$, then

(a) f is injective but not surjective
(b) f is surjective but not injective
(c) f is neither injective nor surjective
(d) f is both injective and surjective

14. Let α and β be two distinct complex numbers, such that $|\alpha| = |\beta|$. If real part of α is positive and imaginary part of β is negative, then the complex number $(\alpha + \beta)/(\alpha - \beta)$ may be

(a) zero (b) real and negative
(c) real and positive (d) purely imaginary

15. The complex number z, satisfies the condition $\left|z - \frac{25}{z}\right| = 24$. The maximum distance from the origin of coordinates to the point z, is

(a) 25 (b) 30
(c) 32 (d) None of these

16. The points A, B and C represent the complex numbers $z_1, z_2, (1-i)z_1 + iz_2$ respectively, on the complex plane (where, $i = \sqrt{-1}$). The ΔABC, is

(a) isosceles but not right angled
(b) right angled but not isosceles
(c) isosceles and right angled
(d) None of the above

17. The system of equations $|z+1-i| = \sqrt{2}$ and $|z| = 3$ has (where, $i = \sqrt{-1}$)

(a) no solution (b) one solution
(c) two solutions (d) None of these

18. Dividing $f(z)$ by $z - i$, we obtain the remainder $1 - i$ and dividing it by $z + i$, we get the remainder $1 + i$. Then, the remainder upon the division of $f(z)$ by $z^2 + 1$, is

(a) $i + z$ (b) $1 + z$
(c) $1 - z$ (d) None of these

19. The centre of the circle represented by $|z+1| = 2|z-1|$ on the complex plane, is

(a) 0 (b) $\frac{5}{3}$
(c) $\frac{1}{3}$ (d) None of these

20. If $x = 9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27} \ldots \infty$, $y = 4^{1/3} \cdot 4^{-1/9} \cdot 4^{1/27} \ldots \infty$ and $z = \sum_{r=1}^{\infty} (1+i)^{-r}$, where $i = \sqrt{-1}$, then $\arg(x + yz)$ is equal to

(a) 0 (b) $-\tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$
(c) $-\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (d) $\pi - \tan^{-1}\left(\frac{\sqrt{2}}{3}\right)$

21. If centre of a regular hexagon is at origin and one of the vertices on Argand diagram is $1 + 2i$, where $i = \sqrt{-1}$, then its perimeter is

(a) $2\sqrt{5}$ (b) $4\sqrt{5}$
(c) $6\sqrt{5}$ (d) $8\sqrt{5}$

22. Let $|z_r - r| \le r, \forall\ r = 1, 2, 3, \ldots, n$, then $\left|\sum_{r=1}^{n} z_r\right|$ is less than

(a) n (b) $2n$
(c) $n(n+1)$ (d) $\frac{n(n+1)}{2}$

23. If $\arg\left(\dfrac{z_1 - \dfrac{z}{|z|}}{\dfrac{z}{|z|}}\right) = \dfrac{\pi}{2}$ and $\left|\dfrac{z}{|z|} - z_1\right| = 3$, then $|z_1|$ equals to

(a) $\sqrt{3}$ (b) $2\sqrt{2}$ (c) $\sqrt{10}$ (d) $\sqrt{26}$

24. If $|z - 2 - i| = |z| \left|\sin\left(\frac{\pi}{4} - \arg z\right)\right|$, where $i = \sqrt{-1}$, then locus of z, is

(a) a pair of straight lines (b) circle
(c) parabola (d) ellipse

25. If $1, z_1, z_2, z_3, \ldots, z_{n-1}$ are the n, nth roots of unity, then the value of $\sum_{r=1}^{n-1} \frac{1}{(3 - z_r)}$, is

(a) $\frac{n \cdot 3^{n-1}}{3^n - 1} + \frac{1}{2}$ (b) $\frac{n \cdot 3^{n-1}}{3^n - 1} - 1$
(c) $\frac{n \cdot 3^{n-1}}{3^n - 1} + 1$ (d) None of these

26. If $z = (3 + 7i)(\lambda + i\mu)$, when $\lambda, \mu \in I \sim \{0\}$ and $i = \sqrt{-1}$, is purely imaginary then minimum value of $|z|^2$ is

(a) 0 (b) 58
(c) $\frac{3364}{3}$ (d) 3364

27. Given, $z = f(x) + ig(x)$, where $i = \sqrt{-1}$ and $f, g : (0, 1) \to (0, 1)$ are real-valued functions, which of the following hold good?

(a) $z = \frac{1}{1 - ix} + i\left(\frac{1}{1 + ix}\right)$ (b) $z = \frac{1}{1 + ix} + i\left(\frac{1}{1 - ix}\right)$
(c) $z = \frac{1}{1 + ix} + i\left(\frac{1}{1 + ix}\right)$ (d) $z = \frac{1}{1 - ix} + i\left(\frac{1}{1 - ix}\right)$

28. If $z^3 + (3 + 2i)z + (-1 + ia) = 0$, where $i = \sqrt{-1}$, has one real root, the value of a lies in the interval $(a \in R)$

(a) $(-2, -1)$ (b) $(-1, 0)$
(c) $(0, 1)$ (d) $(1, 2)$

29. If m and n are the smallest positive integers satisfying the relation $\left(2\, CiS\, \frac{\pi}{6}\right)^m = \left(4\, CiS\, \frac{\pi}{4}\right)^n$, where $i = \sqrt{-1}$, $(m + n)$ equals to

(a) 60 (b) 72 (c) 96 (d) 120

30. Number of imaginary complex numbers satisfying the equation, $z^2 = \bar{z} \cdot 2^{1 - |z|}$ is

(a) 0 (b) 1 (c) 2 (d) 3

Complex Numbers Exercise 2 :
More than One Option Correct Type Questions

- This section contains **15 multiple choice questions**. Each question has four choices (a), (b), (c) and (d) out of which **MORE THAN ONE** may be correct.

31. If $\dfrac{z+1}{z+i}$ is a purely imaginary number (where $i=\sqrt{-1}$), then z lies on a
(a) straight line
(b) circle
(c) circle with radius $=\dfrac{1}{\sqrt{2}}$
(d) circle passing through the origin

32. If z satisfies $|z-1|<|z+3|$, then $\omega = 2z+3-i$ (where, $i=\sqrt{-1}$) satisfies
(a) $|\omega-5-i|<|\omega+3+i|$ (b) $|\omega-5|<|\omega+3|$
(c) $\text{Im}(i\omega)>1$ (d) $|\arg(\omega-1)|<\dfrac{\pi}{2}$

33. If the complex number is $(1+ri)^3=\lambda(1+i)$, when $i=\sqrt{-1}$, for some real λ, the value of r can be
(a) $\cos\dfrac{\pi}{5}$ (b) $\text{cosec}\dfrac{3\pi}{2}$
(c) $\cot\dfrac{\pi}{12}$ (d) $\tan\dfrac{\pi}{12}$

34. If $z\in C$, which of the following relation(s) represents a circle on an Argand diagram?
(a) $|z-1|+|z+1|=3$ (b) $|z-3|=2$
(c) $|z-2+i|=\dfrac{7}{3}$ (d) $(z-3+i)(\bar{z}-3-i)=5$
(where, $i=\sqrt{-1}$)

35. If $1, z_1, z_2, z_3, \ldots, z_{n-1}$ be the n, nth roots of unity and ω be a non-real complex cube root of unity, then $\prod_{r=1}^{n-1}(\omega-z_r)$ can be equal to
(a) $1+\omega$ (b) -1
(c) 0 (d) 1

36. If z is a complex number which simultaneously satisfies the equations
$3|z-12|=5|z-8i|$ and $|z-4|=|z-8|$, where $i=\sqrt{-1}$, then $\text{Im}(z)$ can be
(a) 8 (b) 17
(c) 7 (d) 15

37. If $P(z_1)$, $Q(z_2)$, $R(z_3)$ and $S(z_4)$ are four complex numbers representing the vertices of a rhombus taken in order on the complex plane, which one of the following is hold good?
(a) $\dfrac{z_1-z_4}{z_2-z_3}$ is purely real
(b) $\dfrac{z_1-z_3}{z_2-z_4}$ is purely imaginary
(c) $|z_1-z_3|\neq|z_2-z_4|$
(d) $\text{amp}\left(\dfrac{z_1-z_4}{z_2-z_4}\right)\neq\text{amp}\left(\dfrac{z_2-z_4}{z_3-z_4}\right)$

38. If $|z-3|=\min\{|z-1|,|z-5|\}$, then $\text{Re}(z)$ is equal to
(a) 2 (b) 2.5 (c) 3.5 (d) 4

39. If $\arg(z+a)=\dfrac{\pi}{6}$ and $\arg(z-a)=\dfrac{2\pi}{3}$ $(a\in R^+)$, then
(a) $|z|=a$ (b) $|z|=2a$
(c) $\arg(z)=\dfrac{\pi}{3}$ (d) $\arg(z)=\dfrac{\pi}{2}$

40. If $z=x+iy$, where $i=\sqrt{-1}$, then the equation $\left|\dfrac{(2z-i)}{(z+i)}\right|=m$ represents a circle, then m can be
(a) $\dfrac{1}{2}$ (b) 1 (c) 2 (d) $\in(3,2\sqrt{3})$

41. Equation of tangent drawn to circle $|z|=r$ at the point $A(z_0)$, is
(a) $\text{Re}\left(\dfrac{z}{z_0}\right)=1$ (b) $\text{Im}\left(\dfrac{z}{z_0}\right)=1$
(c) $\text{Im}\left(\dfrac{z_0}{z}\right)=1$ (d) $z\bar{z}_0+z_0\bar{z}=2r^2$

42. z_1 and z_2 are the roots of the equation $z^2-az+b=0$, where $|z_1|=|z_2|=1$ and a, b are non-zero complex numbers, then
(a) $|a|\le 1$ (b) $|a|\le 2$
(c) $\arg(a)=\arg(b^2)$ (d) $\arg(a^2)=\arg(b)$

43. If α is a complex constant, such that $\alpha z^2+z+\bar{\alpha}=0$ has a real root, then
(a) $\alpha+\bar{\alpha}=1$
(b) $\alpha+\bar{\alpha}=0$
(c) $\alpha+\bar{\alpha}=-1$
(d) the absolute value of real root is 1

44. If the equation $z^3+(3+i)z^2-3z-(m+i)=0$, where $i=\sqrt{-1}$ and $m\in R$, has atleast one real root, value of m is
(a) 1 (b) 2 (c) 3 (d) 5

45. If $z^3+(3+2i)z+(-1+ia)=0$, where $i=\sqrt{-1}$, has one real root, the value of a lies in the interval $(a\in R)$
(a) $(-2,1)$ (b) $(-1,0)$ (c) $(0,1)$ (d) $(-2,3)$

Complex Numbers Exercise 3 :
Passage Based Questions

- This section contains **4 passages**. Based upon each of the passage **3 multiple choice questions** have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which **ONLY ONE** is correct.

Passage I
(Q. Nos. 46 to 48)

$\arg(\bar{z}) + \arg(-z) = \begin{cases} \pi, & \text{if } \arg(z) < 0 \\ -\pi, & \text{if } \arg(z) > 0 \end{cases}$, where $-\pi < \arg(z) \leq \pi$.

46. If $\arg(z) > 0$, then $\arg(-z) - \arg(z)$ is equal to

(a) $-\pi$ (b) $-\frac{\pi}{2}$
(c) $\frac{\pi}{2}$ (d) π

47. Let z_1 and z_2 be two non-zero complex numbers, such that $|z_1| = |z_2|$ and $\arg(z_1 z_2) = \pi$, then z_1 is equal to

(a) z_2 (b) $\bar{z}_2$
(c) $-z_2$ (d) $-\bar{z}_2$

48. If $\arg(4z_1) - \arg(5z_2) = \pi$, then $\left|\frac{z_1}{z_2}\right|$ is equal to

(a) 1 (b) 1.25
(c) 1.50 (d) 2.50

Passage II
(Q. Nos. 49 to 51)

Sum of four consecutive powers of i (iota) is zero.
i.e., $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0, \forall n \in I$.

49. If $\sum_{n=1}^{25} i^{n!} = a + ib$, where $i = \sqrt{-1}$, then $a - b$, is

(a) prime number
(b) even number
(c) composite number
(d) perfect number

50. If $\sum_{r=-2}^{95} i^r + \sum_{r=0}^{50} i^{r!} = a + ib$, where $i = \sqrt{-1}$, the unit place digit of $a^{2011} + b^{2012}$, is

(a) 2 (b) 3
(c) 5 (d) 6

51. If $\sum_{r=4}^{100} i^{r!} + \prod_{r=1}^{101} i^r = a + ib$, where $i = \sqrt{-1}$, then $a + 75b$, is

(a) 11 (b) 22
(c) 33 (d) 44

Passage III
(Q. Nos. 52 to 54)

For any two complex numbers z_1 and z_2,

$|z_1 - z_2| \geq \begin{cases} |z_1| - |z_2| \\ |z_2| - |z_1| \end{cases}$

and equality holds iff origin z_1 and z_2 are collinear and z_1, z_2 lie on the same side of the origin.

52. If $\left|z - \frac{1}{z}\right| = 2$ and sum of greatest and least values of $|z|$ is λ, then λ^2, is

(a) 2 (b) 4 (c) 6 (d) 8

53. If $\left|z + \frac{2}{z}\right| = 4$ and sum of greatest and least values of $|z|$ is λ, then λ^2, is

(a) 12 (b) 18 (c) 24 (d) 30

54. If $\left|z - \frac{3}{z}\right| = 6$ and sum of greatest and least values of $|z|$ is 2λ, then λ^2, is

(a) 12 (b) 18 (c) 24 (d) 30

Passage IV
(Q. Nos. 55 to 57)

Consider the two complex numbers z and w, such that $w = \frac{z-1}{z+2} = a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$.

55. If $z = C\,iS\,\theta$, which of the following does hold good?

(a) $\sin\theta = \frac{9b}{1-4a}$
(b) $\cos\theta = \frac{1-5a}{1+4a}$
(c) $(1+5a)^2 + (3b)^2 = (1-4a)^2$
(d) All of these

56. Which of the following is the value of $-\frac{b}{a}$, whenever it exists?

(a) $3\tan\left(\frac{\theta}{2}\right)$ (b) $\frac{1}{3}\tan\left(\frac{\theta}{2}\right)$
(c) $-\frac{1}{3}\cot\theta$ (d) $3\cot\frac{\theta}{2}$

57. Which of the following equals to $|z|$?

(a) $|w|$ (b) $(a+1)^2 + b^2$
(c) $a^2 + (b+2)^2$ (d) $(a+1)^2 + (b+1)^2$

Complex Numbers Exercise 4 :
Single Integer Answer Type Questions

- This section contains **10 questions**. The answer to each question is a **single digit integer**, ranging from 0 to 9 (both inclusive).

58. The number of values of z (real or complex) simultaneously satisfying the system of equations

$$1 + z + z^2 + z^3 + \ldots + z^{17} = 0$$

and $1 + z + z^2 + z^3 + \ldots + z^{13} = 0$ is

59. Number of complex numbers z satisfying $z^3 = \bar{z}$ is

60. Let $z = 9 + ai$, where $i = \sqrt{-1}$ and a be non-zero real. If $\operatorname{Im}(z^2) = \operatorname{Im}(z^3)$, sum of the digits of a^2 is

61. Number of complex numbers z, such that $|z| = 1$ and $\left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = 1$ is

62. If $x = a + ib$, where $a, b \in R$ and $i = \sqrt{-1}$ and $x^2 = 3 + 4i$, $x^3 = 2 + 11i$, the value of $(a + b)$ is

63. If $z = \frac{\pi}{4}(1+i)^4\left(\frac{1-\sqrt{\pi}\,i}{\sqrt{\pi}+i} + \frac{\sqrt{\pi}-i}{1+\sqrt{\pi}\,i}\right)$, where $i = \sqrt{-1}$, then $\left(\frac{|z|}{\operatorname{amp}(z)}\right)$ equals to

64. Suppose A is a complex number and $n \in N$, such that $A^n = (A+1)^n = 1$, then the least value of n is

65. Let $z_r; r = 1, 2, 3, \ldots, 50$ be the roots of the equation $\sum_{r=0}^{50} (z)^r = 0$. If $\sum_{r=1}^{50} \frac{1}{(z_r - 1)} = -5\lambda$, then λ equals to

66. If $P = \sum_{p=1}^{32} (3p+2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p$, where $i = \sqrt{-1}$ and if $(1+i)P = n(n!)$, $n \in N$, then the value of n is

67. The least positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}\sin^{-1}\left(\frac{1+x^2}{2x}\right)$, where $x > 0$ and $i = \sqrt{-1}$ is

Complex Numbers Exercise 5 :
Matching Type Questions

- This section contains **4 questions**. Questions 68 and 69 have three statements (A, B and C) given in **Column I** and four statements (p, q, r and s) in **Column II** and questions 70 and 71 have four statements (A, B, C and D) given in **Column I** and five statements (p, q, r, s and t) in **Column II**. Any given statement in **Column I** can have correct matching with one or more statement(s) given in **Column II**.

68.

	Column I		Column II
(A)	If $\left\|z - \frac{1}{z}\right\| = 2$ and if greatest and least values of $\|z\|$ are G and L respectively, then $G - L$, is	(p)	natural number
(B)	If $\left\|z + \frac{2}{z}\right\| = 4$ and if greatest and least values of $\|z\|$ are G and L respectively, then $G - L$, is	(q)	prime number
(C)	If $\left\|z - \frac{3}{z}\right\| = 6$ and if greatest and least values of $\|z\|$ are G and L respectively, then $G - L$, is	(r)	composite number
		(s)	perfect number

69.

	Column I		Column II
(A)	If $\sqrt{(6+8i)} + \sqrt{(-6+8i)} = z_1, z_2, z_3, z_4$ (where $i = \sqrt{-1}$), then $\|z_1\|^2 + \|z_2\|^2 \| + \|z_3\|^2 + \|z_4\|^2$ is divisible by	(p)	7
(B)	If $\sqrt{(5-12i)} + \sqrt{(-5-12i)} = z_1, z_2, z_3, z_4$ (where $i = \sqrt{-1}$), then $\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 + \|z_4\|^2$ is divisible by	(q)	8
(C)	If $\sqrt{(8+15i)} + \sqrt{(-8-15i)} = z_1, z_2, z_3, z_4$(where $i = \sqrt{-1}$), then $\|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 + \|z_4\|^2$ is divisible by	(r)	13
		(s)	17

70.

	Column I		Column II
(A)	If λ and μ are the unit's place digits of $(143)^{861}$ and $(5273)^{1358}$ respectively, then $\lambda+\mu$ is divisible by	(p)	2
(B)	If λ and μ are the unit's place digits of $(212)^{7820}$ and $(1322)^{1594}$ respectively, then $\lambda+\mu$ is divisible by	(q)	3
(C)	If λ and μ are the unit's place digits of $(136)^{786}$ and $(7138)^{13491}$ respectively, then $\lambda+\mu$ is divisible by	(r)	4
		(s)	5
		(t)	6

71.

	Column I		Column II
(A)	If $\left\lvert z-\frac{6}{z}\right\rvert=5$ and maximum and minimum values of $\lvert z\rvert$ are λ and μ respectively, then	(p)	$\lambda^{\mu}+\mu^{\lambda}=8$
(B)	If $\left\lvert z-\frac{7}{z}\right\rvert=6$ and maximum and minimum values of $\lvert z\rvert$ are λ and μ respectively, then	(q)	$\lambda^{\mu}-\mu^{\lambda}=7$
(C)	If $\left\lvert z-\frac{8}{z}\right\rvert=7$ and maximum and minimum values of $\lvert z\rvert$ are λ and μ respectively, then	(r)	$\lambda^{\mu}+\mu^{\lambda}=7$
		(s)	$\lambda^{\mu}-\mu^{\lambda}=6$
		(t)	$\lambda^{\mu}+\mu^{\lambda}=9$

Complex Numbers Exercise 6 :
Statement I and II Type Questions

- **Directions** (Q. Nos. 72 to 78) are Assertion-Reason type questions. Each of these questions contains two statements:
Statement-1 (Assertion) and **Statement-2** (Reason)
Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice as given below.
(a) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1
(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1
(c) Statement1 is true, Statement-2 is false
(d) Statement-1 is false, Statement-2 is true

72. **Statement-1** $3+7i>2+4i$, where $i=\sqrt{-1}$.

Statement-2 $3>2$ and $7>4$

73. **Statement-1** $(\cos\theta+i\sin\phi)^3=\cos 3\theta+i\sin 3\phi$, $i=\sqrt{-1}$

Statement-2 $\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^2=i$

74. **Statement-1** Let z_1, z_2 and z_3 be three complex numbers, such that $\lvert 3z_1+1\rvert=\lvert 3z_2+1\rvert=\lvert 3z_3+1\rvert$ and $1+z_1+z_2+z_3=0$, then z_1, z_2, z_3 will represent vertices of an equilateral triangle on the complex plane.

Statement-2 z_1, z_2 and z_3 represent vertices of an equilateral triangle, if
$z_1^2+z_2^2+z_3^2+z_1z_2+z_2z_3+z_3z_1=0$.

75. **Statement-1** Locus of z satisfying the equation $\lvert z-1\rvert+\lvert z-8\rvert=5$ is an ellipse.

Statement-2 Sum of focal distances of any point on ellipse is constant for an ellipse.

76. Let z_1, z_2 and z_3 be three complex numbers in AP.

Statement-1 Points representing z_1, z_2 and z_3 are collinear.

Statement-2 Three numbers a, b and c are in AP, if $b-a=c-b$.

77. **Statement-1** If the principal argument of a complex number z is θ, the principal argument of z^2 is 2θ.

Statement-2 $\arg(z^2)=2\arg(z)$

78. Consider the curves on the Argand plane as

$$C_1 : \arg(z)=\frac{\pi}{4},$$
$$C_2 : \arg(z)=\frac{3\pi}{4}$$

and $C_3 : \arg(z-5-5i)=\pi$, where $i=\sqrt{-1}$.

Statement-1 Area of the region bounded by the curves C_1, C_2 and C_3 is $\frac{25}{2}$.

Statement–2 The boundaries of C_1, C_2 and C_3 constitute a right isosceles triangle.

Complex Numbers Exercise 7 :
Subjective Type Questions

- In this section, there are **24 subjective** questions.

79. If z_1, z_2 and z_3 are three complex numbers, then prove that $z_1 \operatorname{Im}(\bar{z}_2 z_3) + z_2 \operatorname{Im}(\bar{z}_3 z_1) + z_3 \operatorname{Im}(\bar{z}_1 z_2) = 0$.

80. The roots z_1, z_2 and z_3 of the equation $x^3 + 3ax^2 + 3bx + c = 0$ in which a, b and c are complex numbers, correspond to the points A, B, C on the Gaussian plane. Find the centroid of the ΔABC and show that it will be equilateral, if $a^2 = b$.

81. If $1, \alpha_1, \alpha_2, \alpha_3$ and α_4 are the roots of $x^5 - 1 = 0$, then prove that

$$\frac{\omega - \alpha_1}{\omega^2 - \alpha_1} \cdot \frac{\omega - \alpha_2}{\omega^2 - \alpha_2} \cdot \frac{\omega - \alpha_3}{\omega^2 - \alpha_3} \cdot \frac{\omega - \alpha_4}{\omega^2 - \alpha_4} = \omega,$$ where ω is a non-real complex root of unity.

82. If z_1 and z_2 both satisfy the relation $z + \bar{z} = 2|z - 1|$ and $\arg(z_1 - z_2) = \frac{\pi}{4}$, find the imaginary part of $(z_1 + z_2)$.

83. If $ax + cy + bz = X$, $cx + by + az = Y$, $bx + ay + cz = Z$, show that

(i) $(a^2 + b^2 + c^2 - bc - ca - ab)(x^2 + y^2 + z^2 - yz - zx - xy) = X^2 + Y^2 + Z^2 - YZ - ZX - XY$

(ii) $(a^3 + b^3 + c^3 - 3abc)(x^3 + y^3 + z^3 - 3xyz) = X^3 + Y^3 + Z^3 - 3XYZ$.

84. For every real number $c \ge 0$, find all complex numbers z which satisfy the equation $|z|^2 - 2iz + 2c(1 + i) = 0$, where $i = \sqrt{-1}$.

85. Find the equations of two lines making an angle of 45° with the line $(2 - i)z + (2 + i)\bar{z} + 3 = 0$, where $i = \sqrt{-1}$ and passing through $(-1, 4)$.

86. For $n \ge 2$, show that

$$\left[1 + \left(\frac{1+i}{2}\right)\right]\left[1 + \left(\frac{1+i}{2}\right)^2\right]\left[1 + \left(\frac{1+i}{2}\right)^{2^2}\right] \dots \left[1 + \left(\frac{1+i}{2}\right)^{2^n}\right] = (1+i)\left(1 - \frac{1}{2^{2^n}}\right),$$ where $i = \sqrt{-1}$.

87. Find the point of intersection of the curves $\arg(z - 3i) = 3\pi/4$ and $\arg(2z + 1 - 2i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$.

88. Show that if a and b are real, then the principal value of $\arg(a)$ is 0 or π, according as a is positive or negative and that of b is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$, according as b is positive or negative.

89. Two different non-parallel lines meet the circle $|z| = r$. One of them at points a and b and the other which is tangent to the circle at c. Show that the point of intersection of two lines is $\frac{2c^{-1} - a^{-1} - b^{-1}}{c^{-2} - a^{-1}b^{-1}}$.

90. A, B and C are the points representing the complex numbers z_1, z_2 and z_3 respectively, on the complex plane and the circumcentre of ΔABC lies at the origin. If the altitude of the triangle through the vertex A meets the circumcircle again at P, prove that P represents the complex number $\left(-\frac{z_2 z_3}{z_1}\right)$.

91. If $|z| \le 1$ and $|\omega| \le 1$, show that $|z - \omega|^2 \le (|z| - |\omega|)^2 + \{\arg(z) - \arg(\omega)\}^2$.

92. Let z, z_0 be two complex numbers. It is given that $|z| = 1$ and the numbers $z, z_0, z\bar{z}_0, 1$ and 0 are represented in an Argand diagram by the points P, P_0, Q, A and the origin, respectively. Show that ΔPOP_0 and ΔAOQ are congruent. Hence, or otherwise, prove that $|z - z_0| = |z\bar{z}_0 - 1|$.

93. Suppose the points $z_1, z_2, \dots, z_n$ $(z_i \ne 0)$ all lie on one side of a line drawn through the origin of the complex planes. Prove that the same is true of the points $\frac{1}{z_1}, \frac{1}{z_2}, \dots, \frac{1}{z_n}$. Moreover, show that

$$z_1 + z_2 + \dots + z_n \ne 0 \text{ and } \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \ne 0.$$

94. If a, b and c are complex numbers and z satisfies $az^2 + bz + c = 0$, prove that $|a||b| = \sqrt{a(\bar{b})^2 c}$ and $|a| = |c| \Leftrightarrow |z| = 1$.

95. Let z_1, z_2 and z_3 be three non-zero complex numbers and $z_1 \ne z_2$. If $\begin{vmatrix} |z_1| & |z_2| & |z_3| \\ |z_2| & |z_3| & |z_1| \\ |z_3| & |z_1| & |z_2| \end{vmatrix} = 0$, prove that

(i) z_1, z_2, z_3 lie on a circle with the centre at origin.

(ii) $\arg\left(\frac{z_3}{z_2}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$.

96. Prove that, if z_1 and z_2 are two complex numbers and $c > 0$, then $|z_1 + z_2|^2 \le (1+c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2$.

97. Find the circumcentre of the triangle whose vertices are given by the complex numbers z_1, z_2 and z_3.

98. Find the orthocentre of the triangle whose vertices are given by the complex numbers z_1, z_2 and z_3.

99. Prove that the roots of the equation $8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}$ and $\cos\frac{5\pi}{7}$.

