1 RELATIONS AND FUNCTIONS

"Function means a quantity, which depends on a variable"

— G.W. Leibnitz

Objectives

After studying the material of this chapter, you should be able to :

- Understand the meaning of Relation and its types.
- Classify the relations – as Reflexive, Symmetric, Transitive and Equivalence Relations.
- Understand the meaning of Function and its types : Onto, Into and One–one.
- Understand the composition of Functions.
- Understand the Invertible Function and the process to find the inverse of a function.
- Understand Binary operations and their applications.

SUB CHAPTER 1.1 Relations

INTRODUCTION

Relations in Mathematics are analogous to the relations in our daily life. Similar to relations like mother and son, brother and sister, we can have mathematical relation between two elements, like the first element is greater than the second or the second element is twice the first. Relations are used extensively in the branches of Mathematics like Group theory, Ring theory and Vector Space. Functions are special types of relations. Many other terms like 'map', 'mapping', 'rule', 'operation', transformation' are used for functions. Functions are fundamental to Calculus and many other branches of Mathematics.

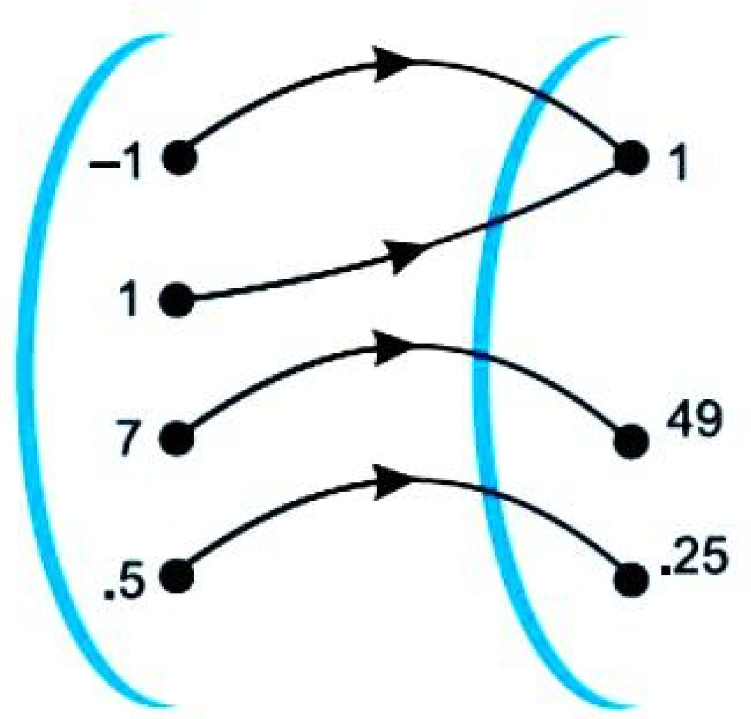


Fig.

All functions are relations

In this chapter we will study the following concepts :

- Relations
- Functions
- Binary Operations.

Chapter at Glance

1.1. RELATION

The word *relation,* used here, has the same usual meaning, which we have in our everyday life. By a relation we mean something, like friendship, marriage, parenthood; etc. *"Is the father of", "is the brother of", "is the friend of"* are relations over the set of human beings.

(I) Let us consider an **Example :**

Let A = {Swara, Lisa, Ramisha, Kaira} and B = {Kiya, Mia, Maire, Nisha}.

Here the relation will be *"is a friend of"* between the elements of the sets A and B.

Let R denote the relation *"is a friend of"*. Then Swara R Nisha, Lisa R Maire, Ramisha R Kiya and Kaira R Mia. These can be written in the form of a set as :

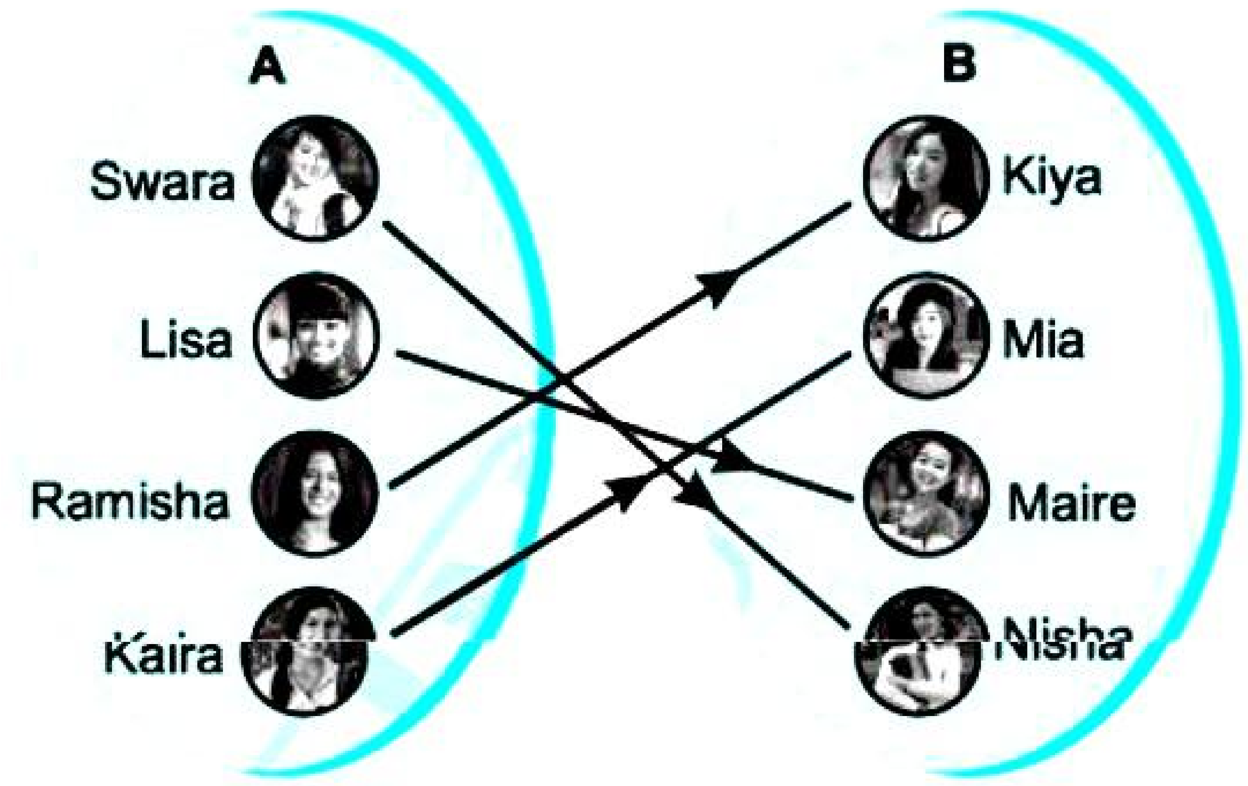


Fig.

$$R = \{(\text{Swara, Nisha}), (\text{Lisa, Maire}), (\text{Ramisha, Kiya}), (\text{Kaira, Mia})\}$$
$$= \{(x, y) : x \in A, y \in B, x \text{ R } y\}.$$

Thus, the above relation *"is a friend of",* from the set A to the set B, gives rise to a subset R of A × B such that $(x, y) \in R$ if and only if x R y.

Similarly, *"is less than", "is greater than", "is equal to", "is square of"* are relations over the set of numbers.

(II) Let us consider another **Example :**

Let **N** be the set of natural numbers.

Here the relation *"has as its square"* from the set **N** to the set **N**.

Let R denote *"has as its square",* then :

1 R 1, 2 R 4, 3 R 9, 4 R 16, ...

These can be written in the form of a set as :

Fig.

$$R = \{(1, 1), (2, 4), (3, 9), (4, 16), \ldots\ldots\}$$
$$= \{(x, y) : x, y \in \mathbf{N} \text{ and } y = x^2\}.$$

These are as shown in the figure.

Thus, the above relation *"has as its square"* from **N** to **N** gives rise to a subset R of **N** × **N** such that $(x, y) \in R$ if and only if $y = x^2$.

From the above examples, we can now define a relation :

Definition

A relation R from a set A to a set B is a subset of A × B.

If R is a relation from A to B, then $R \subseteq A \times B$.

The set of first elements in R is called the **domain** of R and the set of second elements in R is called the **range** of R.

Domain of R = $\{x : (x, y) \in R\}$ and Range = $\{y : (x, y) \in R\}$.

Domain of R is a subset of A and range of R is a subset of B.

The set B is called the **co-domain** of the relation R.

Range ⊆ Co-domain

For Example : Consider the relation R of the set A = {1, 3, 5, 7} to the set B {2, 4, 6, 8} and R = {(1, 2), (3, 4), (5, 6)}.

The domain of R = {1, 3, 5}, Range of R = {2, 4, 6} and Co-domain of R = {2, 4, 6, 8}.

In above example (II), domain = {1, 2, 3, 4, ...} and range = {1, 4, 9, 16,.......}.

In particular, any subset A × A defines a relation in A.

Notes : 1. If $(a, b) \in R$, then we write it as aRb and it is read as *'a is in relation R to b'*.

2. If $(a, b) \notin R$, then we write it as $a \not{R}\, b$ and it is read as *'a is not in relation R to b'*.

Examples : (I) *If a, b ∈ N and R is "a is a divisor of b", then R is a relation on N.*

The subset S of **N** × **N**, which corresponds to the relation, is :

$$S = \{(n, r) : n \in \mathbf{N}, r \in \mathbf{N}\}.$$

For Instance : (1, 3), (3, 15), (4, 4) are in S [∵ *1 divides 3 ; etc.*]

while (2, 5), (3, 7) do not belong to S. [∵ *2 does not divide 5; etc.*]

(II) *If a, b ∈ N and R is "a – b is divisible by a number n ∈ N", then R is a relation on **N**.*

The subset S of $\mathbf{N} \times \mathbf{N}$, which corresponds to the relation, is :

$$S = \{(n, n + rm) : n \in \mathbf{N}, r \in \mathbf{N}\}.$$

For Instance : When $m = 3$, $(2, 8), (5, 11) \in S$ [$\because 2 - 8 = -6$, *which is divisible by 3; etc.*]
while $(3, 8) \notin S$. [$\because 3 - 8 = -5$, *which is not divisible by 3*]

Binary Relation. *Let A be a non-empty set. The subset of A × A is called a binary relation or simply a relation on A.*

1.2. TYPES OF RELATIONS

(*a*) Void / Empty Relation. *A relation R in a set A is called empty (or void) relation if no element of A is related to any element of A i.e.* $R = \phi \subset A \times A$.

For Example : *Consider the set of all students in a Girls' School.*

Here no student in a brother of another student.

Thus R "*is a brother of*" is a void or an empty relation.

(*b*) Universal Relation. *A relation R in a set A is called universal relation if each element of A is related to every element of A i.e.* $R = A \times A$. ***(Jammu B. 2015 W)***

For Example : The difference between the heights of any two living human beings is less than 3 metres is an universal relation.

Note. Both (empty and universal) relations are called **trivial relations.**

(*c*) Identity Relation. The relation $I_A = \{(x, x) : x \in A\}$ is called identity relation on A.

For Example : Let $A = \{1, 2, 3, 4\}$.

Then the identity relation on A is given by :

$$I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}.$$

(*d*) Inverse Relation.

Let R be a relation from a set A to a set B and let (x, y) be the member of the subset D of A × B corresponding to the relation R from A to B.

To the relation R from the set A to the set B, there corresponds relation from the set B to the set A, called the inverse of the relation and denoted by R^{-1} such that the subset B × A corresponding to the relation R^{-1} is :

$$\{(y, x) : (x, y) \in B\} \quad i.e. \quad yR^{-1}x \Leftrightarrow xRy.$$

Examples : (I) The inverse of the relation "*is the father of*" in the set of all men is the relation "*is the son of.*"

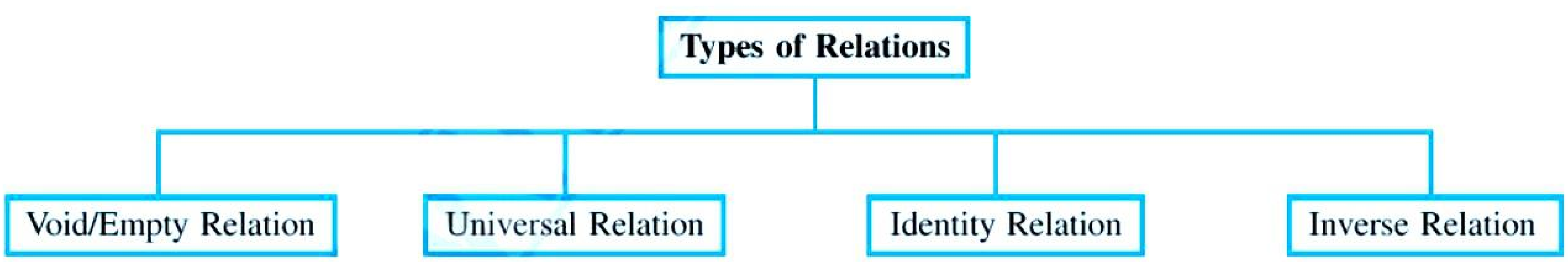
(II) The inverse of the relation "*is less than*" in **R** is the relation "*is greater than*" in **R**.

KEY POINT

Sometimes the inverse of a relation coincides with the relation itself.

Examples : (I) The inverse of the relation "*is perpendicular to*" in the set of straight lines, is a relation, which coincides with itself.

(II) The inverse of the relation "*is not equal to*" in the set **R** is a relation, which coincides with itself.



1.3. CLASSIFICATION OF RELATIONS

(a) Reflexive Relations. We introduce the concept of this type of relations by means of **Examples :**

(I) Consider the relation "*is less than or equal to*" denoted by '$\leq$' in the set of natural numbers.

Here we have : $2 \leq 2$, $3 \leq 3$; and so on.

In general, $x \leq x \ \forall \ x \in \mathbf{N}$.

Thus, this relation '$\leq$' is such that each member of the set bears this relation to itself.

A relation in a set is said to be reflexive if it is such that each member bears this relation to itself *i.e.* if $(a, a) \in R$, for every $a \in A$. This relation '$\leq$' in the set of natural numbers is reflexive.

(II) Consider the relation "*is less than*" symbolised by '<' in the set **N** of natural numbers. This relation is not reflexive. In fact in this case no member bears the relation '*is less than*" to itself. Thus the relation "*is less than*" in the set **N** is not reflexive.

(III) Consider the relation "*is the successor of*" in the set **N** of natural numbers. Since no number is a successor of itself, therefore, this relation is not reflexive.

(IV) Consider the relation "*is a factor of*" in the set of rational numbers is reflexive because every rational number is a factor of itself.

Definition

A relation R in a set A is said to be reflexive iff $(a, a) \in$ **R**, for all $a \in A$.*

(b) Symmetric Relations. We introduce this concept by means of **Examples :**

(I) Consider the set of lines in a plane and the relation "*is perpendicular to*" symbolised by '$\perp$'.

Let l, m be two straight lines such that $l \perp m$.

Surely when $l \perp m$, then $m \perp l$.

In fact $l \perp$ m $\Rightarrow$ m $\perp l$.

Such a relation is called a *symmetric relation.*

(II) Consider the set **N** of natural numbers and the relation "*a factor of*". This relation is not symmetric.

In fact it is not difficult to think of the pairs (a, b) such that "*a is factor of b*" but "*b is not a factor of a*".

For Example : (2, 4) and (3, 9).

Here '2' is a factor of '4' but '4' is not a factor of 2.

And '3' is a factor of '9' but '9' is not a factor of 3.

Of course there do exist an infinite number of pairs such that a member of the pair is a factor of the other member *e.g.*, the pairs :

(2, 2), (3, 3), (4, 4),..............

Definition

*A relation R in a set A is said to be symmetric if $(a_1, a_2) \in R \Rightarrow (a_2, a_1) \in R$ for all $a_1, a_2 \in$ **A**.* **(Jammu B. 2015 W)**

A relation R in a set A is not symmetric if there exists one pair $(a, b) \in$ R such that $(b, a) \notin$ R.

(c) Transitive Relations. Again, we introduce this concept by means of **Examples** :

(I) Consider the relation "*is a factor of*" in the set of natural numbers.

Let us ask the following question :

Given three numbers a, b, c such that a "*is a factor of*" b and b "*is a factor of*" c. Does it follow that a "*is a factor of*" c ?

The answer to this question is affirmative.

For this, we consider 2, 4, 8.

Here 2 "*is a factor of*" 4 and 4 "*is a factor of*" 8.

And 2 "*is also a factor of*" 8.

(II) In respect of relation '$\perp$' in the set of all lines, we see that for lines l, m, n ; $l \perp m, m \perp n \Rightarrow l \parallel n$.

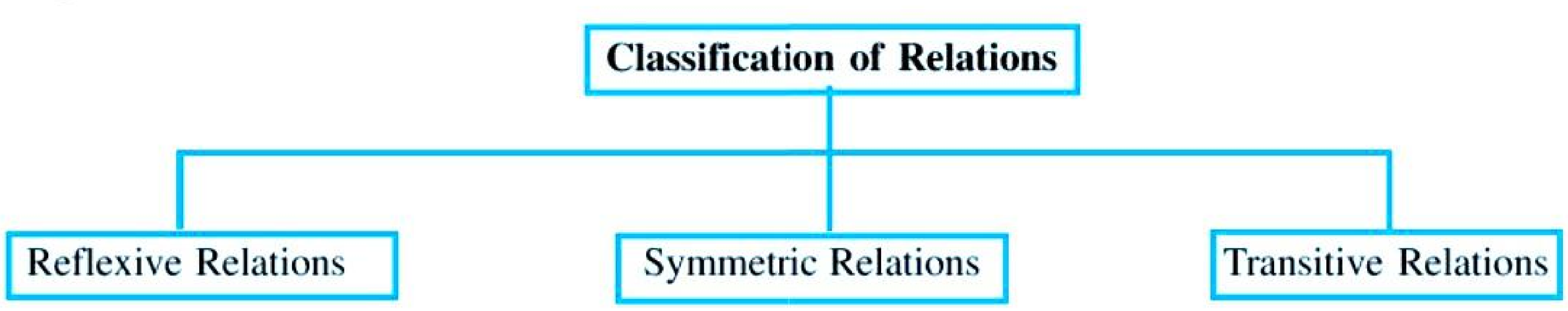
Definition

A relation R in a set A is said to be transitive if $(a_1, a_2) \in R$ and $(a_2, a_3) \in R$

*$\Rightarrow (a_1, a_3) \in R$ for all $a_1, a_2, a_3 \in$ **A**.*

A relation R is not transitive if there exists even one triplet a_1, a_2, a_3 of members of A such that when we have $(a_1, a_2) \in$ R, $(a_2, a_3) \in$ R, we do not have $(a_1, a_3) \in$ R.

(d) Anti-symmetric Relations. *A relation R in a set A is said to be anti-symmetric if* $(a_1, a_2) \in$ R and $(a_2, a_1) \in$ R $\Rightarrow$ $a_1 = a_2$.



1.4. EQUIVALENCE RELATIONS

A relation R in a set A is said to be an equivalence relation if it is :

(i) reflexive (ii) symmetric and (iii) transitive.

*'iff' means if and only if.

Examples : (I) The relation '*is congruent to*' in the set of all triangles in a plane is an equivalence relation.

(II) The relation '*is similar to*' in the set of all triangles in a plane is an equivalence relation.

(III) The relation '*is a divisor of*' in the set of natural numbers is not an equivalence relation.

In fact, this relation is reflexive and transitive but not symmetric.

An Important Property *is that it divides the set into pairwise disjoint subsets*, called **equivalent classes,** whose collection is called a partition of the set.

Note : The union of all equivalence classes gives the whole set.

Example : In the set **N** of natural numbers, we define a relation R as follows :

For $n, m \in \mathbf{N}$; $n \text{ R } m$ if on division by 5, each of the integers n and m leaves some remainder *viz.* 0, 1, 2, 3 and 4.

R is an equivalence relation because :

(I) $a \text{ R } a \ \forall \ a \in \mathbf{N}$ (*Reflexive*)

(II) If $a \text{ R } b$, then $b \text{ R } a \ \forall \ a, b \in \mathbf{N}$ (*Symmetric*)

(III) If $a \text{ R } b$ and $b \text{ R } c$, then $a \text{ R } c \ \forall \ a, b, c \in \mathbf{N}$ (*Transitive*).

Let $A_0 = \{n : n \in \mathbf{N}$ and on division by 5, n leaves the remainder 0],

$A_1 = \{n : n \in \mathbf{N}$ and on division by 5, n leaves the remainder 1$\}$.

Similarly A_2, A_3 and A_4.

Thus $A_0 = \{5, 10, 15, ...\}$, $A_1 = \{1, 6, 11, 16, ...\}$,

$A_2 = \{2, 7, 12, 17, ...\}$, $A_3 = \{3, 8, 13, 18, ...\}$ and $A_4 = \{4, 9, 14, 19, ...\}$.

Evidently, the above five sets are pairwise disjoint and

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{i=0}^{4} A_i = \mathbf{N}.$$

Conversely : A partition of a set defines an equivalence relation. If $S_1, S_2, ..., S_n$ is a partition, this equivalence relation is $a \text{ R } b$ if and only if $a, b \in S_i$ for some $i = 1, 2, ..., n$.

1.5. THEOREMS ON EQUIVALENCE RELATIONS

Theorem I. *If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.*

Proof. Since R is an equivalence relation on a set A, [*Given*]

$\therefore$ R is reflexive, symmetric and transitive.

Now R is reflexive $\Rightarrow (a, a) \in R \quad \forall \ a \in A$

$\Rightarrow (a, a) \in R^{-1} \quad \forall \ a \in A.$

Thus R^{-1} is reflexive.

Let $(a, b) \in R^{-1}$.

Now $(a, b) \in R^{-1} \Rightarrow (b, a) \in R$

$\Rightarrow (a, b) \in R$ [$\because$ *R is symmetric*]

$\Rightarrow (b, a) \in R^{-1}$.

Thus R^{-1} is symmetric.

Let $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$.

Now $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$

$\Rightarrow (b, a) \in R$ and $(c, b) \in R$

$\Rightarrow (c, b) \in R$ and $(b, a) \in R$

$\Rightarrow (c, a) \in R$ [$\because$ *R is transitive*]

$\Rightarrow (a, c) \in R^{-1}$.

Thus R^{-1} is transitive.

Hence, R^{-1} is an equivalence relation on A.

Theorem II. *The intersection of two equivalence relations on a set A is an equivalence relation on A.*

(Assam B. 2013 ; Rajasthan B. 2013)

Proof. Let R and S be two equivalence relations on a set A.

Then either of them is reflexive, symmetric and transitive.

Since $R \subseteq A \times A$; $S \subseteq A \times A \Rightarrow R \cap S \subseteq A \times A$,

$\therefore R \cap S$ is a relation on a set A.

Since R and S are reflexive,

$\therefore (a, a) \in R$ and $(a, a) \in S \ \forall \ a \in A$

$\Rightarrow (a, a) \in R \cap S \quad \forall \ a \in A.$

Thus $R \cap S$ is reflexive.

Now $(a, b) \in (R \cap S)$

$\Rightarrow (a, b) \in R$ and $(a, b) \in S$

$\Rightarrow (b, a) \in R$ and $(b, a) \in S$

$\Rightarrow (b, a) \in (R \cap S)$. [$\because$ *R and S are symmetric*]

Thus $R \cap S$ is symmetric.

Again $(a, b) \in (R \cap S)$ and $(b, c) \in (R \cap S)$

$\Rightarrow (a, b) \in R, (a, b) \in S$ and $(b, c) \in R, (b, c) \in S$

$\Rightarrow [(a, b) \in R, (b, c) \in R]$ and $[(a, b) \in S, (b, c) \in S]$

$\Rightarrow (a, c) \in R$ and $(a, c) \in S$ [$\because$ *R and S are transitive*]

$\Rightarrow (a, c) \in (R \cap S)$.

Thus $R \cap S$ is transitive.

Hence, $R \cap S$ is an equivalence relation.

Cor. The union of two equivalence relations is not necessarily an equivalence relation.

For Example :

Let $A = \{1, 2, 3\}$.

Consider $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

and $S = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$.

Clearly R and S are equivalence relations. [*Verify !*]

But $R \cup S$ is not transitive because :

$(3, 1) \in (R \cup S)$; $(1, 2) \in (R \cup S)$; but $(3, 2) \notin (R \cup S)$.

Hence, $(R \cup S)$ is not an equivalence relation.

ILLUSTRATIVE EXAMPLES

Example 1. Check whether the relation *R* in the set R of real numbers defined by :

$R = \{(a, b) : 1 + ab > 0\}$

is reflexive, symmetric or transitive.

(C.B.S.E. Sample Paper 2019)

Solution. We have : $R = \{(a, b) = 1 + ab > 0\}$.

Reflexive : Since $1 + a.a = 1 + a^2 > 0$,

$\therefore (a, a) \in R \ \forall \ a \in \mathbf{R}$. Thus, *R is reflexive.*

Symmetric : If $(a, b) \in R$, then $1 + ab > 0$

$\Rightarrow 1 + ba > 0 \Rightarrow (b, a) \in R$. Thus, *R is symmetric.*

Transitive : Take $a = -8$, $b = -1$ and $c = \frac{1}{2}$.

Now, $1 + ab = 1 + (-8)(-1) = 1 + 8 = 9 > 0$

$\Rightarrow (a, b) \in R$.

And, $1 + bc = 1 + (-1)\left(\frac{1}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2} > 0$

$\Rightarrow (b, c) \in R$.

But, $1 + ac = 1 + (-8)\left(\frac{1}{2}\right) = 1 - 4 = -3 < 0$.

Thus, *R is not transitive.*

Hence, R is reflexive and symmetric but not transitive.

Example 2. Let A = {1, 2, 3, 4}. Let *R* be the equivalence relation on A × A defined by :

(a, b) R (c, d) iff a + d = b + c. Find {(1, 3)}.

(C.B.S.E. Sample Paper 2018)

Solution. $\{(1, 3)\} = \{(1, 3), (2, 4)\}$.

[$\because$ $1 + 3 = 3 + 1$ & $1 + 4 = 3 + 2$]

Example 3. Let R = {(a, b) : a is a multiple of b}. Show that R is reflexive and transitive but not symmetric.

(Mizoram B. 2017)

Solution. Here $R = \{(a, b) : a$ is a multiple of $b\}$.

R is reflexive [$\because$ *(a, a) ∈ R as a is a multiple of a*]

R is transitive [$\because$ *(a, b) ∈ R, (b, c) ∈ R ⇒ (a, c) ∈ R as a is a multiple of b, b is a multiple of c ⇒ a is a multiple of c*]

R is not symmetric.

[$\because$ *(a, b) ∈ R ⇒ a is a multiple of b ⇒ b may not be a multiple of a*

For Ex. 2 is a multiple of 4 but 4 is not a multiple of 2]

Hence, R is reflexive and transitive but not symmetric.

Example 4. Let A be the set of all students of a Boys' school. Show that the relation *R* in A given by :

R = {(a, b) : a is sister of b}

is an empty relation and the relation R' given by :

R' = {(a, b) : the difference between heights of a and b is less than 3 metres} is an universal relation.

(N.C.E.R.T.)

Solution. (*i*) Here $R = \{(a, b) : a$ is sister of $b\}$.

Since the school is a Boys' school,

$\therefore$ no student of the school can be the sister of any student of the school.

Thus $R = \phi$. Hence, R is an empty relation.

(*ii*) Here $R' = \{(a, b)$: the difference between heights of a and b is less than 3 metres$\}$.

Since the difference between heights of any two students of the school is to be less than 3 metres,

$\therefore R' = A \times A$. Hence, R' is a universal relation.

Example 5. Let Z be the set of all integers and R be the relation on Z defined as :

$R = \{(a, b) : a, b \in Z$ and $a - b$ is divisible by 5$\}$.

Prove that R is an equivalence relation.

(Assam B. 2016; P.B. 2014; C.B.S.E. 2010)

Solution. For $a \in \mathbf{Z}$, $a - a = 0$, which is divisible by 5.

$\therefore \quad (a, a) \in R \ \forall \ a \in \mathbf{Z}$.

Thus *R is reflexive.*

Now let $(a, b) \in R \quad \Rightarrow a - b$ is divisible by 5

$\Rightarrow b - a$ is divisible by 5 $\Rightarrow (b, a) \in R$.

Thus *R is symmetric.*

Again let $(a, b) \in R$, $(b, c) \in R$

$\Rightarrow a - b$ and $b - c$ are divisible by 5

$\Rightarrow (a - b) + (b - c) = a - c$ is divisible by 5

$\Rightarrow (a, c) \in R$.

Thus *R is transitive.*

Hence, R is an equivalence relation.

Example 6. Let L be the set of all lines in the plane and R be the relation in L, defined as :

$R = \{(L_1, L_2) : L_1$ is perpendicular to $L_2\}$.

Show that R is symmetric but neither reflexive nor transitive. (N.C.E.R.T.)

Solution. We have :

$R = \{(L_1, L_2) : L_1$ is perpendicular to $L_2\}$.

Now L_1 can't be perpendicular to itself

i.e. $(L_1, L_1) \notin R$.

Thus *R is not reflexive.*

Now $(L_1, L_2) \in R \quad \Rightarrow L_1$ is perpendicular to L_2

$\Rightarrow L_2$ is perpendicular to L_1

$\Rightarrow (L_2, L_1) \in R$.

Thus *R is symmetric.*

Now $(L_1, L_2) \in R$ and $(L_2, L_3) \in R$

$\Rightarrow L_1$ is perpendicular to L_2

and L_2 is perpendicular to L_3

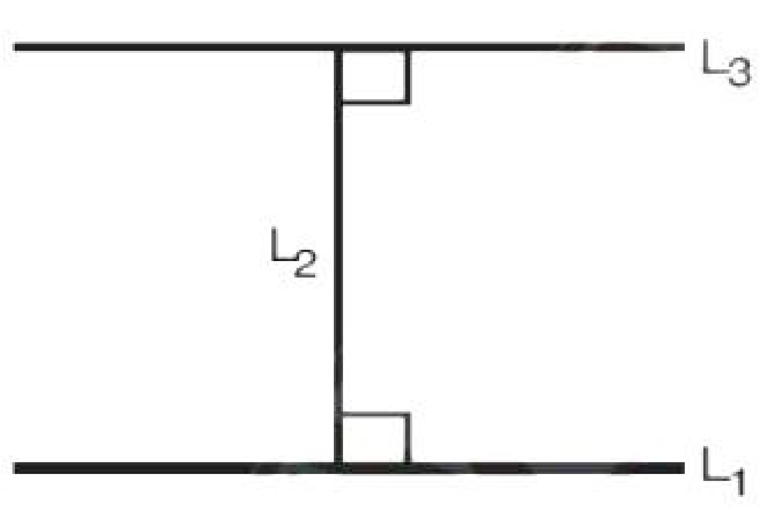


Fig.

$\Rightarrow L_1$ is parallel to L_3

$\Rightarrow L_1$ is not perpendicular to L_3

$\Rightarrow (L_1, L_3) \notin R$.

Thus *R is not transitive.*

Hence, R is symmetric but neither reflexive nor transitive.

Example 7. Let T be the set of all triangles in a plane with R, a relation in T given by :

$R = \{(T_1, T_2) : T_1$ is congruent to $T_2\}$.

Show that R is an equivalence relation.

(N.C.E.R.T.; H.P.B. 2017; Jammu B. 2016)

Solution. We have :

$R = \{(T_1, T_2) : T_1$ is congruent to $T_2\}$.

Now $\quad (T_1, T_1) \in R$.

[$\because$ *Every triangle is congruent to itself*]

Thus *R is reflexive.*

$(T_1, T_2) \in R \quad \Rightarrow \quad T_1$ is congruent to T_2

$\Rightarrow \quad T_2$ is congruent to T_1

$\Rightarrow \quad (T_2, T_1) \in R$.

Thus *R is symmetric.*

$(T_1, T_2) \in R$ and $(T_2, T_3) \in R$

$\Rightarrow T_1$ is congruent to T_2 and T_2 is congruent to T_3

$\Rightarrow T_1$ is congruent to T_3

$\Rightarrow (T_1, T_3) \in R$.

Thus *R is transitive.*

Hence, R is an equivalence relation.

Example 8. Let A = {1, 2, 3,........., 9} and R be the relation in A × A defined by (a, b) R (c, d) if :

$a + d = b + c$ for (a, b), (c, d) in A × A.

Prove that R is an equivalence relation. Also obtain the equivalence class {(2, 5)}. (C.B.S.E. 2014)

Solution. (*i*) We have : $(a, b) \, R \, (c, d) \Rightarrow a + d = b + c$ on the set $A = \{1, 2, 3, ..., 9\}$.

(I) $(a, b) \, R \, (a, b) \Rightarrow a + b = b + a$, which is true.

$[\because a + b = b + a \ \forall a, b \in A]$

Thus *R is reflexive.*

(II) $(a, b) \, R \, (c, d) \Rightarrow a + d = b + c$

$(c, d) \, R \, (a, b) \Rightarrow c + b = d + a$.

But $c + b = b + c$ and $d + a = a + d \ \forall \ a, b, c, d \in A$.

$\therefore \quad (a, b) \, R \, (c, d) = (c, d) \, R \, (a, b)$.

Thus *R is symmetric.*

(III) $(a, b) \, R \, (c, d) \Rightarrow a + d = b + c \ \forall \ a, b, c, d \in A$ ...(1)

$(c, d) \, R \, (e, f) \Rightarrow c + f = d + e \ \forall \ c, d, e, f \in A$ ...(2)

Adding (1) and (2),

$$(a + d) + (c + f) = (b + c) + (d + e)$$

$$\Rightarrow \quad a + f = b + e \Rightarrow (a, b) \, R \, (e, f).$$

Thus *R is transitive.*

Hence, the relation R is an equivalence relation.

(*ii*) $\{(2, 5)\} = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$.

$[\because 2 + 4 = 5 + 1;$ *etc.*$]$

Example 9. Let N denote the set of all natural numbers and R be the relation on N × N defined by :

(a, b) R (c, d) if $ad(b + c) = bc(a + d)$.

Show that R is an equivalence relation.

(C.B.S.E. 2015)

Solution. We have : $(a, b) \, R \, (c, d)$

$\Rightarrow ad\,(b + c) = bc\,(a + d)$ on $\mathbf{N}$.

(I) $\quad (a, b) \, R \, (a, b) \Rightarrow ab(b + a) = ba\,(a + b)$

$\Rightarrow ab(a + b) = ab\,(a + b)$,

which is true.

$[\because ab = ba \ \forall \ a, b \in N]$

Thus R is *reflexive.*

(II) (a, b) R $(c, d) \Rightarrow ad(b + c) = bc\,(a + d)$

$\Rightarrow bc(a + d) = ad\,(b + c)$

$\Rightarrow cb(d + a) = da\,(c + b)$

$[\because bc = cb$ *and* $a + d = d + a$ *; etc.* $\forall\, a, b, c, d \in N]$

$\Rightarrow (c, d)$ R (a, b).

Thus R is *symmetric*.

(III) Let (a, b) R (c, d) and (c, d) R (e, f).

$\therefore\quad ad(b + c) = bc(a + d)$ and $cf(d + e) = de(c + f)$

$$\Rightarrow \frac{b+c}{bc} = \frac{a+d}{ad} \text{ and } \frac{d+e}{de} = \frac{c+f}{cf}$$

$$\Rightarrow \frac{1}{c}+\frac{1}{b} = \frac{1}{d}+\frac{1}{a} \text{ and } \frac{1}{e}+\frac{1}{d} = \frac{1}{f}+\frac{1}{c}$$

$$\Rightarrow \frac{1}{c}-\frac{1}{d} = \frac{1}{a}-\frac{1}{b} \text{ and } \frac{1}{c}-\frac{1}{d} = \frac{1}{e}-\frac{1}{f}$$

$$\Rightarrow \frac{1}{a}-\frac{1}{b} = \frac{1}{e}-\frac{1}{f} \Rightarrow \frac{1}{a}+\frac{1}{f} = \frac{1}{b}+\frac{1}{e}$$

$\Rightarrow be(a + f) = af(b + e) \Rightarrow af(b + e) = be\,(a + f)$

$\Rightarrow (a, b)$ R (e, f).

Thus R *is transitive*.

Hence, R is an equivalence relation.

Example 10. Let R be the relation defined in the set A = {1, 2, 3, 4, 5, 6, 7} by :
R = {(a, b) : both a and b are either odd or even}.
Show that R is an equivalence relation.
Further, show that all the elements of the subset : {1, 3, 5, 7} are related to each other and all the elements of the subset {2, 4, 6} are related to each other, but no element of the subset {1, 3, 5, 7} is related to any element of the subset {2, 4, 6}. *(N.C.E.R.T.)*

Solution. We have :

R = {(a, b) : both a, b are either odd or even}.

(I) Let $a \in$ A.

Both a and a are either odd or even.

$\therefore\ (a, a) \in$ R.

Thus R *is reflexive*.

(II) Let $(a, b) \in$ R $\Rightarrow$ both a and b are either odd or even

$\Rightarrow$ both b and a are either odd or even

$\Rightarrow (b, a) \in$ R.

Thus R *is symmetric*.

(III) Let $(a, b) \in$ R and $(b, c) \in$ R.

$\therefore$ Both a, b and both b, c are either odd or even

$\Rightarrow$ both a, c are either odd or even

$\Rightarrow (a, c) \in$ R.

Thus R *is transitive*.

Hence, R is an equivalence relation.

Further all elements of {1, 3, 5, 7} are related to each other. [$\because$ *All elements of this subset are odd*]

Similarly all elements of {2, 4, 6} are related to each other.

[$\because$ *All elements of this subset are even*]

But no element of {1, 3, 5, 7} is related to any element of {2, 4, 6}.

[$\because$ *Elements of* {*1, 3, 5, 7*} *are odd while elements of* {*2,4,6*} *are even*]

Example 11. Let A = {x ∈ Z : 0 ≤ x ≤ 12}. Show that R = {(a, b) : a, b ∈ A : |a – b| is divisible by 4} is an equivalence relation. Find the set of all elements related to 1. Also, write the equivalence class [2]. *(C.B.S.E. 2018)*

Solution. We have :

R = {$(a, b) : a, b \in$ A : $|a - b|$ is divisible by 4}.

(*i*) Reflexive : For any $a \in$ A,

$\therefore\quad (a, a) \in$ R.

$|a - a| = 0$, which is divisible by 4.

Thus, ***R is reflexive.***

Symmetric : Let $(a, b) \in$ R

$\Rightarrow |a - b|$ is divisible by 4

$\Rightarrow |b - a|$ is divisible by 4.

Thus, ***R is symmetric.***

Transitive : Let $(a, b) \in$ R and $(b, c) \in$ R

$\Rightarrow |a - b|$ is divisible by 4 and $|b - c|$ is divisible by 4

$\Rightarrow \quad |a - b| = 4\lambda \quad$ ...(1)

$\Rightarrow \quad a - b = \pm 4\lambda$

and $\quad |b - c| = 4\mu$

$\Rightarrow \quad b - c = \pm 4\mu \quad$ ...(2)

Adding (1) and (2),

$(a - b) + (b - c) = \pm 4(\lambda + \mu)$

$\Rightarrow \quad a - c = \pm 4(\lambda + \mu)$

$\Rightarrow \quad (a - c) \in$ R.

Thus, ***R is transitive.***

Now, R is reflexive, symmetric and transitive.

Hence, R is an equivalence relation.

(*ii*) Let 'x' be an element of A such that $(x, 1) \in$ R

$\Rightarrow |x - 1|$ is divisible by 4

$\Rightarrow \quad x - 1 = 0, 4, 8, 12$

$\Rightarrow \quad x = 1, 5, 9.$

Hence, the set of all elements of A which are related to 1 is {1, 5, 9}.

(*iii*) Let $(x, 2) \in$ R.

Then, $\quad |x - 2| = 4k,$

where $\quad k \le 3.$

$\therefore \quad x = 2, 6, 10.$

Hence, equivalence class [2] = [2, 6, 10].