Hence, obtain the equations whose roots are

(i) $\sec^2\frac{\pi}{7}, \sec^2\frac{3\pi}{7}, \sec^2\frac{5\pi}{7}$

(ii) $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$

(iii) Evaluate $\sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7}$

100. Solve the equation $z^7 + 1 = 0$ and deduce that

(i) $\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8}$

(ii) $\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8}$

(iii) $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \frac{1}{8}$

(iv) $\tan\frac{\pi}{14}\tan\frac{3\pi}{14}\tan\frac{5\pi}{14} = \frac{1}{\sqrt{7}}$

Also, show that

$$(1+y)^7 + (1-y)^7 = 14\left(y^2 + \tan^2\frac{\pi}{14}\right)\left(y^2 + \tan^2\frac{3\pi}{14}\right)\left(y^2 + \tan^2\frac{5\pi}{14}\right)$$

and then deduce that

$$\tan^2\left(\frac{\pi}{14}\right) + \tan^2\left(\frac{3\pi}{14}\right) + \tan^2\left(\frac{5\pi}{14}\right) = 5$$

101. If the complex number z is to satisfy $|z| = 3, |z - \{a(1+i) - i\}| \le 3$ and $|z + 2a - (a+1)i| > 3$, where $i = \sqrt{-1}$ simultaneously for atleast one z, then find all $a \in R$.

102. Write equations whose roots are equal to numbers

(i) $\sin^2\frac{\pi}{2n+1}, \sin^2\frac{2\pi}{2n+1}, \sin^2\frac{3\pi}{2n+1}, \ldots, \sin^2\frac{n\pi}{2n+1}$.

(ii) $\cot^2\frac{\pi}{2n+1}, \cot^2\frac{2\pi}{2n+1}, \cot^2\frac{3\pi}{2n+1}, \ldots, \cot^2\frac{n\pi}{2n+1}$.

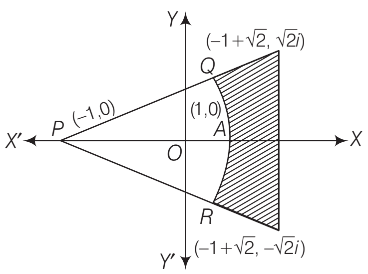
Complex Numbers Exercise 8 :
Questions Asked in Previous 13 Years' Exams

- This section contains questions asked in **IIT-JEE, AIEEE, JEE Main & JEE Advanced** from year **2005** to year **2017**.

103. If ω is a cube root of unity but not equal to 1, then minimum value of $|a + b\omega + c\omega^2|$, (where a, b and c are integers but not all equal), is **[IIT-JEE 2005, 3M]**

(a) 0 (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) 2

104. PQ and PR are two infinite rays. QAR is an arc. Point lying in the shaded region excluding the boundary satisfies **[IIT-JEE 2005, 3M]**



(a) $|z-1| > 2; |\arg(z-1)| < \frac{\pi}{4}$

(b) $|z-1| > 2; |\arg(z-1)| < \frac{\pi}{2}$

(c) $|z+1| > 2; |\arg(z+1)| < \frac{\pi}{4}$

(d) $|z+1| > 2; |\arg(z+1)| < \frac{\pi}{3}$

105. If one of the vertices of the square circumscribing the circle $|z-1| = \sqrt{2}$ is $2 + \sqrt{3}i$, where $i = \sqrt{-1}$. Find the other vertices of the square. **[IIT-JEE 2005, 4M]**

106. If z_1 and z_2 are two non-zero complex numbers, such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to **[AIEEE 2005, 3M]**

(a) $-\pi$ (b) $-\pi/2$
(c) $\pi/2$ (d) 0

107. If $1, \omega, \omega^2$ are the cube roots of unity, then the roots of the equation $(x-1)^3 + 8 = 0$ are **[AIEEE 2005, 3M]**

(a) $-1, 1+2\omega, 1+2\omega^2$ (b) $-1, 1-2\omega, 1-2\omega^2$
(c) $-1, -1, -1$ (d) None of these

108. If $\omega = \dfrac{z}{z - \frac{1}{3}i}$ and $|\omega| = 1$, where $i = \sqrt{-1}$, then z lies on **[AIEEE 2005, 3M]**

(a) a straight line (b) a parabola
(c) an ellipse (d) a circle

109. If $\omega = \alpha + i\beta$, where $\beta \neq 0$, $i = \sqrt{-1}$ and $z \neq 1$, satisfies the condition that $\left(\dfrac{\omega - \overline{\omega} z}{1 - z}\right)$ is purely real, the set of values of z is **[IIT-JEE 2006, 3M]**

(a) $\{z : |z| = 1\}$ (b) $\{z : z = \bar{z}\}$
(c) $\{z : z \neq 1\}$ (d) $\{z : |z| = 1, z \neq 1\}$

110. The value of $\sum_{k=1}^{10} \left(\sin \dfrac{2k\pi}{11} + i \cos \dfrac{2k\pi}{11}\right)$ (where $i = \sqrt{-1}$) is **[AIEEE 2006, 3M]**

(a) i (b) 1
(c) -1 (d) $-i$

111. If $z^2 + z + 1 = 0$, where z is a complex number, the value of $\left(z + \dfrac{1}{z}\right)^2 + \left(z^2 + \dfrac{1}{z^2}\right)^2 + \left(z^3 + \dfrac{1}{z^3}\right)^2 + \ldots + \left(z^6 + \dfrac{1}{z^6}\right)^2$ is **[AIEEE 2006, 6M]**

(a) 18 (b) 54
(c) 6 (d) 12

112. A man walks a distance of 3 units from the origin towards the North-East (N 45° E) direction. From there, he walks a distance of 4 units towards the North-West (N 45° W) direction to reach a point P. Then, the position of P in the Argand plane, is **[IIT-JEE 2007, 3M]**

(a) $3e^{i\pi/4} + 4i$ (b) $(3 - 4i)\, e^{i\pi/4}$
(c) $(4 + 3i)\, e^{i\pi/4}$ (d) $(3 + 4i)\, e^{i\pi/4}$
(where $i = \sqrt{-1}$)

113. If $|z| = 1$ and $z \neq \pm 1$, then all the values of $\dfrac{z}{1 - z^2}$ lie on **[IIT-JEE 2007, 3M]**

(a) a line not passing through the origin
(b) $|z| = \sqrt{2}$
(c) the X-axis
(d) the Y-axis

114. If $|z + 4| \leq 3$, the maximum value of $|z + 1|$ is **[AIEEE 2007, 3M]**

(a) 4 (b) 10
(c) 6 (d) 0

Passage (Q. Nos. 115 to 117)

Let A, B and C be three sets of complex numbers as defined below:
$A = \{z : \text{Im}(z) \geq 1\}$
$B = \{z : |z - 2 - i| = 3\}$
$C = \{z : \text{Re}((1 - i)z) = \sqrt{2}\}$, where $i = \sqrt{-1}$
[IIT-JEE 2008, 4+4+4M]

115. The number of elements in the set $A \cap B \cap C$, is

(a) 0 (b) 1
(c) 2 (d) ∞

116. Let z be any point in $A \cap B \cap C$. Then, $|z + 1 - i|^2 + |z - 5 - i|^2$ lies between

(a) 25 and 29 (b) 30 and 34
(c) 35 and 39 (d) 40 and 44

117. Let z be any point in $A \cap B \cap C$ and ω be any point satisfying $|\omega - 2 - i| < 3$. Then, $|z| - |\omega| + 3$ lies between

(a) -6 and 3 (b) -3 and 6
(c) -6 and 6 (d) -3 and 9

118. A particle P starts from the point $z_0 = 1 + 2i$, $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1. From z_1, the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and then it moves through an angle $\dfrac{\pi}{2}$ in anti-clockwise direction on a circle with centre at origin, to reach a point z_2, then the point z_2 is given by **[IIT-JEE 2008, 3M]**

(a) $6 + 7i$ (b) $-7 + 6i$
(c) $7 + 6i$ (d) $-6 + 7i$

119. If the conjugate of a complex numbers is $\dfrac{1}{i - 1}$, where $i = \sqrt{-1}$. Then, the complex number is **[AIEEE 2008, 3M]**

(a) $\dfrac{-1}{i - 1}$ (b) $\dfrac{1}{i + 1}$
(c) $\dfrac{-1}{i + 1}$ (d) $\dfrac{1}{i - 1}$

120. Let $z = x + iy$ be a complex number, where x and y are integers and $i = \sqrt{-1}$. Then, the area of the rectangle whose vertices are the roots of the equation $z\bar{z}^3 + \bar{z}z^3 = 350$, is **[IIT-JEE 2009, 3M]**

(a) 48 (b) 32
(c) 40 (d) 80

121. Let $z = \cos\theta + i\sin\theta$, where $i = \sqrt{-1}$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$ is **[IIT-JEE 2009, 3M]**

(a) $\dfrac{1}{\sin 2°}$ (b) $\dfrac{1}{3\sin 2°}$
(c) $\dfrac{1}{2\sin 2°}$ (d) $\dfrac{1}{4\sin 2°}$

122. If $\left|z - \dfrac{4}{z}\right| = 2$, the maximum value of $|z|$ is equal to **[AIEEE 2009, 4M]**

(a) $2 + \sqrt{2}$ (b) $\sqrt{3} + 1$
(c) $\sqrt{5} + 1$ (d) 2

123. Let z_1 and z_2 be two distinct complex numbers and $z=(1-t)z_1+iz_2$, for some real number t with $0<t<1$ and $i=\sqrt{-1}$. If arg (w) denotes the principal argument of a non-zero complex number w, then **[IIT-JEE 2010, 3M]**

(a) $|z-z_1|+|z-z_2|=|z_1-z_2|$
(b) $\arg(z-z_1)=\arg(z-z_2)$
(c) $\begin{vmatrix} z-z_1 & \bar{z}-\bar{z}_1 \\ z_2-z_1 & \bar{z}_2-\bar{z}_1 \end{vmatrix}=0$
(d) $\arg(z-z_1)=\arg(z_2-z_1)$

124. Let ω be the complex number $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$, where $i=\sqrt{-1}$, then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$, is equal to **[IIT-JEE 2010, 3M]**

(a) 0 (b) 1
(c) 2 (d) 3

125. Match the statements in **Column I** with those in **Column II**.

[**Note** Here, z takes values in the complex plane and Im (z) and Re (z) denote respectively, the imaginary part and the real part of z.] **[IIT- JEE 2010, 8M]**

	Column I		Column II
(A)	The set of points z satisfying $\|z-i\|z\|\|=\|z+i\|z\|\|$, where $i=\sqrt{-1}$, is contained in or equal to	(p)	an ellipse with eccentricity 4/5
(B)	The set of points z satisfying $\|z+4\|+\|z-4\|=10$ is contained in or equal to	(q)	the set of points z satisfying Im $(z)=0$
(C)	If $\|w\|=2$, the set of points $z=w-\frac{1}{w}$ is contained in or equal to	(r)	the set of points z satisfying $\|\text{Im}(z)\|\le 1$
(D)	If $\|w\|=1$, the set of points $z=w+\frac{1}{w}$ is contained in or equal to	(s)	the set of points satisfying $\|\text{Re}(z)\|\le 2$
		(t)	the set of points z satisfying $\|z\|\le 3$

126. If α and β are the roots of the equation $x^2-x+1=0$, $\alpha^{2009}+\beta^{2009}$ is equal to **[AIEEE 2010, 4M]**

(a) -1 (b) 1
(c) 2 (d) -2

127. The number of complex numbers z, such that $|z-1|=|z+1|=|z-i|$, where $i=\sqrt{-1}$, equals to **[AIEEE 2010, 4M]**

(a) 1 (b) 2
(c) ∞ (d) 0

128. If z is any complex number satisfying $|z-3-2i|\le 2$, where $i=\sqrt{-1}$, then the minimum value of $|2z-6+5i|$, is **[IIT-JEE 2011, 4M]**

129. The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right): z \text{ is a complex number } |z|=1, z\ne\pm 1\right\}$ is **[IIT-JEE 2011, 2M]**

(a) $(-\infty,-1]\cap[1,\infty)$ (b) $(-\infty,0)\cup(0,\infty)$
(c) $(-\infty,-1)\cup(1,\infty)$ (d) $[2,\infty)$

130. The maximum value of $\left|\arg\left(\frac{1}{1-z}\right)\right|$ for $|z|=1, z\ne 1$, is given by **[IIT-JEE 2011, 2M]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

131. Let $w=e^{i\pi/3}$, where $i=\sqrt{-1}$ and a, b, c, x, y and z be non-zero complex numbers such that

$$a+b+c=x$$
$$a+bw+cw^2=y$$
$$a+bw^2+cw=z.$$

The value of $\frac{|x|^2+|y|^2+|z|^2}{|a|^2+|b|^2+|c|^2}$, is **[IIT-JEE 2011, 4M]**

132. Let α and β be real and z be a complex number. If $z^2+\alpha z+\beta=0$ has two distinct roots on the line Re $(z)=1$, then it is necessary that **[AIEEE 2011, 4M]**

(a) $\beta\in(-1,0)$ (b) $|\beta|=1$
(c) $\beta\in(1,\infty)$ (d) $\beta\in(0,1)$

133. If $\omega\,(\ne 1)$ is a cube root of unity and $(1+\omega)^7=A+B\omega$, then (A, B) equals to **[AIEEE 2011, 4M]**

(a) (1, 1) (b) (1, 0)
(c) (−1, 1) (d) (0, 1)

134. Let z be a complex number such that the imaginary part of z is non-zero and $a=z^2+z+1$ is real. Then, a **cannot** take the value **[IIT-JEE 2012, 3M]**

(a) -1 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

135. If $z\ne 1$ and $\frac{z^2}{z-1}$ is real, the point represented by the complex number z lies **[AIEEE 2012, 4M]**

(a) on a circle with centre at the origin
(b) either on the real axis or on a circle not passing through the origin
(c) on the imaginary axis
(d) either on the real axis or on a circle passing through the origin

136. If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals to **[JEE Main 2013, 4M]**

(a) $\frac{\pi}{2}-\theta$ (b) θ
(c) $\pi-\theta$ (d) $-\theta$

137. Let complex numbers α and $\frac{1}{\alpha}$ lie on circles $(x-x_0)^2+(y-y_0)^2=r^2$ and $(x-x_0)^2+(y-y_0)^2=4r^2$, respectively. If $z_0=x_0+iy_0$ satisfies the equation $2|z_0|^2=r^2+2$, then $|\alpha|$ equals to **[JEE Advanced 2013, 2M]**

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{3}$

138. Let $w=\frac{\sqrt{3}+i}{2}$ and $P=\{w^n : n=1,2,3,\dots\}$. Further, $H_1=\left\{z\in C:\mathrm{Re}(z)>\frac{1}{2}\right\}$ and $H_2=\left\{z\in C:\mathrm{Re}(z)<\left(-\frac{1}{2}\right)\right\}$, where C is the set of all complex numbers. If $z_1\in P\cap H_1$, $z_2\in P\cap H_2$ and O represents the origin, then $\angle z_1Oz_2$ equals to

[JEE Advanced 2013, 3M]

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$
(c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

Passage (Q. Nos. 139 to 140)

Let $S=S_1\cap S_2\cap S_3$, where

$S_1=\{z\in C:|z|<4\}$,

$S_2=\left\{z\in C:\mathrm{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right]>0\right\}$

and $S_3=\{z\in C:\mathrm{Re}\ z>0\}$. **[JEE Advanced 2013, 3+3M]**

139. $\min\limits_{z\in S}|1-3i-z|$ equals to

(a) $\frac{2-\sqrt{3}}{2}$ (b) $\frac{2+\sqrt{3}}{2}$
(c) $\frac{3-\sqrt{3}}{2}$ (d) $\frac{3+\sqrt{3}}{2}$

140. Area of S equals to

(a) $\frac{10\pi}{3}$ (b) $\frac{20\pi}{3}$
(c) $\frac{16\pi}{3}$ (d) $\frac{32\pi}{3}$

141. If z is a complex number such that $|z|\geq 2$, then the minimum value of $\left|z+\left(\frac{1}{2}\right)\right|$, is **[JEE Main 2014, 4M]**

(a) is strictly greater than $\frac{5}{2}$
(b) is equal to $\frac{5}{2}$
(c) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$
(d) lies in the interval (1, 2)

142. Let $z_k=\cos\left(\frac{2k\pi}{10}\right)+i\sin\left(\frac{2k\pi}{10}\right)$; $k=1,2,\dots,9$. Then, match the column.

	Column I		Column II
(A)	For each z_k there exists a z_j such that $z_k\cdot z_j=1$	(1)	True
(B)	There exists a $k\in\{1,2,\dots,9\}$ such that $z_1\cdot z=z_k$ has no solution z in the set of complex numbers	(2)	False
(C)	$\frac{\lvert 1-z_1\rvert\lvert 1-z_2\rvert\dots\lvert 1-z_9\rvert}{10}$ equals to	(3)	1
(D)	$1-\sum_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right)$ equals to	(4)	2

[JEE Advanced 2014, 3M]

Codes

	A	B	C	D		A	B	C	D
(a)	1	2	4	3	(b)	2	1	3	4
(c)	1	2	3	4	(d)	2	1	4	3

143. A complex number z is said to be unimodular if $|z|=1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1-2z_2}{2-z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a **[JEE Main 2015, 4M]**

(a) circle of radius z
(b) circle of radius $\sqrt{2}$
(c) straight line parallel to X-axis
(d) straight line parallel to Y-axis

144. Let $\omega\neq 1$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4n+3}+(2+3\omega-3\omega^2)^{4n+3}+(-3+2\omega+3\omega^2)^{4n+3}=0$, then possible value(s) of n is (are) **[JEE Advanced 2015, 2M]**

(a) 1 (b) 2
(c) 3 (d) 4

145. For any integer k, let $\alpha_k=\cos\left(\frac{k\pi}{7}\right)+i\sin\left(\frac{k\pi}{7}\right)$, where $i=\sqrt{-1}$. The value of the expression $\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}$ is

[JEE Advanced 2015, 4M]

146. A value of θ for which $\dfrac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is **[JEE Main 2016, 4M]**

(a) $\dfrac{\pi}{6}$ (b) $\sin^{-1}\left(\dfrac{\sqrt{3}}{4}\right)$

(c) $\sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$ (d) $\dfrac{\pi}{3}$

147. Let $a, b \in R$ and $a^2 + b^2 \neq 0$.

Suppose $S = \left\{ z \in C : z = \dfrac{1}{a+ibt}, t \in R, t \neq 0 \right\}$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in S$, then (x, y) lies on **[JEE Advanced 2016 4M]**

(a) the circle with radius $\dfrac{1}{2a}$ and centre $\left(\dfrac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(b) the circle with radius $-\dfrac{1}{2a}$ and centre $\left(-\dfrac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(c) the X-axis for $a \neq 0, b = 0$

(d) the Y-axis for $a = 0, b \neq 0$

148. Let ω be a complex number such that $2\omega + 1 = z$, when $z = \sqrt{-3}$ if $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to **[JEE Main 2017, 4M]**

(a) 1 (b) $-z$ (c) z (d) -1

Answers

Chapter Exercises

1. (b) **2.** (c) **3.** (d) **4.** (d) **5.** (a) **6.** (d)
7. (b) **8.** (d) **9.** (b) **10.** (c) **11.** (c) **12.** (b)
13. (c) **14.** (d) **15.** (a) **16.** (c) **17.** (a) **18.** (c)
19. (b) **20.** (b) **21.** (c) **22.** (c) **23.** (c) **24.** (c)
25. (d) **26.** (d) **27.** (b) **28.** (b) **29.** (b) **30.** (c)
31. (b,c,d) **32.** (b,c,d) **33.** (b,c,d) **34.** (b,c,d) **35.** (a,c,d) **36.** (a,b)
37. (a,b,c) **38.** (a,d) **39.** (a,c) **40.** (a,b,d) **41.** (a,d)
42. (b,d) **43.** (a,c,d) **44.** (a,d) **45.** (a,b,d)
46. (a) **47.** (d) **48.** (b) **49.** (a) **50.** (c) **51.** (b)
52. (d) **53.** (c) **54.** (a) **55.** (c) **56.** (d) **57.** (b)
58. (1) **59.** (5) **60.** (9) **61.** (8) **62.** (3) **63.** (4)
64. (6) **65.** (5) **66.** (4) **67.** (4)
68. A $\to$ (p, q); B $\to$ (p, r); C $\to$ (p, r, s)
69. A $\to$ (q); B $\to$ (q, r); C $\to$ (q, s)
70. A $\to$ (p, q, r, t); B $\to$ (p, s); C $\to$ (p, r)
71. A $\to$ (r); B $\to$ (p, s); C $\to$ (q, t)
72. (d) **73.** (d) **74.** (c) **75.** (d) **76.** (a) **77.** (d)
78. (d)
82. 2

84. $z = c + i(-1 \pm \sqrt{(1-c^2-2c)})$ for $0 \leq c \leq \sqrt{2}-1$ and no solution for $c > \sqrt{2}-1$

85. $(1-3i)z + (1+3i)\bar{z} - 22 = 0$ and $(3+i)z + (3-i)\bar{z} + 14 = 0$

87. No solution **97.** $\dfrac{\sum |z_1|^2 (z_2 - z_3)}{\sum \bar{z}_1 (z_2 - z_3)}$

98. $\dfrac{\sum z_1^2(\bar{z}_2 - \bar{z}_3) + \sum |z_1|^2 (z_2 - z_3)}{\sum (z_1 \bar{z}_2 - z_2 \bar{z}_1)}$

99. (i) $x^3 - 24x^2 + 80x - 64 = 0$
(ii) $x^3 - 21x^2 + 35x - 7 = 0$
(iii) 4

100. Roots of $z^7 + 1 = 0$ are $-1, \alpha, \alpha^3, \alpha^5, \bar{\alpha}, \bar{\alpha}^3, \bar{\alpha}^5$, where $\alpha = \cos\dfrac{\pi}{7} + i\sin\dfrac{\pi}{7}$

101. $a \in \left(\dfrac{1-\sqrt{71}}{2}, \dfrac{-1-4\sqrt{11}}{5}\right) \cup \left(\dfrac{-1+4\sqrt{11}}{5}, \dfrac{1+\sqrt{71}}{2}\right)$

102. (i) ${}^{2n+1}C_1(1-x)^n - {}^{2n+1}C_3(1-x)^{n-1}x + \ldots + (-1)^n x^n = 0$
(ii) ${}^{2n+1}C_1 x^n - {}^{2n+1}C_3 x^{n-1} + {}^{2n+1}C_5 x^{n-2} - \ldots = 0$

103. (c) **104.** (c) **105.** $(1-\sqrt{3}) + i, -i\sqrt{3}, (\sqrt{3}+1) - i$ **106.** (d)
107. (b) **108.** (a) **109.** (d) **110.** (d) **111.** (d) **112.** (d)
113. (d) **114.** (c) **115.** (b) **116.** (c) **117.** (d) **118.** (d)
119. (c) **120.** (a) **121.** (d) **122.** (c) **123.** (a, c, d) **124.** (b)
125. A $\to$ (q, r); B $\to$ (p); C $\to$ (p, s); D $\to$ (q, r, s, t) **126.** (b) **127.** (a)
128. (5) **129.** (a) **130.** (c) **131.** (3) **132.** (c) **133.** (a)
134. (d) **135.** (d) **136.** (b) **137.** (c) **138.** (c) **139.** (c)
140. (b) **141.** (d) **142.** (c) **143.** (a) **144.** (a, b, d)
145. (4) **146.** (c) **147.** (a,c,d) **148.** (b)

Solutions

1. We have,

$$a + ib = \cos(1 - i) = \cos 1 \cos i + \sin 1 \sin i$$
$$= \cos 1 \cosh 1 + \sin 1\, i \sinh 1$$
$$[\because \cos i = \cosh 1, \sin i \cdot 1 = i \sinh 1]$$
$$= \cos 1 \left(\frac{e + e^{-1}}{2}\right) + i \sin 1 \left(\frac{e - e^{-1}}{2}\right)$$
$$= \frac{1}{2}\left(e + \frac{1}{e}\right) \cos 1 + i \cdot \frac{1}{2}\left(e - \frac{1}{e}\right) \sin 1$$
$$\therefore \quad a = \frac{1}{2}\left(e + \frac{1}{e}\right) \cos 1$$
and $$b = \frac{1}{2}\left(e - \frac{1}{e}\right) \sin 1$$

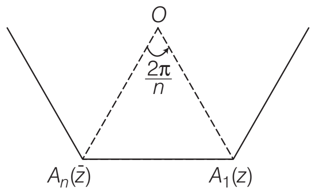
2. Given that, $z^{10} - z^5 - 992 = 0$

Let $$t = z^5$$
$$\Rightarrow \quad t^2 - t - 992 = 0$$
$$\Rightarrow \quad t = \frac{1 \pm \sqrt{1 + 3968}}{2} = \frac{1 \pm 63}{2} = 32, -31$$
$$\therefore \quad z^5 = 32$$
and $$z^5 = -31$$

But the real part is negative, therefore $z^5 = 32$ does not hold.

$\therefore$ Number of solutions is 5.

3. From Coni method,



$$\frac{z - 0}{\bar{z} - 0} = e^{2\pi i/n} \quad \text{or} \quad \frac{z}{\bar{z}} = e^{2\pi i/n} \quad \ldots \text{(i)}$$

But given $$\frac{\text{Im}(z)}{\text{Re}(z)} = \sqrt{2} - 1$$

$$\Rightarrow \quad \frac{\frac{z - \bar{z}}{2i}}{\frac{z + \bar{z}}{2}} = \sqrt{2} - 1 \Rightarrow \frac{1}{i}\left(\frac{\frac{z}{\bar{z}} - 1}{\frac{z}{\bar{z}} + 1}\right) = \sqrt{2} - 1$$
$$\Rightarrow \quad \left(\frac{e^{2\pi i/n} - 1}{e^{2\pi i/n} + 1}\right) = i(\sqrt{2} - 1) \quad \text{[from Eq. (i)]}$$
$$\Rightarrow \quad i \tan\left(\frac{\pi}{n}\right) = i(\sqrt{2} - 1)$$
$$\Rightarrow \quad \tan\left(\frac{\pi}{n}\right) = \tan\left(\frac{\pi}{8}\right)$$
$$\therefore \quad n = 8$$

4. We have, $$\prod_{p=1}^{r} e^{ip\theta} = 1$$
$$\Rightarrow \quad e^{i\theta} \cdot e^{2i\theta} \cdot e^{3i\theta} \ldots e^{ri\theta} = 1$$
$$\Rightarrow \quad e^{i\theta(1 + 2 + 3 + \ldots + r)} = 1 \Rightarrow e^{i\theta\left(\frac{r(r+1)}{2}\right)} = 1$$
or $$\cos\left\{\frac{r(r+1)}{2}\theta\right\} + i \sin\left\{\frac{r(r+1)}{2}\theta\right\} = 1 + i \cdot 0$$

On comparing, we get
$$\cos\left\{\frac{r(r+1)}{2}\theta\right\} = 1 \text{ and } \sin\left\{\frac{r(r+1)}{2}\theta\right\} = 0$$
$$\Rightarrow \quad \frac{r(r+1)}{2}\theta = 2m\pi \text{ and } \frac{r(r+1)}{2}\theta = m_1\pi$$
$$\Rightarrow \quad \theta = \frac{4m\pi}{r(r+1)} \text{ and } \theta = \frac{2m_1\pi}{r(r+1)}$$

where, $m, m_1 \in I$

Hence, $\theta = \dfrac{4n\pi}{r(r+1)}, n \in I$.

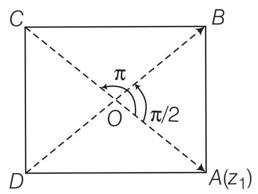
5. Let $z = x + iy$, then

$(3 + i)(z + \bar{z}) - (2 + i)(z - \bar{z}) + 14i = 0$ reduces to
$$(3 + i)\,2x - (2 + i)(2iy) + 14i = 0$$
$$\Rightarrow \quad 6x + 2y + i(2x - 4y + 14) = 0$$

On comparing real and imaginary parts, we get
$$6x + 2y = 0$$
$$\Rightarrow \quad 3x + y = 0 \quad \ldots \text{(i)}$$
and $$2x - 4y + 14 = 0$$
$$\Rightarrow \quad x - 2y + 7 = 0 \quad \ldots \text{(ii)}$$

On solving Eqs. (i) and (ii), we get
$$x = -1 \text{ and } y = 3$$
$$\therefore \quad z = -1 + 3i$$
$$\therefore \quad z\bar{z} = |z|^2 = |-1 + 3i|^2 = (-1)^2 + (3)^2 = 10$$

6. Since, affix of A is z_1.



$\therefore \overrightarrow{\mathbf{OA}} = z_1$ and $\overrightarrow{\mathbf{OB}}$ and $\overrightarrow{\mathbf{OC}}$ are obtained by rotating $\overrightarrow{\mathbf{OA}}$ through $\dfrac{\pi}{2}$ and π. Therefore, $\overrightarrow{\mathbf{OB}} = iz_1$ and $\overrightarrow{\mathbf{OC}} = -z_1$.