EXERCISE 1 (a)

Fast Track Answer Type Questions

FTATQ

1. If A = {0, 1, 3}, what is the number of relations on A ? *(Assam B. 2015)*

2. (*i*) If R = {(1, –1), (2, –2), (3, –1)} is a relation, then find the range of R. *(Meghalaya B. 2015)*

(*ii*) If R = {$(x, y) : x + 2y = 8$} is a relations in **N**, write the range of R. *(A.I.C.B.S.E. 2014)*

3. (*i*) Given A = {1, 2, 3}, then the relation :
R = {(1, 1), (2, 2), (3, 3)} is reflexive.
(True/False) (Jammu B. 2017)

(*ii*) Given a set A = {*a*, *b*, *c*, *d*}, then the relation :
R = {(*a*, *a*), (*b*, *b*), (*c*, *c*), (*d*, *d*)} is reflexive. *(True/False)*
(Jammu B. 2017)

(*iii*) What is the minimum number of ordered pairs to form a non-zero reflexive relation on a set of *n* elements ? *(Kerala B. 2015)*

4. (*a*) Give an example of a relation, which is :
(*i*) Symmetric but neither reflexive nor transitive *(Kashmir B. 2012)*
(*ii*) Transitive but neither reflexive nor symmetric
(*iii*) Reflexive and symmetric but not transitive *(Kashmir B. 2012 ; P.B. 2010)*
(*iv*) Reflexive and transitive but not symmetric. *(P.B. 2010)*

(*b*) State the reason for the relation R, in the set {1, 2, 3} given that R = {(1, 2), (2, 1)}, not to be transitive. *(C.B.S.E. 2011)*

5. (*a*) If R is the relation *"less than"* from :
A = {2, 4, 6, 8, 10} to B = {8, 10, 12}. Write down the elements corresponding to R. *(H.B. 2010)*

(*b*) Let R he the relation *"greater than"* from :
A = {1, 4, 5} to B = {1, 2, 4, 5, 6, 7}. Write down the elements corresponding to R. *(H.B. 2010)*

Very Short Answer Type Questions

VSATQ

6. Let **N** be the set of all natural numbers and R be a relation in **N** defined by :
R = {(*a*, *b*) : *a* is a factor of *b*}.
Show that R is reflexive and transitive. *(Meghalaya B. 2014)*

7. Determine whether each of the following relations are reflexive, symmetric and transitive :
(*i*) Relation R in the set A = {1, 2, 3,...., 13, 14} defined as :
R = {(x, y) : $3x - y = 0$}
(*ii*) Relation R in the set **N** of natural numbers defined as :
R = {(x, y) : $y = x + 5$ and $x < 4$} *(Kashmir B. 2012)*
(*iii*) Relation R in the set **Z** of all integers defined as :
R = {(x, y) : $x - y$ is an integer}
(*iv*) Relation R in the set A of human beings in a town at a particular time given by :
(*a*) R = {(x, y) : x and y work at the same place}
(*b*) R = {(x, y) : x and y live in the same locality}
(*c*) R = {(x, y) : x is exactly 7 cm taller than y}
(*d*) R = {(x, y) : x is wife of y}
(*e*) R = {(x, y) : x is father of y}. *(N.C.E.R.T.)*

8. Show that the relation R in the set **R** of real numbers, defined as :
(*a*) R = {(a, b) : $a \le b$ }
is reflexive and transitive but not symmetric
(Karnataka B. 2017; Assam B. 2017; H.B. 2013;Kashmir B. 2011)
(*b*) R = {(a, b) : $a \ge b$}
is reflexive and transitive but not symmetric. *(H.B. 2013)*

9. Prove that the relation in the set **N** of natural numbers defined as R = {(a, b) : $a \div b$} is reflexive and transitive but not symmetric. *(H.B. 2013)*

10. Check whether the relation R in the set **R** of real numbers defined as :
R = {(a, b) : $a \le b^3$}
is reflexive, symmetric or transitive. *(N.C.E.R.T. ; Jammu B. 2013)*

11. (*a*) Check whether the relation R in the set {1, 2, 3, 4, 5, 6} defined as :
R = {(a, b) : $b = a + 1$}
is reflexive, symmetric or transitive. *(N.C.E.R.T. ; Jammu B. 2013)*
(*b*) Prove that on the set of integers, **Z**, the relation R defined as $a\text{R}b \Leftrightarrow a = \pm b$ is an equivalence relation. *(Nagaland B. 2016)*

12. Show that the relation R in the set {1, 2, 3} defined as:
(*a*) R = {(1, 2), (2, 1)}
is symmetric, but neither reflexive nor transitive *(N.C.E.R.T.)*
(*b*) R = {(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)}
is reflexive, but neither symmetric nor transitive *(N.C.E.R.T.; Kashmir B. 2011 ; P.B. 2010)*
(*c*) R = {(1, 3), (3, 2), (1, 2)}
is transitive, but neither reflexive nor symmetric. *(P.B. 2010)*

Short Answer Type Questions

SATQ

13. If 'R' is relation *'less than'* from :
Set A = {1, 2, 3, 4, 5} to set B = {1, 4, 6},
write down the Cartesian Product corresponding to 'R'. Also find the inverse relation to 'R'. *(P.B. 2012)*

14. Prove that the following relation R in **Z** of integers is an equivalence relation :
R = {(x, y) : $x - y$ is an integer}. *(P.B. 2011)*

15. (*a*) Show that the relation R in **Z** of integers given by :
R = {(a, b) : 2 divides $a - b$}
is an equivalence relation. *(N.C.E.R.T.)*
(*b*) Consider the set of integers **Z.** Define a relation R on **Z** as :
R = {(x, y) : $x - y = 3k$, where k is some integer}.
Prove that R is an equivalence relation. *(Nagaland B. 2017)*
(*c*) Show that the relation R defined by :
R = {(a, b) : $a - b$ is divisible by 3; $a, b \in$ **N**}
is an equivalence relation. *(Assam B. 2013)*
(*d*) A relation R is defined on the set of all natural numbers **N** by :

$(x, y) \in R \Rightarrow (x - y)$ is divisible by 5 for all $x, y \in \mathbf{N}$.
Prove that R is an equivalence relation on **N**. *(W. Bengal B. 2016)*

16. Show that the relation in the set A = {1, 2, 3, 4, 5}, given by :
R = {(a, b) : | a – b | is even} is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other, but no element of {1, 3, 5} is related to any element of {2,4}. *(N.C.E.R.T.)*

17. (a) Show that the relation R in the set :
$A = \{x : x \in \mathbf{Z}, 0 \le x \le 12\}$, given by :
R = {(a, b) : | a – b | is divisible by 4}
is an equivalence relation.
Find the set of all elements related to 1. *(A.I.C.B.S.E. 2010)*

(b) Consider the set A = {1, 2, 4, 5, 7, 9, 10}.
Define the relation R on A as :
"aRb if and only if a – b is an even number."
Then : (i) write the elements of the set R
(ii) find the domain and range of R. *(Nagaland B. 2016)*

18. Show that the relation R defined on the set A of all lines as :
$R = \{(L_1, L_2) : L_1$ and L_2 are parallel lines$\}$
is an equivalence relation. *(P.B. 2016)*

19. Show that the relation R defined on the set of all polygons as :
$R = \{(P_1, P_2) : P_1$ and P_2 have same number of sides$\}$
is an equivalence relation. *(P.B. 2016)*

Long Answer Type Questions

LATQ

20. Show that the relation R defined by :
$(a, b)\ R\ (c, d) \Rightarrow a + d = b + c$ in the set **N** is an equivalence relation. *(A.I.C.B.S.E. 2010)*

21. (a) Let R be a relation on the set A of ordered pairs of positive integers defined by :
(x, y) R (u, v) if and only if xv = yu.
Show that R is an equivalence relation. *(N.C.E.R.T.)*

(b) Let **N** be the set of natural numbers and R be the relation in **N** × **N** defined by :
(a, b) R (c, d) iff ad = bc. for all (a, b), (c, d) ∈ **N** × **N.**
Show that R is an equivalence relation. *(Kashmir B. 2011)*

22. Show that the relation R, defined by the set A of all triangles as :
$R = \{(T_1, T_2) : T_1$ is similar to $T_2\}$
is an equivalence relation. *(H.P.B. 2017; P.B. 2016; Jammu B. 2013)*
Consider three right angle triangles T_1 with sides 3, 4, 5; T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10.
Which triangles among T_1, T_2 and T_3 are related ? *(N.C.E.R.T.)*

23. Let L be the set of all lines in XY–plane and R be the relation in L defined as :
$R = \{(L_1, L_2) : L_1$ is parallel to $L_2\}$.
Show that R is an equivalence relation.
Find the set of all lines related to the line $y = 2x + 4$.
(N.C.E.R.T. ; Assam B. 2018; H.P.B. 2017; Tripura B. 2016; Uttarakhand B. 2015)

24. Let L be the set of all lines in the XY-plane and R be the relation in L defined by :
$R = \{(l_i, l_j) = l_i$ is parallel to $l_j, \forall i, j\}$.
Show that R is an equivalace relation. Find the set of all lines related to the line $y = 7x + 5$. *(Assam B. 2015)*

25. Prove that the relation in the set :
$A = \{x : x \in \mathbf{W}, 0 \le x \le 12\}$ given by :
R = {(a + b) : (a – b) is a multiple of 4} is an equivalance relation.
Also, find the set of all elements related to 2.

Answers

1. 9.
2. (i) {–1, –2} (ii) {1, 2, 3}.
3. (i) – (ii) True (iii) n.
4. (a) Let A = {1, 2, 3}.
(i) R = {(2, 3), (3, 2)}
(ii) R = {(1, 3), (3, 2), (1, 2)}
(iii) R = {(1, 1), (2, 2), (3, 3), (2, 3), (3, 2), (1, 2), (2, 1)}
(iv) R = {(1, 1), (2, 2), (3, 3), (1, 2)}.
(b) **For Ex.** 2 is a multiple of 4 but 4 is not a multiple of 2.
5. (a) {(2, 8), (2, 10), (2, 12), (4, 8), (4, 10), (4, 12), (6, 8), (6, 10), (6, 12), (8, 10), (8, 12), (10, 12)}
(b) {(4, 1), (4, 2), (5, 1), (5, 2), (5, 4)}.
7. (i) – (ii) Neither reflexive nor symmetric nor transitive
(iii) Reflexive, symmetric and transitive
(iv) (a) – (b) Reflexive, symmetric and transitive
(c) – (e) Neither reflexive nor symmetric nor transitive.
10. Neither reflexive nor symmetric nor transitive.
11. (a) Neither reflexive nor symmetric nor transitive.
13. {(1, 4), (1, 6), (2, 4), (2, 6), (3, 4), (3, 6), (4, 6), (5, 6)};
Inverse relation corresponding to *'greater than'* from B to A is :

{(4, 1), (4, 2), (4, 3), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5)}.

17. (*a*) {1, 5, 9}

(*b*) {(1, 5), (1, 7), (1, 9), (2, 4), (2, 10), (4, 2), (4, 10), (5, 1), (5, 7) (5, 8), (7, 1), (7, 5), (7, 9), (9, 1), (9, 5), (9, 7), (10, 2), (10, 4)}; (*i*) A (*ii*) A.

22. Triangles T_1 and T_3

23. Set of all lines $y = 2x + c, c \in \mathbf{R}$.

24. Set of all lines $y = 7x + c, c \in \mathbf{R}$.

25. {2, 6, 10}.

Hints to Selected Questions

7. (*i*) $R = \{(x, y) : y = 3x\}$
$= \{(1, 3), (2, 6), (3, 9), (4, 12)\}$.

8. (*a*) Since $a \le a \ \forall \ a \in \mathbf{R}$, $(a, a) \in \mathbf{R}$.

Thus R is reflexive.

Also R is transitive.

R is not symmetric. [$\because$ *(3, 5) ∈ R but (5, 3) ∉ R*]

10. R is not reflexive. [$\because$ $a \le a^3$ *is not true*]

Also R is neither symmetric nor transitive.

11. (*a*) R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}.

16. Since $|1 - 3| = 2$, $|3 - 5| = 2$ and $|1 - 5| = 4$ are all even,

$\therefore$ all elements of {1, 3, 5} are related to each other.

And no element of {1, 3, 5} is related to any element of {2, 4}. [$\because$ $|1 - 2| = 1$; *etc.*]

21. Proceed as in **Ex. 11.**

22. Since $\frac{6}{3} = \frac{8}{4} = \frac{10}{5}$,

$\therefore$ triangles T_1 and T_3 are similar

$\Rightarrow$ $(T_1, T_3) \in R$

$\Rightarrow$ T_1 is related to T_3.

23. Reqd. set = $\{l : l$ is a line parallel to $y = 2x + 4\}$
$= \{l = l$ is a line $y = 2x + c, c \in \mathbf{R}\}$.

SUB CHAPTER 1.2 Functions

1.6. INTRODUCTION

The word *'function'* was introduced by **Leibnitz.** The meaning of the word function has since undergone many stages of generalisation.

Roughly speaking, a function *is a law of correspondence* under which to each element of one set there corresponds one and only one element of another set.

We first give an intuitive idea of the function.

Let A = {*a, b, c*}, B = {3, 11, 17, 21, 24}. If we associate *a* with 11, we write $a \to 11$.

Let us consider the following association between the elements of A and B :

$$a \to 11, b \to 21, c \to 3.$$

In this association, each element of A has been associated with some element of B. Moreover, no element of A has been associated with more than one element of B. Such an association between elements of A and B is called a *function* from the set A to the set B. Functions are usually denoted by *f, g, h* ; etc.

We write $f : A \to B$ or $A \xrightarrow{f} B$ if '*f*' is a function from the set A to the set B.

The set A is said to be the **domain** of the function and the set B is said to be the **domain** of the function.

The element *y* of the set B, which is associated with an element *x* of the set A, is usually denoted by $f(x)$ read as "*f* of *x*".

So we can write the function as $y = f(x)$.

The element $f(x)$ is said to be the **value** of the function '*f*' at *x* or the **image** of *x* under '*f*'. The set of all values of '*f*' is called the **range** (or **image set**) of the function '*f*'.

The element *x* to which is associated $f(x)$ itself is called the **pre–image** of $f(x)$.

A = {*a, b, c*} is the **domain,** B = {3, 11, 17, 21, 24} is the **colomain** and the set {11, 21, 3} is the image set or **range** of the function '*f*'.

Definition

Let X and Y be two non–empty sets. A function 'f' defined from X to Y, is a rule (or a collection of rules) which associates (or associate) to each element x in X a unique element y in Y.

Symbolically : We write it as $f : X \to Y$, and is read as '*function*' defined from the set X to the set Y.

Notes. The word '*function*' is also termed as '*mapping*' or '*correspondence*' or '*transformation.*'

(I) The unique element y of Y is called the **value** of 'f' at x (the image of x under 'f'). It is written as $f(x)$. Thus $y = f(x)$.

(II) The element x of X is called **pre–image (or inverse image)** of y.

(III) The set X is called the **domain** of 'f'.

(IV) The set Y is called the **co–domain.**

(V) The set consisting of all images of the elements of X under 'f' is called the **range** of 'f'. This is denoted by $f(x)$.

Thus range of 'f' = $\{f(x) \ \forall \ x \in X\}$.

This is a subset of Y, which may or may not be equal to Y.

For Examples : (1) *Let* $X = \{1, 2, 3, 4\}$

and $Y = \{2, 4, 6, 8, 10\}$.

Here the rule *i.e.* 'f', which associates to each element x in X, the element $2x$ in Y is a function from X to Y. The rule written as $f(x) = 2x$ is depicted by the adjoining diagram.

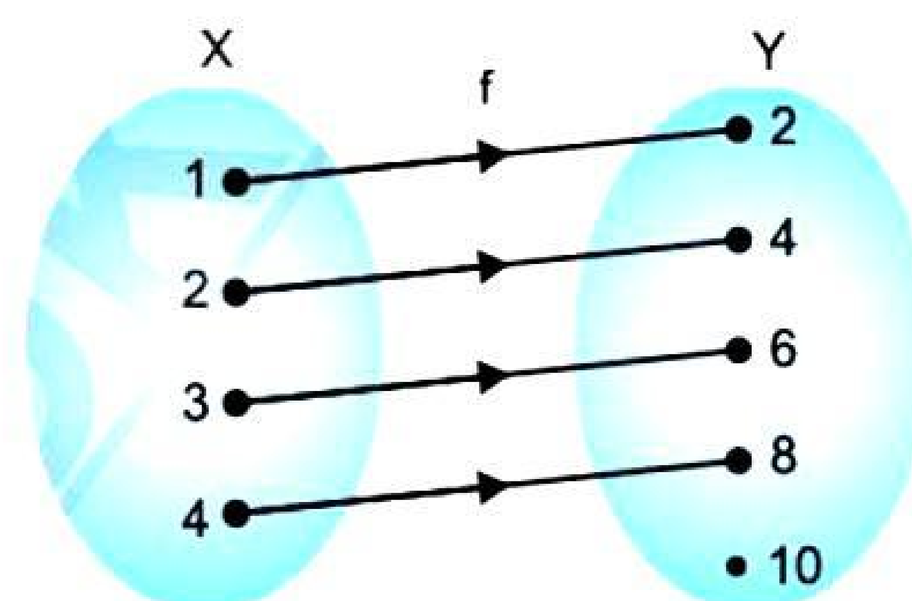


Fig.

(2) *Let* $X = \{a, b, c, d, e\}$ *and* $Y = \{p, q, r, s\}$.

Here the rule, which is depicted in the following diagram is not a function from X to Y because the elements a and c in X have been associated with two elements p and q each of Y. Also, the element b has no image under 'f' in Y.

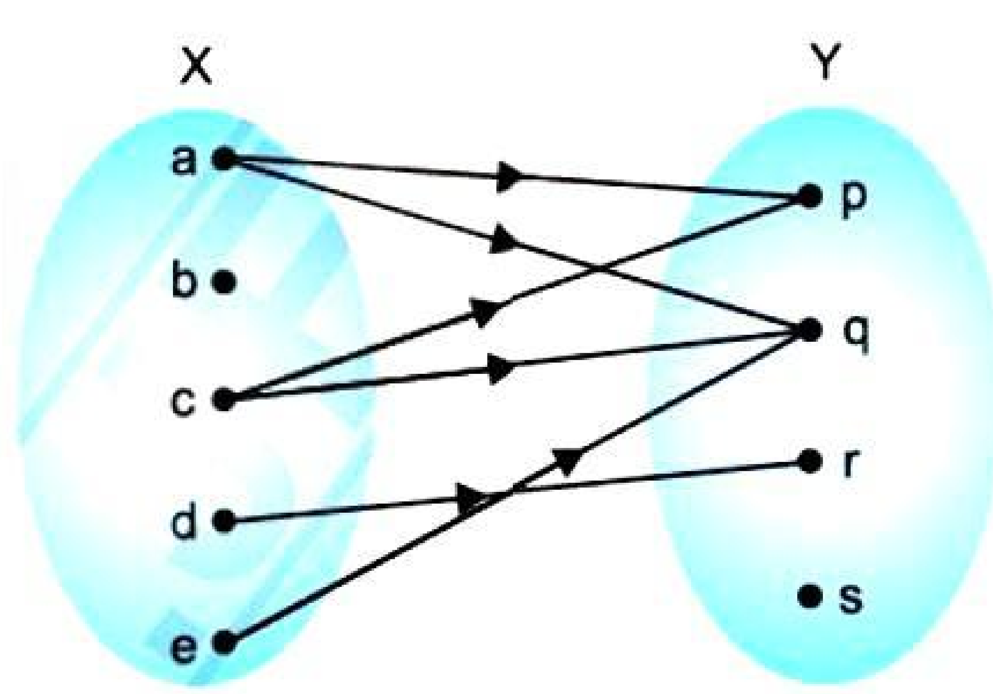


Fig.

KEY POINT

1. To each element x in X, there exists a unique element y in Y such that $y = f(x)$.
2. Different elements of X may be associated with the same element of Y.
3. There may exist some elements of Y, which are not associated with any element of X.

In Class XI, we have discussed some special functions like identity function, constant function, polynomial function, rational function, modulus function, signum function; etc. along with their graphs. We also discussed addition, subtraction, multiplication and division of two functions. Now we extend our study about function by studying different types of functions; etc.

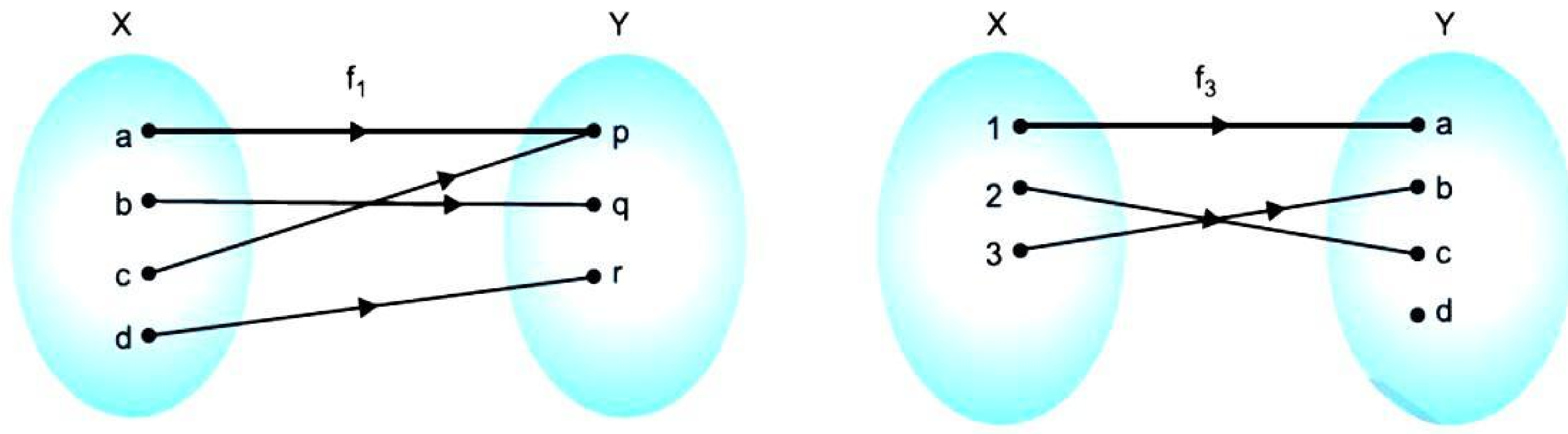
1.7. TYPES OF FUNCTION

(*a*) **Onto Function.** *A function 'f' from X to Y is said to be onto (or* **surjective**) *iff each element in Y is the image of at least one element in X.* Here the range of 'f' = Y.

For Example : *Let* $X = \{a, b, c, d\}$ *and* $Y = \{p, q, r\}$.

The function $f : X \to Y$ defined by $f(a) = f(c) = p, f(b) = q, f(d) = r$ is a mapping from X onto Y.

The following diagram depicts the given function :



onto function

into function

Fig.