Hence, centroid of $\Delta ABC = \dfrac{z_1 + iz_1 + (-z_1)}{3}$
$$= \frac{i}{3}z_1 = \frac{z_1}{3}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)$$

If A, B and C are taken in clockwise, then centroid of ΔABC
$$= \frac{1}{3}z_1\left(\cos\frac{\pi}{2} - i \sin\frac{\pi}{2}\right)$$
$$\therefore \quad \text{Centroid of } \Delta ABC = \frac{z_1}{3}\left(\cos\frac{\pi}{2} \pm i \sin\frac{\pi}{2}\right)$$

7. Given that, $z = \dfrac{\sqrt{3} - i}{2} = i\left(\dfrac{-1 - i\sqrt{3}}{2}\right) = i\omega^2$

$\therefore \quad z^{101} = (i\omega^2)^{101} = i^{101}\,\omega^{202} = i\omega$

Now, $i^{101} + z^{101} = i + i\omega = i(-\omega^2)$

$\therefore \quad (i^{101} + z^{101})^{103} = -\,i^{103}\,\omega^{206} = -\,i^3\,\omega^2 = i\omega^2 = z$

8. The complex slope of the line $a\bar{z} + \bar{a}z + 1 = 0$ is $\alpha = -\dfrac{a}{\bar{a}}$

and the complex slope of the line $b\bar{z} + \bar{b}z - 1 = 0$ is $\beta = -\dfrac{b}{\bar{b}}$

Since, both lines are mutually perpendicular, then

$\therefore \quad \alpha + \beta = 0$

$\Rightarrow \quad -\dfrac{a}{\bar{a}} - \dfrac{b}{\bar{b}} = 0$

$\Rightarrow \quad a\bar{b} + \bar{a}b = 0$

9. We have, $\alpha = \cos\left(\dfrac{8\pi}{11}\right) + i\sin\left(\dfrac{8\pi}{11}\right)$

Now, $\text{Re}\,(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$

$$= \frac{\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^3 + \bar{\alpha}^4 + \bar{\alpha}^5}{2}$$

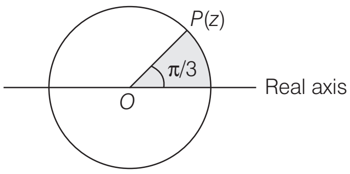
$$= \frac{-1 + (1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^3 + \bar{\alpha}^4 + \bar{\alpha}^5}{2}$$

$$= \frac{-1 + 0}{2} \qquad \text{[sum of 11, 11th roots of unity]}$$

$$= -\frac{1}{2}$$

10. $|z| \le 4$ ...(i)

and $0 \le \arg(z) \le \dfrac{\pi}{3}$ ...(ii)



which implies the set of points in an argand plane, is a sector of a circle.

11. Since, $x^2 + x + 1 = (x - \omega)(x - \omega^2)$, where ω is the cube root of unity and $f(x) = g(x^3) + x\,h(x^3)$ is divisible by $x^2 + x + 1$. Therefore, ω and ω^2 are the roots of $f(x) = 0$.

$\Rightarrow \quad f(\omega) = 0$ and $f(\omega^2) = 0$

$\Rightarrow \quad g(\omega^3) + \omega\,h(\omega^3) = 0$

and $\quad g((\omega^2)^3) + \omega^2 h(\omega^2)^3 = 0$

$\Rightarrow \quad g(1) + \omega\,h(1) = 0$

and $\quad g(1) + \omega^2 h(1) = 0$

$\Rightarrow \quad g(1) = h(1) = 0$

Hence, $g(x)$ and $h(x)$ both are divisible by $(x - 1)$.

12. Since, z_1, z_2 and $z_1 - z_2$ are collinear.

$$\therefore \quad \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_1 - z_2 & \overline{z_1 - z_2} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_1 - z_2 & \bar{z}_1 - \bar{z}_2 & 1 \end{vmatrix} = 0$$

Applying $R_3 \to R_3 - R_1 + R_2$, then $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 0$

Expand w.r.t. R_3, then

$z_1\bar{z}_2 - \bar{z}_1 z_2 = 0$

$\Rightarrow \quad z_1\bar{z}_2 - (\overline{z_1\bar{z}_2}) = 0$

$\Rightarrow \quad \text{Im}\,(z_1\bar{z}_2) = 0$

$\Rightarrow \quad \text{Im}\,((a + ib)\,\overline{(c + id)}) = 0$

$\Rightarrow \quad \text{Im}\,((a + ib)(c - id)) = 0$

$\Rightarrow \quad bc - ad = 0 \Rightarrow ad - bc = 0$

13. Let $z = a + ib$

$\therefore f(a + ib) = \sqrt{(a^2 + b^2)}$

$\Rightarrow \quad f(z) = f(\bar{z}) = f(-z) = f(-\bar{z}) = \sqrt{(a^2 + b^2)}$

$\therefore \quad f$ is not injective (i.e., it is many-one).

but $|z| > 0$ i.e. $f(z) > 0 \Rightarrow f(z) \in R^+$ (Range)

$\Rightarrow R^+ \subset R$

$\therefore \; f$ is not surjective (i.e., into).

Hence, f is neither injective nor surjective.

14. Let $\alpha = re^{i\theta}$, $\beta = re^{i\phi}$ $\quad [\because |\alpha| = |\beta|$, given$]$

where, $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and $\phi \in (-\pi, 0)$

$$\therefore \quad \frac{\alpha + \beta}{\alpha - \beta} = \frac{re^{i\theta} + re^{i\phi}}{re^{i\theta} - re^{i\phi}} = \frac{e^{i\left(\frac{\theta + \phi}{2}\right)} \cdot 2\cos\left(\frac{\theta - \phi}{2}\right)}{e^{i\left(\frac{\theta + \phi}{2}\right)} \cdot 2i\sin\left(\frac{\theta - \phi}{2}\right)}$$

$$= -\,i\cot\left(\frac{\theta - \phi}{2}\right) = \text{Purely imaginary}$$

15. We have, $|z| = \left|z + \dfrac{25}{z} - \dfrac{25}{z}\right| \le \left|z + \dfrac{25}{z}\right| + \left|\dfrac{25}{-z}\right|$

$\Rightarrow \quad |z| \le 24 + \dfrac{25}{|z|}$

$\Rightarrow \quad |z|^2 - 24|z| - 25 \le 0 \Rightarrow (|z| - 25)(|z| + 1) \le 0$

$\therefore \quad |z| - 25 \le 0 \qquad [\because |z| + 1 > 0]$

$\Rightarrow \quad |z| \le 25$ or $|z - 0| \le 25$

Hence, the maximum distance from the origin of coordinates to the point z is 25.

16. $\because \; A \equiv z_1, B \equiv z_2, C \equiv (1 - i)\,z_1 + iz_2$

$\therefore \quad AB = |z_1 - z_2|$

$BC = |z_2 - (1 - i)\,z_1 - iz_2| = |(1 - i)(z_2 - z_1)|$

$= \sqrt{2}\,|z_1 - z_2|$

and $\quad CA = |(1 - i)\,z_1 + iz_2 - z_1| = |-i(z_1 - z_2)|$

$= |-i||z_1 - z_2| = |z_1 - z_2|$

It is clear that, $AB = CA$ and $(AB)^2 + (CA)^2 = (BC)^2$

$\therefore \; \Delta\,ABC$ is isosceles and right angled.

17. Centre and radius of circle $|z| = 3$ ...(i)

$$\text{are } C_1 \equiv 0,\ r_1 = 3$$

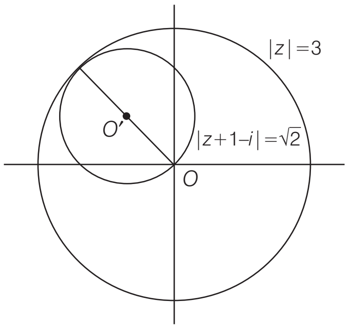
and centre and radius of circle

$$|z + 1 - i| = \sqrt{2} \quad \text{...(ii)}$$

and $\quad C_2 = -1 + i, r_2 = \sqrt{2}$

$\because \quad |C_1C_2| = |-1 + i| = \sqrt{2}$

and $\quad |C_1C_2| < r_1 - r_2$



Hence, circle (ii) completely inside circle (i)

$\therefore$ Number of solutions = 0

18. We have, $f(z) = g(z)(z^2 + 1) + h(z)$

where, degree of $h(z) <$ degree of $(z^2 + 1)$

$\Rightarrow \quad h(z) = az + b\ ;\ a, b \in C$

$\therefore \quad f(z) = g(z)(z^2 + 1) + az + b\ ;\ a, b \in C$

$\Rightarrow \quad f(z) = g(z)(z - i)(z + i) + az + b\ ;\ a, b \in C$ ...(i)

Now, $\quad f(i) = 1 - i$ [given]

$\Rightarrow \quad ai + b = 1 - i$ [from Eq. (i)] ...(ii)

and $\quad f(-i) = 1 + i$ [given]

$\Rightarrow \quad a(-i) + b = 1 + i$ [from Eq. (i)] ...(iii)

On solving Eqs. (ii) and (iii) for a and b, we get

$$a = -1 \text{ and } b = 1$$

$\therefore$ Required remainder, $h(z) = az + b = -z + 1 = 1 - z$

19. We have, $|z + 1| = 2|z - 1|$

Put $\quad z = x + iy$, we get

$$(x + 1)^2 + y^2 = 4[(x - 1)^2 + y^2]$$

$\Rightarrow \quad 3x^2 + 3y^2 - 10x + 3 = 0$

$\Rightarrow \quad x^2 + y^2 - \dfrac{10}{3}x + 1 = 0$ ...(i)

On comparing Eq. (i) with the standard equation

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$\Rightarrow \quad g = -\dfrac{10}{6} = -\dfrac{5}{3}$ and $f = 0$

$\therefore$ Required centre of circle $\equiv (-g, -f) \equiv \left(\dfrac{5}{3}, 0\right)$

i.e. $\quad \dfrac{5}{3} + 0 \cdot i = \dfrac{5}{3}$

20. $\because \quad x = 9^{1/3} \cdot 9^{1/9} \cdot 9^{1/27} \ldots \infty$

$$= 9^{1/3 + 1/9 + 1/27 + \ldots \infty} = 9^{\frac{1/3}{1 - 1/3}} = 9^{1/2} = 3$$

$$y = 4^{1/3} \cdot 4^{-1/9} \cdot 4^{1/27} \ldots \infty = 4^{1/3 - 1/9 + 1/27 \ldots \infty}$$

$$= 4^{\frac{1/3}{1 + 1/3}} = 4^{1/4} = \sqrt{2}$$

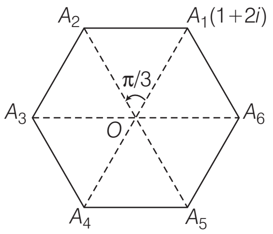
and $z = \displaystyle\sum_{r=1}^{\infty} (1 + i)^{-r} = \frac{1}{(1 + i)} + \frac{1}{(1 + i)^2} + \frac{1}{(1 + i)^3} + \ldots \infty$

$$= \frac{\dfrac{1}{(1 + i)}}{1 - \dfrac{1}{(1 + i)}} = \frac{1}{i} = -i$$

Now, $x + yz = 3 - i\sqrt{2}$

$\therefore \arg(x + yz) = \arg(3 - i\sqrt{2}) = -\tan^{-1}\left(\dfrac{\sqrt{2}}{3}\right)$

21. $\because \quad A_1 \equiv 1 + 2i$



$\therefore \quad A_2 = (1 + 2i)\, e^{i\pi/3}$

$$= (1 + 2i)\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = \frac{1}{2} + \frac{i\sqrt{3}}{2} + i - \sqrt{3}$$

$$= \left(\frac{1}{2} - \sqrt{3}\right) + i\left(\frac{\sqrt{3}}{2} + 1\right)$$

$\therefore |A_1A_2| = \left|1 + 2i - \left(\dfrac{1}{2} - \sqrt{3}\right) - i\left(\dfrac{\sqrt{3}}{2} + 1\right)\right|$

$$= \left|\frac{1}{2} + \sqrt{3} + i\left(1 - \frac{\sqrt{3}}{2}\right)\right|$$

$$= \sqrt{\left(\frac{1}{2} + \sqrt{3}\right)^2 + \left(1 - \frac{\sqrt{3}}{2}\right)^2} = \sqrt{5}$$

$\therefore \quad$ Perimeter $= 6\,|A_1A_2| = 6\sqrt{5}$

22. We have,

$$\left|\sum_{r=1}^{n} z_r\right| = \left|\sum_{r=1}^{n} (z_r - r) + r\right| \le \sum_{r=1}^{n} (|z_r - r| + |r|)$$

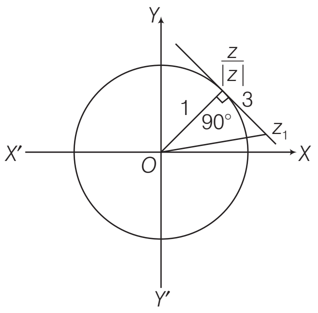
$$= \sum_{r=1}^{n} |z_r - r| + \sum_{r=1}^{n} |r| \le \sum_{r=1}^{n} r + \sum_{r=1}^{n} |r|$$

$$= \frac{n(n + 1)}{2} + \frac{n(n + 1)}{2} = n(n + 1)$$

$\therefore \quad \left|\displaystyle\sum_{r=1}^{n} z_r\right| \le n(n + 1)$

23. We have, $\arg\left(\dfrac{z_1 - \dfrac{z}{|z|}}{\dfrac{z}{|z|}}\right) = \dfrac{\pi}{2}$ and $\left|\dfrac{z}{|z|} - z_1\right| = 3$

which implies the following diagram



$\Rightarrow \quad \left|\frac{z}{|z|} - z_1\right| = 3 \Rightarrow |z_1| = \sqrt{9+1} = \sqrt{10}$

24. Let $z = x + iy = r(\cos\theta + i\sin\theta)$

$\therefore\ |z| = r, \arg(z) = \theta$

Given, $|z - 2 - i| = |z|\left|\sin\left(\frac{\pi}{4} - \arg(z)\right)\right|$

$\Rightarrow\ |x + iy - 2 - i| = r\left|\sin\left(\frac{\pi}{4} - \theta\right)\right|$

$\Rightarrow\ |(x-2) + i(y-1)| = r\left|\frac{1}{\sqrt{2}}(\cos\theta - \sin\theta)\right|$

$\Rightarrow\ \sqrt{(x-2)^2 + (y-1)^2} = \frac{1}{\sqrt{2}}|x - y|$

On squaring both sides, we get

$$2(x^2 + y^2 - 4x - 2y + 5) = x^2 + y^2 - 2xy$$

$\Rightarrow \qquad (x+y)^2 = 2(4x + 2y - 5)$

which is a parabola.

25. Since, $1, z_1, z_2, z_3, \ldots, z_{n-1}$ are the n, nth roots of unity.

$\therefore\ (z^n - 1) = (z-1)(z-z_1)(z-z_2)(z-z_3)\ldots(z-z_{n-1})$

$$= (z-1)\prod_{r=1}^{n-1}(z - z_r)$$

Taking log on both sides, we get

$$\log_e(z^n - 1) = \log_e(z-1) + \sum_{r=1}^{n-1}\log_e(z - z_r)$$

On differentiating both sides w.r.t.z, we get

$$\frac{nz^{n-1}}{(z^n-1)} - \frac{1}{(z-1)} = \sum_{r=1}^{n-1}\frac{1}{(z-z_r)}$$

Putting $z = 3$, we get

$$\sum_{r=1}^{n-1}\frac{1}{(3-z_r)} = \frac{n\cdot 3^{n-1}}{(3^n-1)} - \frac{1}{2}$$

26. We have,

$$z = (3+7i)(\lambda + i\mu)$$
$$= (3\lambda - 7\mu) + i(7\lambda + 3\mu)$$

Since, z is purely imaginary.

$\therefore \qquad 3\lambda - 7\mu = 0$

$\Rightarrow \qquad \frac{\lambda}{\mu} = \frac{7}{3}$

$\because \qquad \lambda, \mu \in I - \{0\}$

For minimum value $\lambda = 7, \mu = 3$

$\therefore \qquad |z|^2 = |(3+7i)(\lambda + i\mu)|^2$

$$= |3+7i|^2|\lambda + i\mu|^2 = 58(\lambda^2 + \mu^2)$$
$$= 58(7^2 + 3^2) = (58)^2 = 3364$$

27. We have,

$$z = f(x) + i\,g(x)$$

where, $i = \sqrt{-1}$ and $f, g : (0,1) \to (0,1)$ are real-valued functions.

(a) $z = \frac{1}{1-ix} + i\left(\frac{1}{1+ix}\right)$

$$= \frac{1+ix}{1+x^2} + \frac{x+i}{1+x^2} = \frac{1+x}{1+x^2} + i\frac{(1+x)}{1+x^2}$$

$\Rightarrow f(x) = \frac{1+x}{1+x^2}$ and $g(x) = \frac{1+x}{1+x^2}$

But for $x = 0.5$, $f(0.5) > 1$ and $g(0.5) > 1$, which is out of range.

Hence, (a) is not a correct option.

(b) $\quad z = \frac{1}{1+ix} + i\left(\frac{1}{1-ix}\right)$

$$= \frac{1-ix}{1+x^2} + \frac{(i-x)}{1+x^2} = \left(\frac{1-x}{1+x^2}\right) + i\left(\frac{1-x}{1+x^2}\right)$$

$\Rightarrow\ f(x) = \frac{1-x}{1+x^2}$ and $g(x) = \frac{1-x}{1+x^2}$

Clearly, $f(x), g(x) \in (0,1)$, if $x \in (0,1)$

Hence, (b) is the correct option.

(c) $\ z = \frac{1-ix}{1+x^2} + \frac{i(1-ix)}{1+x^2} = \frac{(1+x)}{(1+x^2)} + \frac{i(1-x)}{(1+x^2)}$

Hence, (c) is not a correct option.

(d) $\ z = \frac{1}{1-ix} + i\left(\frac{1}{1-ix}\right) = \frac{1+ix}{1+x^2} + \frac{i(1+ix)}{(1+x^2)}$

$$= \frac{(1-x)}{(1+x^2)} + \frac{i(1+x)}{(1+x^2)}$$

Hence, (d) is not a correct option.

28. Let $z = \alpha$ be a real roots of equation.

$$z^3 + (3+2i)z + (-1+ia) = 0$$

$\Rightarrow \qquad \alpha^3 + (3+2i)\alpha + (-1+ia) = 0$

$\Rightarrow \qquad (\alpha^3 + 3\alpha - 1) + i(a + 2\alpha) = 0$

On comparing the real and imaginary parts, we get

$$\alpha^3 + 3\alpha - 1 = 0 \text{ and } a + 2\alpha = 0$$

$\Rightarrow \qquad \alpha = -\frac{a}{2}$

$\Rightarrow \qquad -\frac{a^3}{8} - \frac{3a}{2} - 1 = 0$

$\Rightarrow \qquad a^3 + 12a + 8 = 0$

Let $\quad f(a) = a^3 + 12a + 8$

$\Rightarrow \quad f(-1) < 0$ and $f(0) > 0$

$\therefore \quad a \in (-1, 0)$

29. $CiS\,\dfrac{\pi}{6} = \cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}$

$$= \left(\frac{\sqrt{3}+i}{2}\right) = \frac{1}{i}\left(\frac{-1+i\sqrt{3}}{2}\right) = \frac{\omega}{i} = -i\omega$$

$$\therefore \left(2\,CiS\,\frac{\pi}{6}\right)^m = (-2i\omega)^m = ((-2i\omega)^3)^{m/3} = (8i)^{m/3}$$

and $\left(4\,CiS\,\dfrac{\pi}{4}\right)^n = \left(4\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)\right)^n = (2\sqrt{2}\,(1+i)^n$

$$= (8(1+i)^2)^{n/2} = (16i)^{n/2}$$

Thus, $\quad (8i)^{m/3} = (16i)^{n/2}$

which is satisfy only when $m = 48$ and $n = 24$

$\therefore \quad m + n = 72$

30. We have, $\quad z^2 = \bar{z}\cdot 2^{1-|z|}$

Taking modulus on both sides, we get

$$|z|^2 = |z|\cdot 2^{1-|z|}$$

$\Rightarrow \quad |z|(|z| - 2^{1-|z|}) = 0 \quad$ ...(i)

and $\quad \arg(z^2) = \arg(\bar{z}\cdot 2^{1-|z|})$

$\Rightarrow \quad 2\arg(z) = \arg(\bar{z}) = -\arg(z)$

$\Rightarrow \quad 3\arg(z) = 0$

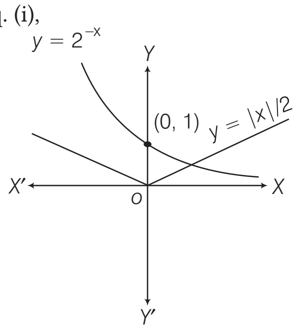
$\therefore \quad \arg(z) = 0$

Then, $\quad y = 0 \quad [\because z = x + iy]$

From Eq. (i), $|z| = 0 \Rightarrow x = 0 \quad [\because y = 0]$

One solution is $z = 0 + i\cdot 0 = 0$.

Also, from Eq. (i),



$$|z| = 2^{1-|z|} \Rightarrow |x| = 2^{1-x}$$

$\Rightarrow \quad \dfrac{|x|}{2} = 2^{-x} = y$ (say)

Hence, total number of solutions = 2

31. $\because \dfrac{z+1}{z+i}$ is a purely imaginary number.

$$\therefore \left(\overline{\frac{z+1}{z+i}}\right) = -\left(\frac{z+1}{z+i}\right) \Rightarrow \frac{\bar{z}+1}{\bar{z}-i} = -\left(\frac{z+1}{z+i}\right)$$

$\Rightarrow \quad (\bar{z}+1)(z+i) + (\bar{z}-i)(z+1) = 0$

$\Rightarrow \quad 2z\bar{z} + \bar{z}(1+i) + z(1-i) = 0$

$\Rightarrow \quad z\bar{z} + \left(\dfrac{1+i}{2}\right)\bar{z} + \left(\dfrac{1-i}{2}\right)z = 0$

which is a circle and passing through the origin

and radius $= \sqrt{\left|\dfrac{1+i}{2}\right|^2 - 0} = \left|\dfrac{1+i}{2}\right| = \dfrac{1}{\sqrt{2}}$

32. Given, $\quad |z-1| < |z+3|$

$\Rightarrow \quad |z-1|^2 < |z+3|^2$

$\Rightarrow \quad |z|^2 + 1 - 2\,\mathrm{Re}(z) < |z|^2 + 9 + 2\,\mathrm{Re}(3z)$

$\Rightarrow \quad 2\,\mathrm{Re}(4z) > -8$

$\Rightarrow \quad \mathrm{Re}(4z) > -4$

$\Rightarrow \quad \dfrac{4z + 4\bar{z}}{2} > -4$

$\therefore \quad z + \bar{z} > -2$

and $\quad \omega = 2z + 3 - i$

$\therefore \quad \omega + \bar{\omega} = 2z + 3 - i + 2\bar{z} + 3 + i$

$\qquad = 2(z + \bar{z}) + 6 > -4 + 6$

$\Rightarrow \quad \omega + \bar{\omega} > 2$

Option (a) $|\omega - 5 - i| < |\omega + 3 + i|$

$\Rightarrow \quad |2z + 3 - i - 5 - i| < |2z + 3 - i + 3 + i|$

$\Rightarrow \quad |2z - 2 - 2i| < |2z + 6|$

$\Rightarrow \quad |z - 1 - i| < |z + 3|$

which is false.

Option (b) $|\omega - 5| < |\omega + 3|$

$\Rightarrow \quad |2z + 3 - i - 5| < |2z + 3 - i + 3|$

$\Rightarrow \quad |2z - 2 - i| < |2z + 6 - i|$

$\Rightarrow \quad \left|z - 1 - \dfrac{i}{2}\right| < \left|z + 3 - \dfrac{i}{2}\right|$

$\Rightarrow \quad |z - 1| < |z + 3|$

which is true.

Option (c) $\quad \mathrm{Im}(i\omega) > 1$

$\Rightarrow \quad \dfrac{i\omega - \overline{i\omega}}{2i} > 1$

$\Rightarrow \quad \dfrac{i\omega + i\bar{\omega}}{2i} > 1$

$\Rightarrow \quad \omega + \bar{\omega} > 2$

which is true.

Option (d) $\quad |\arg(\omega - 1)| < \dfrac{\pi}{2}$

$\Rightarrow \quad |\arg(2z + 3 - i - 1)| < \dfrac{\pi}{2}$

$\Rightarrow \quad |\arg(2z + 2 - i)| < \dfrac{\pi}{2}$

$\Rightarrow \quad \left|\tan^{-1}\left(\dfrac{\mathrm{Im}(2z+2-i)}{\mathrm{Re}(2z+2-i)}\right)\right| < \dfrac{\pi}{2}$

$\therefore \quad \mathrm{Re}(2z + 2 - i) > 0$

$\Rightarrow \quad \dfrac{(2z + 2 - i) + (2\bar{z} + 2 + i)}{2} > 0$

$\Rightarrow \quad z + \bar{z} + 2 > 0$

$\Rightarrow \quad z + \bar{z} > -2$

which is true.

33. $\therefore \quad (1 + ri)^3 = \lambda(1 + i)$

$\Rightarrow \quad 1 + (ri)^3 + 3(1)^2 ri + 3(1)(ri)^2 = \lambda(1 + i)$

$$\Rightarrow \quad 1 - r^3 i + 3ri - 3r^2 = \lambda + i\lambda$$

On comparing real and imaginary parts, we get

$$1 - 3r^2 = \lambda$$

and $\quad -r^3 + 3r = \lambda$

Then, $\quad -r^3 + 3r = 1 - 3r^2$

$$\Rightarrow \quad r^3 - 3r^2 - 3r + 1 = 0$$

$$\Rightarrow \quad (r^3 + 1) - 3r(r + 1) = 0$$

$$\Rightarrow \quad (r + 1)(r^2 - r + 1 - 3r) = 0$$

$$\Rightarrow \quad (r + 1)(r^2 - 4r + 1) = 0$$

$$\therefore \quad r = -1, 2 \pm \sqrt{3}$$

$$\Rightarrow \quad r = \operatorname{cosec} \frac{3\pi}{2}, \tan \frac{\pi}{12}, \cot \frac{\pi}{12}$$

34. **Option** (a) $|z - 1| + |z + 1| = 3$

Here, $|1 - (-1)| < 3$

i.e. $2 < 3$, which is an ellipse.

Option (b) $|z - 3| = 2$

It is a circle with centre 3 and radius 2.

Option (c) $|z - 2 + i| = \frac{7}{3}$

It is a circle with centre $(2 - i)$ and radius $\frac{7}{3}$.

Option (d) $\quad (z - 3 + i)(\bar{z} - 3 - i) = 5$

$$\Rightarrow \quad (z - 3 + i)\overline{(z - 3 + i)} = 5$$

$$\Rightarrow \quad |z - 3 + i|^2 = 5$$

$$\Rightarrow \quad |z - 3 + i| = \sqrt{5}$$

It is a circle with centre at $(3 - i)$ and radius $\sqrt{5}$.

35. Since, $1, z_1, z_2, z_3, \ldots, z_{n-1}$ are the n, nth roots of unity.

Therefore,

$$z^n - 1 = (z - 1)(z - z_1)(z - z_2)\ldots(z - z_{n-1})$$

$$\Rightarrow \quad \frac{z^n - 1}{z - 1} = (z - z_1)(z - z_2)\ldots(z - z_{n-1})$$

$$= \prod_{r=1}^{n-1} (z - z_r)$$

Now, putting $z = \omega$, we get

$$\prod_{r=1}^{n-1} (\omega - z_r) = \frac{\omega^n - 1}{\omega - 1}$$

$$= \begin{cases} 0, & \text{if } n = 3r, r \in Z \\ 1, & \text{if } n = 3r + 1, r \in Z \\ 1 + \omega, & \text{if } n = 3r + 2, r \in Z \end{cases}$$

36. $\because \quad 3|z - 12| = 5|z - 8i|$

$\therefore \; 9|z - 12|^2 = 25|z - 8i|^2$

$$\Rightarrow \quad 9(z - 12)(\bar{z} - 12) = 25(z - 8i)(\bar{z} + 8i)$$

$$\Rightarrow \quad 9(z\bar{z} - 12(z + \bar{z}) + 144) = 25(z\bar{z} + 8i(z - \bar{z}) + 64)$$

$$\Rightarrow \quad 16z\bar{z} + 108(z + \bar{z}) + 200(z - \bar{z})i + 304 = 0$$

$$\Rightarrow \quad 16(x^2 + y^2) + 216x - 400y + 304 = 0$$

$$\Rightarrow \quad 2(x^2 + y^2) + 27x - 50y + 38 = 0 \quad \ldots\text{(i)}$$

and $\quad |z - 4| = |z - 8| \;\Rightarrow |z - 4|^2 = |z - 8|^2$

$$\Rightarrow \quad |z|^2 + 16 - 2\operatorname{Re}(4z) = |z|^2 + 64 - 2\operatorname{Re}(8z)$$

$$\Rightarrow \quad 8\operatorname{Re}(z) = 48$$

$$\therefore \quad \operatorname{Re}(z) = 6$$

$$\Rightarrow \quad x = 6 \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$2(36 + y^2) + 162 - 50y + 38 = 0$$

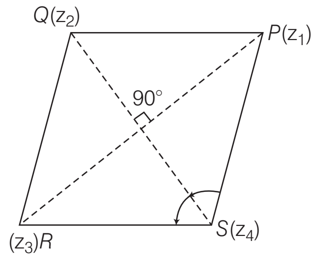
$$\Rightarrow \quad y^2 - 25y + 136 = 0$$

$$\Rightarrow \quad (y - 17)(y - 8) = 0$$

$$\Rightarrow \quad y = 17, 8$$

$$\therefore \quad \operatorname{Im}(z) = 17, 8$$

37.



Option (a) $\because \; PS \| QR$

$$\therefore \quad \arg\left(\frac{z_1 - z_4}{z_2 - z_3}\right) = 0$$

$\Rightarrow \quad \dfrac{z_1 - z_4}{z_2 - z_3}$ is purely real.

Option (b) $\because$ Diagonals of rhombus are perpendicular.

Then, $\arg\left(\dfrac{z_1 - z_3}{z_2 - z_4}\right) = \dfrac{\pi}{2}$

$\Rightarrow \quad \dfrac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary.

Option (c) $\because \quad PR \neq QS$

$$\therefore \quad |z_1 - z_3| \neq |z_2 - z_4|$$

Option (d) $\because \quad \angle QSP = \angle RSQ$

$$\therefore \quad \operatorname{amp}\left(\frac{z_2 - z_4}{z_1 - z_4}\right) = \operatorname{amp}\left(\frac{z_3 - z_4}{z_2 - z_4}\right)$$

$$\Rightarrow \quad -\operatorname{amp}\left(\frac{z_1 - z_4}{z_2 - z_4}\right) = -\operatorname{amp}\left(\frac{z_2 - z_4}{z_3 - z_4}\right)$$

$$\Rightarrow \quad \operatorname{amp}\left(\frac{z_1 - z_4}{z_2 - z_4}\right) = \operatorname{amp}\left(\frac{z_2 - z_4}{z_3 - z_4}\right)$$

38. $\because \; |z - 3| = \min\{|z - 1|, |z - 5|\}$

Case I If $|z - 3| = |z - 1|$

On squaring both sides, we get

$$|z - 3|^2 = |z - 1|^2$$

$$\Rightarrow \quad |z|^2 + 9 - 2\operatorname{Re}(3z) = |z|^2 + 1 - 2\operatorname{Re}(z)$$

$$\Rightarrow \quad 4\operatorname{Re}(z) = 8$$

$$\Rightarrow \quad \operatorname{Re}(z) = 2$$

***Case* II** If $|z-3| = |z-5|$

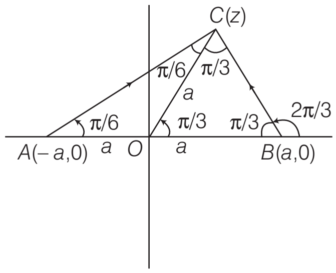
On squaring both sides, we get

$$|z-3|^2 = |z-5|^2$$

$\Rightarrow$ $|z|^2 + 9 - 2\,\text{Re}\,(3z) = |z|^2 + 25 - 2\,\text{Re}\,(5z)$

$\Rightarrow$ $4\,\text{Re}\,(z) = 16 \Rightarrow \text{Re}\,(z) = 4$

39.



From figure, it is clear that z lies on the point of intersection of the rays from A and B.

$\because \angle ACB = 90°$ and OBC is an equilateral triangle.

Hence, $OC = a$

$\Rightarrow$ $|z - 0| = a$ or $|z| = a$

and $\arg(z) = \arg(z-0) = \dfrac{\pi}{3}$

40. $\because$ $\left|\dfrac{2z - i}{z + i}\right| = m$

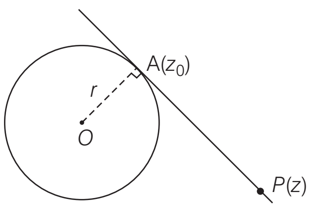
$\Rightarrow$ $\left|\dfrac{z - i/2}{z+i}\right| = \dfrac{m}{2}$

For circle, $\dfrac{m}{2} \neq 1$

$\Rightarrow$ $m \neq 2$ and $m > 0$

41. $\therefore$ $A(z_0)$ lie on $|z| = r$

$\Rightarrow$ $|z_0| = r \Rightarrow |z_0|^2 = r^2 \Rightarrow z_0\bar{z}_0 = r^2$



Let $P(z)$ be any point on tangent, then

$\therefore$ $\angle PAO = \dfrac{\pi}{2}$

Complex slope of AP + Complex slope of $OA = 0$

$\Rightarrow$ $\dfrac{z - z_0}{\bar{z} - \bar{z}_0} + \dfrac{z_0 - 0}{\bar{z}_0 - 0} = 0$

$\Rightarrow$ $z\bar{z}_0 + z_0\bar{z} = 2z_0\bar{z}_0$

$\Rightarrow$ $z\bar{z}_0 + z_0\bar{z} = 2r^2$

$\Rightarrow$ $z\bar{z}_0 = \bar{z}z_0$

Also, $\dfrac{z\bar{z}_0}{r^2} + \dfrac{z_0\bar{z}}{r^2} = 2$

$\Rightarrow$ $\dfrac{z\bar{z}_0}{z_0\bar{z}_0} + \dfrac{z_0\bar{z}}{z_0\bar{z}_0} = 2$

$\Rightarrow$ $\dfrac{\dfrac{z}{z_0} + \left(\dfrac{\bar{z}}{\bar{z}_0}\right)}{2} = 1$

$\therefore$ $\text{Re}\left(\dfrac{z}{z_0}\right) = 1$

42. $\because$ $z_1 + z_2 = a, z_1z_2 = b$

and given $|z_1| = |z_2| = 1$

Let $z_1 = e^{i\theta}$ and $z_2 = e^{i\phi}$

$\because$ $|a| = |z_1 + z_2| \le |z_1| + |z_2| = 1 + 1 = 2$

$\therefore$ $|a| \le 2$

Also, $\arg(a) = \arg(z_1 + z_2) = \arg(e^{i\theta} + e^{i\phi}) = \dfrac{\theta + \phi}{2}$

and $\arg(b) = \arg(z_1z_2) = \arg(e^{i\theta + \phi}) = \theta + \phi$

$\therefore$ $2\arg(a) = \arg(b) \Rightarrow \arg(a^2) = \arg(b)$

43. $\because$ $\alpha z^2 + z + \bar{\alpha} = 0$ ...(i)

Then, $\overline{\alpha z^2 + z + \bar{\alpha}} = \bar{0}$

$\Rightarrow$ $\bar{\alpha}(\bar{z})^2 + \bar{z} + \alpha = 0$

$\Rightarrow$ $\bar{\alpha}z^2 + z + \alpha = 0$ $[\because \bar{z} = z]$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$(\alpha - \bar{\alpha})z^2 - (\alpha - \bar{\alpha}) = 0$$

$\Rightarrow$ $\alpha - \bar{\alpha} = 0$ and $z^2 = 1$

$\therefore$ $\alpha = \bar{\alpha}$ and $z = \pm 1$

Put $z = \pm 1$ in Eq. (i), we get

$$\alpha + \bar{\alpha} = \pm 1$$

and absolute value of real root = 1

i.e., $|z| = |\pm 1| = 1$

44. Let $z = \alpha$ be a real root of equation

$$z^3 + (3+i)z^2 - 3z - (m+i) = 0$$

$\Rightarrow$ $\alpha^3 + (3+i)\alpha^2 - 3\alpha - (m+i) = 0$

$\Rightarrow$ $(\alpha^3 + 3\alpha^2 - 3\alpha - m) + i(\alpha^2 - 1) = 0$

On comparing real and imaginary parts, we get

$$\alpha^3 + 3\alpha^2 - 3\alpha - m = 0$$

and $\alpha^2 - 1 = 0 \Rightarrow \alpha = \pm 1$

For $\alpha = 1$, we get

$$1 + 3 - 3 - m = 0 \Rightarrow m = 1$$

For $\alpha = -1$, we get

$$-1 + 3 + 3 - m = 0 \Rightarrow m = 5$$

45. Let $z = \alpha$ be a real root of equation

$$z^3 + (3+2i)z + (-1 + ia) = 0$$

$\Rightarrow$ $\alpha^3 + (3+2i)\alpha + (-1+ia) = 0$

$\Rightarrow$ $(\alpha^3 + 3\alpha - 1) + i(a + 2\alpha) = 0$

On comparing real and imaginary parts, we get

$$\alpha^3 + 3\alpha - 1 = 0$$

and $a + 2\alpha = 0$

$$\Rightarrow \quad \alpha = -\frac{a}{2}$$

$$\Rightarrow \quad -\frac{a^3}{8} - \frac{3a}{2} - 1 = 0 \Rightarrow a^3 + 12a + 8 = 0$$

Let $\quad f(a) = a^3 + 12a + 8$

$\therefore \quad f(-1) < 0,\ f(0) > 0,\ f(-2) < 0$

$f(1) > 0$ and $f(3) > 0$

$\Rightarrow \quad a \in (-2,1)$ or $a \in (-1,0)$ or $a \in (-2,3)$

Sol. (Q. Nos. 46 to 48)

46. $\because$ $\arg(z) > 0$

$\therefore \quad \arg(\bar{z}) + \arg(-z) = -\pi$

$\Rightarrow \quad -\arg(z) + \arg(-z) = -\pi$

$\Rightarrow \quad \arg(-z) - \arg(z) = -\pi$

47. $\because$ $\arg(z_1 z_2) = \pi$

$\Rightarrow \quad \arg(z_1) + \arg(z_2) = \pi$

$\Rightarrow \quad \arg(z_1) - \arg(\bar{z}_2) = \pi$

Given, $\quad |z_1| = |z_2|$

$\therefore \quad |z_1| = |\bar{z}_2| = |z_2|$

Then, $\quad z_1 + \bar{z}_2 = 0$

$\Rightarrow \quad z_1 = -\bar{z}_2$

48. $\arg(4z_1) - \arg(5z_2) = \pi$

is possible only when $|4z_1| = |5z_2|$

$$\Rightarrow \quad \left|\frac{z_1}{z_2}\right| = \frac{5}{4} = 1.25$$

and also $\quad 4z_1 + 5z_2 = 0$

$$\Rightarrow \quad \frac{z_1}{z_2} = -\frac{5}{4}$$

$$\therefore \quad \left|\frac{z_1}{z_2}\right| = \frac{5}{4} = 1.25$$

Sol. (Q. Nos. 49 to 51)

49. $\because$ $n!$ is divisible by 4, $\forall\ n \ge 4$.

$$\therefore \quad \sum_{n=4}^{25} i^{n!} = \sum_{n=1}^{22} i^{(n+3)!}$$

$$= i^0 + i^0 + i^0 + \ldots (22 \text{ times}) = 22 \qquad \ldots(\text{i})$$

$$\therefore \quad \sum_{n=1}^{25} i^{n!} = i^{1!} + i^{2!} + i^{3!} + \sum_{n=4}^{25} i^{n!}$$

$$= i + i^2 + i^6 + 22 \qquad [\text{from Eq. (i)}]$$

$$= i - 1 - 1 + 22 = 20 + i$$

$\therefore \quad a = 20,\ b = 1$

$\therefore \quad a - b = 20 - 1 = 19$

which is a prime number.

50. $\therefore \displaystyle \sum_{r=-2}^{95} i^r + \sum_{r=0}^{50} i^{r!}$

$$= \sum_{r=1}^{98} i^{r-3} + \left(i^{0!} + i^{1!} + i^{2!} + i^{3!} + \sum_{r=4}^{50} i^{r!}\right)$$

$$= (i^{-2} + i^{-1} + 0) + \left(i^1 + i^1 + i^2 + i^6 + \sum_{r=1}^{47} i^{(r+3)!}\right)$$

$$= (-1 - i) + (i + i - 1 - 1 + (i^0 + i^0 + i^0 + \ldots 47 \text{ times}))$$

$$= (-1 - i) + (2i - 2 + 47)$$

$$= 44 + i = a + ib \qquad [\text{given}]$$

$\therefore \quad a = 44,\ b = 1$

Unit place digit of $a^{2011} = (44)^{2011}$

$= (44)\,((44)^2)^{1005} = (44)\,(1936)^{1005}$

$=$ (Unit place of 44) $\times$ (Unit place digit of $(1936)^{1005}$)

$=$ Unit place of $(4 \times 6) = 4$

and unit place digit of $b^{2012} = (1)^{2012} = 1$

Hence, the unit place digit of $a^{2011} + b^{2012} = 4 + 1 = 5$.

51. $\because \displaystyle \sum_{r=4}^{100} i^{r!} + \prod_{r=1}^{101} i^r$

$$= \sum_{r=1}^{97} i^{(r+3)!} + i^1 \cdot i^2 \cdot i^3 \ldots i^{101}$$

$$= (i^0 + i^0 + i^0 + \ldots 97 \text{ times}) + i^{1+2+3+\ldots+101}$$

$$= 97 + i^{5151} = 97 + i^3 = 97 - i$$

$\therefore \quad a = 97$ and $b = -1$

Hence, $a + 75b = 97 - 75 = 22$

Sol. (Q. Nos. 52 to 54)

If $\left|z \pm \dfrac{a}{z}\right| = b$, where $a, b > 0$

$$\therefore \quad \left|z \pm \frac{a}{z}\right| \le |z| + \frac{a}{|z|}$$

$$\Rightarrow \quad b \le |z| + \frac{a}{|z|}$$

$$\Rightarrow \quad |z|^2 - b|z| + a \ge 0$$

$$\therefore \quad |z| \le \frac{b - \sqrt{b^2 - 4a}}{2}$$

$$\text{and} \quad |z| \ge \frac{b + \sqrt{b^2 - 4a}}{2} \qquad \ldots(\text{i})$$

$$\text{Also,} \quad \left|z \pm \frac{a}{z}\right| \ge \left||z| - \frac{a}{|z|}\right|$$

$$\Rightarrow \quad b \ge \left||z| - \frac{a}{|z|}\right|$$

$$\Rightarrow \quad -b \le |z| - \frac{a}{|z|} \le b$$

$$\Rightarrow \quad -b|z| \le |z|^2 - a \le b|z|$$

Case I $-b|z| \le |z|^2 - a$

$\Rightarrow \quad |z|^2 + b|z| - a \ge 0$

$\therefore \qquad |z| \le \dfrac{-b - \sqrt{b^2 + 4a}}{2}$

and $\qquad |z| \ge \dfrac{-b + \sqrt{b^2 - 4a}}{2}$

Case II $|z|^2 - a \le b|z|$

$\Rightarrow \quad |z|^2 - b|z| - a \le 0$

$\therefore \quad \dfrac{b - \sqrt{b^2 + 4a}}{2} \le |z| \le \dfrac{b + \sqrt{b^2 + 4a}}{2}$

From Case I and Case II, we get

$$\frac{-b + \sqrt{b^2 + 4a}}{2} \le |z| \le \frac{b + \sqrt{b^2 + 4a}}{2} \qquad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{-b + \sqrt{b^2 + 4a}}{2} \le |z| \le \frac{b + \sqrt{(b^2 + 4a)}}{2}$$

$\therefore$ The greatest value of $|z|$ is $\dfrac{b + \sqrt{b^2 + 4a}}{2}$

and the least value of $|z|$ is $\dfrac{-b + \sqrt{b^2 + 4a}}{2}$.

52. Here, $a = 1$ and $b = 2$

$\lambda =$ Sum of the greatest and least values of $|z|$

$= \sqrt{b^2 + 4a} = \sqrt{4 + 4} = \sqrt{8}$

$\therefore \quad \lambda^2 = 8$

53. Here, $a = 2$ and $b = 4$

$\lambda =$ Sum of the greatest and least value of $|z|$.

$= \sqrt{b^2 + 4a} = \sqrt{16 + 8} = \sqrt{24}$

$\therefore \qquad \lambda^2 = 24$

54. Here, $a = 3$ and $b = 6$

$\lambda =$ Sum of the greatest and least value of $|z|$

$= \sqrt{b^2 + 4a} = \sqrt{36 + 12} = \sqrt{48} = 4\sqrt{3}$

$\Rightarrow \quad \lambda = 2\sqrt{3}$

$\therefore \quad \lambda^2 = 12$

Sol. (Q. Nos. 55 to 57)

$\because \quad W = \dfrac{z - 1}{z + 2} = a + ib$

55. $\because \quad z = CiS\,\theta = e^{i\theta}$

$\therefore \quad \dfrac{e^{i\theta} - 1}{e^{i\theta} + 2} = a + ib$

$\Rightarrow (\cos\theta + i\sin\theta - 1) = (a + ib)(\cos\theta + i\sin\theta + 2)$

On comparing real and imaginary parts, we get

$$\cos\theta - 1 = a\cos\theta + 2a - b\sin\theta$$

$\Rightarrow \quad (1 - a)\cos\theta + b\sin\theta = 2a + 1 \qquad$ ...(i)

and $\sin\theta = a\sin\theta + b\cos\theta + 2b$

$(1 - a)\sin\theta - b\cos\theta = 2b \qquad$ ...(ii)

On squaring and adding Eqs. (i) and (ii), we get

$$(1 - a)^2 + b^2 = (2a + 1)^2 + (2b)^2$$

$\Rightarrow \qquad 3a^2 + 3b^2 + 6a = 0$

$\Rightarrow \qquad a^2 + b^2 + 2a = 0$

From option (c),

$$(1 + 5a)^2 + (3b)^2 = (1 - 4a)^2$$

$\Rightarrow \qquad 9a^2 + 9b^2 + 18a = 0$

$\therefore \qquad a^2 + b^2 + 2a = 0$

56. From Eq. (i), we get

$$(1 - a)\left(\frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2}\right) + b\left(\frac{2\tan\theta/2}{1 + \tan^2\theta/2}\right) = 2a + 1$$

$\Rightarrow \quad (1 - a) - (1 - a)\tan^2\dfrac{\theta}{2} + 2b\tan\dfrac{\theta}{2}$

$$= (2a + 1) + (2a + 1)\tan^2\frac{\theta}{2}$$

$\Rightarrow \quad (2 + a)\tan^2\dfrac{\theta}{2} - 2b\tan\dfrac{\theta}{2} + 3a = 0$

$\therefore \qquad \tan\dfrac{\theta}{2} = \dfrac{2b \pm \sqrt{4b^2 - 12a(2 + a)}}{2(2 + a)}$

$= \dfrac{2b \pm \sqrt{4b^2 - 12(-b^2)}}{-2b^2/a} \qquad [\because a^2 + b^2 + 2a = 0]$

$= \dfrac{(2b \pm 4b)a}{-2b^2} = \dfrac{6ba}{-2b^2}$ or $\dfrac{-2ab}{-2b^2} = -\dfrac{3a}{b}$ or $\dfrac{a}{b}$

$\therefore \qquad \cot\dfrac{\theta}{2} = -\dfrac{b}{3a}$ or $\dfrac{b}{a}$ or $-\dfrac{b}{a} = 3\cot\dfrac{\theta}{2}$ or $-\cot\dfrac{\theta}{2}$

57. $\therefore \quad a^2 + b^2 + 2a = 0 \Rightarrow (a + 1)^2 + b^2 = 1$

Now, $|z| = 1 = (a + 1)^2 + b^2$

58. $\because \qquad 1 + z + z^2 + z^3 + \ldots + z^{17} = 0$

$\therefore \qquad \dfrac{1\cdot(1 - z^{18})}{(1 - z)} = 0$

$\Rightarrow \qquad 1 - z^{18} = 0, 1 - z \ne 0$

$\therefore \qquad z^{18} = 1, z \ne 1 \qquad$ ...(i)

and $\quad 1 + z + z^2 + z^3 + \ldots + z^{13} = 0$

$\therefore \qquad \dfrac{1\cdot(1 - z^{14})}{(1 - z)} = 0$

$\Rightarrow \qquad 1 - z^{14} = 0, 1 - z \ne 0$

$\therefore \qquad z^{14} = 1, z \ne 1 \qquad$ ...(ii)

From Eqs. (i) and (ii), we get

$$z^{14}\cdot z^4 = 1 \Rightarrow 1\cdot z^4 = 1$$

$\therefore \qquad z^4 = 1$

Then, $\qquad z = 1, -1, i, -i$

$\because \qquad z \ne 1$

$\because \qquad z = -1, i, -i$

Hence, only $z = -1$ satisfy both Eqs. (i) and (ii).

$\therefore$ Number of values of z is 1.

59. We have, $z^3 = \bar{z}$ ...(i)

$\Rightarrow \quad |z|^3 = |\bar{z}| = |z|$

$\Rightarrow \quad |z|(|z|^2 - 1) = 0$

$\Rightarrow \quad |z| = 0$ and $|z|^2 = 1$

Now, $|z|^2 = 1$

$\Rightarrow \quad z\bar{z} = 1 \Rightarrow \bar{z} = \dfrac{1}{z}$

On putting this value in Eq. (i), we get

$$z^3 = \frac{1}{z}$$

$\Rightarrow \quad z^4 = 1$ ...(ii)

Clearly, Eq. (ii) has 4 solutions.

Therefore, the required number of solutions is 5.

60. We have, $z = 9 + ai$

$\Rightarrow \quad z^2 = (81 - a^2) + 18ai$

$z^3 = (729 - 27a^2) + (243a - a^3)\, i$

According to the question, we have

$\mathrm{Im}(z^2) = \mathrm{Im}(z^3)$

$\Rightarrow \quad 18a = 243a - a^3 \Rightarrow a(a^2 - 225) = 0$

$\Rightarrow \quad a = 0$ or $a^2 = 225$

But $\quad a \neq 0$

$\therefore \quad a^2 = 225$

$\therefore$ The sum of digits of $a^2 = 2 + 2 + 5 = 9$

61. Let $z = x + iy$

$\because \quad |z| = 1$ ...(i)

$\therefore \quad x^2 + y^2 = 1$

and $\quad \left|\dfrac{z}{\bar{z}} + \dfrac{\bar{z}}{z}\right| = 1$

$\Rightarrow \quad \left|\dfrac{x+iy}{x-iy} + \dfrac{x-iy}{x+iy}\right| = 1$

$\Rightarrow \quad \left|\dfrac{(x+iy)^2 + (x-iy)^2}{x^2+y^2}\right| = 1$

$\Rightarrow \quad \left|\dfrac{2(x^2-y^2)}{1}\right| = 1$ [from Eq. (i)]

$\Rightarrow \quad x^2 - y^2 = \pm\dfrac{1}{2}$ ...(ii)

From Eqs. (i) and (ii), we get

$$2x^2 = 1 \pm \frac{1}{2} = \frac{1}{2}, \frac{3}{2}$$

$\Rightarrow \quad x^2 = \dfrac{1}{4}, \dfrac{3}{4} \Rightarrow x = \pm\dfrac{1}{2}, \pm\dfrac{\sqrt{3}}{2}$

For $x = \dfrac{1}{2}, y = \pm\dfrac{\sqrt{3}}{2}$ [from Eq. (i)]

For $x = -\dfrac{1}{2}, y = \pm\dfrac{\sqrt{3}}{2}$ [from Eq. (i)]

For $x = \dfrac{\sqrt{3}}{2}, y = \pm\dfrac{1}{2}$ [from Eq. (i)]

For $x = -\dfrac{\sqrt{3}}{2}, y = \pm\dfrac{1}{2}$ [from Eq. (i)]

$\therefore$ Solutions are $\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}, -\dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}, \dfrac{\sqrt{3}}{2} \pm \dfrac{i}{2}, -\dfrac{\sqrt{3}}{2} \pm \dfrac{i}{2}$

Hence, number of solutions is 8.

62. We have, $x = a + ib$

$\Rightarrow \quad x^2 = (a^2 - b^2) + 2iab = 3 + 4i$ [given]

$\therefore \quad a^2 - b^2 = 3$ and $ab = 2$ ...(i)

and $x^3 = x \cdot x^2 = (a + ib)\,[(a^2 - b^2) + 2iab]$

$= (a^3 - ab^2 - 2ab^2) + i[2a^2b + b(a^2 - b^2)]$

$= (a^3 - 3ab^2) + i(3a^2b - b^3) = 2 + 11i$ [given]

$\therefore \quad a^3 - 3ab^2 = 2$

and $3a^2b - b^3 = 11$ ...(ii)

From Eq. (i), we get

$$a^2 + b^2 = \sqrt{(a^2 - b^2)^2 + 4a^2b^2} = 5$$

Then, $\quad 2a^2 = 8, 2b^2 = 2$

$\therefore \quad a^2 = 4, b^2 = 1$

$\Rightarrow \quad a = 2, b = 1$

and $\quad a = -2, b = -1$ [$\because ab = 2$]

Finally, $a = 2, b = 1$ satisfies Eq. (ii).