Note. If the function is not onto, it is called **into.**

(*b*) **One–one Function.** *A function 'f' from X to Y is said to be one–one* (*or* **injective**) *iff different elements of X have different images in Y* [*Also see Art. 1.8*]

i.e. iff $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in X$ or iff $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ for all $x_1, x_2 \in X$.

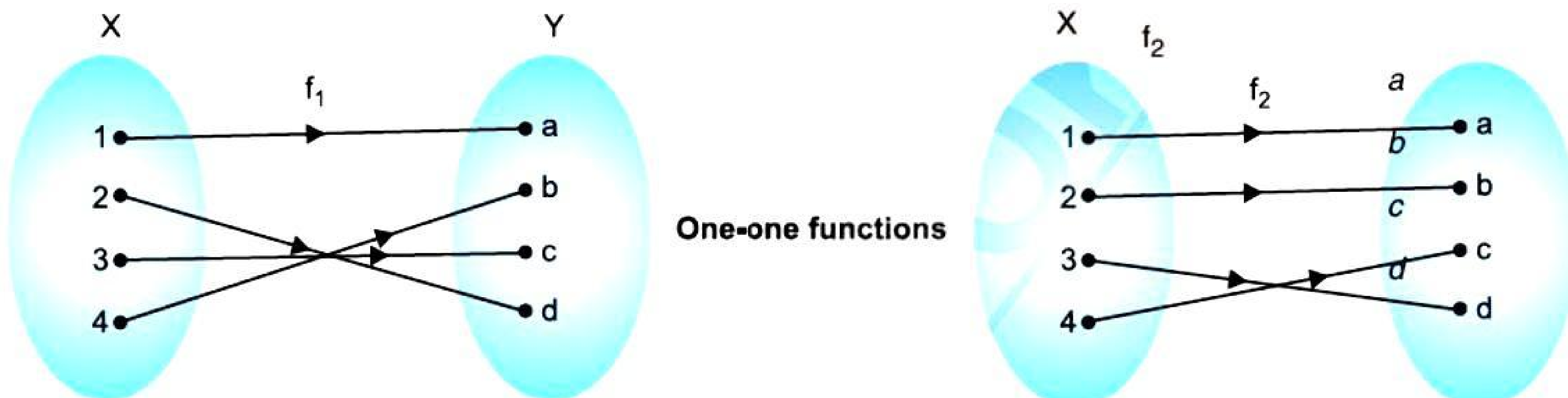


Fig.

If the function is not one–one, then '*f*' is called **many–one.**

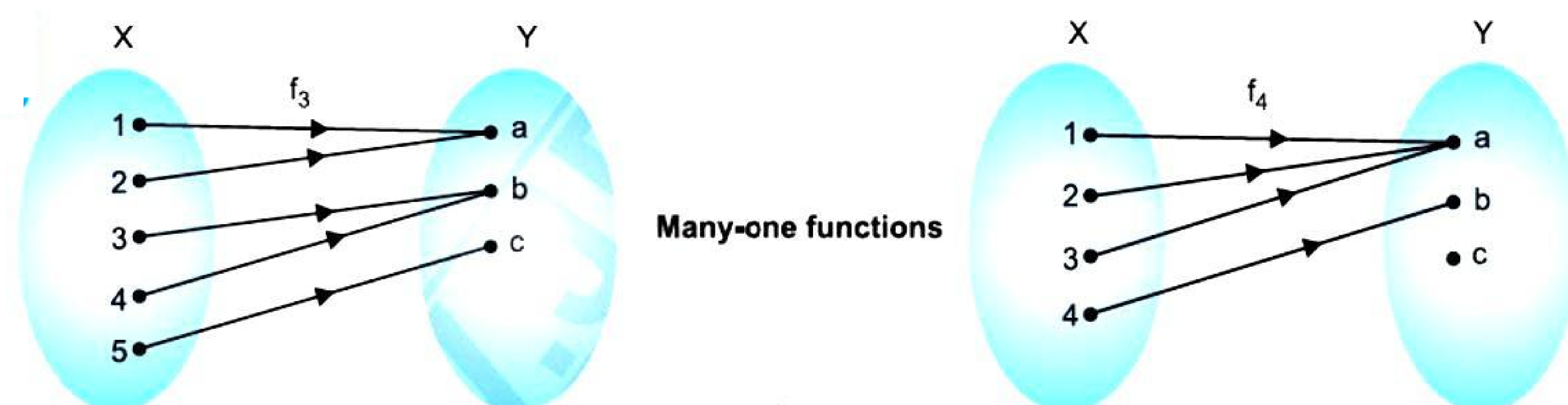


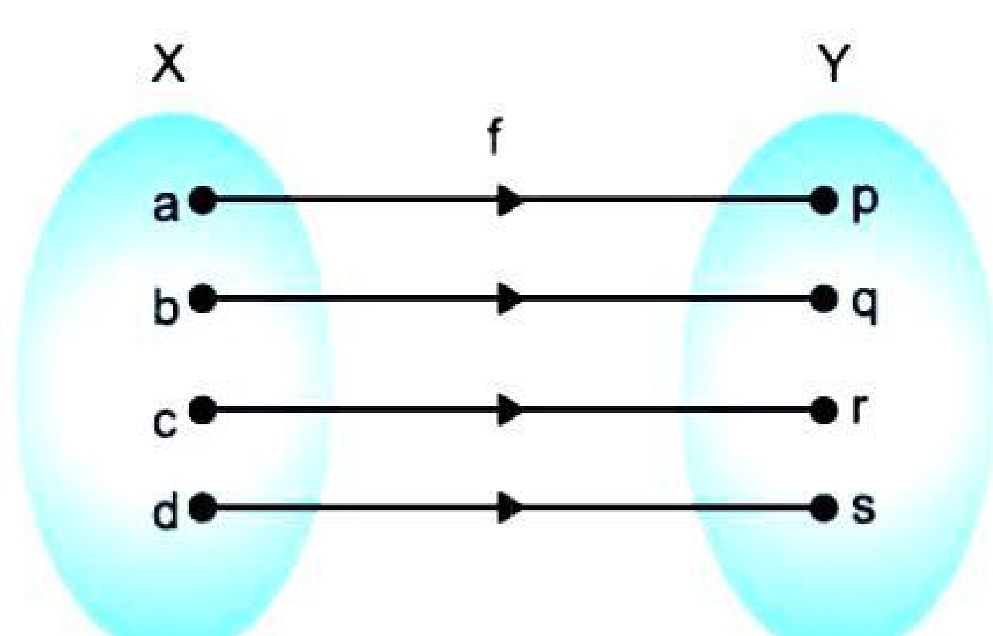
Fig.

(*c*) **One–one Correspondence.** *A function 'f' from X to Y is said to be one–one correspondence* (*or* **bijective**) *iff 'f' is both one–one* (**injective**) *and onto* (**surjective**).

For Examples : (1) *Let* $X = \{a, b, c, d\}$ *and* $Y = \{p, q, r, s\}$.

The function $f : X \to Y$ defined by $f(a) = p, f(b) = q, f(c) = r, f(d) = s$ is one–one onto.

The following diagram depicts the given function :



one-one and onto function

Fig.

(2) *Let* $X = \{a, b, c, d\}$ *and* $Y = \{p, q, r, s, t\}$.

The function $f : X \rightarrow Y$ defined by $f(a) = p, f(b) = q, f(c) = r, f(d) = t$

is one–one but not onto because s has no pre–image.

The following diagram depicts the given function :

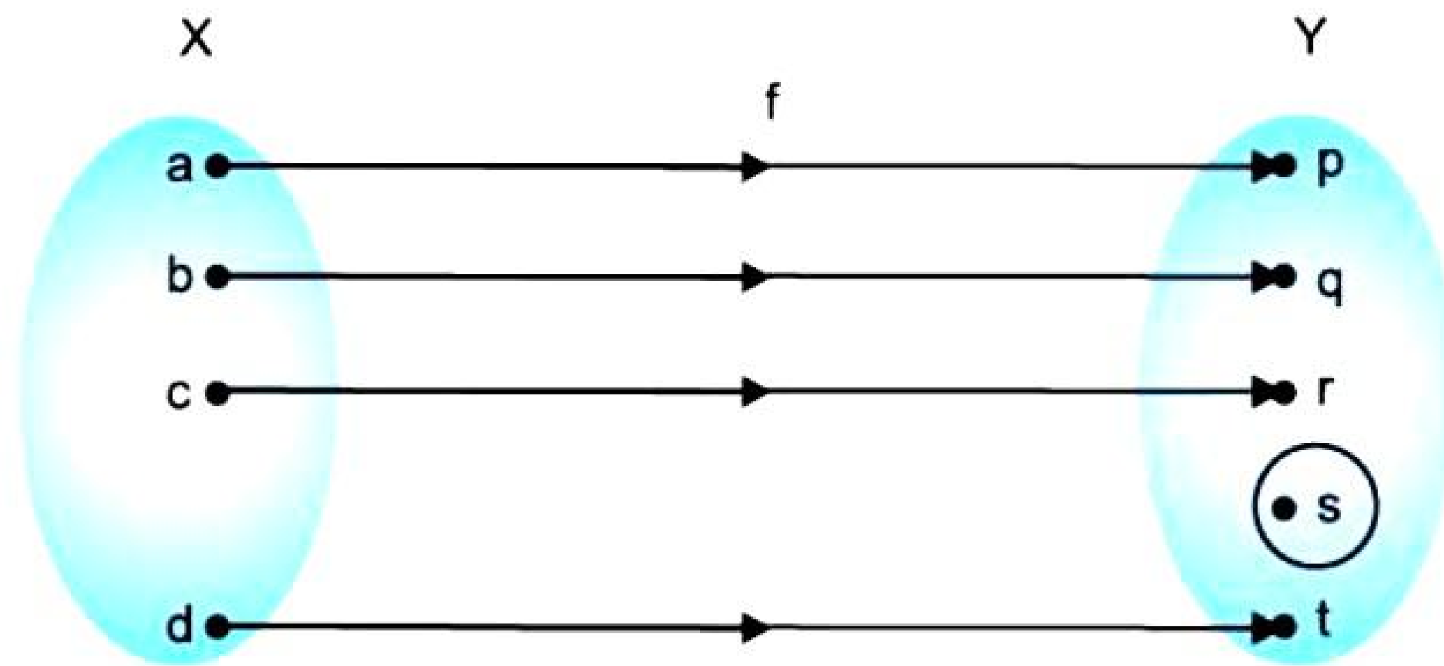


Fig.

(3) *Let* $X = \{a, b, c, d\}$ *and* $Y = \{p, q, r, s\}$.

The function $f : X \rightarrow Y$ defined by $f(a) = p, f(b) = r, f(c) = p, f(d) = q$ is neither one–one nor onto because the distinct elements a and c have the same image p and the element s has no pre–image.

The following diagram depicts the given function :

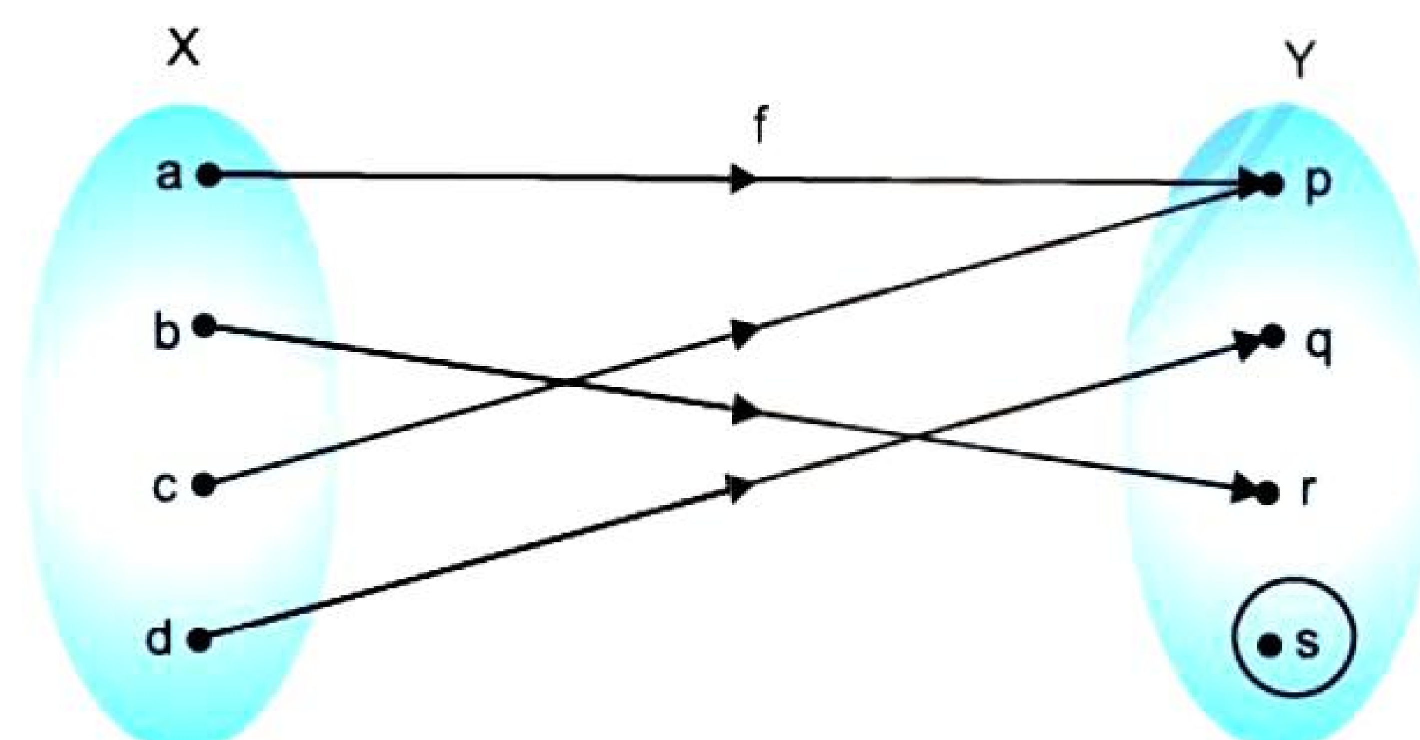


Fig.

(4) *Let* $X = \{a, b, c, d\}$ *and* $Y = \{p, q\}$.

The function $f : X \rightarrow Y$ defined by $f(a) = q, f(b) = p, f(c) = q, f(d) = q$ is many–one onto.

The following diagram depicts the given function :

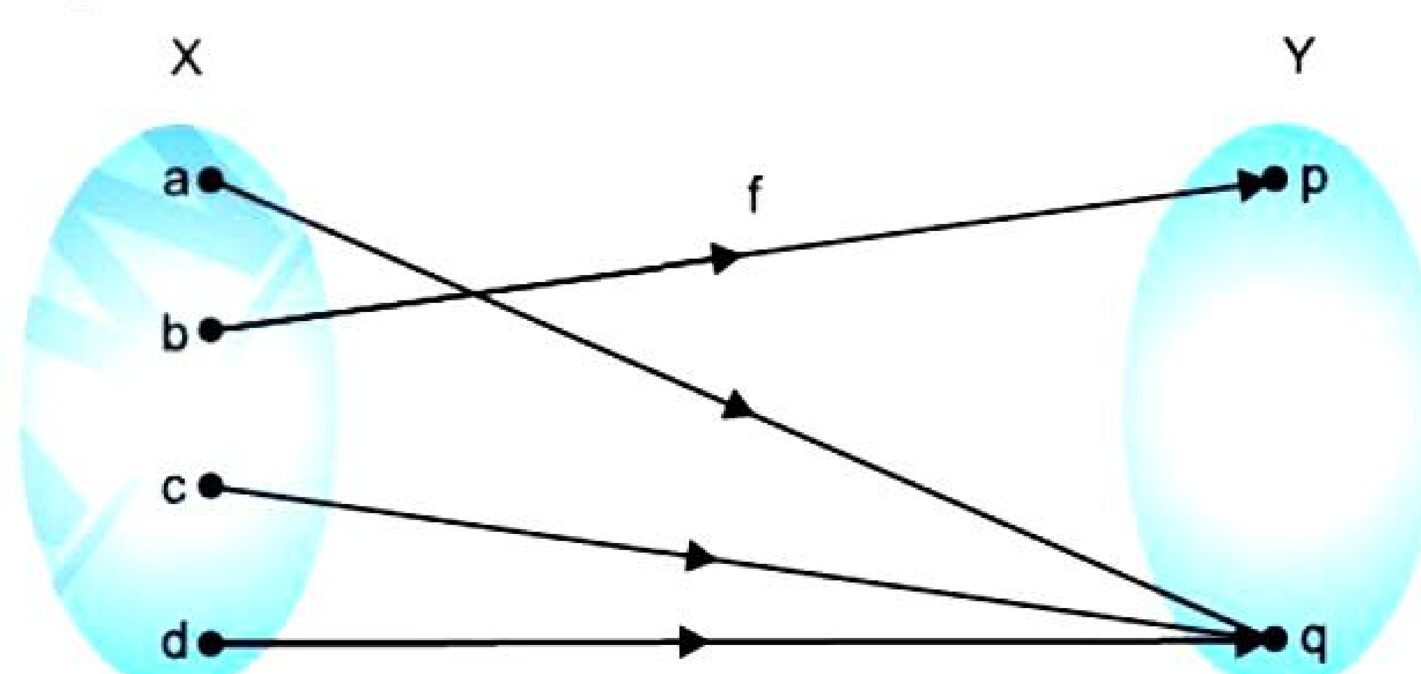
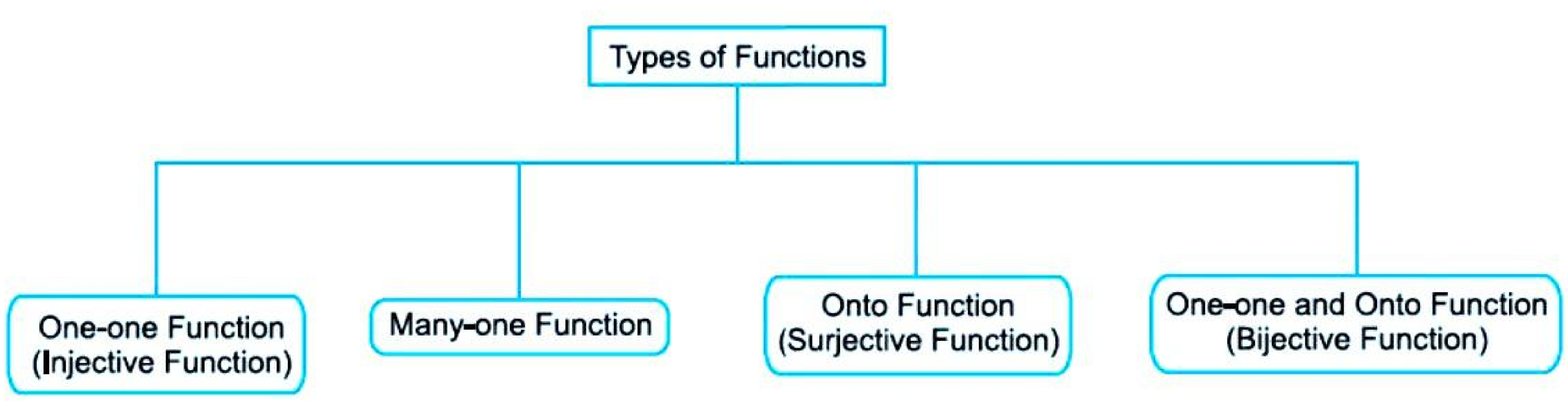


Fig.



1.8. ONE-ONE FUNCTION

Definition

A function 'f' from X to Y is said to be one–one iff distinct elements of X have distinct images in Y i.e. iff $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ *for all* $x_1, x_2 \in X$.
Or equivalently, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for $x_1, x_2 \in$ X.

Graphically. A function *'f'* is one–one if the line parallel to *x*–axis does not meet the graph of $y = f(x)$ in more than one point.

GUIDE-LINES FOR SHOWING ONE–ONE

Step (i) : *Obtain the domain of 'f'.*

Step (ii) : *Solve the equation for x in terms of y.*

Step (iii) : *Make use of the domain of 'f'.*

Step (iv) : *If for each value of y, there corresponds a unique value of x, then the function is one–one.*

Frequently Asked Questions

FAQs

Example 1. Which one of the following graphs represents the function of x ? Why ?