Hence, $\quad a + b = 2 + 1 = 3$

63. $\because \quad (1+i)^4 = [(1+i)^2]^2$

$= (1 + i^2 + 2i)^2 = (1 - 1 + 2i)^2$

$= 4i^2 = -4$ ...(i)

and $\dfrac{1 - \sqrt{\pi}\, i}{\sqrt{\pi} + i} + \dfrac{\sqrt{\pi} - i}{1 + \sqrt{\pi}\, i}$

$= \dfrac{(1 - \sqrt{\pi}\, i)(\sqrt{\pi} - i)}{\pi + 1} + \dfrac{(\sqrt{\pi} - i)(1 - \sqrt{\pi}\, i)}{1 + \pi}$

$= \dfrac{\sqrt{\pi} - i - \pi i - \sqrt{\pi} + \sqrt{\pi} - \pi i - i - \sqrt{\pi}}{\pi + 1}$

$= \dfrac{-2\pi i - 2i}{\pi + 1} = -2i$ ...(ii)

Given, $\quad z = \dfrac{\pi}{4}(1+i)^4 \left(\dfrac{1 - \sqrt{\pi}\, i}{\sqrt{\pi} + i} + \dfrac{\sqrt{\pi} - i}{1 + \sqrt{\pi}\, i}\right)$

$= \dfrac{\pi}{4}(-4)(-2i) = 2\pi i$ [from Eqs. (i) and (ii)]

Now, $\left(\dfrac{|z|}{\text{amp (z)}}\right) = \dfrac{2\pi}{\pi/2} = 4$

64. $\because \quad A^n = 1$

$\Rightarrow \quad A = (1)^{1/n} = e^{2\pi r i/n}, r = 0, 1, 2, \ldots, n-1$

$\therefore \quad A = 1, e^{2\pi i/n}, e^{4\pi i/n}, e^{6\pi i/n}, \ldots, e^{2\pi(n-1)i/n}$

and $\quad (A+1)^n = 1 \Rightarrow A + 1 = (1)^{1/n} = e^{2\pi pi/n}$

$$\Rightarrow \quad A = e^{2\pi pi/n} - 1 = e^{p\pi i/n} \cdot 2i \sin\left(\frac{\pi p}{n}\right),$$

$$p = 0, 1, 2, \ldots, n-1$$

$$\therefore \quad A = 0, e^{\pi i/n} \cdot 2i \sin\left(\frac{\pi}{n}\right), e^{2\pi i/n}\, 2i \sin\left(\frac{2\pi}{n}\right), \ldots,$$

$$e^{\pi i(n-1)/n} \cdot 2i \sin\left(\frac{\pi(n-1)}{n}\right)$$

For $n = 6$,

$$e^{4\pi i/n} = e^{4\pi i/6} = e^{2\pi i/3}$$

$$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

and $e^{\pi i/n} \cdot 2i \sin\left(\frac{\pi}{n}\right) = e^{\pi i/6} \cdot 2i \sin\left(\frac{\pi}{6}\right)$

$$= \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \cdot i$$

$$= \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) i = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

Hence, the least value of n is 6.

65. Given, $z_1, z_2, z_3, \ldots, z_{50}$ are the roots of the equation

$\sum_{r=0}^{50} (z)^r = 0$, then

$$\sum_{r=0}^{50} (z)^r = (z - z_1)(z - z_2)(z - z_3)\ldots(z - z_{50}) = \prod_{r=1}^{50} (z - z_r)$$

Taking log on both sides on base e, we get

$$\log_e\left(\sum_{r=0}^{50} (z)^r\right) = \sum_{r=1}^{50} \log_e(z - z_r)$$

On differentiating both sides w.r.t. z, we get

$$\frac{\sum_{r=0}^{50} r(z)^{r-1}}{\sum_{r=0}^{50} (z)^r} = \sum_{r=1}^{50} \frac{1}{(z - z_r)}$$

On putting $z = 1$ in both sides, we get

$$\frac{\sum_{r=0}^{50} r}{\sum_{r=0}^{50} 1} = \sum_{r=1}^{50} \frac{1}{(1 - z_r)}$$

$$\Rightarrow \quad \frac{(1 + 2 + 3 + \ldots + 50)}{51} = -\sum_{r=1}^{50} \frac{1}{(z_r - 1)}$$

$$= -(-5\lambda) \qquad \text{[given]}$$

$$\Rightarrow \quad \frac{\frac{50}{2} \times 51}{51} = 5\lambda$$

$$\Rightarrow \quad \lambda = 5$$

66. $\because \sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)$

$$= -i \sum_{q=1}^{10}\left(\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}\right)$$

$$= -i\left\{\sum_{q=0}^{10}\left(\cos\frac{2q\pi}{11} + i\sin\frac{2q\pi}{11}\right) - 1\right\}$$

$= -i$ {(sum of 11, 11th roots of unity) $- 1$}

$= -i(0 - 1) = i$

$$\because \quad P = \sum_{p=1}^{32} (3p + 2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11} - i\cos\frac{2q\pi}{11}\right)\right)^p$$

$$= \sum_{p=1}^{32} (3p + 2)(i)^p$$

$$= 3\sum_{p=1}^{32} p(i)^p + 2\sum_{p=1}^{32} (i)^p$$

$$= 3\sum_{p=1}^{32} p(i)^p + 0 = 3S \text{ (say)}$$

where, $S = \sum_{p=1}^{32} p(i)^p$

$$S = 1 \cdot i + 2 \cdot i^2 + 3 \cdot i^3 + \ldots + 31 \cdot i^{31} + 32 \cdot i^{32}$$

$$iS = 1 \cdot i^2 + 2 \cdot i^3 + \ldots + 31 \cdot i^{32} + 32 i^{33}$$

$$(1 - i)S = (i + i^2 + i^3 + \ldots + i^{32}) - 32i^{33}$$

$$= (0) - 32i$$

$$\therefore \quad S = -\frac{32i \cdot (1 + i)}{(1 - i) \cdot (1 + i)}$$

$$= -16(i - 1) = 16(1 - i)$$

$$\therefore \quad P = 3S = 48(1 - i)$$

Given, $(1 + i)P = n(n!) \Rightarrow (1 + i) \cdot 48(1 - i) = n(n!)$

$$\Rightarrow \quad 96 = n(n!) \Rightarrow 4(4!) = n(n!)$$

$$\therefore \quad n = 4$$

67. $\because \quad \frac{1 + i}{1 - i} = \frac{(1 + i)^2}{(1 - i)(1 + i)} = \frac{1 + i^2 + 2i}{2} = i$

Given, $\left(\frac{1 + i}{1 - i}\right)^n = \frac{2}{\pi}\sin^{-1}\left(\frac{1 + x^2}{2x}\right)$

$$\Rightarrow \quad i^n = \frac{2}{\pi}\sin^{-1}\left(\frac{1 + x^2}{2x}\right)$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{1 + x^2}{2x}\right) = \frac{\pi}{2}(i)^n$$

$$\Rightarrow \quad \frac{1 + x^2}{2x} = \sin\left(\frac{\pi}{2}(i)^n\right) \qquad \ldots\text{(i)}$$

Now, AM $\geq$ GM

$$\frac{x + \frac{1}{x}}{2} \geq 1 \Rightarrow \frac{x^2 + 1}{2x} \geq 1$$

$$\Rightarrow \quad \sin\left(\frac{\pi}{2}(i)^n\right) \geq 1 \qquad [\because -1 \leq \sin\theta \leq 1]$$

$$\therefore \quad \sin\left(\frac{\pi}{2}(i)^n\right) = 1$$

$$\Rightarrow \quad n = 4, 8, 12, 16, \ldots$$

$\therefore$ Least positive integer, $n = 4$

68. (A) $\rightarrow$ (p,q), (B) $\rightarrow$ (p,r),(C) $\rightarrow$ (p,r,s)

If $\left|z \pm \frac{a}{z}\right| = b$, where $a > 0$ and $b > 0$, then

$$\frac{-b+\sqrt{b^2+4a}}{2} \le |z| \le \frac{b+\sqrt{b^2+4a}}{2}$$

(A) Here, $a = 1$ and $b = 2$

Then, $-1+\sqrt{2} \le |z| \le 1+\sqrt{2}$

$\therefore \quad G = 1+\sqrt{2}$

and $\quad L = -1+\sqrt{2}$

$\Rightarrow \quad G - L = 2$ [natural number and prime number]

(B) Here, $a = 2$ and $b = 4$

Then, $-2+\sqrt{6} \le |z| \le 2+\sqrt{6}$

$\therefore \quad G = 2+\sqrt{6}$

and $\quad L = -2+\sqrt{6}$

$\Rightarrow \quad G - L = 4$ [natural number and composite number]

(C) Here, $a = 3$ and $b = 6$

Then, $-3+2\sqrt{3} \le |z| \le 3+2\sqrt{3}$

$\therefore \quad G = 3+2\sqrt{3}$

and $\quad L = -3+2\sqrt{3}$

$\Rightarrow \quad G - L = 6$

[natural number, composite number and perfect number]

69. (A) $\rightarrow$ (q), B $\rightarrow$ (q, r), C $\rightarrow$ (q, s)

We know that,

$$\sqrt{z} = \pm\left(\sqrt{\frac{|z|-\mathrm{Re}(z)}{2}} + i\sqrt{\frac{|z|-\mathrm{Re}(z)}{2}}\right)$$

If $\mathrm{Im}(z) > 0 = \pm\left(\sqrt{\frac{|z|+\mathrm{Re}(z)}{2}} - i\sqrt{\frac{|z|-\mathrm{Re}(z)}{2}}\right)$

If $\mathrm{Im}(z) < 0$

(A) $\sqrt{6+8i} = \pm\left(\sqrt{\frac{10+6}{2}} + i\sqrt{\frac{10-6}{2}}\right)$

$= \pm(2\sqrt{2} + i\sqrt{2})$

$= \pm\sqrt{2}(2+i)$

and $\sqrt{-6+8i} = \pm\left(\sqrt{\frac{10-6}{2}} + i\sqrt{\frac{10+6}{2}}\right)$

$= \pm(\sqrt{2} + i2\sqrt{2}) = \pm\sqrt{2}(1+2i)$

$\therefore \quad z = \sqrt{6+8i} + \sqrt{-6+8i}$

$= \pm\sqrt{2}(2+i) \pm \sqrt{2}(1+2i)$

$= 3\sqrt{2}(1+i), \sqrt{2}(1-i), -3\sqrt{2}(1+i), \sqrt{2}(-1+i)$

$\therefore \quad z_1 = 3\sqrt{2}(1+i), z_2 = \sqrt{2}(1-i),$

$z_3 = -3\sqrt{2}(1+i)$

and $\quad z_4 = \sqrt{2}(-1+i)$

$\therefore \ |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2$

$= 36 + 4 + 36 + 4 = 80$ which is divisible by 8.

(B) $\sqrt{5-12i} = \pm\left(\sqrt{\frac{13+5}{2}} - i\sqrt{\frac{13-5}{2}}\right) = \pm(3-2i)$

and $\sqrt{-5-12i} = \pm\left(\sqrt{\frac{13-5}{2}} - i\sqrt{\frac{13+5}{2}}\right) = \pm(2-3i)$

$\therefore \quad z = \sqrt{5-12i} + \sqrt{-5-12i} = \pm(3-2i) \pm (2-3i)$

$= 5-5i, -1-i, -5+5i, 1+i$

$\therefore \quad z_1 = 5-5i, \ z_2 = -1-i,$

$z_3 = -5+5i$ and $z_4 = 1+i$

$\therefore \ |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 50 + 2 + 50 + 2$

$= 104 = 8 \times 13$

(C) $\sqrt{8+15i} = \pm\left(\sqrt{\frac{17+8}{2}} + i\sqrt{\frac{17-8}{2}}\right)$

$= \pm\left(\frac{5}{\sqrt{2}} + \frac{3i}{\sqrt{2}}\right) = \pm\frac{1}{\sqrt{2}}(5+3i)$

and $\sqrt{-8-15i} = \pm\left(\sqrt{\frac{17-8}{2}} - i\sqrt{\frac{17+8}{2}}\right)$

$= \pm\left(\frac{3}{\sqrt{2}} - \frac{5}{\sqrt{2}}i\right) = \pm\frac{1}{\sqrt{2}}(3-5i)$

$\therefore \quad z = \sqrt{8+15i} + \sqrt{-8-15i}$

$= \pm\frac{1}{\sqrt{2}}(5+3i) \pm \frac{1}{\sqrt{2}}(3-5i)$

$z = \frac{1}{\sqrt{2}}(8-2i), \frac{1}{\sqrt{2}}(-2-8i),$

$\frac{1}{\sqrt{2}}(-8+2i), \frac{1}{\sqrt{2}}(2+8i)$

$\therefore \quad z_1 = \sqrt{2}(4-i), \ z_2 = \sqrt{2}(-1-4i)$

$z_3 = \sqrt{2}(-4+i)$ and $z_4 = \sqrt{2}(1+4i)$

$\therefore \ |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 34 + 34 + 34 + 34$

$= 136 = 17 \times 8$

70. (A) $\rightarrow$ (p, q, r, t);(B) $\rightarrow$ (p, s);(C) $\rightarrow$ (p, r)

(A) Here, the last digit of 143 is 3. The remainder when 861 is divided by 4 is 1. Then, press switch number 1 and we get 3. Hence, the digit in the units place of $(143)^{861}$ is 3.

$\therefore \quad \lambda = 3$

Next, the last digit of 5273 is 3. The remainder when 1358 is divided by 4 is 2. Then, press switch number 2 and we get 9. Hence, the digit in the units place of $(5273)^{1358}$ is 9.

$\therefore \quad \mu = 9$

Hence, $\quad \lambda + \mu = 3 + 9 = 12$

which is divisible by 2, 3, 4 and 6.

(B) Here, the last digit of 212 is 2. The remainder when 7820 is divided by 4 is 0. Then, press switch number 0 and we get 6. Hence, the digit in the unit's place of $(212)^{7820}$ is 6.

$\therefore \quad \lambda = 6$

Next, the last digit of 1322 is 2. The remainder when 1594 is divided by 4 is 2. Then, press switch number 2 and we get 4.

Hence, the digit in the unit's place of $(1322)^{1594}$ is 4.

$\therefore \qquad \mu = 4$

Hence, $\lambda + \mu = 6 + 4 = 10$, which is divisible by 2 and 5.

(C) Here, the last digit of 136 is 6. Therefore, the unit's place of $(136)^{786}$ is 6.

$\therefore \qquad \lambda = 6$

Next, the last digit of 7138 is 8. The remainder when 13491 is divided by 4 is 3. Then, press switch number 3 and we get 2. Hence, unit's place of $(7138)^{13491}$ is 2.

$\therefore \qquad \mu = 2$

Hence, $\qquad \lambda + \mu = 6 + 2 = 8$

which is divisible by 2 and 4.

71. (A) $\to$ (r); (B) $\to$ (p,s); (C) $\to$ (q,t)

If $\left|z - \frac{a}{z}\right| = b$, where $a > 0$ and $b > 0$, then

$$\frac{-b + \sqrt{b^2 + 4a}}{2} \le |z| \le \frac{b + \sqrt{b^2 + 4a}}{2}$$

$$\therefore \quad \lambda = \frac{b + \sqrt{b^2 + 4a}}{2} \text{ and } \mu = \frac{-b + \sqrt{b^2 + 4a}}{2}$$

(A) Here, $a = 6$ and $b = 5$

$\therefore \qquad \lambda = 6$ and $\mu = 1$

$\Rightarrow \qquad \lambda^\mu + \mu^\lambda = 6^1 + 1^6 = 7$

and $\qquad \lambda^\mu - \mu^\lambda = 6^1 - 1^6 = 5$

(B) Here, $a = 7$ and $b = 6$

$\therefore \qquad \lambda = 7$ and $\mu = 1$

$\therefore \qquad \lambda^\mu + \mu^\lambda = 7^1 + 1^7 = 8$

and $\qquad \lambda^\mu - \mu^\lambda = 7^1 - 1^7 = 6$

(C) Here, $a = 8$ and $b = 7$

$\therefore \qquad \lambda = 8$ and $\mu = 1$

$\Rightarrow \qquad \lambda^\mu + \mu^\lambda = 8^1 + 1^8 = 9$

and $\qquad \lambda^\mu - \mu^\lambda = 8^1 - 1^8 = 7$

72. Statement-1 is false because $3 + 7i > 2 + 4i$ is meaningless in the set of complex number as set of complex number does not hold ordering. But Statement-2 is true.

73. Statement-1 is false as

$$(\cos\theta + i\sin\phi)^n \ne \cos n\theta + i\sin n\phi$$

Now, $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= i \qquad$ [by De-Moivre's theorem]

$\therefore$ Statement-2 is true.

74. We have,

$$|3z_1 + 1| = |3z_2 + 1| = |3z_3 + 1|$$

$\therefore$ z_1, z_2 and z_3 are equidistant from $\left(-\frac{1}{3}, 0\right)$ and circumcentre of triangle is $\left(-\frac{1}{3}, 0\right)$.

Also, $\qquad 1 + z_1 + z_2 + z_3 = 0$

$\Rightarrow \qquad \frac{1 + z_1 + z_2 + z_3}{3} = 0$

$\Rightarrow \qquad \frac{z_1 + z_2 + z_3}{3} = -\frac{1}{3}$

$\therefore$ Centroid of the triangle is $\left(-\frac{1}{3}, 0\right)$.

So, the circumcentre and centroid of the triangle coincide. Hence, required triangle is an equilateral triangle.

Therefore, Statement-1 is true. Also, z_1, z_2 and z_3 represent vertices of an equilateral triangle, if $z_1^2 + z_2^2 + z_3^2 - (z_1z_2 + z_2z_3 + z_3z_1) = 0$.

Therefore, Statement-2 is false.

75. We have,

$$|z - 1| + |z - 8| = 5 \qquad \text{...(i)}$$

Here, $\qquad z_1 = 1, z_2 = 8$ and $2a = 5$

Now, $\qquad |z_1 - z_2| = |1 - 8| = |-7| = 7$

$\therefore \qquad 2a = 5 < 7$

Therefore, locus of Eq. (i) does not represent an ellipse. Hence, Statement-1 is false. Statement-2 is true by the property of ellipse.

76. Since, z_1, z_2 and z_3 are in AP.

$\therefore \qquad 2z_2 = z_1 + z_3$

$\Rightarrow \qquad z_2 = \frac{z_1 + z_3}{2}$

It is clear that, z_2 is the mid-point of z_1 and z_3.

$\therefore$ z_1, z_2 and z_3 are collinear.

Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation of Statement-1.

77. Principal argument of a complex number depend upon quadrant and principal argument lies in $(-\pi, \pi]$.

Hence, Statement-1 is always not true and Statement-2 is obviously true.

78. We have, $\quad C_1 : \arg(z) = \frac{\pi}{4}$

$\Rightarrow \qquad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} \qquad$ [let $z = x + iy$]

$\Rightarrow \qquad \frac{y}{x} = \tan\frac{\pi}{4} = 1$

$\Rightarrow \qquad y = x$

$\therefore \qquad C_1 : y = x \qquad$ ...(i)

$C_2 : \arg(z) = \frac{3\pi}{4}$

$\Rightarrow \qquad \tan^{-1}\left(\frac{y}{x}\right) = \frac{3\pi}{4} \qquad$ [let $z = x + iy$]

$\Rightarrow \qquad \frac{y}{x} = \tan\frac{3\pi}{4} = -1$

$\Rightarrow \qquad y = -x$

$\therefore \qquad C_2 : y = -x \qquad$ ...(ii)

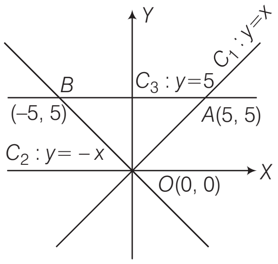
and $C_3 : \arg(z - 5 - 5i) = \pi$

$\Rightarrow \qquad \tan^{-1}\left(\frac{y - 5}{x - 5}\right) = \pi \qquad$ [let $z = x + iy$]

$$\Rightarrow \quad \frac{y-5}{x-5} = \tan \pi = 0 \quad \Rightarrow y = 5$$

$\therefore \quad C_3 : y = 5$ ...(iii)

We get the following figure.



$\therefore$ Area of the region bounded by C_1, C_2 and C_3

$$= \frac{1}{2} \left\| \begin{matrix} 5-0 & 5-0 \\ -5-0 & 5-0 \end{matrix} \right\| = 25$$

$\therefore$ Statement-1 is false.

Now, $\quad OA = 5\sqrt{2}, OB = 5\sqrt{2}$ and $AB = 10$

$\because \quad (OA)^2 + (OB)^2 = (AB)^2$ and $OA = OB$

Therefore, the boundary of C_1, C_2 and C_3 constitutes right isosceles triangle.

Hence, Statement-2 is true.

79. Since, $\text{Im}(\bar{z}_2 z_3) = \dfrac{\bar{z}_2 z_3 - (\overline{\bar{z}_2 z_3})}{2i} = \dfrac{1}{2i}\{\bar{z}_2 z_3 - z_2 \bar{z}_3\}$

$$z_1 \text{Im}(\bar{z}_2 z_3) = \frac{1}{2i}\{z_1 \bar{z}_2 z_3 - z_1 z_2 \bar{z}_3\} \quad \text{...(i)}$$

Similarly, $z_2 \text{Im}(\bar{z}_3 z_1) = \dfrac{1}{2i}\{z_2 \bar{z}_3 z_1 - z_2 \bar{z}_1 z_3\}$ ...(ii)

and $\quad z_3 \text{Im}(\bar{z}_1 z_2) = \dfrac{1}{2i}\{z_3 \bar{z}_1 z_2 - z_3 z_1 \bar{z}_2\}$ ...(iii)

On adding Eqs. (i), (ii) and (iii), we get

$$z_1 \text{Im}(\bar{z}_2 z_3) + z_2 \text{Im}(\bar{z}_3 z_1) + z_3 \text{Im}(\bar{z}_1 z_2) = 0$$

Therefore, this is proved.

80. Since, z_1, z_2 and z_3 are the roots of

$$x^3 + 3ax^2 + 3bx + c = 0,$$

we get $\quad z_1 + z_2 + z_3 = -3a$

$$\Rightarrow \quad \frac{z_1 + z_2 + z_3}{3} = -a$$

and $\quad z_1 z_2 + z_2 z_3 + z_3 z_1 = 3b$

Hence, the centroid of the ΔABC is the point of affix $(-a)$.

Now, the triangle will be equilateral, if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

$$\Rightarrow \quad (z_1 + z_2 + z_3)^2 = 3(z_1 z_2 + z_2 z_3 + z_3 z_1)$$

$$\Rightarrow \quad (-3a)^2 = 3(3b)$$

Therefore, the condition is $a^2 = b$.

81. $\because \quad x^5 - 1 = 0$ has roots $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$.

$$\therefore (x^5 - 1) = (x-1)(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)$$

$$\Rightarrow \frac{x^5 - 1}{x - 1} = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4) \quad \text{...(i)}$$

On putting $x = \omega$ in Eq. (i), we get

$$\frac{\omega^5 - 1}{\omega - 1} = (\omega - \alpha_1)(\omega - \alpha_2)(\omega - \alpha_3)(\omega - \alpha_4)$$

$$\Rightarrow \quad \frac{\omega^2 - 1}{\omega - 1} = (\omega - \alpha_1)(\omega - \alpha_2)(\omega - \alpha_3)(\omega - \alpha_4) \quad \text{...(ii)}$$

and putting $x = \omega^2$ in Eq. (i), we get

$$\frac{\omega^{10} - 1}{\omega^2 - 1} = (\omega^2 - \alpha_1)(\omega^2 - \alpha_2)(\omega^2 - \alpha_3)(\omega^2 - \alpha_4)$$

$$\Rightarrow \quad \frac{\omega - 1}{\omega^2 - 1} = (\omega^2 - \alpha_1)(\omega^2 - \alpha_2)(\omega^2 - \alpha_3)(\omega^2 - \alpha_4) \quad \text{...(iii)}$$

On dividing Eq. (ii) by Eq. (iii), we get

$$\frac{\omega - \alpha_1}{\omega^2 - \alpha_1} \cdot \frac{\omega - \alpha_2}{\omega^2 - \alpha_2} \cdot \frac{\omega - \alpha_3}{\omega^2 - \alpha_3} \cdot \frac{\omega - \alpha_4}{\omega^2 - \alpha_4} = \frac{(\omega^2 - 1)^2}{(\omega - 1)^2}$$

$$= \frac{\omega^4 + 1 - 2\omega^2}{\omega^2 + 1 - 2\omega} = \frac{\omega + 1 - 2\omega^2}{\omega^2 + 1 - 2\omega}$$

$$= \frac{-\omega^2 - 2\omega^2}{-\omega - 2\omega} = \frac{-3\omega^2}{-3\omega} = \omega$$

82. Let $z = x + iy$, then $\dfrac{z + \bar{z}}{2} = x$

$\therefore$ From given relation, we get

$$\Rightarrow \quad x = |x + iy - 1|$$

$$\Rightarrow \quad x = |(x-1) + iy|$$

$$\Rightarrow \quad x^2 = (x-1)^2 + y^2 \quad \Rightarrow 2x = 1 + y^2$$

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Then, $\quad 2x_1 = 1 + y_1^2$ ...(i)

and $\quad 2x_2 = 1 + y_2^2$ ...(ii)

On subtracting Eq. (ii) from Eq. (i), we get

$$2(x_1 - x_2) = y_1^2 - y_2^2$$

$$2(x_1 - x_2) = (y_1 + y_2)(y_1 - y_2) \quad \text{...(iii)}$$

But, given that $\arg(z_1 - z_2) = \pi/4$

Then, $\quad \tan^{-1}\left(\dfrac{y_1 - y_2}{x_1 - x_2}\right) = \dfrac{\pi}{4} \quad \Rightarrow \quad \dfrac{y_1 - y_2}{x_1 - x_2} = 1$

$\therefore \quad y_1 - y_2 = x_1 - x_2$ ...(iv)

From Eqs. (iii) and (iv), we get

$$y_1 + y_2 = 2 \quad [\because y_1 - y_2 \neq 0]$$

$\therefore \quad \text{Im}(z_1 + z_2) = 2$

Hence, the imaginary part $(z_1 + z_2)$ is 2.

83. (i) LHS $= (a^2 + b^2 + c^2 - bc - ca - ab)$

$(x^2 + y^2 + z^2 - yz - zx - xy)$

$$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$$

$$= \{(a + b\omega + c\omega^2)(x + y\omega + z\omega^2)\}\{(a + b\omega^2 + c\omega)(x + y\omega^2 + z\omega)\}$$

$$= \{ax + cy + bz + \omega(bx + ay + cz) + \omega^2(cx + by + az)\} \times \{ax + cy + bz + \omega^2(bx + ay + cz) + \omega(cx + by + az)\}$$

$$= (X + \omega Z + \omega^2 Y)(X + \omega^2 Z + \omega Y)$$

$$= \text{RHS}$$

(ii) LHS $= (a^3 + b^3 + c^3 - 3abc)(x^3 + y^3 + z^3 - 3xyz)$

$$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \times (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

$$= (a + b + c)(x + y + z)(a^2 + b^2 + c^2 - ab - bc - ca) \times (x^2 + y^2 + z^2 - xy - yz - zx) \text{ [using (i) part]}$$

$$= (ax + ay + az + bx + by + bz + cx + cy + cz)(X^2 + Y^2 + Z^2 - YZ - ZX - XY)$$

$$= \{(ax + cy + bz) + (cx + by + az) + (bx + ay + cz)\}(X^2 + Y^2 + Z^2 - YZ - ZX - XY)$$

$$= (X + Y + Z)(X^2 + Y^2 + Z^2 - YZ - ZX - XY)$$

$$= X^3 + Y^3 + Z^3 - 3XYZ = \text{RHS}$$

84. Let $z = x + iy$

$\therefore \quad |z|^2 = x^2 + y^2$

$\therefore \quad x^2 + y^2 - 2i(x + iy) + 2c(1 + i) = 0$

$(x^2 + y^2 + 2y + 2c) + i(-2x + 2c) = 0$

On comparing the real and imaginary parts, we get

$$x^2 + y^2 + 2y + 2c = 0 \quad \text{...(i)}$$

and $$-2x + 2c = 0 \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$y^2 + 2y + c^2 + 2c = 0$$

$$\Rightarrow \quad y = \frac{-2 \pm \sqrt{4 - 4(c^2 + 2c)}}{2} = -1 \pm \sqrt{(1 - c^2 - 2c)}$$

$\because$ x and y are real.