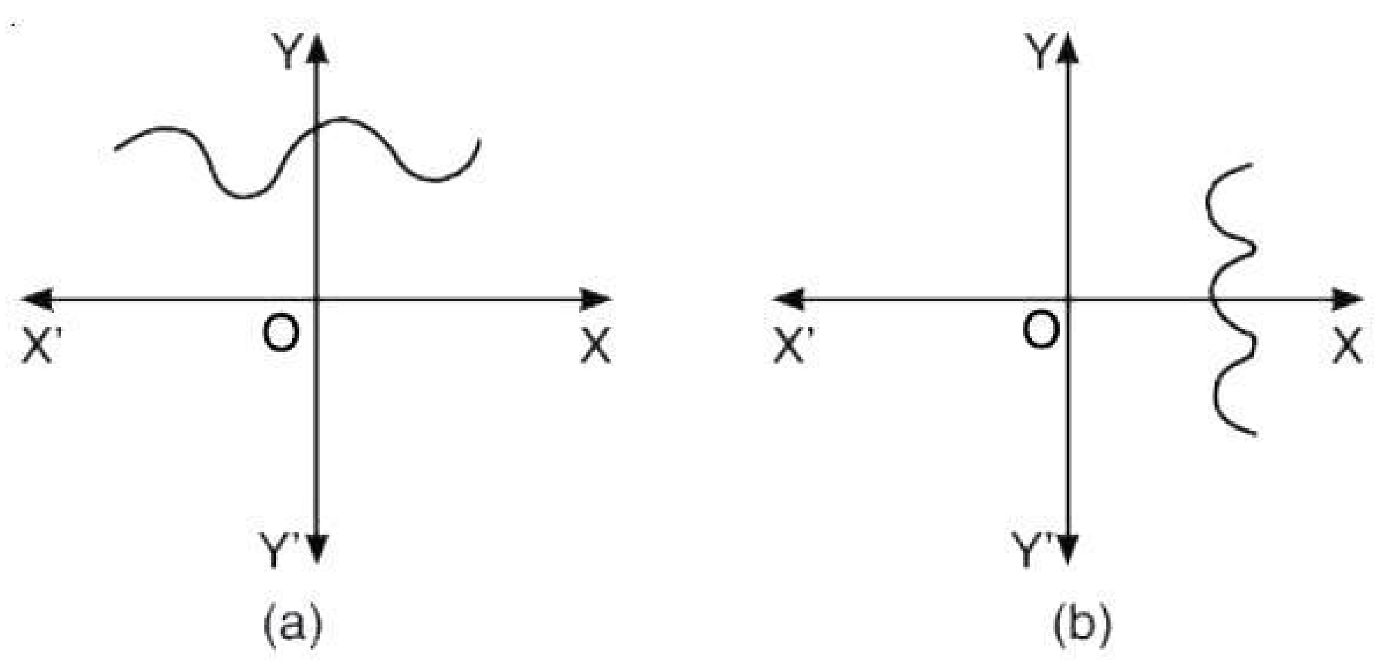


Fig.

Solution. (*a*). Because in (*b*) at some *x* there are many values of *y*.

Example 2. Show that a one-one function :

f : {1, 2, 3} → {1, 2, 3} must be onto. (*N.C.E.R.T.*)

Solution. Since *'f'* is one–one, [*Given*]

∴ under *'f'*, all the three elements of {1, 2, 3} should correspond to three different elements of the co-domain {1, 2, 3}.

Hence, *'f'* is onto.

Example 3. Let A be the set of all 50 students of class XII in a school. Let f : A → N be the function defined by :

f (x) = Roll number of the student x.

Show that 'f' is one-one but not onto. (*N.C.E.R.T.*)

Solution. Since no two students of class XII have the same roll number,

∴ *'f'* is one-one.

Without any loss of generality, let the roll numbers of students be from 1 to 50.

Thus 51 ∈ **N** but is not the roll number of any student of the class.

∴ 51 is not the image of any element of A under *'f'*

⇒ *'f'* is not onto.

Hence, *'f'* is one-one but not onto.

Example 4. Prove that the function f : R → R given by :

f(x) = 2x is one–one and onto. (*N.C.E.R.T.*)

Solution.

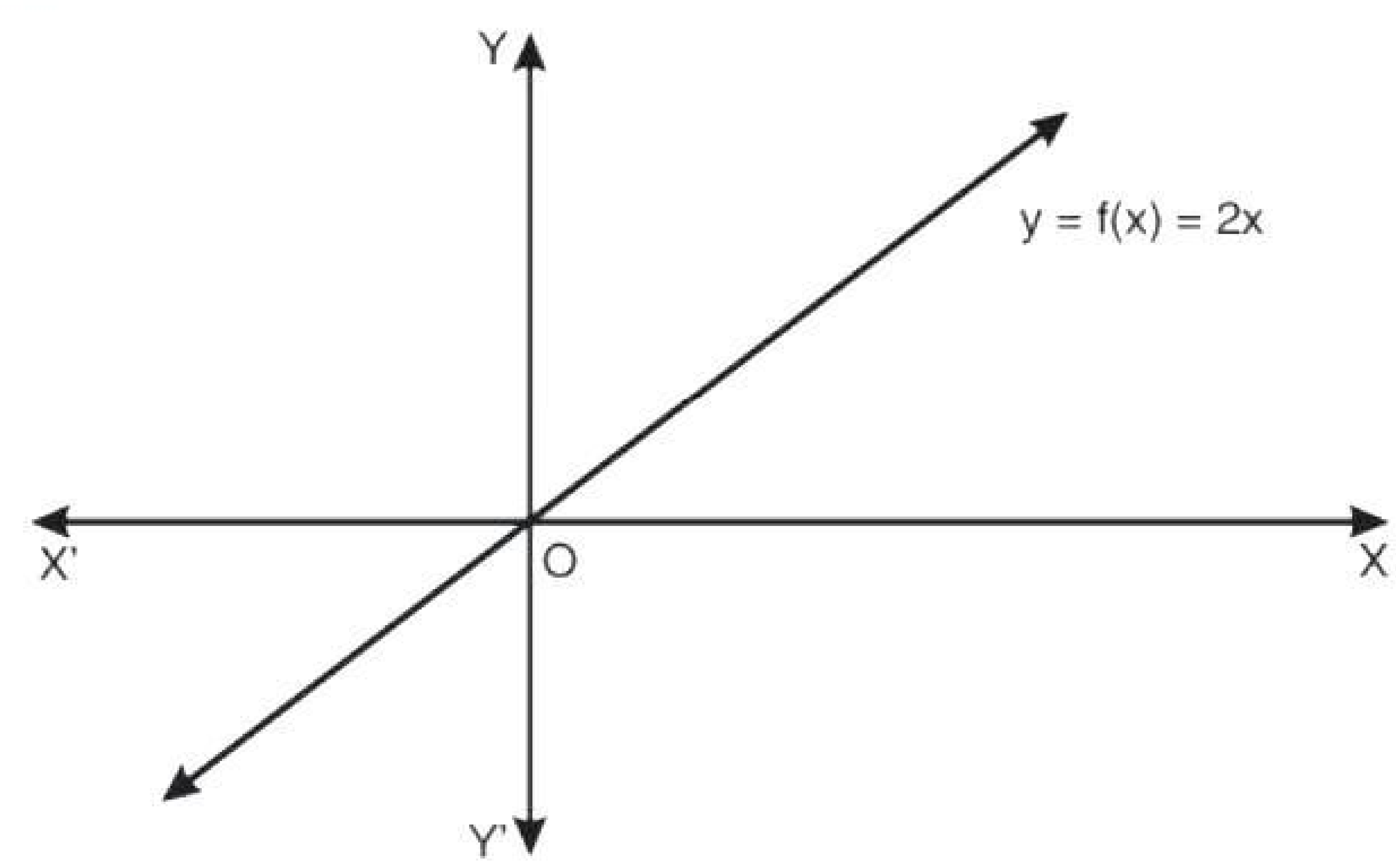


Fig.

Let $x_1, x_2 \in \mathbf{R}$.

Now $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2$
$\Rightarrow x_1 = x_2$
⇒ *'f'* is one–one.

Let $y \in \mathbf{R}$. Let $y = f(x_0)$.

Then $2x_0 = y \Rightarrow x_0 = \dfrac{y}{2}$.

Now $y \in \mathbf{R} \Rightarrow \dfrac{y}{2} \in \mathbf{R} \Rightarrow x_0 \in \mathbf{R}$.

$$f(x_0) = 2x_0 = 2\left(\frac{y}{2}\right) = y.$$

∴ For each $y \in \mathbf{R}$, there exists $x_0 \in \mathbf{R}$ such that $f(x_0) = y$.

∴ *'f'* is onto.

Hence, *'f'* is one–one and onto.

Example 5. Show that the function :

f : N → N

given by f(1) = f(2) = 1 and f (x) = x – 1, for every x > 2 is onto but not one- one. (*N.C.E.R.T.; Magalaya B. 2017*)

Solution. Since $f(1) = f(2) = 1$,

∴ $f(1) = f(2)$, where $1 \neq 2$.

∴ 'f' is not one–one.

Let $y \in \mathbf{N}$, $y \neq 1$, we can choose x as $y + 1$ such that

$$f(x) = x - 1 = y + 1 - 1 = y.$$

Also $1 \in \mathbf{N}$, $f(1) = 1$.

Thus 'f' is onto.

Hence, 'f' is onto but not one–one.

Example 6. Show that the function f : R → R defined by f(x) = x² is neither one–one nor onto. (N.C.E.R.T.)

Solution.

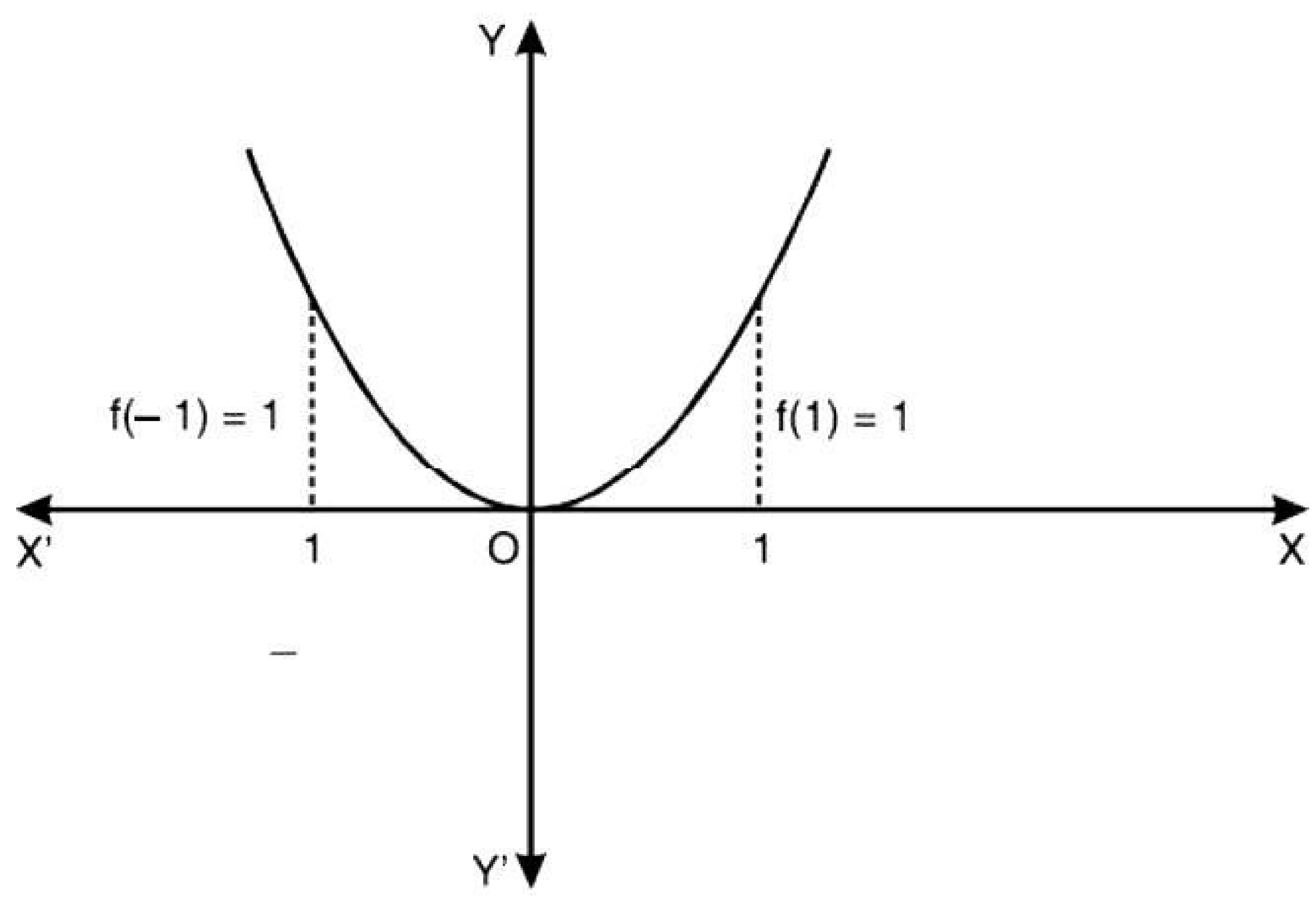


Fig.

Here, $f(-1) = f(1) = 1$ but $-1 \neq 1$.

∴ 'f' is not one–one.

Also -2 is in the co-domain **R** but is not the image of any element 'x' in the domain **R.**

[∵ *-2 is not the square of any real number*]

∴ 'f' is not onto.

Hence, 'f' is neither one–one nor onto.

Example 7. Show that f : N → N given by :

$$f(x) = \begin{cases} x+1, \text{ if } x \text{ is odd} \\ x-1, \text{ if } x \text{ is even} \end{cases}$$

is both one–one and onto.

(N.C.E.R.T.; H.B. 2018; A.I.C.B.S.E. 2012; Jammu B. 2012)

Solution. One–One.

Here we discuss the following possible cases :

(I) *When x_1 is odd and x_2 is even.*

Here $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 - 1$

$\Rightarrow x_2 - x_1 = 2$, which is impossible.

(II) *When x_1 is even and x_2 is odd.*

Here $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 + 1$

$\Rightarrow x_1 - x_2 = 2$, which is impossible.

(III) *When x_1 and x_2 are both odd.*

Here $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1$

$\Rightarrow x_1 = x_2$.

∴ 'f' is one–one.

(IV) *When x_1 and x_2 are both even.*

Here $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1$

$\Rightarrow x_1 = x_2$.

∴ 'f' is one–one.

Onto. Let 'x' be an arbitrary natural number.

When x is an odd natural number, then there exists an even natural number $(x + 1)$ such that :

$$f(x + 1) = (x + 1) - 1 = x.$$

When x is an even natural number, then there exists an odd natural number $(x - 1)$ such that :

$$f(x - 1) = (x - 1) + 1 = x.$$

∴ Each $x \in \mathbf{N}$ has its pre-image in **N**.

Thus 'f' is onto.

Hence, 'f' is both one–one and onto.

EXERCISE 1 (b)

Fast Track Answer Type Questions

FTATQ

1. If $f(x) = \log\left(\frac{1-x}{1+x}\right)$, $-1 < x < 1$, then show that $f(-x) = -f(x)$. (H.B. 2012)

2. What is the range of the function $f(x) = \frac{|x-1|}{x-1}$? (C.B.S.E. 2010)

3. (*a*) Show that an onto function $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is always one–one. (N.C.E.R.T.)

   (*b*) Let A = {1, 2, 3} and B = {5, 6, 7, 8} be two sets. The mapping f : A → B is defined as :
   $f = \{(1, 5), (2, 7), (3, 6)\}$. (Tripura B. 2016)
   Examine if f is one-one.

   (*c*) $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 2x$ is one-one function. (True/False) (Kashmir B. 2016)

4. Show that the function $f : \mathbf{N} \to \mathbf{N}$ given by :
   (*i*) $f(x) = 2x$ (N.C.E.R.T.)
   (*ii*) $f(x) = 3x$ (P.B. 2010)
   is one-one but not onto.

5. (*a*) If f is a function from **R** → **R** such that $f(x) = x^2$ $\forall x \in \mathbf{R}$, then show that 'f' is not one-one. (Meghalya B. 2015)

   (*b*) Give example of many-one function. (Kashmir B. 2016)

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. Show that 'f' is one–one. (N.C.E.R.T.)

7. Show that $f(x) = 3x + 5$, for all $x \in \mathbf{Q}$, is one-one. (H.B. 2018)

Short Answer Type Questions

SATQ

8. (*a*) Show that the function $f : \mathbf{R} \to \mathbf{R}$ given by : $f(x) = ax + b$, where $a, b \in \mathbf{R}$, $a \neq 0$ is a bijection. **(C.B.S.E. 2010 C)**

(*b*) A function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 5x + 6$, prove that f is one-one and onto. **(Magalaya B. 2016)**

9. State whether the following function is one–one onto or bijective :

$f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 + x^2$.

(N.C.E.R.T.; Karnataka B. 2017)

10. (*a*) Show that $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$ is bijective. **(N.C.E.R.T.; P.B. 2016)**

(*b*) A function $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = 4x^3 + 5$, $x \in \mathbf{R}$.

Examine if f is one-one and onto. **(Magalaya B. 2016)**

11. Show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by :

(*i*) $f(x) = \dfrac{4x-3}{5}$, $x \in \mathbf{R}$

(*ii*) $f(x) = \dfrac{3x-1}{2}$, $x \in \mathbf{R}$

is one-one and onto function. **(C.B.S.E. 2010 C)**

12. Consider the function $f(x) = \dfrac{x-3}{x+1}$ defined from $\mathbf{R} - \{-1\}$ to $\mathbf{R} - \{1\}$.

Prove that f is both one-one and onto.

(Nagaland B. 2018)

13. Show that the function $f : \mathbf{R} \to \{x \in \mathbf{R} : -1 < x < 1\}$ defined by $f(x) = \dfrac{x}{1+|x|}$, $x \in \mathbf{R}$ is one-one and onto function. **HOTS** **(N.C.E.R.T. ; Kashmir B. 2012)**

14. Check the injectivity and surjectivity of the following functions :

(*a*) (*i*) $f : \mathbf{N} \to \mathbf{N}$, given by $f(x) = x^2$

(*ii*) $f : \mathbf{N} \to \mathbf{N}$, given by $f(x) = x^3$ **(P.B. 2016)**

(*b*) (*i*) $f : \mathbf{Z} \to \mathbf{Z}$, given by $f(x) = x^2$

(*ii*) $f : \mathbf{Z} \to \mathbf{Z}$, given by $f(x) = x^3$ **(P.B. 2016)**

(*c*) $f : \mathbf{R} \to \mathbf{R}$, given by $f(x) = x^2$ **(N.C.E.R.T.)**

15. Let A and B be sets. Show that :

$f : A \times B \to B \times A$ such that $f(a, b) = (b, a)$

is a bijective function. **(N.C.E.R.T.; Kashmir B. 2015)**

16. Prove that Modulus Function $f : \mathbf{R} \to \mathbf{R}$, given by :

$f(x) = |x|$

is neither one–one nor onto, where $|x|$ is x, if x is positive and $|x|$ is $-x$, if x is negative.

(N.C.E.R.T. ; Kashmir B. 2011)

17. Show that the Signum Function $f : \mathbf{R} \to \mathbf{R}$, given by :

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is neither one–one nor onto.

(N.C.E.R.T. ; H.B. 2018; Jammu B. 2011)

18. Let A = $\{a_1, a_2, a_3, a_4, a_5\}$ and B = $\{b_1, b_2, b_3, b_4\}$, when a_i's and b_i's are school going students. Define a relation from a set A to set B by x R y iff y is a true friend of x.

If R = $\{(a_1, b_1), (a_2, b_1), (a_3, b_3), (a_4, b_2), (a_5, b_2)\}$.

Is R a bijective function ?

Answers

2. $\{-1, 1\}$.

3. (*b*) Yes (*c*) True.

5. (*b*) $f(x) = x^2 \ \forall \ x \in \mathbf{R}$.

9. Neither one-one nor onto.

10. (*b*) f is one-one onto.

14. (*a*) (*i*) – (*ii*) Injective but nor surjective

(*b*) (*i*) Neither injective nor surjective

(*ii*) Injective but not surjective

(*c*) Neither injective nor surjective.

18. Not bijective as it is neither one-one nor onto.

Hints to Selected Questions

1. $f(-x) = \log\left(\dfrac{1+x}{1-x}\right) = -\log\left(\dfrac{1-x}{1+x}\right) = -f(x)$.

8. (*a*) $f(x_1) = f(x_2) \Rightarrow ax_1 + b = ax_2 + b \Rightarrow x_1 = x_2$

$\Rightarrow$ 'f' is one-one.

Let $y \in \mathbf{R}$. Let $y = f(x_0)$.

Then $ax_0 + b = y \Rightarrow x_0 = \dfrac{y-b}{a} \Rightarrow x_0 \in \mathbf{R}$.

$$f(x_0) = ax_0 + b = a\left(\frac{y-b}{a}\right) + b = y \in \mathbf{R}.$$

$\therefore$ 'f' is onto.

Combining, 'f' is a bijaction.

9. 'f' is not one-one $[\because f(1) = f(-1)]$

Also 'f' is not onto. $[\because R_f = [1, \infty) \neq \mathbf{R}]$

10. (a) f is one-one onto.

13. $f(x) = \dfrac{x}{1+|x|} = \begin{cases} \dfrac{x}{1+x}, & \text{if } x \geq 0 \\ \dfrac{x}{1-x}, & \text{if } x < 0. \end{cases}$

15. Let $(a_1, b_1), (a_2, b_2) \in A \times B$ such that

$f(a_1, b_1) = f(a_2, b_2)$

$\Rightarrow (b_1, a_1) = (b_2, a_2) \Rightarrow b_1 = b_2$ and $a_1 = a_2$

$\Rightarrow (a_2, b_2) = (a_1, b_1) \Rightarrow$ 'f' is one-one.

And corresponding to each ordered pair $(y, x) \in B \times A$ there exists $(x, y) \in (A \times B)$ such that :

$f(x, y) = (y, x)$

$\Rightarrow$ 'f' is onto.

16. Since $f(x) = f(-x)$, $\therefore$ 'f' is not one-one.

Now $R_f = [0, \infty) \neq \mathbf{R} \Rightarrow$ 'f' is not onto.

17. 'f' is not one-one.

$R_f = \{-1, 0, 1\} \neq \mathbf{R} \Rightarrow$ 'f' is not onto.

1.9. COMPOSITION OF FUNCTIONS

Definition

Let A, B and C be three sets. Let $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of f and g, denoted by gof, is defined as the function $gof : A \to C$ given by :

$$gof(x) = g(f(x)) \ \forall \ x \in A.$$

This is also known as **Function of a function** or **Resultant of functions.**

gof is represented by diagram as below :

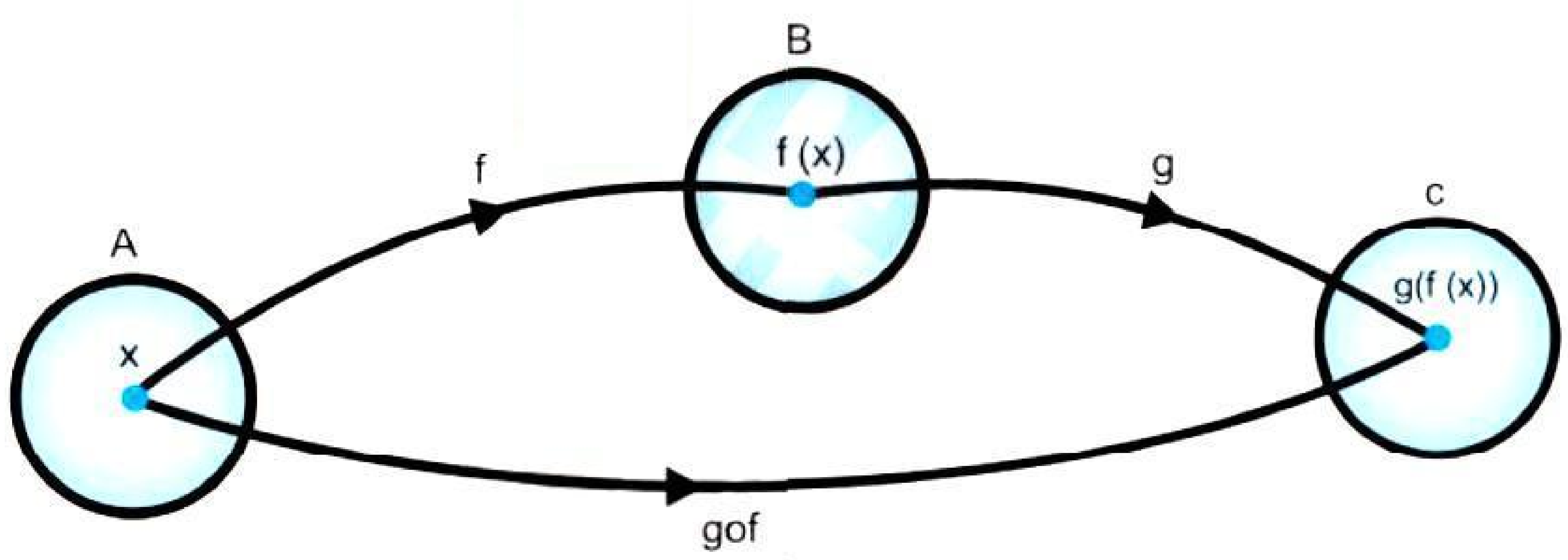


Fig.

1.9.1 THEOREMS

Theorem I. *If $f : A \to B$ and $g : B \to C$ are one–one, then $gof : A \to C$ is also one–one.* **(N.C.E.R.T. ; Kashmir B. 2011)**

Proof. $gof(x_1) = gof(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$ [$\because$ *g is one–one*]

$\Rightarrow x_1 = x_2.$ [$\because$ *f is one–one*]

Hence, *gof* is one–one.

Theorem II. *If $f : A \to B$ and $g : B \to C$ are onto, then $gof : A \to C$ is also onto.* **(N.C.E.R.T.)**

Proof. Let $z \in C$ be an arbitrary element.

Then there exists a pre-image y of z under g such that $g(y) = z$. [$\because$ *g is onto*]

And for $y \in B$, there exists an element x in A such that $f(x) = y$. [$\because$ *f is onto*]

Thus $gof(x) = g(f(x)) = g(y) = z.$

Hence, *gof* is onto.

Theorem III. *Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Then $ho(gof) = (hog)of$.*

Proof. $ho(gof)(x) = h((gof)(x)) = h(g(f(x)) \ \forall \ x \in A$

$(hog)of(x) = (hog)(f(x)) = h(g(f(x)) \ \forall \ x \in A.$

Hence, $ho(gof) = (hog)of.$

Frequently Asked Questions

FAQs

Example 1. If f : R → R is defined by f (x) = 3x + 2, find f (f (x)). *(C.B.S.E. 2011 (F), 10 C)*

Solution. $f(f(x)) = 3f(x) + 2$

$= 3(3x+2) + 2 = 9x + 8.$

Example 2. If f : R → R is defined by :

$f(x) = \left(3 - x^3\right)^{1/3}$, find fof (x).

(H.P.B. 2016; A.I.C.B.S.E. 2010)

Solution. $fof(x) = f(f(x))$

$$= \left(3 - (f(x))^3\right)^{1/3} = \left(3 - \left(\left(3 - x^3\right)^{1/3}\right)^3\right)^{1/3}$$

$$= \left(3 - \left(3 - x^3\right)\right)^{1/3} = \left(x^3\right)^{1/3} = x.$$

Example 3. Find gof and fog, if :

$f : R \to R$ and $g : R \to R$

are given by f (x) = cos x and g (x) = $3x^2$.

Show that gof ≠ fog.

(N.C.E.R.T.; Jammu B. 2017; Karnataka B.2014)

Solution. We have : $f(x) = \cos x$ and $g(x) = 3x^2$.

$\therefore \quad gof(x) = g(f(x)) = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x$

and $\quad fog(x) = f(g(x)) = f(3x^2) = \cos 3x^2$.

Hence, $\quad gof \neq fog.$

Example 4. Let f : {2, 3, 4, 5} → {3, 4, 5, 9} and g : {3, 4, 5, 9} → {7, 11, 15} be functions defined as : f (2) = 3, f (3) = 4, f (4) = f (5) = 5 and g (3) = g (4) = 7, g (5) = g (9) = 11.

Find gof. *(N.C.E.R.T.)*

Solution. $gof(2) = g(f(2)) = g(3) = 7$

$gof(3) = g(f(3)) = g(4) = 7$

$gof(4) = g(f(4)) = g(5) = 11$

and $\quad gof(5) = g(f(5)) = g(5) = 11.$

Example 5. Find gof and fog if :

f (x) = |x| and g (x) = |5x – 2|.

(H.P.B. Model Question Paper 2018; H.P.B. 2018, 16, 14, 13 S; Jammu B. 2017)

Solution. We have : $f(x) = |x|$ and $g(x) = |5x - 2|$.

(*i*) $\quad (gof)(x) = g(f(x)) = g(|x|)$

$= |5|x| - 2|.$

(*ii*) $\quad (fog)(x) = f(g(x)) = f(|5x - 2|)$

$= ||5x - 2|| = |5x - 2|.$

Example 6. Let $f(x) = \dfrac{4x+3}{6x-4}, x \neq \dfrac{2}{3}$. Show that fof (x) = x.

(Jammu B. 2017; P.B. 2015; H.B. 2015; Meghalaya B. 2018, 14)

Solution. $fof(x) = f(f(x)) = f\left(\dfrac{4x+3}{6x-4}\right)$

$$= \frac{4\left(\dfrac{4x+3}{6x-4}\right)+3}{6\left(\dfrac{4x+3}{6x-4}\right)-4} = \frac{16x+12+18x-12}{24x+18-24x+16}$$

$$= \frac{34x}{34} = x, \text{ which is true.}$$

Example 7. Show that the function f : R → R defined by $f(x) = \dfrac{x}{x^2+1}$, ∀ x ∈ R is neither one-one nor onto. Also, if g : R → R is defined as g(x) = 2x – 1, find fog(x). *(C.B.S.E. 2018)*

Solution. We have : $f(x) = \dfrac{x}{x^2+1}$.

(*i*) **One-one** $\quad f(x_1) = f(x_2)$

$\Rightarrow \quad \dfrac{x_1}{x_1^2+1} = \dfrac{x_2}{x_2^2+1}$

$\Rightarrow \quad x_1x_2^2 + x_1 = x_2x_1^2 + x_2$

$\Rightarrow \quad x_1x_2(x_2 - x_1) = x_2 - x_1$

$\Rightarrow \quad x_1x_2 = 1$

$\Rightarrow \quad x_1 = \dfrac{1}{x_2}$

$\Rightarrow \quad x_1 \neq x_2.$

Thus, f is **not one-one.**

Onto :

$f(x) = y$

$\Rightarrow \quad \dfrac{x}{x^2+1} = y$

$\Rightarrow \quad x = yx^2 + y$

$\Rightarrow \quad x^2y + y - x = 0$

⇒ x cannot be expressed in y.

Thus, f is **not onto.**

Hence, f is neither one-one nor into.

(*ii*) Since $\quad g(x) = 2x - 1,$

$\therefore \quad fog(x) = f(g(x)) = f(2x - 1)$

$$= \frac{2x-1}{(2x-1)^2+1} = \frac{2x-1}{4x^2-4x+2}.$$

Example 8. If the function $f : R \to R$ be given by $f(x) = x^2 + 2$ and $g : R \to R$ be given by $g(x) = \frac{x}{x-1}, x \neq 1$, find fog and gof and hence find fog (2) and gof (–3).

(A.I.C.B.S.E. 2014)

Solution. We have :

$$f(x) = x^2 + 2 \text{ and } g(x) = \frac{x}{x-1}.$$

$$\therefore \quad fog\,(x) = f(g(x)) = (g(x))^2 + 2$$

$$= \left(\frac{x}{x-1}\right)^2 + 2 = \frac{x^2 + 2(x-1)^2}{(x-1)^2}$$

$$= \frac{x^2 + 2(x^2 - 2x + 1)}{(x-1)^2} = \frac{3x^2 - 4x + 2}{(x-1)^2}.$$

Hence, $fog\,(2) = \frac{3(4) - 4(2) + 2}{(2-1)^2} = \frac{12 - 8 + 2}{(1)^2} = \frac{6}{1} = 6.$

And $\quad gof\,(x) = g(f(x)) = \frac{f(x)}{f(x) - 1}$

$$= \frac{x^2 + 2}{(x^2 + 2) - 1} = \frac{x^2 + 2}{x^2 + 1}.$$

Hence, $\quad gof\,(-3) = \frac{(-3)^2 + 2}{(-3)^2 + 1} = \frac{9 + 2}{9 + 1} = \frac{11}{10}.$

Example 9. Let $f : \{1, 2, 3\} \to \{a, b, c\}$ be one-one and onto function given by $f(1) = a, f(2) = b$ and $f(3) = c$. Show that there exists a function :

$g : \{a, b, c\} \to \{1, 2, 3\}$ such that $gof = I_x$ and $fog = I_y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$. *(N.C.E.R.T.)*

Solution. Consider $g : \{a, b, c\} \to \{1, 2, 3\}$ such that :

$$g(a) = 1, g(b) = 2 \text{ and } g(c) = 3.$$

For $x = 1$,

$$gof = g(f(x)) = g(f(1)) = g(a) = 1.$$

Similarly for $x = 2$ and 3.

Then $gof = I_X$, which is identity function in X.

Similarly $fog = I_Y$, which is identity function in Y.

Example 10. If $f : R - \left\{\frac{7}{5}\right\} \to R - \left\{\frac{3}{5}\right\}$ be defined as :

$$f(x) = \frac{3x + 4}{5x - 7} \text{ and}$$

$g : R - \left\{\frac{3}{5}\right\} \to R - \left\{\frac{7}{5}\right\}$ be defined as :

$$g(x) = \frac{7x + 4}{5x - 3}.$$

Show that $fog = I_A$ and $gof = I_B$, where :

$A = R - \left\{\frac{3}{5}\right\}, B = R - \left\{\frac{7}{5}\right\}, I_A(x) = x \;\forall\; x \in A,$

$I_B(x) = x \;\forall\; x \in B$ are identity functions on sets A and B respectively.

(N.C.E.R.T.; Jharkhand B. 2016)

Solution. $gof\,(x) = g(f(x)) = g\left(\frac{3x+4}{5x-7}\right)$

$$= \frac{7\left(\frac{3x+4}{5x-7}\right) + 4}{5\left(\frac{3x+4}{5x-7}\right) - 3} = \frac{21x + 28 + 20x - 28}{15x + 20 - 15x + 21}$$

$$= \frac{41x}{41} = x.$$

Also $\quad fog\,(x) = f(g(x)) = f\left(\frac{7x+4}{5x-3}\right)$

$$= \frac{3\left(\frac{7x+4}{5x-3}\right) + 4}{5\left(\frac{7x+4}{5x-3}\right) - 7} = \frac{21x + 12 + 20x - 12}{35x + 20 - 35x + 21}$$

$$= \frac{41x}{41} = x.$$

Hence, $gof\,(x) = x \;\forall\; x \in B \;\Rightarrow\; gof = I_B$

and $\quad fog\,(x) = x \;\forall\; x \in A \;\Rightarrow\; fog = I_A.$

EXERCISE 1 (c)

Fast Track Answer Type Questions

FTATQ

1. If functions $f : A \to B$ and $g : B \to C$ are defined by : $f(x) = \sqrt{x}$ and $g(x) = x^2$ respectively, find $gof\,(x)$. *(Uttarakhand B. 2015)*
2. Consider functions f and g such that gof is defined and is one-one. Are f and g both necessarily one-one ? *(N.C.E.R.T.)*
3. Are f and g both necessarily onto, if gof is onto ? *(N.C.E.R.T.)*
4. Give examples of two functions : $f : \mathbf{N} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$ such that gof is injective but g is not injective. *(N.C.E.R.T.)*
5. Give examples of the functions : $f : \mathbf{N} \to \mathbf{N}$ and $g : \mathbf{N} \to \mathbf{N}$ such that gof is onto but f is not onto. *(N.C.E.R.T.)*

Very Short Answer Type Questions

VSATQ

6. (*i*) If $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2 - 2x + 3$, then find $f(f(x))$. *(H.B. 2016)*

(*ii*) If $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$. *(Mizoram B. 2018; H.P.B. 2018)*

7. If $f: \mathbf{R} \to \mathbf{R}$ be given by $f(x) = (3 - x^3)^{1/3}$, find $fof\ (x)$. *(Assam B. 2018; H.P.B. 2016)*

8. (*i*) If $f(x) = \dfrac{x}{x-1}, x \neq 1$, then show that $fof\,(x) = x$. *(Kerela B. 2018; H.B. 2011)*

(*ii*) If $f(x) = \log\left(\dfrac{1-x}{1+x}\right)$, $-1 < x < 1$, then show that $f(-x) = -f(x)$. *(H.B. 2011)*

9. If $f(x) = \dfrac{3x-1}{x+1}, x \neq -1$, then find $fof\,(x)$. *(P.B. 2015)*

10. If $f(x) = \dfrac{2x+3}{3x-2}, x \neq \dfrac{2}{3}$, then find $fof\,(x)$. *(P.B. 2015)*

11. Consider a function $f(x) = \dfrac{3x+4}{x-2}, x \neq 2$. Find a function $g(x)$ on a suitable domain such that $(gof)\,(x) = x = (fog)\,(x)$ *(Kerela B. 2015)*

12. Let f, g and h be functions from $\mathbf{R}$ to $\mathbf{R}$. Show that :

(*i*) $(f+g)\,o\,h = foh + goh$

(*ii*) $(f.\,g)\,o\,h = (foh).\,(goh)$. *(N.C.E.R.T.)*

Short Answer Type Questions

SATQ

13. Find fog and gof, if :

(*i*) $f(x) = 8x^3$; $g(x) = x^{1/3}$ *(H.P.B. 2018, 16, 14 ; Meghalaya B. 2016)*

(*ii*) $f(x) = x^2$; $g(x) = x + 1$ *(H.B. 2014 ; P.B. 2010)*

(*iii*) $f(x) = 4x - 1$; $g(x) = x^3 + 2$ *(H.B. 2012)*

(*iv*) $f(x) = |x+1|$; $g(x) = 2x - 1$. *(Kerala B. 2017)*

14. Describe fog and gof, where :

(*i*) $f(x) = e^x$, $g(x) = \log x$

(*ii*) $f(x) = \sqrt{1-x^2}$, $g(x) = \log x$. *(N.C.E.R.T.)*

15. Let $f(x) = 2x^2$ and $g(x) = 3x - 4$; $x \in \mathbf{R}$. Find the following :

(*i*) $fof\,(x)$ (*ii*) $gog\,(x)$

(*iii*) $fog\,(x)$ (*iv*) $gof\,(x)$. *(Bihar B. 2014)*

16. If $f(x) = \dfrac{x-1}{x+1}, x \neq -1$, then show that :

$f(f(x)) = -\dfrac{1}{x}, x \neq 0$. *(N.C.E.R.T.)*

17. Let $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ be two functions defined by $f(x) = |x|$ and $g(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x. Find $(fog)\,(5.75)$ and $(gof)\ (-\sqrt{5})$. *(Assam B. 2017)*

18. Let $f: \mathbf{R} \to \mathbf{R}$ be the Signum Function defined as :

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

and $g: \mathbf{R} \to \mathbf{R}$ be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x. Then does fog and gof coincide in $(0, 1]$? *(N.C.E.R.T.)*

19. Find gof and fog, if $f: \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ are given by :
$f(x) = \cos x$ and $g(x) = 3x^2$.
Show that $gof \neq fog$. *(H.P.B. 2010 S)*

20. Consider three functions :
$f: \mathbf{N} \to \mathbf{N}$, $g: \mathbf{N} \to \mathbf{N}$ and $h: \mathbf{N} \to \mathbf{R}$
defined as $f(x) = 2x$, $g(y) = 3y + 4$ and $h(z) = \sin z$ $\forall\, x, y, z \in \mathbf{N}$.
Show that $ho(gof) = (hog)\,of$. *(Uttarakhand B. 2013)*

Answers

1. x.

2. g is not one-one.

3. f is not onto.

4. $f(x) = x$ and $g(x) = |x|$.

5. $f(x) = x + 1$ and $g(x) = \begin{cases} x-1, & \text{if } x > 1 \\ 1, & \text{if } x = 1. \end{cases}$

6. (*i*) $x^4 - 4x^3 + 8x^2 - 8x + 6$ (*ii*) $x^4 - 6x^3 + 10x^2 - 3x$.

7. x.

9. $\dfrac{2x-1}{x}$. **10.** x.

11. $g(x) = \dfrac{4+2x}{x-3}, x \neq 3$.

13. (*i*) $8x, 2x$ (*ii*) $(x+1)^2, x^2+1$

(*iii*) $4x^3 + 7, 64x^3 - 48x^2 + 12x + 1$

(*iv*) $|2x|, 2|x+1| - 1$.

14. (*i*) x, x (*ii*) $\sqrt{1-(\log x)^2}, \dfrac{1}{2}\log(1-x^2)$.

15. (*i*) $8x^4$ (*ii*) $9x - 16$ (*iii*) $18x^2 - 48x + 32$ (*iv*) $6x^2 - 4$.

17. 5 and 2.

18. No.

19. $(gof)\,(x) = 3\cos^2 x$; $(fog)\,(x) = \cos 3x^2$.

Hints to Selected Questions

8. (*i*) $(fof)(x) = f(f(x)) = \dfrac{f(x)}{f(x)-1}$

$$= \dfrac{\dfrac{x}{x-1}}{\dfrac{x}{x-1}-1} = \dfrac{x}{1} = x.$$

11. $f(g(x)) = x \Rightarrow \dfrac{3\,g(x)+4}{g(x)-2} = x$

$$\Rightarrow \quad g(x) = \frac{4+2x}{x-3}, x \neq 3.$$

12. (*i*) $[(f+g)oh](x) = (f+g)(h(x))$

$= f(h(x)) + g(h(x))$

$= (foh)(x) + (goh)(x)$

$= [(foh) + (goh)](x).$

16. $f(f(x)) = f\left(\dfrac{x-1}{x+1}\right)$; etc.

1.10. INVERTIBLE FUNCTIONS

(Kashmir B. 2017)

Definition

> *A function $f : X \to Y$ is said to be invertible if there exists a function $g : Y \to X$ such that $gof = I_X$ and $fog = I_Y$.*

The function 'g' is the inverse of 'f', which is denoted by f^{-1} and is read as **f–inverse.**

For Example : Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 3x + 4$.

Then $g : \mathbf{R} \to \mathbf{R}$, defined by $g(x) = \dfrac{x-4}{3}$ is the inverse of f.

1.10.1. THEOREMS

Theorem I. *A function $f : X \to Y$ is invertible if 'f' is one–one and onto.*

Proof. Given : $f : X \to Y$ is invertible.