$\therefore \quad 1 - c^2 - 2c \geq 0$ or $c^2 + 2c + 1 \leq 2$

$$(c + 1)^2 \leq (\sqrt{2})^2 \Rightarrow -\sqrt{2} - 1 \leq c \leq \sqrt{2} - 1$$

$\therefore \quad 0 \leq c \leq \sqrt{2} - 1 \quad$ [$\because$ given $c \geq 0$]

Hence, the solution is $z = x + iy = c + i(-1 \pm \sqrt{1 - c^2 - 2c})$

for $0 \leq c \leq \sqrt{2} - 1$

and $z = x + iy \equiv$ no solution for $c > \sqrt{2} - 1$

85. Let $z = x + iy$

$$\therefore \quad \text{Re}(z) = x = \frac{z + \bar{z}}{2} \quad \text{...(i)}$$

and $$\text{Im}(z) = y = \frac{z - \bar{z}}{2i} \quad \text{...(ii)}$$

The equation $(2 - i)z + (2 + i)\bar{z} + 3 = 0$ can be written as

$$2(z + \bar{z}) - i(z - \bar{z}) + 3 = 0$$

or $$4x + 2y + 3 = 0$$

$\therefore$ Slope of the given line, $m = -2$

Let slope of the required line be m_1, then

$$\tan 45^\circ = \left|\frac{m_1 - m}{1 + m_1 m}\right| \Rightarrow 1 = \left|\frac{m_1 + 2}{1 - 2m_1}\right| \Rightarrow \pm 1 = \frac{m_1 + 2}{1 - 2m_1}$$

$$\therefore \quad m_1 = -\frac{1}{3}, 3$$

$\therefore$ Equation of straight lines through $(-1, 4)$ and having slopes $-\frac{1}{3}$ and 3 are $y - 4 = -\frac{1}{3}(x + 1)$ and $y - 4 = 3(x + 1)$

$\Rightarrow \quad x + 3y - 11 = 0 \quad$ and $\quad 3x - y + 7 = 0$

Using Eqs. (i) and (ii), then equations of lines are

$$\frac{z + \bar{z}}{2} + \frac{3(z - \bar{z})}{2i} - 11 = 0$$

and $$\frac{3(z + \bar{z})}{2} - \frac{(z - \bar{z})}{2i} + 7 = 0$$

i.e., $\quad (1 - 3i)z + (1 + 3i)\bar{z} - 22 = 0$

and $\quad (3 + i)z + (3 - i)\bar{z} + 14 = 0$

86. Putting $\frac{1 + i}{2} = x$ in LHS, we get

$$\text{LHS} = (1 + x)(1 + x^2)(1 + x^{2^2}) \ldots (1 + x^{2^n})$$

$$= \frac{(1 - x)(1 + x)(1 + x^2)(1 + x^{2^2}) \ldots (1 + x^{2^n})}{(1 - x)}$$

$$= \frac{(1 - x^2)(1 + x^2)(1 + x^{2^2}) \ldots (1 + x^{2^n})}{(1 - x)}$$

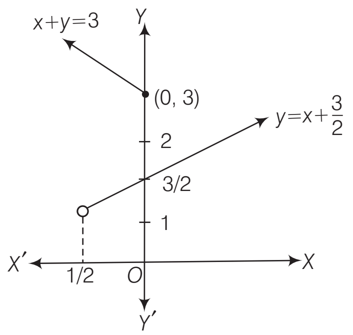
$$= \frac{(1 - x^{2^2})(1 + x^{2^2}) \ldots (1 + x^{2^n})}{(1 - x)}$$

$$= \frac{(1 - x^{2^n})(1 + x^{2^n})}{(1 - x)} = \frac{1 - (x^2)^{2^n}}{(1 - x)}$$

$$= \frac{1 - \left(\frac{i}{2}\right)^{2^n}}{1 - \left(\frac{1 + i}{2}\right)} \quad \left[\because x = \frac{1 + i}{2}\right]$$

$$= \frac{1 - \frac{1}{2^{2^n}}(1)}{\left(\frac{1 - i}{2}\right)} \cdot \frac{(1 + i)}{(1 + i)} = (1 + i)\left(1 - \frac{1}{2^{2^n}}\right) = \text{RHS}$$

87. Since, $\arg(z - 3i) = 3\pi/4$ is a ray which is start from $3i$ and makes an angle $3\pi/4$ with positive real axis as shown in the figure.



$\therefore$ Equation of ray in cartesian form is

$$y - 3 = \tan(3\pi/4)(x - 0)$$

or $\quad y - 3 = -x$ or $x + y = 3$

and $$\arg(2z + 1 - 2i) = \pi/4$$

$$\Rightarrow \quad \arg\left(2\left(z + \frac{1}{2} - i\right)\right) = \pi/4$$

or $$\arg(2) + \arg\left(z + \frac{1}{2} - i\right) = \pi/4$$

or $$0 + \arg\left(z + \frac{1}{2} - i\right) = \pi/4$$

or $\quad \arg\left(z-\left(-\frac{1}{2}+i\right)\right)=\pi/4$

which is a ray that start from point $-\frac{1}{2}+i$ and makes an angle $\pi/4$ with positive real axis as shown in the figure.

$\therefore$ Equation of ray in cartesian form is

$$y-1=1\,[x-(-1/2)] \Rightarrow y=x+3/2$$

From the figure, it is clear that the system of equations has no solution.

88. Let $\quad a=r\cos\alpha$ and $0=r\sin\alpha \quad$ ...(i)

So that, $\quad a^2+0^2=r^2$

$\therefore \quad r=|a|$

Then, $\quad a=|a|\cos\alpha \quad$ [from Eq. (i)]

$\therefore \quad \cos\alpha=\pm 1$

Then, $\cos\alpha=1$ or -1 according as a is + ve or − ve and $\sin\alpha=0$.

Hence, $\alpha=0$ or π according as a is + ve and − ve.

Again, let $0=r_1\cos\beta$ or $b=r_1\sin\beta \quad$ ...(ii)

So that, $\quad 0^2+b^2=r_1^2$

$\therefore \quad r_1=|b|$

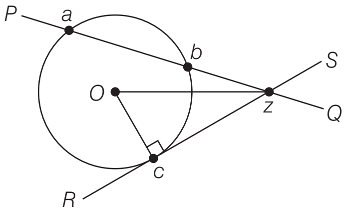
From Eq. (ii), we get $b=|b|\sin\beta$

$\therefore \quad \sin\beta=\pm 1$

Then, $\sin\beta=1$ or -1 according as b is + ve or − ve and $\cos\beta=0$.

Hence, $\beta=\frac{\pi}{2}$ or $-\frac{\pi}{2}$ according as b is +ve or −ve.

89. Let two non-parallel straight lines PQ, RS meet the circle $|z|=r$ in the points a, b and c.



Then, $|a|=r, |b|=r$ and $|c|=r$ or $|a|^2=|b|^2=|c|^2=r^2$

$\therefore \quad a\bar{a}=b\bar{b}=c\bar{c}=r^2,$

then $\quad \bar{a}=\frac{r^2}{a},\ \bar{b}=\frac{r^2}{b}$ and $\bar{c}=\frac{r^2}{c}$

Points a, b and z are collinear, then $\begin{vmatrix} z & \bar{z} & 1\\ a & \bar{a} & 1\\ b & \bar{b} & 1\end{vmatrix}=0$

$\therefore \quad z(\bar{a}-\bar{b})-\bar{z}(a-b)+a\bar{b}-\bar{a}b=0$

$\Rightarrow \quad z\left(\frac{r^2}{a}-\frac{r^2}{b}\right)-\bar{z}(a-b)+\frac{r^2a}{b}-\frac{r^2b}{a}=0$

On dividing both sides by $r^2(b-a)$, we get

$$\frac{z}{ab}+\frac{\bar{z}}{r^2}=a^{-1}+b^{-1} \quad \text{...(i)}$$

For RS, replace $a=b=c$ in Eq. (i), then

$$\frac{z}{c^2}+\frac{\bar{z}}{r^2}=2c^{-1} \quad \text{...(ii)}$$

On subtracting Eq. (i) from Eq. (ii), we get

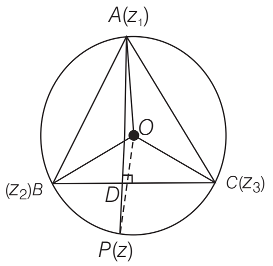
$$z\,(c^{-2}-a^{-1}b^{-1})=2c^{-1}-a^{-1}-b^{-1}$$

Hence, $\quad z=\frac{2c^{-1}-a^{-1}-b^{-1}}{c^{-2}-a^{-1}b^{-1}}$

which is a required point.

90. $\because AD\perp BC$

$\therefore$ AP is also perpendicular to BC.



Then, $\quad \arg\left(\frac{z_1-z}{z_3-z_2}\right)=\frac{\pi}{2}$

$\therefore \quad \text{Re}\left(\frac{z_1-z}{z_3-z_2}\right)=0$

$\Rightarrow \quad \dfrac{\dfrac{z_1-z}{z_3-z_2}+\dfrac{\bar{z}_1-\bar{z}}{\bar{z}_3-\bar{z}_2}}{2}=0$

$\Rightarrow \quad \frac{z_1-z}{z_3-z_2}+\frac{\bar{z}_1-\bar{z}}{\bar{z}_3-\bar{z}_2}=0 \quad$ ...(i)

But O is the circumcentre of ΔABC, then

$$OP=OA=OB=OC$$

$$|z|=|z_1|=|z_2|=|z_3|$$

On squaring the above relation, we get

$$|z|^2=|z_1|^2=|z_2|^2=|z_3|^2$$

$\Rightarrow \quad z\bar{z}=z_1\bar{z}_1=z_2\bar{z}_2=z_3\bar{z}_3$

From first two relations $\frac{\bar{z}_1}{\bar{z}}=\frac{z}{z_1}$ ...(ii)

From first and third relation $\frac{\bar{z}_2}{\bar{z}}=\frac{z}{z_2}$ ...(iii)

and from first and fourth relation $\frac{\bar{z}_3}{\bar{z}}=\frac{z}{z_3}$ ...(iv)

From Eq. (i), we get $\dfrac{z_1-z}{z_3-z_2}+\dfrac{\dfrac{\bar{z}_1}{\bar{z}}-1}{\dfrac{\bar{z}_3}{\bar{z}}-\dfrac{\bar{z}_2}{\bar{z}}}=0$ ...(v)

From Eqs. (ii), (iii), (iv) and (v), we get

$$\frac{z_1-z}{z_3-z_2}+\frac{\dfrac{z}{z_1}-1}{\dfrac{z}{z_3}-\dfrac{z}{z_2}}=0$$

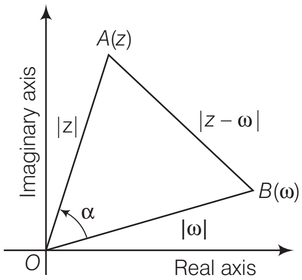
$\Rightarrow \quad \left(\frac{z_1-z}{z_3-z_2}\right)\left\{1+\frac{z_2z_3}{zz_1}\right\}=0 \quad \left[\because \frac{z_1-z}{z_3-z_2}\neq 0\right]$

$\Rightarrow \quad 1+\frac{z_2z_3}{zz_1}=0 \quad \Rightarrow z=-\frac{z_2z_3}{z_1}$

91. From the figure,

$$\alpha = (\arg(z) - \arg(\omega)) \qquad \text{...(i)}$$

and for every α, $\sin^2\frac{\alpha}{2} \le \left(\frac{\alpha}{2}\right)^2$ ...(ii)



In ΔOAB, from cosine rule

$$(AB)^2 = (OA)^2 + (OB)^2 - 2OA\cdot OB\cos\alpha$$

$$\Rightarrow |z-\omega|^2 = |z|^2 + |\omega|^2 - 2|z||\omega|\cos\alpha$$

$$\Rightarrow |z-\omega|^2 = (|z|-|\omega|)^2 + 2|z||\omega|(1-\cos\alpha)$$

$$\Rightarrow |z-\omega|^2 = (|z|-|\omega|)^2 + 4|z||\omega|\sin^2\frac{\alpha}{2}$$

$$\Rightarrow |z-\omega|^2 \le (|z|-|\omega|)^2 + 4|z||\omega|\left(\frac{\alpha}{2}\right)^2 \qquad \text{[from Eq. (ii)]}$$

$$\Rightarrow |z-\omega|^2 \le (|z|-|\omega|)^2 + \alpha^2 \qquad [\because |z|\le 1, |\omega|\le 1]$$

$$|z-\omega|^2 \le (|z|-|\omega|)^2 + (\arg(z) - \arg(\omega))^2 \qquad \text{[from Eq. (i)]}$$

I. Aliter

Let $z = r(\cos\theta + i\sin\theta)$ and $\omega = r_1(\cos\theta_1 + i\sin\theta_1)$,

then $|z| = r$ and $|\omega| = r_1$

Also, $\arg(z) = \theta$ and $\arg(\omega) = \theta_1$

and $r \le 1$ and $r_1 \le 1$ $[\because$ given $|z|\le 1, |\omega|\le 1]$

We have, $z - \omega = (r\cos\theta - r_1\cos\theta_1) + i(r\sin\theta - r_1\sin\theta_1)$

$$\therefore \quad |z-\omega|^2 = (r\cos\theta - r_1\cos\theta_1)^2 + (r\sin\theta - r_1\sin\theta_1)^2$$

$$\Rightarrow \quad |z-\omega|^2 = r^2 + r_1^2 - 2rr_1\cos(\theta-\theta_1)$$

$$= (r-r_1)^2 + 2rr_1 - 2rr_1\cos(\theta-\theta_1)$$

$$= (r-r_1)^2 + 2rr_1(1-\cos(\theta-\theta_1))$$

$$= (r-r_1)^2 + 4rr_1\sin^2\left(\frac{\theta-\theta_1}{2}\right)$$

$$\le (r-r_1)^2 + 4rr_1\left(\frac{\theta-\theta_1}{2}\right)^2 \qquad [\because |\sin\theta| \le |\theta|]$$

$$= (r-r_1)^2 + rr_1(\theta-\theta_1)^2$$

$$\le (r-r_1)^2 + (\theta-\theta_1)^2 \qquad [\because r, r_1 \le 1]$$

$$\Rightarrow \quad |z-\omega|^2 \le (|z|-|\omega|)^2 + (\arg z - \arg\omega)^2$$

II. Aliter

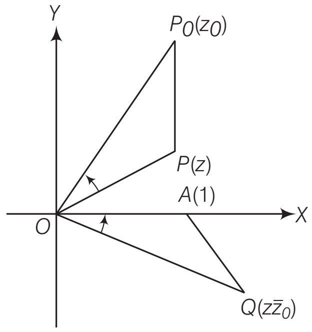
Let $z = r\cos\theta$

and $\omega = r_1\cos\theta_1$

$$\therefore \quad r^2 + r_1^2 - 2rr_1\cos(\theta-\theta_1) \le r^2 + r_1^2 - 2rr_1 + (\theta-\theta_1)^2$$

$$\Rightarrow rr_1\sin^2\left(\frac{\theta-\theta_1}{2}\right) \le \left(\frac{\theta-\theta_1}{2}\right)^2 \qquad \left[\begin{array}{l}\because r, r_1 \le 1 \\ \text{and } \sin^2 x \le x^2\end{array}\right]$$

92. Given, $OA = 1$ and $|z| = 1$



$$\therefore \quad OP = |z-0| = |z| = 1$$

$$\therefore \quad OP = OA$$

$$OP_0 = |z_0 - 0| = |z_0|$$

and $\quad OQ = |z\bar{z}_0 - 0| = |z\bar{z}_0| = |z||\bar{z}_0| = 1|\bar{z}_0| = |z_0|$

$$\therefore \quad OP_0 = OQ$$

Also, $\angle P_0OP = \arg\left(\frac{z_0 - 0}{z - 0}\right) = \arg\left(\frac{z_0}{z}\right) = \arg\left(\frac{\bar{z}\,z_0}{z\,\bar{z}}\right)$

$$= \arg\left(\frac{\bar{z}\,z_0}{|z|^2}\right) = \arg\left(\frac{\bar{z}\,z_0}{1}\right) = -\arg(\overline{\bar{z}\,z_0})$$

$$= -\arg(z\,\bar{z}_0) = \arg\left(\frac{1}{z\,\bar{z}_0}\right)$$

$$= \arg\left(\frac{1-0}{z\,\bar{z}_0 - 0}\right) = \angle AOQ$$

Thus, the triangles POP_0 and AOQ are congruent.

$$\therefore \quad PP_0 = AQ$$

$$|z - z_0| = |z\,\bar{z}_0 - 1|$$

93. Let the equation of line passing through the origin be

$$\bar{a}z + a\bar{z} = 0 \qquad \text{...(i)}$$

According to the question, $z_1, z_2, \ldots, z_n$ all lie on one side of line (i)

$\therefore \quad \bar{a}z_i + a\bar{z}_i > 0$ or < 0 for all $i = 1, 2, 3, \ldots, n$ ...(ii)

$\Rightarrow \bar{a}\sum_{i=1}^{n} z_i + a\sum_{i=1}^{n}\bar{z}_i > 0$ or < 0 ...(iii)

$\Rightarrow \quad \sum_{i=1}^{n} z_i \ne 0 \qquad \left\{\text{If } \sum_{i=1}^{n} z_i = 0, \text{ then } \sum_{i=1}^{n}\bar{z}_i = 0,\right.$ hence $\left.\bar{a}\sum_{i=1}^{n} z_i + a\sum_{i=1}^{n}\bar{z}_i = 0\right\}$

From Eq. (ii), we get

$\bar{a}z_i + a\bar{z}_i > 0$ or < 0 for all $i = 1, 2, 3, \ldots, n$

$\Rightarrow \quad \frac{\bar{a}z_i\bar{z}_i}{\bar{z}_i} + \frac{a\bar{z}_iz_i}{z_i} > 0$ or < 0

$\Rightarrow \quad |z_i|^2\left\{\frac{\bar{a}}{\bar{z}_i} + \frac{a}{z_i}\right\} > 0$ or < 0

$\Rightarrow \quad \frac{\bar{a}}{\bar{z}_i} + \frac{a}{z_i} > 0$ or < 0 for all $i = 1, 2, 3, \ldots, n$

$\Rightarrow \quad \frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_n}$ lie on one side of the line $\bar{a}z + a\bar{z} = 0$

or $\quad \bar{a} \sum_{i=1}^{n} \frac{1}{\bar{z}_i} + a \sum_{i=1}^{n} \frac{1}{z_i} > 0$ or < 0

Therefore, $\sum_{i=1}^{n} \frac{1}{z_i} \neq 0 \left\{ \text{If } \sum_{i=1}^{n} \frac{1}{z_i} = 0, \text{ then } \sum_{i=1}^{n} \frac{1}{\bar{z}_i} = 0 \right.$

$$\left. \Rightarrow \quad \bar{a} \sum_{i=1}^{n} \frac{1}{\bar{z}_i} + a \sum_{i=1}^{n} \frac{1}{z_i} = 0 \right\}$$

94. Given, $|a||b| = \sqrt{a\bar{b}^2 c}$; $|a| = |c|$; $az^2 + bz + c = 0$, then we have to prove that $|z| = 1$

On squaring, we get

$|a|^2 |b|^2 = a\,\bar{b}^2 c$ and $|a|^2 = |c|^2$

$\Rightarrow \; a\,\bar{a}\,b\,\bar{b} = a\,\bar{b}^2 c \quad$ and $\quad a\,\bar{a} = c\,\bar{c}$

$\Rightarrow \quad \bar{a}\,b = \bar{b}\,c \quad$ and $\quad a\,\bar{a} = c\,\bar{c}$ ...(i)

If z_1 and z_2 are the roots of $az^2 + bz + c = 0$

Then, $\bar{z}_1$ and $\bar{z}_2$ are the roots of $\bar{a}(\bar{z})^2 + \bar{b}\bar{z} + \bar{c} = 0$ ...(A)

$$\left.\begin{aligned} \therefore \quad & z_1 + z_2 = -\frac{b}{a}, z_1 z_2 = \frac{c}{a} \\ \text{and} \quad & \bar{z}_1 + \bar{z}_2 = -\frac{\bar{b}}{\bar{a}}, \bar{z}_1\bar{z}_2 = \frac{\bar{c}}{\bar{a}} \end{aligned}\right\} \quad \text{...(ii)}$$

$\because \quad \frac{1}{z_1} + \frac{1}{z_2} = \frac{z_1 + z_2}{z_1 z_2} = \frac{-b/a}{c/a} = -\frac{b}{c} = -\frac{\bar{b}}{\bar{a}} = \overline{z_1} + \overline{z_2}$

[from Eqs. (i) and (ii)]

and $\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} = \frac{\bar{z}_1 + \bar{z}_2}{\bar{z}_1 \bar{z}_2} = \frac{-\bar{b}/\bar{a}}{\bar{c}/\bar{a}}$

$= -\frac{\bar{b}}{\bar{c}} = -\frac{\bar{a}\,bc}{ca\,\bar{a}} = -\frac{b}{a} = z_1 + z_2$

[from Eqs. (i) and (ii)]

Now, it is clear that $z_1 = \frac{1}{\bar{z}_1}$ and $z_2 = \frac{1}{\bar{z}_2}$

Then, $|z_1|^2 = 1$ and $|z_2|^2 = 1$

Hence, $\quad |z| = 1$

Conversely For $az^2 + bz + c = 0$, we have to prove

$|z| = 1 \Rightarrow |a||b| = \sqrt{a\,\bar{b}^2 c}$

and $\quad |a| = |c|$

$|z| = 1 \Rightarrow |z|^2 = 1 \Rightarrow z\,\bar{z} = 1 \Rightarrow z = \frac{1}{\bar{z}}$

From Eq. (A), we get

$$\bar{a}\left(\frac{1}{z}\right)^2 + \bar{b}\left(\frac{1}{z}\right) + \bar{c} = 0 \text{ or } \bar{c}z^2 + \bar{b}z + \bar{a} = 0$$

Also, $az^2 + bz + c = 0$, on comparing

$$\frac{\bar{c}}{a} = \frac{\bar{b}}{b} = \frac{\bar{a}}{c}$$

$\therefore \quad a\,\bar{a} = c\,\bar{c}$ and $\bar{a}\,b = \bar{b}$ cs

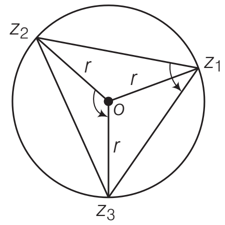
$\Rightarrow \; |a| = |c|$ and $|a||b| = \sqrt{a\,\bar{b}^2 c}$

95. (i) Let $z_1 = r_1(\cos\alpha + i\sin\alpha)$,

$z_2 = r_2(\cos\beta + i\sin\beta)$ and $z_3 = r_3(\cos\gamma + i\sin\gamma)$

$\therefore \; |z_1| = r_1, |z_2| = r_2, |z_3| = r_3$

and $\arg(z_1) = \alpha, \arg(z_2) = \beta, \arg(z_3) = \gamma$



From the given condition,

$$\begin{vmatrix} r_1 & r_2 & r_3 \\ r_2 & r_3 & r_1 \\ r_3 & r_1 & r_2 \end{vmatrix} = 0$$

$\Rightarrow \quad r_1^3 + r_2^3 + r_3^3 - 3r_1r_2r_3 = 0$

$\Rightarrow \frac{1}{2}(r_1 + r_2 + r_3)\{(r_1 - r_2)^2 + (r_2 - r_3)^2 + (r_3 - r_1)^2\} = 0$

Since, $\quad r_1 + r_2 + r_3 \neq 0,$

then $\quad (r_1 - r_2)^2 + (r_2 - r_3)^2 + (r_3 - r_1)^2 = 0$

It is possible only when

$$r_1 - r_2 = r_2 - r_3 = r_3 - r_1 = 0$$

$\therefore \quad r_1 = r_2 = r_3$

and $\quad |z_1| = |z_2| = |z_3| = r$ [say]

Hence, z_1, z_2, z_3 lie on a circle with the centre at the origin.

(ii) Again, in $\Delta\, oz_2z_3$ by Coni method

$\arg\left(\frac{z_3 - 0}{z_2 - 0}\right) = \angle z_2 oz_3 \Rightarrow \arg\left(\frac{z_3}{z_2}\right) = \angle z_2 oz_3$ ...(i)

In $\Delta\, z_2z_1z_3$ by Coni method

$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \angle z_2\, z_1\, z_3 = \frac{1}{2}\angle z_2 oz_3$ [property of circle]

$= \frac{1}{2}\arg\left(\frac{z_3}{z_1}\right)$ [from Eq. (i)]

$\therefore \quad \arg\left(\frac{z_3}{z_1}\right) = 2\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

Hence, $\arg\left(\frac{z_3}{z_1}\right) = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)^2$

96. We know that,

$$\text{Re}(z_1\bar{z}_2) \le |z_1\bar{z}_2|$$

$\therefore \; |z_1|^2 + |z_2|^2 + 2\,\text{Re}(z_1\bar{z}_2) \le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2|$

$\Rightarrow \quad |z_1 + z_2|^2 \le |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$ ...(i)

Also, AM $\ge$ GM

$\therefore \quad \frac{(\sqrt{c}\,|z_1|)^2 + \left(\frac{1}{\sqrt{c}}|z_2|\right)^2}{2} \ge \left\{\sqrt{c}\cdot|z_1|^2\cdot\frac{1}{\sqrt{c}}|z_2|^2\right\}^{1/2} \quad [\because c > 0]$

$\Rightarrow \; c|z_1|^2 + \frac{1}{c}|z_2|^2 \ge 2|z_1||z_2|$

$\therefore \; |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \le |z_1|^2 + |z_2|^2 + c|z_1|^2 + \frac{1}{c}|z_2|^2$

$\Rightarrow |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \le (1 + c)|z_1|^2 + (1 + c^{-1})(|z_2|^2)$

...(ii)

From Eqs. (i) and (ii), we get

$$|z_1 + z_2|^2 \le (1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2$$

Aliter

Here, $(1 + c)|z_1|^2 + (1 + c^{-1})|z_2|^2 - |z_1 + z_2|^2$

$$= (1 + c) z_1\bar{z}_1 + \left(1 + \frac{1}{c}\right) z_2\bar{z}_2 - (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$

$$= (1 + c) z_1\bar{z}_1 + \left(1 + \frac{1}{c}\right) z_2\bar{z}_2 - z_1\bar{z}_1 - z_1\bar{z}_2 - z_2\bar{z}_1 - z_2\bar{z}_2$$

$$= c z_1\bar{z}_1 + \frac{1}{c} z_2\bar{z}_2 - z_1\bar{z}_2 - z_2\bar{z}_1$$

$$= \frac{1}{c}\{c^2 z_1\bar{z}_1 + z_2\bar{z}_2 - cz_1\bar{z}_2 - cz_2\bar{z}_1\}$$

$$= \frac{1}{c}\{cz_1(c\bar{z}_1 - \bar{z}_2) - z_2(c\bar{z}_1 - \bar{z}_2)\}$$

$$= \frac{1}{c}(cz_1 - z_2)(c\bar{z}_1 - \bar{z}_2) = \frac{1}{c}(cz_1 - z_2)\overline{(cz_1 - z_2)}$$

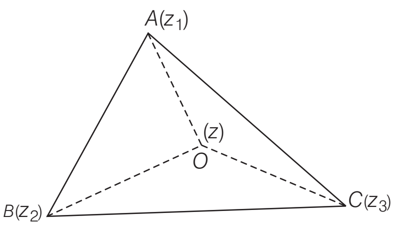
$$= \frac{1}{c}|cz_1 - z_2|^2 \ge 0 \text{ as } c > 0$$

$$\therefore \quad (1 + c)|z_1|^2 + \left(1 + \frac{1}{c}\right)|z_2|^2 - |z_1 + z_2|^2 \ge 0$$

Hence, $|z_1 + z_2|^2 \le (1 + c)|z_1|^2 + \left(1 + \dfrac{1}{c}\right)|z_2|^2$

97. If z be the complex number corresponding to the circumcentre O, then we have

$$OA = OB = OC$$



$$\Rightarrow \quad |z - z_1| = |z - z_2| = |z - z_3|$$

$$\Rightarrow \quad |z - z_1|^2 = |z - z_2|^2 = |z - z_3|^2$$

$$\Rightarrow \quad (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2) = (z - z_3)(\bar{z} - \bar{z}_3) \quad \text{...(i)}$$

From first two members of Eq. (i), we get

$$\bar{z}(z_2 - z_1) = \bar{z}_1(z - z_1) - \bar{z}_2(z - z_2) \quad \text{...(ii)}$$

and from last two members of Eq. (i), we get

$$\bar{z}(z_3 - z_2) = \bar{z}_2(z - z_2) - \bar{z}_3(z - z_3) \quad \text{...(iii)}$$

Eliminating $\bar{z}$ from Eqs. (ii) and (iii), we get

$$(z_2 - z_1)[\bar{z}_2(z - z_2) - \bar{z}_3(z - z_3)] = (z_3 - z_2)[\bar{z}_1(z - z_1) - \bar{z}_2(z - z_2)]$$