Then there exists $g : Y \to X$ such that

$$fog = I_Y \text{ and } gof = I_X.$$

To Prove : *f is one– one.*

Let x_1 and $x_2 \in X$.

Now $f(x_1) = f(x_2)$

$\Rightarrow \quad g(f(x_1)) = g(f(x_2))$

$\Rightarrow \quad (gof)(x_1) = (gof)(x_2)$

$\Rightarrow \quad I_X(x_1) = I_X(x_2) \quad \Rightarrow \quad x_1 = x_2.$

$\therefore$ f is one–one.

To Prove : *f is onto.*

To each $y \in Y$, there exists $x \in X$ such that $g(y) = x$

$\Rightarrow \quad f(g(y)) = f(x) \quad \Rightarrow \quad (fog)(y) = f(x)$

$\Rightarrow \quad I_Y(y) = f(x) \quad \Rightarrow \quad y = I_Y.$

$\therefore$ f is onto.

Conversely : Given : $f : X \to Y$ is one–one and onto.

To Prove : *f is invertible.*

Since f is one–one and onto,

$\therefore$ to each $y \in Y$, there exists one and only one $x \in X$ such that $f(x) = y$.

$\therefore$ We define $g : Y \to X$ such that :

$$g(y) = x \quad \text{iff} \quad f(x) = y.$$

Now $(gof)(x) = g(f(x)) = g(y) = x \;\forall\; x \in X$

$\Rightarrow \quad gof = I_Y$.

Also $(fog)(y) = f(g(y)) = f(x) = y \ \forall \ y \in Y$

$\Rightarrow \quad fog = I_Y$.

Hence, f is invertible.

Theorem II. *If $f : X \to Y$ is one–one and onto, then $f^{-1} : X \to Y$ is also one-one and onto.*

Proof. We have : $f : X \to Y$ is one–one and onto.

To Prove : *f^{-1} is one–one.*

Let $\quad y_1, y_2$ be two different elements of Y.

Then $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$, where $x_1, x_2 \in X$.

Now $\quad y_1 = f(x_1)$ and $y_2 = f(x_2)$.

Thus $\quad x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2)$

i.e. $\quad f^{-1}(y_1) \neq f^{-1}(y_2) \quad \Rightarrow \quad y_1 \neq y_2$.

Hence, f^{-1} is one–one.

To Prove : *f^{-1} is onto.*

Let $x \in X$.

Then there exists $y \in Y$ such that :

$y = f(x) \quad$ *i.e.* $\quad x = f^{-1}(y)$

$\Rightarrow x$ is the image of $y \in Y$.

Hence, f^{-1} is onto.

KEY POINT

If f is invertible, then f is one–one and onto.

If f is one–one and onto, then f is invertible.

Theorem III. *If $f : X \to Y$ is one–one onto, then inverse of 'f' is unique.*

Or

If $f : X \to Y$ is an invertible function, then 'f' has unique inverse. ***(N.C.E.R.T., Kashmir B. 2013)***

Proof. Let g_1 and g_2 be two inverses of f.

Let $\quad y \in Y$.

Let $g_1(y) = x_1$ and $g_2(y) = x_2$.

Since g_1 the inverse of f,

$\therefore \quad g_1(y) = x_1 \quad \Rightarrow \quad f(x_1) = y$ ...(1)

Similarly $\quad g_2(y) = x_2 \quad \Rightarrow \quad f(x_2) = y$ ...(2)

From (1) and (2), $f(x_1) = f(x_2)$.

Since f is one–one,

$\therefore \quad f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2$

$\Rightarrow \quad g_1(y) = g_2(y) \ \forall \ y \in Y$

$\Rightarrow \quad g_1 = g_2$.

Hence, inverse of 'f' is unique.

Theorem IV. *Let $f : X \to Y$ and $g : Y \to Z$ be two invertible functions. Then gof is also invertible and $(gof)^{-1} = f^{-1} og^{-1}$.*

Proof. In order to show that gof is invertible with :

$(gof)^{-1} = f^{-1} og^{-1}$, it is sufficient to show that :

$(f^{-1}og^{-1}) \, o \, (gof) = I_X$ and $(gof) \, o \, (f^{-1} og^{-1}) = I_Z$.

Now $\quad (f^{-1} o \, g^{-1}) \, o \, (gof) = ((f^{-1} o \, g^{-1}) \, og) \, of$

$= f^{-1} o \, (g^{-1} \, og) \, of$

$= (f^{-1} o \, I_Y) \, of$ [*Def. of g^{-1}*]

$= I_X$.

Similarly $\quad (gof) \, o \, (f^{-1} o \, g^{-1}) = I_Z$.

ILLUSTRATIVE EXAMPLES

Example 1. Prove that function R → R, f(x) = $\frac{3-2x}{7}$ is one-one onto. Also, find f^{-1}. *(P.B. 2018)*

Solution. Let $x_1, x_2, \in$ **R.**

Now, $f(x_1) = f(x_2)$

$\Rightarrow \quad \frac{3-2x_1}{7} = \frac{3-2x_2}{7}$

$\Rightarrow \quad 3 - 2x_1 = 3 - 2x_2 \Rightarrow - 2x_1 - 2x_2$

$\Rightarrow \quad x_1 = x_2.$

Thus, '*f*' is one-one.

Let $y \in$ **R.** Let $y = f(x_0)$.

Then, $\frac{3-2x_0}{7} = y \Rightarrow 3 - 2x_0 = 7y$

$\Rightarrow \quad 2x_0 = 3 - 7y \Rightarrow x_0 = \frac{3-7y}{2}.$

Now, $y \in \mathbf{R} \Rightarrow \frac{3-2x_0}{7} \in \mathbf{R} \Rightarrow x_0 \in \mathbf{R}$

$\therefore \quad f(x_0) = \frac{3-2x_0}{7} = \frac{3-2\frac{3-7y}{2}}{7}$

$= \frac{3-3+7y}{7} = \frac{7y}{7} = y.$

Thus, '*f*' is onto.

Hence, '*f*' is one-one onto.

Also, $y = \frac{3-2x}{7} \Rightarrow 7y = 3-2x$

$\Rightarrow \quad 2x = 3-7y \Rightarrow x = \frac{3-7y}{2}$

$\Rightarrow \quad f^{-1}(y) = \frac{3-7y}{2}.$

Hence, $f^{-1}(x) = \frac{3-7x}{2}$ for all $x \in$ **R**.

Example 2. Prove that the function f : [0, ∞) → R, given by f(x) = $9x^2 + 6x - 5$ is not invertible.

Modify the co–domains of the function f to make it invertible, and hence, find f^{-1}.

(C.B.S.E. Sample Paper 2019)

Solution. Let $y \in$ **R.**

Then for any x, $f(x) = y$ if $y = 9x^2 + 6x - 5$

$\Rightarrow \quad y = (3x)^2 + 2(3x)(1) + (1)^2 - 6$

$= (3x + 1)^2 - 6$

$\Rightarrow \quad (3x + 1)^2 = y + 6$

$\Rightarrow \quad 3x + 1 = \pm\sqrt{y+6}$

$\Rightarrow \quad x = \frac{\pm\sqrt{y+6}-1}{3}$

$\Rightarrow \quad x = \frac{\sqrt{y+6}-1}{3}.$

$\left[\because \frac{-\sqrt{y+6}-1}{3} \notin [0,\infty) \text{ for any } y\right]$

Now, for $y = - 6 \in$ **R**, $x = -\frac{1}{3} \notin [0, \infty)$.

Hence, f is not invertible.

Modification : Since $x \geq 0$, $\therefore \frac{\sqrt{y+6}-1}{3} \geq 0$

$\Rightarrow \quad \sqrt{y+6}-1 \geq 0 \Rightarrow \sqrt{y+6} \geq 1.$

Squaring, $y + 6 \geq 1 \Rightarrow y \geq - 5.$

Thus redefining $f : [0, \infty) \to [- 5, \infty)$ whereas

$f(x) = 9x^2 + 6x - 5$ as onto function.

Let $x_1, x_2 \in [0, \infty)$ such that $f(x_1) = f(x_2)$

$\Rightarrow (3x_1 + 1)^2 = (3x_2 + 1)^2 \Rightarrow (3x_1 + 1)^2 - (3x_2 + 1)^2 = 0$

$\Rightarrow [(3x_1 + 1) + (3x_2 + 1)] [(3x_1 + 1) - (3x_2 + 1)] = 0$

$\Rightarrow [3(x_1 + x_2) + 2] [3(x_1 - x_2)] = 0$

$\Rightarrow \quad 3(x_1 - x_2) = 0 \qquad [\because 3(x_1 + x_2) + 2 > 0]$

$\Rightarrow \quad x_1 - x_2 = 0$

$\Rightarrow \quad x_1 = x_2.$

Thus, '*f*' *is one-one*.

Now, f is bijective and hence, f is invertible

and $f^{-1} : [- 9, \infty) \to [0, \infty)$

$f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}.$

Hence, $f^{-1}(x) = \frac{\sqrt{x+6}-1}{3}.$

Example 3. Consider f : $R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that '*f*' is invertible with

$$f^{-1}(y) = \left(\frac{\sqrt{y+6}-1}{3}\right).$$

Hence, find (i) $f^{-1}(10)$

(ii) y if $f^{-1}(y) = \frac{4}{3}$,

where R_+ is the set of all non-negative real numbers.

(C.B.S.E. 2017)

Solution. We have : $f = \mathbf{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 - 6x - 5$.

Let $x_1, x_2 \in \mathbf{R}_+$ such that $f(x_1) = f(x_2)$

$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow \quad 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$

$\Rightarrow \quad (x_1 - x_2)[9(x_1 + x_2) + 6] = 0$

$\Rightarrow \quad x_1 - x_2 = 0 \Rightarrow x_1 = x_2$ or $x_1 + x_2 = \frac{-6}{9}$,

which is not possible as $x_1, x_2 \in \mathbf{R}_+$.

Thus '*f*' is one-one.

Let $y = f(x) \ \forall \ y \in [-5, \infty)$.

$\therefore \ 9x^2 + 6x - 5 = y \quad \Rightarrow (3x+1)^2 - 1 - 5 = y$

$\Rightarrow \quad 3x + 1 = \sqrt{y+6} \quad \Rightarrow x = \frac{\sqrt{y+6}-1}{3}$.

Now x is defined as $x \in \mathbf{R}_+$ if $y + 6 \geq 1 \Rightarrow y \geq -5$.

Thus '*f*' is onto.

$\therefore$ *f* is one-one and onto

$\Rightarrow$ *f* is invertible and f^{-1} exists.

$$\therefore \quad f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}.$$

$$(i) \quad f^{-1}(10) = \frac{\sqrt{10+6}-1}{3} = \frac{4-1}{3} = \frac{3}{3} = 1.$$

$$(ii) \quad f^{-1}(y) = \frac{4}{3} = x$$

$$\therefore \quad y = f(x) = f\left(\frac{4}{3}\right)$$

$$= 9\left(\frac{4}{3}\right)^2 + 6\left(\frac{4}{3}\right) - 5$$

$$= 16 + 8 - 5 = 19.$$

Example 4. Let A = R – {3} and B = R – {1}. Consider the function f : A → B defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Show that 'f' is one-one and onto and hence find f^{-1}.

(C.B.S.E. 2012)

Solution. Let $x_1, x_2 \in \mathbf{R} - \{3\}$.

Now $\quad f(x_1) = f(x_2) \Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$

$\Rightarrow \quad (x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$

$\Rightarrow \quad x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow \quad x_1 = x_2$.

Thus '*f*' is one-one.

Let $y \in \mathbf{R} - \{1\}$. Let $y = f(x_0)$.

Then $\quad \frac{x_0 - 2}{x_0 - 3} = y \Rightarrow x_0 - 2 = x_0y - 3y$

$\Rightarrow \quad x_0(y-1) = 3y - 2 \Rightarrow x_0 = \frac{3y-2}{y-1}$.

Now $y \in \mathbf{R} - \{1\} \Rightarrow \frac{1}{y-1} \in \mathbf{R} - \{3\} \Rightarrow x_0 \in \mathbf{R} - \{3\}$.

$$\therefore \quad f(x_0) = \frac{x_0 - 2}{x_0 - 3} = \frac{\frac{3y-2}{y-1} - 2}{\frac{3y-2}{y-1} - 3}$$

$$= \frac{3y - 2 - 2y + 2}{3y - 2 - 3y + 3} = \frac{y}{1} = y.$$

Thus '*f*' is onto.

Hence, '*f*' is one-one and onto function

$\Rightarrow$ '*f*' is invertible.

Also $\quad y = \frac{x-2}{x-3} \Rightarrow xy - 3y = x - 2$

$\Rightarrow \quad x(y-1) = 3y - 2 \Rightarrow x = \frac{3y-2}{y-1}$

$\Rightarrow \quad f^{-1}(y) = \frac{3y-2}{y-1}$.

Hence, $\quad f^{-1}(x) = \frac{3x-2}{x-1}$ for all $x \in \mathbf{R} - \{1\}$.

Example 5. Let $Y = \{n^2 : n \in N\} \subset N$.

Consider f : N → Y as $f(n) = n^2$.

Show that 'f' is invertible. Find the inverse of 'f '.

(N.C.E.R.T. ; Jammu B. 2012)

Solution. Let $y \in Y$, where y is arbitrary.

Here y is of the form n^2, for $n \in \mathbf{N}$

$\Rightarrow \quad n = \sqrt{y}$.

This motivates a function :

$g : Y \to \mathbf{N}$, defined by $g(y) = \sqrt{y}$.

Now $gof(n) = g(f(n)) = g(n^2) = \sqrt{n^2} = n$

and $\quad fog(y) = f(g(y)) = f(\sqrt{y}) = (\sqrt{y})^2 = y$.

Thus $\quad gof = I_N$ and $fog = I_Y$.

Hence, 'f' is invertible with $f^{-1} = g$.

Example 6. Let f : N → R be a function defined as : f (x) = 4x² + 12x + 15. Show that f : N → Range f is invertible. Find the inverse of 'f'.

(N.C.E.R.T.; Kashmir B. 2017; Karnataka B. 2017)

Solution. Let $y \in R_f$, where y is arbitrary.

Then $\quad y = 4x^2 + 12x + 15$ for $x \in \mathbf{N}$

$\Rightarrow \quad y = (2x+3)^2 + 6 \quad \Rightarrow \quad 2x + 3 = \sqrt{y-6}$

$\Rightarrow \quad x = \dfrac{\sqrt{y-6} - 3}{2}$.

This motivates a function :

g : Range of $f \to \mathbf{N}$, defined by $g(y) = \dfrac{\sqrt{y-6} - 3}{2}$.

Now

$$gof(x) = g(f(x)) = g(4x^2 + 12x + 15)$$
$$= g((2x+3)^2 + 6)$$
$$= \frac{\sqrt{(2x+3)^2 + 6 - 6} - 3}{2} = \frac{2x + 3 - 3}{2} = x$$

and

$$fog(y) = f(g(y)) = f\left(\frac{\sqrt{y-6} - 3}{2}\right)$$
$$= \left(\frac{2(\sqrt{y-6} - 3)}{2} + 3\right)^2 + 6$$
$$= \left((\sqrt{y-6}) - 3 + 3\right)^2 + 6$$
$$= (\sqrt{y-6})^2 + 6 = y - 6 + 6 = y.$$

Thus $\quad gof = I_N$ and $fog = I_{R_f}$.

Hence, 'f' is invertible with $f^{-1} = g$.

Example 7. Let f : W → W be defined by:

$$\mathbf{f(n)} = \begin{cases} \mathbf{n - 1, \text{ if } n \text{ is odd}} \\ \mathbf{n + 1, \text{ if } n \text{ is even}} . \end{cases}$$

Show that 'f' is invertible. Find the inverse of 'f'. (Here 'W' is the set of whole numbers)

(N.C.E.R.T.; A.I.C.B.S.E. 2015)

Solution. We have : $f : W \to \mathbf{W}$ defined by :

$$f(n) = \begin{cases} n - 1, \text{ if } n \text{ is odd} \\ n + 1, \text{ if } n \text{ is even.} \end{cases}$$

f is one–one.

When n_1 and n_2 are both odd,

then $f(n_1) = f(n_2) \quad \Rightarrow \quad n_1 - 1 = n_2 - 1 \Rightarrow \quad n_1 = n_2$.

When n_1 and n_2 are both even,

then $f(n_1) = f(n_2) \quad \Rightarrow \quad n_1 + 1 = n_2 + 1 \Rightarrow \quad n_1 = n_2$.

$\therefore$ In both cases, $f(n_1) = f(n_2) \quad \Rightarrow \quad n_1 = n_2$.

When n_1 is odd and n_2 is even,

then $f(n_1) = n_1 - 1$, which is even

and $\; f(n_2) = n_2 + 1$, which is odd.

$\therefore \quad n_1 \neq n_2 \quad \Rightarrow f(n_1) \neq f(n_2)$.

Similarly when n_1 is even and n_2 is odd,

$n_1 \neq n_2 \quad \Rightarrow \quad f(n_1) \neq f(n_2)$.

In each case, 'f' is one–one.

f is onto.

When n is odd whole number, then there exists an even whole number

$n - 1 \in \mathbf{W}$ such that $f(n-1) = (n-1) + 1 = n$.

When n is even whole number, then there exists an odd whole number

$n + 1 \in \mathbf{W}$ such that $f(n+1) = (n+1) - 1 = n$.

Also $f(1) = 0 \in \mathbf{W}$.

$\therefore$ each number of **W** has its pre-image in **W**.

Thus 'f' is *onto.*

Hence, 'f' is one–one onto $\quad \Rightarrow \quad$ *'f' is invertible.*

To obtain $\mathbf{f^{-1}}$.

Let $n_1, n_2 \in \mathbf{W}$ such that $f(n_1) = n_2$.

$\therefore \quad n_1 + 1 = n_2$, if n_1 is even

$\quad\quad n_1 - 1 = n_2$, if n_1 is odd.

Thus $\quad n_1 = \begin{cases} n_2 - 1, \text{ if } n_2 \text{ is odd} \\ n_2 + 1, \text{ if } n_2 \text{ is even} \end{cases}$

$\Rightarrow \quad f^{-1}(n_2) = \begin{cases} n_2 - 1, \text{ if } n_2 \text{ is odd} \\ n_2 + 1, \text{ if } n_2 \text{ is even}. \end{cases}$

Then $f^{-1}(n_1) = \begin{cases} n_1 + 1, \text{ if } n_1 \text{ is even} \\ n_1 - 1, \text{ if } n_1 \text{ is odd} \end{cases}$.

Hence, $\quad f = f^{-1}$.

EXERCISE 1 (d)

Very Short Answer Type Questions

VSATQ

1. Let S = {1, 2, 3}. Determine whether the function

$f: S \to S$

defined below has inverse. Find f^{-1}, if it exists :

(*i*) $f = \{(1, 1), (2, 2), (3, 3)\}$

(*ii*) $f = \{(1, 2), (2, 1), (3, 1)\}$

(*iii*) $f = \{(1, 3), (3, 2), (2, 1)\}$. (N.C.E.R.T.)

2. Let S = {*a*, *b*, *c*} and T = {1, 2, 3}. Find f^{-1} of the following function F from S to T, if it exists :

(*i*) $F = \{(a, 3), (b, 2), (c, 1)\}$

(*ii*) $F = \{(a, 2), (b, 1), (c, 1)\}$. (N.C.E.R.T.)

3. Are the following functions invertible in their respective domains ? If so, find the inverse in each case :

(*i*) $f(x) = x + 1$

(*ii*) $f(x) = \dfrac{x-1}{x+1}, x \neq -1$. (P.B. 2012)

4. Let $f: \mathbf{N} \to Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in \mathbf{N},\ y = 4x + 3 \text{ for some } x \in \mathbf{N}\}$. Show that '$f$' is invertible. Find the inverse.

(N.C.E.R.T.; Karnataka B. 2014; H.P.B. 2015, 13, 11)

5. Consider $f: \mathbf{R} \to \mathbf{R}$ given by the following. Show that 'f' is invertible. Find the inverse of 'f'.

(*i*) $f(x) = 4x + 3$

(N.C.E.R.T.; Jammu B. 2018, 13 ; H.B. 2017; H.P.B. 2015, 12)

(*ii*) $f(x) = 5x + 2$. (Kerala B. 2013)

6. (*a*) If $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \dfrac{3x+5}{2}$ is an invertible function, find f^{-1}.

(*b*) Show that $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \dfrac{4x-3}{5}$, $x \in \mathbf{R}$ is invertible function and find f^{-1}. (P.B. 2014 S)

7. If $f: \mathbf{R} \to \mathbf{R}$:

$$f(x) = \frac{3x+6}{8}$$

is an invertible function and find f^{-1}. (P.B. 2017)

Short Answer Type Questions

SATQ

8. (*a*) If $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq \dfrac{2}{3}$, show that $f o f(x) = x$, for all $x \neq \dfrac{2}{3}$. What is the inverse of 'f' ?