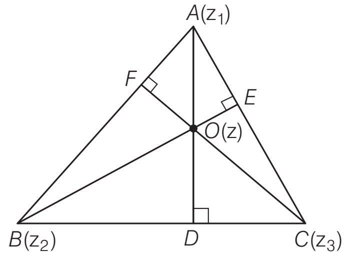
or $\quad z[z_2(z_2 - z_1) - \bar{z}_3(z_2 - z_1) - \bar{z}_1(z_3 - z_2) + \bar{z}_2(z_3 - z_2)]$

$$= z_2\bar{z}_2(z_2 - z_1) - z_3\bar{z}_3(z_2 - z_1) - z_1\bar{z}_1(z_3 - z_2) + z_2\bar{z}_2(z_3 - z_2)$$

or $\quad z\sum \bar{z}_1(z_2 - z_3) = \sum z_1\bar{z}_1(z_2 - z_3)$

or $\quad z = \dfrac{\sum |z_1|^2(z_2 - z_3)}{\sum \bar{z}_1(z_2 - z_3)}$

98. Let z be the complex number corresponding to the orthocentre O, since $AD \perp BC$, we get



$$\arg\left(\frac{z - z_1}{z_2 - z_3}\right) = \frac{\pi}{2}$$

i.e. $\dfrac{z - z_1}{z_2 - z_3}$ is purely imaginary.

i.e. $\quad \text{Re}\left(\dfrac{z - z_1}{z_2 - z_3}\right) = 0 \quad \text{or} \quad \dfrac{z - z_1}{z_2 - z_3} + \dfrac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_3} = 0 \quad$ ...(i)

Similarly, $\quad \dfrac{z - z_2}{z_3 - z_1} + \dfrac{\bar{z} - \bar{z}_2}{\bar{z}_3 - \bar{z}_1} = 0 \quad [\because BE \perp CA]$ ...(ii)

From Eq. (i), we get

$$\bar{z} = \bar{z}_1 - \frac{(z - z_2)(\bar{z}_2 - \bar{z}_3)}{(z_2 - z_3)} \quad \text{...(iii)}$$

From Eq. (ii), we get

$$\bar{z} = \bar{z}_2 - \frac{(z - z_2)(\bar{z}_3 - \bar{z}_1)}{(z_3 - z_1)} \quad \text{...(iv)}$$

Eliminating $\bar{z}$ from Eqs. (iii) and (iv), we get

$$\bar{z}_1 - \bar{z}_2 = \frac{(z - z_1)}{(z_2 - z_3)}(\bar{z}_2 - \bar{z}_3) - \frac{(z - z_2)(\bar{z}_3 - \bar{z}_1)}{(z_3 - z_1)}$$

or $(z - z_1)(\bar{z}_2 - \bar{z}_3)(z_3 - z_1) - (z - z_2)(\bar{z}_3 - \bar{z}_1)(z_2 - z_3)$

$$= (\bar{z}_1 - \bar{z}_2)(z_2 - z_3)(z_3 - z_1)$$

or $z\{(\bar{z}_2 - \bar{z}_3)(z_3 - z_1) - (\bar{z}_3 - \bar{z}_1)(z_2 - z_3)\}$

$$= (\bar{z}_1 - \bar{z}_2)(z_2 - z_3)(z_3 - z_1) + z_1(\bar{z}_2 - \bar{z}_3)(z_3 - z_1) - z_2(\bar{z}_3 - \bar{z}_1)(z_2 - z_3)$$

$\Rightarrow z[\bar{z}_2 z_3 - \bar{z}_2 z_1 - z_3\bar{z}_3 + \bar{z}_3 z_1 - \bar{z}_3 z_2 + z_3\bar{z}_3 + \bar{z}_1 z_2 - \bar{z}_1 z_3]$

$$= (\bar{z}_1 - \bar{z}_2)\{z_2 z_3 - z_2 z_1 - z_3^2 + z_3 z_1\} + (\bar{z}_2 - \bar{z}_3)(z_3 z_1 - z_1^2) + (\bar{z}_3 - \bar{z}_1)(z_2 z_3 - z_2^2)$$

$$= -\{z_1^2(\bar{z}_2 - \bar{z}_3) + z_2^2(\bar{z}_3 - \bar{z}_1) + z_3^2(\bar{z}_1 - \bar{z}_2)\} + \{\bar{z}_1 z_2 z_3 - z_2 z_1\bar{z}_1 + z_3 z_1\bar{z}_1 + \bar{z}_2 z_1 z_3 - z_1 z_3\bar{z}_3 + z_2 z_3\bar{z}_3 - \bar{z}_1 z_2 z_3\} - z\sum(z_1\bar{z}_2 - z_2\bar{z}_1)$$

$$= -\sum z_1^2(\bar{z}_2 - \bar{z}_3) - \sum z_1\bar{z}_1(z_2 - z_3)$$

Hence, $z = \dfrac{\sum z_1^2(\bar{z}_2 - \bar{z}_3) + \sum |z_1|^2(z_2 - z_3)}{\sum(z_1\bar{z}_2 - z_2\bar{z}_1)}$

99. Let $\theta = \dfrac{1}{7}(2n + 1)\pi$, where $n = 0, 1, 2, 3, \ldots, 6$

$\therefore \quad 7\theta = (2n + 1)\pi \quad \text{or} \quad 4\theta = (2n + 1)\pi - 3\theta$

or $\quad \cos 4\theta = -\cos 3\theta$

or $\quad 2\cos^2 2\theta - 1 = -(4\cos^3\theta - 3\cos\theta)$

or $2(2\cos^2\theta - 1)^2 - 1 = -(4\cos^3\theta - 3\cos\theta)$

or $8\cos^4\theta + 4\cos^3\theta - 8\cos^2\theta - 3\cos\theta + 1 = 0$

Now, if $\cos\theta = x$, then we have

$8x^4 + 4x^3 - 8x^2 - 3x + 1 = 0$

or $(x+1)(8x^3 - 4x^2 - 4x + 1) = 0$

$$x + 1 \neq 0 \qquad [\because \theta \neq \pi]$$

$$\therefore \quad 8x^3 - 4x^2 - 4x + 1 = 0 \qquad \text{...(i)}$$

Hence, the roots of this equation are

$$\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}.$$

$$\left[\text{since} \cos\frac{9\pi}{7} = \cos\frac{5\pi}{7}, \cos\frac{11\pi}{7} = \cos\frac{3\pi}{7}, \cos\frac{13\pi}{7} = \cos\frac{\pi}{7} \text{ and Eq. (i) is cubic}\right]$$

(i) On putting $\frac{1}{x^2} = y$ or $x = \frac{1}{\sqrt{y}}$ in Eq. (i), then Eq. (i) becomes

$$\Rightarrow \quad \frac{8}{y\sqrt{y}} - \frac{4}{y} - \frac{4}{\sqrt{y}} + 1 = 0$$

$$\Rightarrow \quad \left(1 - \frac{4}{y}\right)^2 = \left[\frac{4}{\sqrt{y}}\left(1 - \frac{2}{y}\right)\right]^2$$

$$\text{or} \quad 1 + \frac{16}{y^2} - \frac{8}{y} = \frac{16}{y}\left(1 + \frac{4}{y^2} - \frac{4}{y}\right)$$

$$\text{or} \quad y^3 - 24y^2 + 80y - 64 = 0 \qquad \text{...(ii)}$$

$$\text{where} \quad y = \frac{1}{x^2} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

Thus, the roots of $x^3 - 24x^2 + 80x - 61 = 0$

are $\sec^2\frac{\pi}{7}, \sec^2\frac{3\pi}{7}, \sec^2\frac{5\pi}{7}$

(ii) Again, putting $y = 1 + z$ i.e. $z = y - 1$

$= \sec^2\theta - 1 = \tan^2\theta$, Eq. (ii) reduces to

$$(1+z)^3 - 24(1+z)^2 + 80(1+z) - 64 = 0$$

$$\text{or} \quad z^3 - 21z^2 + 35z - 7 = 0 \qquad \text{...(iii)}$$

Hence, $\tan^2\frac{\pi}{7}, \tan^2\frac{3\pi}{7}, \tan^2\frac{5\pi}{7}$ are the roots of

$$x^3 - 21x^2 + 35x - 7 = 0$$

(iii) Putting $x = \frac{1}{u}$ in Eq. (i), then Eq. (i) reduces to

$u^3 - 4u^2 - 4u + 8 = 0$ whose roots are

$$\sec\frac{\pi}{7}, \sec\frac{3\pi}{7}, \sec\frac{5\pi}{7}.$$

Therefore, sum of the roots is

$$\sec\frac{\pi}{7} + \sec\frac{3\pi}{7} + \sec\frac{5\pi}{7} = 4$$

100. Let roots of $z^7 + 1 = 0$ are $-1, \alpha, \alpha^3, \alpha^5, \bar{\alpha}, \bar{\alpha}^3, \bar{\alpha}^5$,

where $\alpha = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$

$$\therefore \quad (z^7 + 1) = (z+1)(z-\alpha)(z-\bar{\alpha})(z-\alpha^3)(z-\bar{\alpha}^3)(z-\alpha^5)(z-\bar{\alpha}^5)$$

$$\Rightarrow \frac{(z^7+1)}{(z+1)} = (z-\alpha)(z-\bar{\alpha})(z-\alpha^3)(z-\bar{\alpha}^3)(z-\alpha^5)(z-\bar{\alpha}^5)$$

$$\Rightarrow \quad z^6 - z^5 + z^4 - z^3 + z^2 - z + 1 = \left(z^2 + 1 - 2z\cos\frac{\pi}{7}\right)\left(z^2 + 1 - 2z\cos\frac{3\pi}{7}\right)\left(z^2 + 1 - 2z\cos\frac{5\pi}{7}\right) \qquad \text{...(A)}$$

Dividing by z^3 on both sides, we get

$$\left(z^3 + \frac{1}{z^3}\right) - \left(z^2 + \frac{1}{z^2}\right) + \left(z + \frac{1}{z}\right) - 1 = \left(z + \frac{1}{z} - 2\cos\frac{\pi}{7}\right)\left(z + \frac{1}{z} - 2\cos\frac{3\pi}{7}\right)\left(z + \frac{1}{z} - 2\cos\frac{5\pi}{7}\right)$$

On putting $z + \frac{1}{z} = 2x$, we get

$$(8x^3 - 6x) - (4x^2 - 2) + 2x - 1 = 8\left(x - \cos\frac{\pi}{7}\right)\left(x - \cos\frac{3\pi}{7}\right)\left(x - \cos\frac{5\pi}{7}\right)$$

$$\text{or} \quad 8x^3 - 4x^2 - 4x + 1 = 8\left(x - \cos\frac{\pi}{7}\right)\left(x - \cos\frac{3\pi}{7}\right)\left(x - \cos\frac{5\pi}{7}\right) \qquad \text{...(i)}$$

So, $8x^3 - 4x^2 - 4x + 1 = 0$ and this equation has roots $\cos\frac{\pi}{7}, \cos\frac{3\pi}{7}, \cos\frac{5\pi}{7}$

$$\therefore \quad \cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7} = -\frac{1}{8} \qquad [\text{ proved (i) part}]$$

On putting $x = 1$ in Eq. (i), we get

$$1 = 8\left(1 - \cos\frac{\pi}{7}\right)\left(1 - \cos\frac{3\pi}{7}\right)\left(1 - \cos\frac{5\pi}{7}\right)$$

$$\text{or} \quad 1 = 8\left(8\sin^2\frac{\pi}{14}\sin^2\frac{3\pi}{14}\sin^2\frac{5\pi}{14}\right)$$

Since, $\sin\theta > 0$ for $0 < \theta < \pi/2$, we get

$$\therefore \quad \sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \frac{1}{8} \qquad \text{...(ii) [proved (iii) part]}$$

Again, putting $x = -1$ in Eq. (i), we get

$$-7 = -8\left(1 + \cos\frac{\pi}{7}\right)\left(1 + \cos\frac{3\pi}{7}\right)\left(1 + \cos\frac{5\pi}{7}\right)$$

$$7 = 8\left(8\cos^2\frac{\pi}{14}\cos^2\frac{3\pi}{14}\cos^2\frac{5\pi}{14}\right)$$

Since, $\cos\theta > 0$ for $0 < \theta < \pi/2$, we get

$$\cos\frac{\pi}{14}\cos\frac{3\pi}{14}\cos\frac{5\pi}{14} = \frac{\sqrt{7}}{8} \qquad \text{...(iii) [proved (ii) part]}$$

On dividing Eq. (ii) by Eq. (iii), we get

$$\tan\frac{\pi}{14}\tan\frac{3\pi}{14}\tan\frac{5\pi}{14} = \frac{1}{\sqrt{7}} \qquad [\text{ proved (iv) part}]$$

On putting $z = \dfrac{(1+y)}{(1-y)}$ in Eq. (A), we get

$$\frac{(1+y)^7 + (1-y)^7}{2(1-y)^6} = \frac{2^6 \cos^2\frac{\pi}{14}\cos^2\frac{3\pi}{14}\cos^2\frac{5\pi}{14}}{(1-y)^6}\left(y^2+\tan^2\frac{\pi}{14}\right)\left(y^2+\tan^2\frac{3\pi}{14}\right)\left(y^2+\tan^2\frac{5\pi}{14}\right)$$

$$\therefore \quad (1+y)^7 + (1-y)^7 = 2^7 \cdot \frac{7}{64}\left(y^2+\tan^2\frac{\pi}{14}\right)\left(y^2+\tan^2\frac{3\pi}{14}\right)\left(y^2+\tan^2\frac{5\pi}{14}\right)$$

Using result (ii), we get

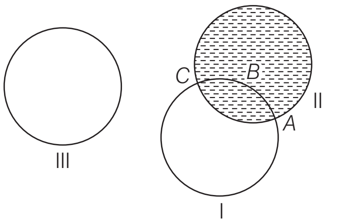
$$(1+y)^7 + (1-y)^7 = 14\left(y^2+\tan^2\frac{\pi}{14}\right)\left(y^2+\tan^2\frac{3\pi}{14}\right)\left(y^2+\tan^2\frac{5\pi}{14}\right)$$

Equating the coefficient of y^4 on both sides, we get

$${}^7C_4 + {}^7C_4 = 14\left[\tan^2\frac{\pi}{14} + \tan^2\frac{3\pi}{14} + \tan^2\frac{5\pi}{14}\right]$$

Therefore, $\tan^2\dfrac{\pi}{14} + \tan^2\dfrac{3\pi}{14} + \tan^2\dfrac{5\pi}{14} = 5$

101. Equation $|z| = 3$ represents boundary of a circle and equation $|z - \{a(1+i) - i\}| \le 3$ represents the interior and the boundary of a circle and equation $|z + 2a - (a+1)i| > 3$ represents the exterior of a circle. Then, any point which satisfies all the three conditions will lie on first circle, on or inside the second circle and outside the third circle.



For the existence of such a point first two circles must cut or atleast touch each other and first and third circles must not intersect each other. The arcABC of first circle lying inside the second but outside the third circle, represents all such possible points.

Let $z = x + iy$, then equation of circles are

$$x^2 + y^2 = 9 \qquad \text{...(i)}$$

$$(x-a)^2 + (y-a+1)^2 = 9 \qquad \text{...(ii)}$$

and $(x+2a)^2 + (y-a-1)^2 = 9$ ...(iii)

Circles (i) and (ii) should cut or touch, then distance between their centres $\le$ sum of their radii

$$\Rightarrow \quad \sqrt{(a-0)^2 + (a-1-0)^2} \le 3 + 3$$

$$\Rightarrow \quad a^2 + (a-1)^2 \le 36$$

$$\Rightarrow \quad 2a^2 - 2a - 35 \le 0$$

$$\Rightarrow \quad a^2 - a - \frac{35}{2} \le 0 \text{ or } \left(a - \frac{1+\sqrt{71}}{2}\right)\left(a - \frac{1-\sqrt{71}}{2}\right) \le 0$$

$$\therefore \quad \frac{1-\sqrt{71}}{2} \le a \le \frac{1+\sqrt{71}}{2} \qquad \text{...(iv)}$$

Again, circles (i) and (iii) should not cut or touch, then distance between their centres $>$ sum of their radii

$$\sqrt{(-2a-0)^2 + (a+1-0)^2} > 3 + 3$$

or $\qquad \sqrt{5a^2 + 2a + 1} > 6$

$$\Rightarrow \quad 5a^2 + 2a + 1 > 36$$

or $\qquad 5a^2 + 2a - 35 > 0$

$$\Rightarrow \quad a^2 + \frac{2a}{5} - 7 > 0$$

or $\quad \left(a - \dfrac{-1-4\sqrt{11}}{5}\right)\left(a - \dfrac{-1+4\sqrt{11}}{5}\right) > 0$

$$\therefore \quad a \in \left(-\infty, \frac{-1-4\sqrt{11}}{5}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \infty\right) \qquad \text{...(v)}$$

Hence, the common values of a satisfying Eqs. (iv) and (v) are

$$a \in \left(\frac{1-\sqrt{71}}{2}, \frac{-1-4\sqrt{11}}{5}\right) \cup \left(\frac{-1+4\sqrt{11}}{5}, \frac{1+\sqrt{71}}{2}\right)$$

102. (i) From De-moivre's theorem, we know that

$$\sin(2n+1)\alpha = {}^{2n+1}C_1(1-\sin^2\alpha)^n \sin\alpha - {}^{2n+1}C_3(1-\sin^2\alpha)^{n-1}\sin^3\alpha + \dots + (-1)^n \sin^{2n+1}\alpha$$

It follows that the numbers

$$\sin\frac{\pi}{2n+1}, \sin\frac{2\pi}{2n+1}, \dots, \sin\frac{n\pi}{2n+1}$$

are the roots of the equation.

${}^{2n+1}C_1(1-x^2)^n x - {}^{2n+1}C_3(1-x^2)^{n-1}x^3 + \dots + (-1)^n x^{2n+1} = 0$ of the $(2n+1)$ th degree

Consequently, the numbers

$\sin^2\dfrac{\pi}{2n+1}, \sin^2\dfrac{2\pi}{2n+1}, \dots, \sin^2\dfrac{n\pi}{2n+1}$ are the roots of the equation

${}^{2n+1}C_1(1-x)^n - {}^{2n+1}C_3(1-x)^{n-1}x + \dots + (-1)^n x^n = 0$ of the nth degree

(ii) From De-moivre's theorem, we know that

$$\sin(2n+1)\alpha = {}^{2n+1}C_1(\cos\alpha)^{2n}\sin\alpha - {}^{2n+1}C_3(\cos\alpha)^{2n-2}\sin^3\alpha + \dots + (-1)^n \sin^{2n+1}\alpha$$

or $\sin(2n+1)\alpha = \sin^{2n+1}\alpha$

$$\{{}^{2n+1}C_1 \cot^{2n}\alpha - {}^{2n+1}C_3 \cot^{2n-2}\alpha + {}^{2n+1}C_5 \cot^{2n-4}\alpha - \dots\}$$

It follows that $\alpha = \dfrac{\pi}{2n+1}, \dfrac{2\pi}{2n+1}, \dfrac{3\pi}{2n+1}, \dots, \dfrac{n\pi}{2n+1}$

Therefore, equality holds

$${}^{2n+1}C_1 \cot^{2n}\alpha - {}^{2n+1}C_3 \cot^{2n-2}\alpha + {}^{2n+1}C_5 \cot^{2n-4}\alpha - \dots = 0$$

It follows that the numbers

$\cot^2\dfrac{\pi}{2n+1}, \cot^2\dfrac{2\pi}{2n+1}, \dots, \cot^2\dfrac{n\pi}{2n+1}$ are the roots of the equation

$${}^{2n+1}C_1 x^n - {}^{2n+1}C_3 x^{n-1} + {}^{2n+1}C_5 x^{n-2} - \dots = 0$$

of the nth degree.

103. Let $y = |a + b\omega + c\omega^2|$. For y to be minimum, y^2 must be minimum.

$\therefore \quad y^2 = |a + b\omega + c\omega^2|^2 = (a + b\omega + c\omega^2)\overline{(a + b\omega + c\omega^2)}$

$= (a + b\omega + c\omega^2)(a + b\overline{\omega} + c\overline{\omega}^2)$

$y^2 = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)3$

$= (a^2 + b^2 + c^2 - ab - bc - ca)$

$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$

Since, a, b and c are not equal at a time, so minimum value of y^2 occurs when any two are same and third is differ by 1.

$\Rightarrow$ Minimum of $y = 1$ (as a, b, c are integers)

104. Equation of ray PQ is $\arg(z + 1) = \frac{\pi}{4}$

Equation of ray PR is $\arg(z + 1) = -\frac{\pi}{4}$

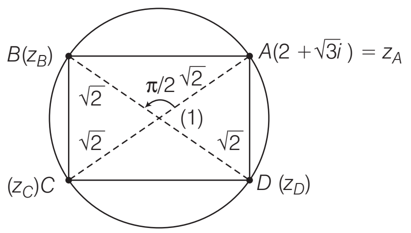
Shaded region is $-\frac{\pi}{4} < \arg(z + 1) < \frac{\pi}{4} \Rightarrow |\arg(z + 1)| < \frac{\pi}{4}$

$\therefore \quad |PQ| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$

So, arc QAR is of a circle of radius 2 units with centre at $P(-1, 0)$. All the points in the shaded region are exterior to this circle $|z + 1| = 2$.

i.e. $|z + 1| > 2$ and $|\arg(z + 1)| < \frac{\pi}{4}$.

105. In ΔAOB from Coni method, $\frac{z_B - 1}{z_A - 1} = e^{i\pi/2} = i$



$z_B - 1 = (z_A - 1)\, i$

$\therefore \quad z_B = 1 + (2 + \sqrt{3}i - 1)\, i = 1 + (1 + i\sqrt{3})\, i$

$= 1 + i - \sqrt{3} = 1 - \sqrt{3} + i$

$z_C = 2 - z_A = 2 - (2 + \sqrt{3}i) = -\sqrt{3}\, i$

and $\quad z_D = 2 - z_B = 2 - (1 - \sqrt{3} + i) = 1 + \sqrt{3} - i$

Hence, other vertices are $(1 - \sqrt{3}) + i, -\sqrt{3}i, (1 + \sqrt{3}) - i$.

106. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$\therefore |z_1 + z_2| = [(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2]^{1/2}$

$= [|r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)]^{1/2} = [(r_1 + r_2)^2]^{1/2}$

$\therefore |z_1 + z_2| = |z_1| + |z_2|$

Therefore, $\cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \quad \theta_1 - \theta_2 = 0$

$\Rightarrow \quad \theta_1 = \theta_2$

Thus, $\arg(z_1) - \arg(z_2) = 0$

107. $(x - 1)^3 = -8 \Rightarrow x - 1 = (-8)^{1/3}$

$\Rightarrow \quad x - 1 = -2, -2\omega, -2\omega^2$

$\Rightarrow \quad x = -1, 1 - 2\omega, 1 - 2\omega^2$

108. $\left|\frac{z}{z - \frac{i}{3}}\right| = 1 \Rightarrow |z| = \left|z - \frac{i}{3}\right|$

Clearly, locus of z is perpendicular bisector of line joining points having complex number $0 + i\ 0$ and $0 + \frac{i}{3}$.

Hence, z lies on a straight line.

109. Given, $\left(\frac{\omega - \overline{\omega}z}{1 - z}\right)$ is purely real $\Rightarrow z \neq 1$

$\therefore \quad \left(\frac{\omega - \overline{\omega}z}{1 - z}\right) = \overline{\left(\frac{\omega - \overline{\omega}z}{1 - z}\right)} = \frac{\overline{\omega} - \omega\overline{z}}{1 - \overline{z}}$

$\Rightarrow \quad (\overline{\omega} - \overline{\omega}z)(1 - \overline{z}) = (1 - z)(\overline{\omega} - \omega\overline{z})$

$\Rightarrow \quad (z\overline{z} - 1)(\omega - \overline{\omega}) = 0$

$\Rightarrow \quad (|z|^2 - 1)(2i\beta) = 0 \qquad [\because \omega = \alpha + i\beta]$

$\therefore \quad |z|^2 - 1 = 0$

$\Rightarrow \quad |z| = 1$ and $z \neq 1 \qquad [\because \beta \neq 0]$

110. $\sum_{k=1}^{10} \sin\left(\frac{2k\pi}{11}\right) + i\cos\left(\frac{2k\pi}{11}\right)$

$= i\sum_{k=1}^{10}\left\{\cos\left(\frac{2k\pi}{11}\right) - i\sin\left(\frac{2k\pi}{11}\right)\right\} = i\sum_{k=1}^{10} e^{\frac{-2k\pi i}{11}}$

$= i\left(\sum_{k=0}^{10} e^{\frac{-2k\pi i}{11}} - 1\right) = i(0 - 1)$ [$\because$ sum of 11, 11th roots of unity $= 0$]

$= -i$

111. $\because \ z^2 + z + 1 = 0$

$\therefore \quad z = \omega, \omega^2$

$\because \quad z + \frac{1}{z} = \omega + \frac{1}{\omega} = \omega + \omega^2 = -1$

$\Rightarrow \quad z^2 + \frac{1}{z^2} = \omega^2 + \frac{1}{\omega^2} = \omega^2 + \omega = -1$

$\therefore \quad z^3 + \frac{1}{z^3} = \omega^3 + \frac{1}{\omega^3} = 1 + 1 = 2$

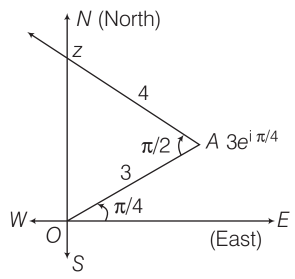
$z^4 + \frac{1}{z^4} = \omega^4 + \frac{1}{\omega^4} = \omega + \frac{1}{\omega} = -1$

$z^5 + \frac{1}{z^5} = \omega^5 + \frac{1}{\omega^5} = \omega^2 + \omega = -1$

and $\quad z^6 + \frac{1}{z^6} = \omega^6 + \frac{1}{\omega^6} = 2$

$\therefore$ Required sum $= (-1)^2 + (-1)^2 + (2)^2 + (-1)^2 + (-1)^2 + (2)^2 = 12$

112. Let $OA = 3$, so that the complex number associated with A is $3e^{i\pi/4}$. If z is the complex number associated with P, then



$$\frac{z-3e^{i\pi/4}}{0-3e^{i\pi/4}}=\frac{4}{3}e^{-\pi/2}=-\frac{4i}{3}$$

$$\Rightarrow \quad 3z-9e^{i\pi/4}=12ie^{i\pi/4} \Rightarrow z=(3+4i)\,e^{i\pi/4}$$

113. Let $z=\cos\theta+i\sin\theta$

$$\Rightarrow \frac{z}{1-z^2}=\frac{\cos\theta+i\sin\theta}{1-(\cos 2\theta+i\sin 2\theta)}$$

$$=\frac{\cos\theta+i\sin\theta}{2\sin^2\theta-2i\sin\theta\cos\theta}$$

$$=\frac{\cos\theta+i\sin\theta}{-2i\sin\theta(\cos\theta+i\sin\theta)}=\frac{i}{2\sin\theta}$$

Hence, $\frac{z}{1-z^2}$ lies on the imaginary axis i.e. $x=0$ or on Y-axis.

Aliter

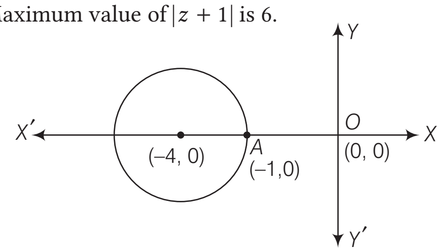
Let $E=\frac{z}{1-z^2}=\frac{z}{z\bar{z}-z^2}=\frac{1}{\bar{z}-z}=-\frac{1}{z-\bar{z}}=-\frac{1}{\left(\frac{z-\bar{z}}{2i}\right)2i}$

$$=\frac{i}{2\,\text{Im}\,|z|}$$ which is imaginary.

114. $|z+4|\le 3$

$\Rightarrow z$ lies inside or on the circle of radius 3 and centre at $(-4, 0)$.

$\therefore$ Maximum value of $|z+1|$ is 6.



115. Let A = set of points on and above the line $y=1$ in the argand plane.

B = set of points on the circle $(x-2)^2+(y-1)^2=3^2$

$C=\text{Re}\,(1-i)\,z=\text{Re}\,[(1-i)(x+iy)]=x+y$

$$\Rightarrow \quad x+y=\sqrt{2}$$

Hence, $(A\cap B\cap C)$ has only one point of intersection.

116. The points $(-1+i)$ and $(5+i)$ are the extremities of diameter of the given circle.

Hence, $|z+1-i|^2+|z-5-i|^2=36$

117. $\because |z-w|\le ||z|-|w||$

and $|z-w|$ = distance between z and w

z is fixed, hence distance between z and w would be maximum for diametrically opposite points.

$\Rightarrow \quad |z-w|<6 \quad \Rightarrow \quad ||z|-|w||<6$

$\Rightarrow \quad -6<|z|-|w|<6 \qquad \Rightarrow -3<|z|-|w|+3<9$

118. $\because \quad z_0=1+2i$

$\therefore \quad z_1=6+5i \quad \Rightarrow z_2=-6+7i$

119. Put $(-i)$ in place of i.

Hence, $\quad \frac{-1}{i+1}$

120. $\because \quad z\bar{z}\,(\bar{z}^2+z^2)=350$

Put $\quad z=x+iy$

$\Rightarrow \quad (x^2+y^2)\cdot 2(x^2-y^2)=350$

$\Rightarrow \quad (x^2+y^2)(x^2-y^2)=175=25\times 7$

$\Rightarrow \quad x^2+y^2=25,\ x^2-y^2=7$

$\Rightarrow \quad x^2=16,\ y^2=9$

$\therefore \quad x=\pm 4,\ y=\pm 3;\ x, y\in I$

Area of rectangle $=8\times 6=48$ sq units

121. $\sum_{m=1}^{15}\text{Im}\,(z^{2m-1})=\sum_{m=1}^{15}\text{Im}\,[e^{(2m-1)\,i\theta}]=\sum_{m=1}^{15}\sin(2m-1)\,\theta$

$$=\sum_{m=1}^{15}\frac{2\sin(2m-1)\,\theta\sin\theta}{2\sin\theta}$$

$$=\sum_{m=1}^{15}\frac{\cos(2m-2)\,\theta-\cos 2m\theta}{2\sin\theta}$$

$$=\frac{\cos 0^\circ-\cos 30\theta}{2\sin\theta}=\frac{1-\cos 60^\circ}{2\sin 2^\circ} \qquad (\because \theta=2^\circ)$$

$$=\frac{1-\frac{1}{2}}{2\sin 2^\circ}=\frac{1}{4\sin 2^\circ}$$

122. $\left|z-\frac{4}{z}\right|\ge\left||z|-\frac{4}{|z|}\right| \quad \Rightarrow \ 2\ge\left||z|-\frac{4}{|z|}\right|$

$\Rightarrow \quad -2\le|z|-\frac{4}{|z|}\le 2 \quad \Rightarrow \ -2|z|\le|z|^2-4\le 2|z|$

$\Rightarrow \quad |z|^2+2|z|-4\ge 0$

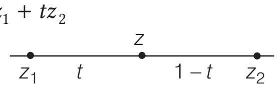
and $\quad 1^2-2|z|-4\le 0$

$\Rightarrow \quad (|z|+1)^2\ge 5$ and $(|z|-1)^2\le 5$

$-\sqrt{5}\le|z|-1\le\sqrt{5}$ and $|z|+1\ge\sqrt{5}$

$\Rightarrow \quad \sqrt{5}-1\le|z|\le\sqrt{5}+1$

123. As $z=(1-t)\,z_1+tz_2$



$\Rightarrow \ z_1, z$ and z_2 are collinear.

Thus, options (a) and (d) are correct.

Also, $\quad \frac{z-z_1}{z_2-z_1}=\frac{\bar{z}-\bar{z}_1}{\bar{z}_2-\bar{z}_1}$

Hence, option (c) is correct.

124. $\omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$

ω is one of the cube root of unity.

$$\therefore \quad \begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0$$

Applying $R_1\to R_1+R_2+R_3$, we get

$$\begin{vmatrix} z & z & z \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix}=0 \qquad [\because 1+\omega+\omega^2=0]$$

Now, applying $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$, we get

$$\begin{vmatrix} z & 0 & 0 \\ \omega & z+\omega^2-\omega & 1-\omega \\ \omega^2 & 1-w^2 & z+\omega-\omega^2 \end{vmatrix} = 0$$

$\Rightarrow \quad z[(z+\omega^2-\omega)(z+\omega-\omega^2)-(1-\omega)(1-\omega^2)] = 0$

$\Rightarrow \quad z[z^2-(\omega^2-\omega)^2-(1-\omega^2-\omega+\omega^3)] = 0$

$\Rightarrow \quad z[z^2-(\omega^4+\omega^2-2\omega^3)-1+\omega^2+\omega-\omega^3] = 0$

$\Rightarrow \quad z^3 = 0$

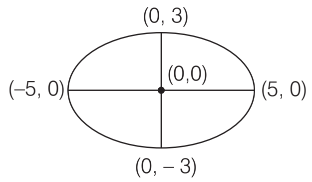
$\therefore \quad z = 0$

125. $|z-i|z|| = |z+i|z||$

(A) Putting $z = x+iy$, we get $y\sqrt{x^2+y^2} = 0$

i.e. $\text{Im}(z) = 0$

(B) $2ae = 8, 2a = 10 \Rightarrow 10e = 8 \Rightarrow e = \dfrac{4}{5}$



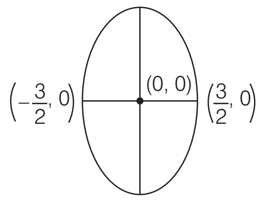
$\therefore \quad b^2 = 25\left(1-\dfrac{16}{25}\right) = 9$

$\Rightarrow \quad \dfrac{x^2}{25}+\dfrac{y^2}{9} = 1$

(C) $z = 2(\cos\theta+i\sin\theta) - \dfrac{1}{2(\cos\theta+i\sin\theta)}$

$= 2(\cos\theta+i\sin\theta) - \dfrac{1}{2}(\cos\theta-i\sin\theta)$

$z = \dfrac{3}{2}\cos\theta+\dfrac{5}{2}i\sin\theta$



Let $z = x+iy$, then

$x = \dfrac{3}{2}\cos\theta$ and $y = \dfrac{5}{2}\sin\theta$

$\Rightarrow \quad \left(\dfrac{2x}{3}\right)^2+\left(\dfrac{2y}{5}\right)^2 = 1$

$\Rightarrow \quad \dfrac{4x^2}{9}+\dfrac{4y^2}{25} = 1$

$\Rightarrow \quad \dfrac{x^2}{9/4}+\dfrac{y^2}{25/4} = 1$

$\Rightarrow \quad \dfrac{9}{4} = \dfrac{25}{4}(1-e^2)$

$\therefore \quad e^2 = 1-\dfrac{9}{25} = \dfrac{16}{25} \Rightarrow e = \dfrac{4}{5}$

(D) Let $w = \cos\theta+i\sin\theta$, then

$z = x+iy = w+\dfrac{1}{w}$

$\Rightarrow \quad x+iy = 2\cos\theta$

$\therefore \quad x = 2\cos\theta$ and $y = 0$

126. $\because \quad x^2-x+1 = 0$

$\therefore \quad x = \dfrac{1\pm\sqrt{(1-4)}}{2} = \dfrac{1\pm i\sqrt{3}}{2}$

$= \dfrac{1+i\sqrt{3}}{2}$ and $\dfrac{1-i\sqrt{3}}{2}$

$\therefore \quad x = -\omega^2, -\omega$

$\therefore \quad \alpha = -\omega^2, \beta = -\omega$

$\Rightarrow \quad \alpha^{2009}+\beta^{2009} = -\omega^{4018}-\omega^{2009}$

$= -\omega-\omega^2 = -(\omega+\omega^2)$

$= -(-1) = 1$

127. $|z-1| = |z+1| = |z-i|$

$\Rightarrow \quad |z-1|^2 = |z+1|^2 = |z-i|^2$

$\Rightarrow \quad (z-1)(\bar{z}-1) = (z+1)(\bar{z}+1) = (z-i)(\bar{z}+i)$

$\Rightarrow \quad z\bar{z}-z-\bar{z}+1 = z\bar{z}+z+\bar{z}+1 = z\bar{z}+iz-i\bar{z}+1$

$\Rightarrow \quad -z-\bar{z} = z+\bar{z} = i(z-\bar{z})$

From first two relations,

$2(z+\bar{z}) = 0 \Rightarrow \text{Re}(z) = 0$ ...(i)

From last two relations,

$z+\bar{z} = i(z-\bar{z}) \Rightarrow 2\,\text{Re}(z) = -2\,\text{Im}(z)$

From Eq. (i), $\quad \text{Im}(z) = 0$

$\therefore \quad z = \text{Re}(z)+i\,\text{Im}(z) = 0+i\cdot 0 = 0$

Hence, number of solutions is one.

128. We have, $\quad |z-3-2i| \le 2$

$\Rightarrow \quad |2z-6-4i| \le 4$ ...(i)

Now, $\quad |2z-6-4i| = |(2z-6+5i)-9i|$

$\ge ||2z-6+5i|-9|$ ...(ii)

From Eqs. (i) and (ii), we get

$||2z-6+5i|-9| \le 4$

$\Rightarrow \quad -4 \le |2z-6+5i|-9 \le 4$

$\Rightarrow \quad 5 \le |2z-6+5i| \le 13$

Hence, the minimum value of $|2z-6+5i|$ is 5.

129. $\because \quad |z| = 1 \quad \therefore \quad z = e^{i\theta}$

$\therefore \quad \text{Re}\left(\dfrac{2iz}{1-z^2}\right) = \text{Re}\left(\dfrac{2ie^{i\theta}}{1-e^{2i\theta}}\right) = \text{Re}\left(\dfrac{2i}{e^{-i\theta}-e^{i\theta}}\right)$

$= \text{Re}\left(\dfrac{2i}{-2i\sin\theta}\right) = \text{Re}\left(-\dfrac{1}{\sin\theta}\right)$

$= -\dfrac{1}{\sin\theta} = -\text{cosec}\,\theta$

$\because \quad \text{cosec}\,\theta \le -1 \Rightarrow \text{cosec}\,\theta \ge 1$

$\Rightarrow \quad -\text{cosec}\,\theta \ge 1 \Rightarrow -\text{cosec}\,\theta \le -1$

$\Rightarrow \quad -\text{cosec}\,\theta \in (-\infty, -1] \cap [1, \infty)$

$\therefore \quad \text{Re}\left(\dfrac{2iz}{1-z^2}\right) \in (-\infty, -1] \cap [1, \infty)$

130. $\because \; |z| = 1$. Let $z = e^{i\theta}$

$\therefore \quad z - 1 = e^{i\theta} - 1 = e^{i\theta/2} \cdot 2i \sin(\theta/2)$

$\Rightarrow \quad \dfrac{1}{z-1} = \dfrac{1}{2ie^{i\theta/2} \cdot \sin(\theta/2)} = -\dfrac{ie^{-i\theta/2}}{2\sin(\theta/2)}$

$\Rightarrow \quad \dfrac{1}{1-z} = \dfrac{i \cdot e^{-i\theta/2}}{2\sin(\theta/2)} \quad \therefore \; \arg\left(\dfrac{1}{1-z}\right) = \left(\dfrac{\pi}{2} - \dfrac{\theta}{2}\right)$

$\Rightarrow \quad \left|\arg\left(\dfrac{1}{1-z}\right)\right| = \left|\dfrac{\pi}{2} - \dfrac{\theta}{2}\right|$

$\therefore$ Maximum value of $\left|\arg\left(\dfrac{1}{1-z}\right)\right| = \dfrac{\pi}{2}$

131. $\because \; |x|^2 = x\bar{x} = (a + b + c)\overline{(a + b + c)}$

$= (a + b + c)(\bar{a} + \bar{b} + \bar{c})$

$= |a|^2 + |b|^2 + |c|^2 + a\bar{b} + \bar{a}b + b\bar{c} + \bar{b}c + c\bar{a} + \bar{c}a \quad \ldots(\text{i})$

$|y|^2 = y\bar{y} = (a + b\omega + c\omega^2)\overline{(a + b\omega + c\omega^2)}$

$= (a + b\omega + c\omega^2)(\bar{a} + \bar{b}\bar{\omega} + \bar{c}\bar{\omega}^2)$

$= (a + b\omega + c\omega^2)(\bar{a} + \bar{b}\omega^2 + \bar{c}\omega)$

$= |a|^2 + |b|^2 + |c|^2 + a\bar{b}\omega^2 + \bar{a}b\omega$

$+ b\bar{c}\omega^2 + \bar{b}c\omega + c\bar{a}\omega^2 + \bar{c}a\omega \quad \ldots(\text{ii})$

and $|z|^2 = z\bar{z} = (a + b\omega^2 + c\omega)\overline{(a + b\omega^2 + c\omega)}$

$= (a + b\omega^2 + c\omega)(\bar{a} + \bar{b}\bar{\omega}^2 + \bar{c}\bar{\omega})$

$= (a + b\omega^2 + c\omega)(\bar{a} + \bar{b}\omega + \bar{c}\omega^2)$

$= |a|^2 + |b|^2 + |c|^2 + a\bar{b}\omega + \bar{a}b\omega^2$

$+ b\bar{c}w + \bar{b}c\omega^2 + c\bar{a}\omega + \bar{c}a\omega^2 \quad \ldots(\text{iii})$

On adding Eqs. (i), (ii) and (iii), we get

$|x|^2 + |y|^2 + |z|^2 = 3(|a|^2 + |b|^2 + |c|^2)$

$+ 0 + 0 + 0 + 0 + 0 + 0 \; (\because 1 + \omega + \omega^2 = 0)$

$\therefore \quad \dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = 3$

132. $\because \; \text{Re}(z) = 1 \quad \therefore \; \dfrac{z + \bar{z}}{2} = 1 \;\Rightarrow\; z + \bar{z} = 2$

Since, $\alpha, \beta \in R$

$\therefore$ The complex roots are conjugate to each other, if z_1, z_2 are two distinct roots, then $z_1 = \bar{z}_2$ or $\bar{z}_1 = z_2$

$\therefore$ Product of the roots $= z_1 z_2 = \beta$

$\Rightarrow \quad z_1\bar{z}_1 = \beta$

$\therefore \quad \beta = |z_1|^2 = [\text{Re}(z_1)]^2 + \text{Im}\,|z_1|^2$

$= 1 + \text{Im}\,|z_1|^2 > 1$

[$\because$ roots are distinct $\therefore \text{Im}\,(z_1) \neq 0$]

$\therefore \quad \beta > 1$ or $\beta \in (1, \infty)$

133. $\because \; (1 + \omega)^7 = (-\omega^2)^7 = -\omega^{14} = -\omega^2 = 1 + \omega$

Given, $(1 + \omega)^7 = A + B\omega \;\Rightarrow\; 1 + \omega = A + B\omega$

On comparing, we get $A = 1, B = 1$

$\therefore \quad (A, B) = (1, 1)$

134. Given, $z^2 + z + 1 = a \;\Rightarrow\; z^2 + z + 1 - a = 0$

$\therefore \quad z = \dfrac{-1 \pm \sqrt{(4a - 3)}}{2}$

Hence, $\quad a \neq \dfrac{3}{4}$ [for $a = 3/4$, z will be purely real]

135. Let $z = x + iy$, then

$\dfrac{z^2}{z-1} = \dfrac{(x + iy)^2}{(x + iy - 1)} = \dfrac{(x^2 - y^2 + 2ix)}{(x - 1 + iy)}$

$= \dfrac{(x^2 - y^2 + 2ixy)(x - 1 - iy)}{(x - 1 + iy)(x - 1 - iy)}$

$= \dfrac{(x-1)(x^2 - y^2) + 2xy^2 + i[2xy(x-1) - y(x^2 - y^2)]}{(x-1)^2 + y^2}$

Now, $\quad \text{Im}\left(\dfrac{z^2}{z-1}\right) = 0$

$\Rightarrow \quad 2xy(x - 1) - y(x^2 - y^2) = 0$

$\Rightarrow \quad y(2x^2 - 2x - x^2 + y^2) = 0$

$\Rightarrow \quad y(x^2 + y^2 - 2x) = 0$

$\Rightarrow \quad y = 0$ or $x^2 + y^2 - 2x = 0$

Hence, z lies on the real axis or on a circle passing through the origin.

136. Given, $|z| = 1$ and $\arg(z) = \theta \quad \ldots(\text{i})$

$\Rightarrow \quad |z|^2 = 1 \;\Rightarrow\; z\bar{z} = 1$

$\Rightarrow \quad \bar{z} = \dfrac{1}{z} \quad \ldots(\text{ii})$

$\therefore \quad \arg\left(\dfrac{1+z}{1+\bar{z}}\right) = \arg\left(\dfrac{1+z}{1+1/z}\right)$ [from Eq. (ii)]

$= \arg(z) = \theta$ [from Eq. (i)]

Aliter I

Given, $\quad |z| = 1$ and $\arg(z) = \theta$

$\Rightarrow \quad z = e^{i\theta}$

$\therefore \quad \arg\left(\dfrac{1+z}{1+\bar{z}}\right) = \arg\left(\dfrac{1+e^{i\theta}}{1+e^{-i\theta}}\right) = \arg(e^{i\theta}) = \arg(z) = \theta$

Aliter II Given, $|z| = 1$ and $\arg(z) = \theta$

Let $z = \omega$ (cube root of unity)

$\therefore \quad \arg\left(\dfrac{1+z}{1+\bar{z}}\right) = \arg\left(\dfrac{1+\omega}{1+\bar{\omega}}\right) = \arg\left(\dfrac{1+\omega}{1+\omega^2}\right) \quad (\because \bar{\omega} = \omega^2)$

$= \arg\left(\dfrac{-\omega^2}{-\omega}\right) \quad (\because 1 + \omega + \omega^2 = 0)$

$= \arg(\omega) = \arg(z) = \theta$

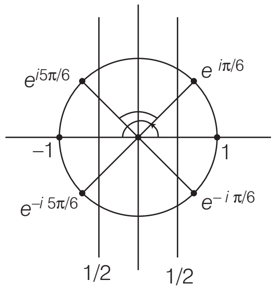
137. $\quad z_0 = 2\alpha - \dfrac{1}{\alpha}$

$\because \quad 2|z_0|^2 = r^2 + 2$

$\therefore \quad 2\left|2\alpha - \dfrac{1}{\alpha}\right|^2 = r^2 + 2 \Rightarrow 2\left|2\alpha - \dfrac{1}{\alpha}\right|^2 = \left|\alpha - \dfrac{1}{\alpha}\right|^2 + 2$

$\Rightarrow 7|\alpha|^2 + \dfrac{1}{|\alpha|^2} - 8 = 0 \;\Rightarrow\; |\alpha|^2 = 1$ or $\dfrac{1}{7} \;\Rightarrow\; |\alpha| = 1$ or $\dfrac{1}{\sqrt{7}}$

138.



$$\omega = \frac{\sqrt{3}+i}{2} = e^{i\pi/6}, \quad P = e^{in\pi/6}$$

As $z_1 \in P \cap H_1 \Rightarrow z_1 = 1, e^{i\pi/6}, e^{-i\pi/6}$

As $z_2 \in P \cap H_2 \Rightarrow z_2 = -1, e^{i5\pi/6}, e^{-i5\pi/6}$

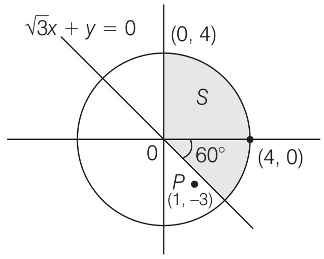
$\angle z_1Oz_2 = 2\pi/3$, where $z_1 = e^{i\pi/6}, z_2 = e^{i5\pi/6}$

Sol. (Q. Nos. 139-140)

Let $z = x + iy, S_1 : x^2 + y^2 < 16$

Now, $\text{Im}\left[\frac{(x-1)+i(y+\sqrt{3})}{1-\sqrt{3}i}\right] > 0$

$\Rightarrow \quad S_2 : \sqrt{3}x + y > 0 \Rightarrow S_3 : x > 0$



139. min $|1 - 3i - z| = \min|z - 1 + 3i|$

= perpendicular distance of the point $(1, -3)$ from the straight line $\sqrt{3}x + y = 0 = \left|\frac{\sqrt{3}-3}{2}\right| = \frac{3-\sqrt{3}}{2}$

140. Area of $S = \left(\frac{1}{4}\right)\pi \times 4^2 + \left(\frac{1}{6}\right)\pi \times 4^2 = \frac{20\pi}{3}$

141. Since, $|z| \geq 2$ is the region lying on or outside circle centered at $(0, 0)$ and radius 2. Therefore, $|z + (1/2)|$ is the distance of z from $(-1/2, 0)$, which lies inside the circle.

Hence, minimum value of $|z + (1/2)|$

= distance of $(-1/2, 0)$ from $(-2, 0)$

$$= \sqrt{\left(-\frac{1}{2}+2\right)^2 + |0-0|^2} = 3/2$$

Aliter

$\because \quad |z + (1/2)| \geq \left||z| - \frac{1}{2}\right| \geq \left|2 - \frac{1}{2}\right| \qquad [\because |z| \geq 2]$

$\therefore \quad |z + (1/2)| \geq 3/2$

142. Clearly, $z_k^{10} = 1, \forall k$, where $z_k \neq 1$

(A) $z_k \cdot z_j = e^{i(2\pi/10)(k+j)} = 1$, if $(k + j)$ is multiple of 10 i.e. possible for each k.

(B) $z_1 \cdot z = z_k$ is clearly incorrect.

(C) Expression $= \frac{\left|\lim\limits_{z\to 1}\frac{z^{10}-1}{z-1}\right|}{10} = 1$

(D) $1 + \Sigma z_k = 0 \Rightarrow 1 + \sum\limits_{k=1}^{9}\cos\left(\frac{2k\pi}{10}\right) = 0$

$\therefore$ Expression = 2

143. $\because \quad \left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$

$\Rightarrow |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

$\Rightarrow (z_1 - 2z_2)\overline{(z_1 - 2z_2)} = (2 - z_1\bar{z}_2)\overline{(2 - z_1\bar{z}_2)}$

$\Rightarrow (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1z_2)$

$\Rightarrow z_1\bar{z}_1 - 2z_1\bar{z}_2 - 2\bar{z}_1z_2 + 4z_2\bar{z}_2 = 4 - 2\bar{z}_1z_2 - 2z_1\bar{z}_2 + z_1\bar{z}_1z_2\bar{z}_2$

$\Rightarrow |z_1|^2 + 4|z_2|^2 + 4 + |z_1|^2|z_2|^2$

$\Rightarrow (|z_1|^2 - 4)(1 - |z_2|^2) = 0$

$\because \quad |z_2| \neq 1$

$\therefore \quad |z_1|^2 = 4$ or $|z_1| = 2$

$\Rightarrow$ Point z_1 lies on circle of radius 2.

144. Let $a = 3, b = -3, c = 2$, then

$(a + b\omega + c\omega^2)^{4n+3} + (c + a\omega + b\omega^2)^{4n+3} + (b + c\omega + a\omega^2)^{4n+3} = 0$

$$\Rightarrow (a + b\omega + c\omega^2)^{4n+3}\left\{1 + \left(\frac{c + a\omega + b\omega^2)^{4n+3}}{a + b\omega + c\omega^2}\right)^{4n+3} + \left(\frac{b + c\omega + a\omega^2)}{a + b\omega + c\omega^2}\right)^{4n+3}\right\} = 0$$

$\Rightarrow \quad (a + b\omega + c\omega^2)^{4n+3}(1 + \omega^{4n+3} + (\omega^2)^{4n+3}) = 0$

$\Rightarrow \quad 4n + 3$ should be an integer other than multiple of 3.

$\therefore \quad n = 1, 2, 4, 5$

145. $\because \quad \alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right) = e^{i\pi k/7}$

$\therefore \alpha_{k+1} - \alpha k = e^{i\pi(k+1)/7} - e^{i\pi k/7} = e^{i\pi k/7}(e^{i\pi/7} - 1)$

$$= e^{i\pi k/7} \cdot e^{i\pi/14} \cdot 2i\sin\left(\frac{\pi}{14}\right)$$

$\Rightarrow |\alpha_{k+1} - \alpha_k| = 2\sin\left(\frac{\pi}{14}\right)$

$$\sum_{k=1}^{12}|\alpha_{k+1} - \alpha_k| = 12 \times 2\sin\left(\frac{\pi}{14}\right) = 24\sin\left(\frac{\pi}{14}\right)$$

and $\alpha_{4k-1} - \alpha_{4k-2} = e^{i\pi(4k-1)/7} - e^{i\pi(4k-2)/7} = e^{i\pi(4k-2)/7}(e^{i\pi/7} - 1)$

$$= e^{i\pi(4k-2)/7} \cdot e^{i\pi/14} \cdot 2i\sin\left(\frac{\pi}{14}\right)$$

$\Rightarrow \quad |\alpha_{4k-1} - \alpha_{4k-2}| = 2\sin\left(\frac{\pi}{14}\right)$

$\therefore \quad \sum\limits_{k=1}^{3}|\alpha_{4k-1} - \alpha_{4k-2}| = 3 \times 2\sin\left(\frac{\pi}{14}\right) = 6\sin\left(\frac{\pi}{14}\right)$

Hence, $$\frac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}=4$$

146. Let $z=\dfrac{2+3i\sin\theta}{1-2i\sin\theta}$

$\because$ z is purely imaginary

$\therefore$ $\bar{z}=-z$

$$\Rightarrow \overline{\left(\frac{2+3i\sin\theta}{1-2i\sin\theta}\right)}=-\left(\frac{2+3i\sin\theta}{1-2i\sin\theta}\right)$$

$$\Rightarrow \left(\frac{2-3i\sin\theta}{1+2i\sin\theta}\right)=-\left(\frac{2+3i\sin\theta}{1-2i\sin\theta}\right)$$

$\Rightarrow$ $(2-3i\sin\theta)(1-2i\sin\theta)+(1+2i\sin\theta)(2+3i\sin\theta)=0$

$\Rightarrow$ $4-12\sin^2\theta=0$ or $\sin^2\theta=\dfrac{1}{3}$

$\therefore$ $\theta=\sin^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

147. $\because$ $$x+iy=\frac{1}{a+ibt}$$

$$\Rightarrow x+iy=\frac{a-ibt}{a^2+b^2t^2}$$

$$\Rightarrow x=\frac{a}{(a^2+b^2t^2)},\ y=-\frac{bt}{(a^2+b^2t^2)}$$

or $$x^2+y^2=\frac{1}{a^2+b^2t^2}=\frac{x}{a}$$

or $$x^2+y^2-\frac{x}{a}=0$$

$\therefore$ Locus of z is a circle with centre $\left(\dfrac{1}{2a},0\right)$

and radius $=\dfrac{1}{2a}$, $a>0$.

Also for $b=0$, $a\neq 0$, we get $y=0$.

$\therefore$ locus is X-axis and for $a=0$, $b\neq 0$, we get $x=0$

$\therefore$ locus is Y-axis.

148. Let $\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & -\omega^2-1 & \omega^2\\ 1 & \omega^2 & \omega^7\end{vmatrix}=\begin{vmatrix}1 & 1 & 1\\ 1 & \omega & \omega^2\\ 1 & \omega^2 & \omega\end{vmatrix}$

($\because 1+\omega+\omega^2=0$ and $\omega^3=1$)

Applying $C_1\to C_1+C_2+C_3$, then we get

$$\Delta=\begin{vmatrix}3 & 1 & 1\\ 0 & \omega & \omega^2\\ 0 & \omega^2 & \omega\end{vmatrix}\qquad(\because\ 1+\omega+\omega^2=0)$$

$=3(\omega^2-\omega^4)$

$=3(-1-\omega-\omega)$ $\quad(\because\ \omega^3=1$ and $1+\omega+\omega^2=0)$

$=-3(1+2\omega)$

$=-3z=3k$ (given) $\quad(\because\ 1+2\omega=z)$

$\therefore$ $k=-z$