(N.C.E.R.T.; Kashmir B. 2016, 12, 11; Jammu B. 2015 W; C.B.S.E.(F) 2012 ; P.B. 2012)

(*b*) Show that the function f in $A = \mathbf{R} - \left\{\dfrac{2}{3}\right\}$ defined as $f(x) = \dfrac{4x+3}{6x-4}$ is one-one and onto. Hence, find f^{-1}.

(C.B.S.E. 2013)

9. Show that $f: [-1, 1] \to \mathbf{R}$, given by :

$f(x) = \dfrac{x}{x+2}$, $x \neq -2$, is one–one.

Find the inverse of the function :

$f: [-1, 1] \to$ Range f.

(N.C.E.R.T.; Assam B. 2018;Jammu B. 2013 ; H.P.B. 2012)

10. Consider $f: \{1, 2, 3\} \to \{a, b, c\}$, given by :

$f(1) = a, f(2) = b$ and $f(3) = c$.

Find f^{-1} and show that $(f^{-1})^{-1} = f$. (N.C.E.R.T.)

11. Consider $f: \{1, 2, 3\} \to \{a, b, c\}$ and

$g: \{a, b, c\} \to \{\text{apple, ball, cat}\}$

defined as $f(1) = a, f(2) = b, f(3) = c$,

$g(a)$ = apple, $g(b)$ = ball, $g(c)$ = cat.

Show that f, g and gof are invertible.

Find f^{-1}, g^{-1} and $(gof)^{-1}$ and show that :

$(gof)^{-1} = f^{-1} o\, g^{-1}$. HOTS (N.C.E.R.T.)

12. Prove that $f: \mathbf{R} \to \mathbf{R}$ is a bijection given by :

$f(x) = x^3 + 3$.

Find $f^{-1}(x)$.

Long Answer Type Questions

LATQ

13. Let $f : \mathbf{N} \to \mathbf{N}$ be a function defined by :
$$f(x) = 9x^2 + 6x - 5.$$
Show that $f : \mathbf{N} \to S$, where S is the range of 'f', is invertible. Find the inverse of 'f' and hence, find $f^{-1}(43)$ and $f^{-1}(163)$.

14. Consider $f : \mathbf{R} - \left\{-\frac{4}{3}\right\} \to \mathbf{R} - \left\{\frac{4}{3}\right\}$ given by :
$$f(x) = \frac{4x+3}{3x+4}.$$
Show that f is bijective. Find the inverse of 'f' and hence find $f^{-1}(0)$ and x such that $f^{-1}(x) = 2$.

(A.I.C.B.S.E. 2017)

15. If the function $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x - 3$ and $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = x^3 + 5$, then find fog and show that fog is invertible. Also, find $(fog)^{-1}$, hence find $(fog)^{-1}(9)$. *(C.B.S.E. Sample Paper 2018)*

Answers

1. (i) Invertible; $f^{-1} = \{(1, 1), (2, 2), (3, 3)\} = f$
(ii) Not invertible
(iii) Invertible; $f^{-1} = \{(3, 1), (2, 3), (1, 2)\}$.

2. (i) $F^{-1} = \{(3, a), (2, b), (1, c)\}$
(ii) F^{-1} does not exist.

3. (i) Invertible ; $f^{-1}(x) = x - 1$
(ii) Invertible; $f^{-1}(x) = \frac{1+x}{1-x}$.

4. $f^{-1}(x) = \frac{x-3}{4}$.

5. (i) $f^{-1}(x) = \frac{x-3}{4}$ (ii) $f^{-1}(x) = \frac{x-2}{5}$.

6. (a) $f^{-1}(x) = \frac{2x-5}{3}$ (b) $f^{-1}(x) = \frac{5x+3}{4}$.

7. $f^{-1}(x) = \frac{8x-6}{3}$.

8. (a) – (b) $f^{-1} = f$.

9. $f^{-1}(x) = \frac{2x}{1-x}, x \neq 1$.

12. $f^{-1}(x) = (x-3)^{1/3}$.
Truthfulness and honesty among people may have bijective relation as people who are honest are usually truthful and vice-versa.

13. $f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$; $f^{-1}(43) = 2$; $f^{-1}(163) = 4$.

14. $f^{-1}(y) = \frac{3-4y}{3y-4}$; $f^{-1}(0) = -\frac{3}{4}$; $x = \frac{11}{10}$.

15. $(fog)(x) = 2x^3 + 7$, $(fog)^{-1}(x) = \left(\frac{x-7}{2}\right)^{1/3}$,
$(fog)^{-1}(9) = 1$.

Hints to Selected Questions

3. (i) – (ii) 'f' is invertible if it is one-one onto.

4. $y = f(x) = 4x + 3 \Rightarrow x = \frac{y-3}{4} \in \mathbf{R}$

$\Rightarrow f^{-1}(y) = \frac{y-3}{4} \Rightarrow f^{-1}(x) = \frac{x-3}{4}$.

6. (a) $y = f(x) = \frac{3x+5}{2} \Rightarrow x = \frac{2y-5}{3} \in \mathbf{R}$

$\Rightarrow f^{-1}(y) = \frac{2y-5}{3} \Rightarrow f^{-1}(x) = \frac{2x-5}{3}$.

9. $y = f(x) = \frac{x}{x+2} \Rightarrow x = \frac{2y}{1-y}$

$\Rightarrow f^{-1}(y) = \frac{2y}{1-y} \Rightarrow f^{-1}(x) = \frac{2x}{1-x}$.

10. Clearly 'f' is one-one onto

$\Rightarrow f^{-1} : \{a, b, c\} \to \{1, 2, 3\}$

$\Rightarrow (f^{-1})$ is one-one onto

and $(f^{-1})^{-1} = \{(1, a), (2, b), (3, c)\}$

$\Rightarrow (f^{-1})^{-1} = f$.

13. $y = f(x) \Rightarrow y = 9x^2 + 6x - 5$

$\Rightarrow y = (3x+1)^2 - 6 \Rightarrow \sqrt{y+6} = 3x + 1$

$\Rightarrow x = \frac{\sqrt{y+6}-1}{3} \Rightarrow f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$.

$\therefore f^{-1}(43) = \frac{\sqrt{43+6}-1}{3} = \frac{7-1}{3} - \frac{6}{3} = 2$

and $f^{-1}(163) = \frac{\sqrt{163+6}-1}{3} = \frac{13-1}{3} = \frac{12}{3} = 4$.

14. $y = \frac{4x+3}{3x+4} \Rightarrow x = \frac{3-4y}{3y-4}$

$\Rightarrow \quad f^{-1}(y) = \frac{3-4y}{3y-4}.$

(*i*) $f^{-1}(0) = \frac{3-0}{0-4} = -\frac{3}{4}.$

(*ii*) $f^{-1}(x) = 2 \Rightarrow \frac{3-4x}{3x-4} = 2$

$\Rightarrow \quad 3 - 4x = 6x - 8 \Rightarrow 10x = 11$

$\Rightarrow \quad x = \frac{11}{10}.$

1.3 Binary Operations

SUB CHAPTER

1.11. BINARY OPERATIONS

A function $f : A \to A$, where A is a set, is considered as a **unitary operation** in the sense that an element of A is associated to each singleton subset $\{a\}$ of A.

If an element of A is associated uniquely with each subset of two elements of A (the order of the elements being taken into account), we obtain a **binary operation** on A.

In general, for $n = 1, 2, ...$, an n-ary operation on the set A is a map :

$f : A \times A \times .... \times A$ (n times) $= A^n \to A$.

For simplicity, we consider here unitary and binary operations only.

Examples : (I) Let A = $\mathbf{R}^+$ (set of all positive real numbers).

Here map $x \to \frac{1}{x}$; A → A

i.e., taking reciprocals of positive real numbers is a unitary operation.

(II) Let A = **R** (set of all real numbers).

Here map $(x, y) = x + y : \mathbf{R}^2 \to \mathbf{R}$

i.e., addition of two real numbers is a binary operation.

Definition

Multiplication is also binary operation on **R**, while division is not a binary operation on **R** because division by 0 is not defined. But division is a binary operation on **R** – {0}.

Notation. We denote the binary operation by 'o' †.

Instead of $(m, n) \to m + n + mn : \mathbf{I}^2 \times \mathbf{I}$, we write :

$$mon = m + n + mn \text{ on } \mathbf{I}.$$

It should be noted with care when we define a binary operation, the **order of the elements is taken into account.** In other words, the map, which defines the binary operation on A, is on the set A^2 of all ordered pairs of the elements of A.

Definition

Let A be non-empty set. Then the rule denoted by 'o' is called binary operation on A if to each ordered pair (a, b) of the elements of A, it associates a unique element, denoted by aob of A.

In the set of numbers, we generally use '+', '.', '–', '÷', to denote binary operations.

Properties of Binary Operations :

(I) Closure. If $a \, o \, b \in A$ for all $a, b \in A$, then 'o' *is closed.*

(II) Commutative. *If $aob = boa$ for $a, b \in A$, then 'o' is commutative.*

In case aob, boa are different, then 'o' is not commutative.

Examples : (I) The operation of division on **R** – {0} is not commutative.

(II) Let the binary operation '*' on **I** be defined as :

$$m * n = m - n + mn,$$

then $1 * 2 = 1 - 2 + 1 \,.\, 2 = 1$

† Other notations for binary operations are '*' '⊙', '⊗' ; etc.

while $2 * 1 = 2 - 1 + 2 . 1 = 3$.

$\therefore 1 * 2 \neq 2 * 1 \Rightarrow$ '*' is not a commutative binary operation.

Example : The operation * : R × R → R defined by :
a*b = a + 2b is not commutative. *(N.C.E.R.T.)*

Solution. Since $3 * 5 = 3 + 2 (5) = 3 + 10 = 13$ and $5 * 3 = 5 + 2 (3) = 5 + 6 = 11$.

$\therefore 3 * 5 \neq 5 * 3$.

Hence, the operation '*' is not commutative.

(III) Associative.

If ao(boc) = (aob)oc for a, b, c ∈ A, then 'o' is associative.

Examples : (I) Addition and Multiplication on **R** are associative binary operations.

(II) Division on **R** – {0} is not an associative binary operation.

Example : The operation * : R × R → R defined by a * b = a + 2b is not associative. *(N.C.E.R.T.)*

Solution. Since $(7 * 5) * 3 = (7 + 10) * 3 = 17 + 6 = 23$

and $7 * (5 * 3) = 7 * (5 + 6) = 7 + 22 = 29$.

$\therefore (7 * 5) * 3 \neq 7 * (5 * 3)$.

Hence, the operation '*' is not associative.

KEY POINT

The concepts of commutativity and associativity are independent.

(IV) Existence of Identity Element.

If '*o*' is binary operation on A and there is $e \in A$ such that $aoe = a = eoa$, where '*e*' is the identity element of the operation.

Examples : (I) For binary operation '+' in **R**, 0 is the identity element. *(N.C.E.R.T.)*

(II) For binary operation '.' in **R**, 1 is the identity element. *(N.C.E.R.T.)*

Further there is no element *e* in **R** with $a - e = a = e - a$.

Also there is no element *e* in $\mathbf{R}^*$ (set of non-zero real numbers) with $a \div e = a = e \div a$.

Thus '–' and '÷' do not have identity elements.

KEY POINT

Zero is identity for the additive operation on **R** but is not identity for the additive operation on **N**.

(V) Existence of Inverse Element. An element *b* in a set A is said to be inverse element of an element $a \in A$ with respect to the binary operation *o* if :

$$aob = e = boa.$$

Example : – a is the inverse of a for additive operation '+' on R and $\frac{1}{a}$ is the inverse of a ≠ 0 for multiplicative operation '×' on R. *(N.C.E.R.T.)*

Since $a + (-a) = a - a = 0$ and $(-a) + a = 0$,

$\therefore -a$ is the inverse of '*a*' for addition on **R**.

Again for $a \neq 0$, $a \times \frac{1}{a} = 1 = \frac{1}{a} \times a$, $\frac{1}{a}$ is the inverse of *a* for multiplication on **R**.

KEY POINT

– *a* is not the inverse of *a* for addition operation '+' on **N**.

Sometimes, it is convenient to write down a binary operation by means of the table.

If X is the set {*a*, *b*} and the binary operation '*o*' is defined as :

$aoa = a$, $bob = b$, $aob = b$, $boa = a$.

Clearly it is not commutative.

These can be written in the form :

o	*a*	*b*
a	*a*	*b*
b	*a*	*b*

Algebraic Structure. *A non-empty set with one or more binary operations defined on it, is called an algebraic structure.*

1 .11.1. SUPREMUM AND INFIMUM OPERATIONS

(*i*) The operation $\vee : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $(x, y) \to$ maximum of *x* and *y* is called supremum operation.

(*ii*) The operation $\wedge : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $(x, y) \to$ minimum of *x* and *y* is called infimum operation.

For Examples : (I) $\vee(3, 5) = 5$, $\vee(3, -5) = 3$

(II) $\wedge(3, 5) = 3$, $\wedge(3, -5) = -5$.

Theorem. $\vee : R \times R \to R$ *given by* $(x, y) \to$ *maximum of* x, y *and* $\wedge : R \times R \to R$ *given by* $(x, y) \to$ *minimum of* x *and* y *are binary operations.*

Proof. Since $\vee$ carries each pair (x, y) in $\mathbf{R} \times \mathbf{R}$ to unique element *viz.* maximum of x and $y \in \mathbf{R}$,

$\therefore$ '$\vee$' is a binary operation.

On similar lines of arguments,

'$\wedge$' is a binary operation.

ILLUSTRATIVE EXAMPLES

Example 1. If a * b denotes the larger of 'a' and 'b' and if aob = (a * b) + 3, then write the value of (5) o (10), where '*' and 'o' are binary operations. *(C.B.S.E. 2018)*

Solution. Given : $aob = (a * b) + 3$

$\therefore$ $5 \text{ o } (10) = (5 * 10) + 3 = 10 + 3 = 13.$

Example 2. Let '*' be a binary operation on the set of all non-zero real numbers, given by $a * b = \frac{ab}{5}$ for all $a, b \in R - \{0\}$. Find the value of 'x', given that 2 * (x * 5) = 10. *(C.B.S.E. 2014)*

Solution. $2 * (x * 5) = 10$

$\Rightarrow$ $2 * \frac{(x)(5)}{5} = 10$ $\Rightarrow$ $2 * x = 10$

$\Rightarrow$ $\frac{(2)(x)}{5} = 10$. Hence, $x = 25$.

Example 3. Let '*' be a binary operation on N given by :

a * b = LCM (a, b) for all a, b ∈ N.

Find (*i*) 5 * 7 *(C.B.S.E. 2012)*

(*ii*) 20 * 16. *(Karnataka B. 2017)*

Solution. (*i*) 5 * 7 = LCM (5, 7) = 35.

(*ii*) 20 * 16 = LCM (20, 16) = 80.

Example 4. The binary operation * : R × R → R is defined as :

a * b = 2a + b.

Find (2 * 3) * 4. *(A.I.C.B.S.E. 2012)*

Solution. $(2 * 3) * 4 = (2(2) + 3) * 4 = 7 * 4 = 2(7) + 4 = 14 + 4 = 18.$

Example 5. Find the identity element in Z with respect to the operation '*' defined by :

a * b = a + b + 1 ∀ a, b ∈ Z.

(Meghalaya B. 2015; Uttarakhand B. 2015)

Solution. Let 'e' be the identity element.

Then $a * e = a \Rightarrow a + e + 1 = a$

$\Rightarrow e = -1$, which is the identity element.

Example 6. Show that addition, subtraction and multiplication are binary operations on R but division is not a binary operation on R. Further, show that division is a binary operation on the set R*, of non-zero real numbers. *(N.C.E.R.T.)*

Solution. $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is given by :

$(x, y) \to x + y$

$- : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is given by :

$(x, y) \to x - y$

$\times : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is given by :

$(x, y) \to xy$.

Since '+', '–' and '×' are functions,

$\therefore$ these are binary operations on **R**.

But $\div : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by :

$(x, y) \to \frac{x}{y}$

is not a function and consequently it is not a binary operation. [$\because$ *For* $y = 0$, $\frac{x}{y}$ *is not defined*]

However, $\div : \mathbf{R}^* \times \mathbf{R}^* \to \mathbf{R}^*$ given by :

$(x, y) \to \frac{x}{y}$

is a function and hence a binary operation in **R***, where **R*** is the set of non-zero real numbers (*i.e.* $\mathbf{R} - \{0\}$).

Example 7. Show that subtraction and divison are not binary operations on N. *(N.C.E.R.T.)*

Solution. (*i*) $- : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ is given by :

$(x, y) \to x - y$, which is not a binary operation.

[$\because$ *Image of (4, 6) under '–' is* $4 - 6 = -2 \notin N$]

(*ii*) $\div : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ is given by :

$(x, y) \to x \div y$, which is not a binary operation.

[$\because$ *Image of (4, 6) under '÷' is* $4 \div 6 = \frac{2}{3} \notin N$]

Example 8. Show that ∨ : R × R → R given by (a, b) → max. {a, b} and ∧ : R × R → R given by (a, b) → min. {a, b} are binary operations. *(N.C.E.R.T.)*

Solution. (*i*) $\vee : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $(a, b) \to$ max. $\{a, b\}$, which is a unique element *viz* max. of $a, b \in \mathbf{R}$.

Hence, $\vee$ is a binary operation.

(*ii*) $\wedge : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by $(a, b) \to$ min. $\{a, b\}$, which is a unique element *viz.* min. of $a, b \in \mathbf{R}$.

Hence, $\wedge$ is a binary operation.

Example 9. Show that * : R × R → R given by a * b → a + 2b is not associative. *(N.C.E.R.T.)*

Solution. Take 3, 5 and 8 as real numbers.

Now $(3 * 5) * 8 = (3 + 10) * 8 = 13 * 8 = 13 + 16 = 29$

and $3 * (5 * 8) = 3 * (5 + 16) = 3 * 21 = 3 + 42 = 45.$

Thus $(3 * 5) * 8 \neq 3 * (5 * 8).$ $[\because 29 \neq 45]$

Hence, '*' is not associative.

Example 10. Determine whether the binary operation '*' on the set N of natural numbers defined by a * b = 2^{ab} is associative or not. *(C.B.S.E. Sample Paper 2018)*

Solution. Let a, b and $c \in \mathbf{N}$.

Then $(a * b) * c = 2^{ab} * c$

$= 2^{2^{ab}c}.$

And $a * (b * c) = a * 2^{bc}$

$= 2^{a\, 2^{bc}}.$

Thus $(a * b) * c \neq a * (b * c).$

Hence, '*' is not associative.

Example 11. Let '*' be a binary operation on Q defined by :

$$\mathbf{a * b = \frac{3ab}{5}}.$$

Show that '*' is commutative as well as associative. Also find its identity element, if it exists. *(C.B.S.E. 2010)*

Solution. Here $a * b = \dfrac{3ab}{5}$; $a, b, \in \mathbf{Q}$ is a binary operation :

(I) Commutativity.

For $a, b, \in \mathbf{Q}$,

$$a * b = \frac{3ab}{5} = \frac{3ba}{5}$$

[$\because$ *Rational numbers are commutative under multiplication*]

$$= b * a.$$

Hence, '*' is commutative on **Q.**

(II) Associativity.

For $a, b, c \in \mathbf{Q}$,

$$(a * b) * c = \frac{3ab}{5} * c$$

$$= \frac{3\,\frac{3ab}{5}\,c}{5} = \frac{9abc}{25}.$$

And $a * (b * c) = a * \dfrac{3bc}{5}$

$$= \frac{3a\,\frac{3bc}{5}}{5} = \frac{9abc}{25}.$$

Thus $(a * b) * c = a * (b * c).$

Hence, '*' is associative on **Q.**

(III) Let 'e' be the identity element.

Then $a * e = a = e * a \Rightarrow \dfrac{3ae}{5} = a = \dfrac{3ea}{5}$

$$\Rightarrow \qquad e = \frac{5}{3}.$$

Hence, the identity element $= \dfrac{5}{3}$.

Example 12. Examine which of the following is a binary operation :

(i) $\mathbf{a * b = \dfrac{a+b}{2}}$**; a, b ∈ N**

(ii) $\mathbf{a * b = \dfrac{a+b}{2}}$**; a, b ∈ Q.**

For binary operation, check the commutative and associative property.

Solution. (*i*) $a * b = \dfrac{a+b}{2}$; $a, b \in \mathbf{N}$ is a binary operation.

(I) Commutativity.

For $a, b \in \mathbf{N}$,

$$a * b = \frac{a+b}{2} = \frac{b+a}{2}$$

[$\because$ *Natural numbers are commutative under addition*]

$$= b * a.$$

Thus $a * b = b * a$; $a, b \in \mathbf{N}$.

Hence, '*' is commutative on **N.**

(II) Associativity.

For $a, b, c \in \mathbf{N}$,

$$(a * b) * c = \frac{a+b}{2} * c$$

$$= \frac{\left(\frac{a+b}{2}\right) + c}{2} = \frac{a+b+2c}{4}.$$

And $a * (b * c) = a * \dfrac{b+c}{2}$

$$= \frac{a + \left(\frac{b+c}{2}\right)}{2} = \frac{2a+b+c}{4}.$$

Thus $(a * b) * c \neq a * (b * c)$.

Hence, '*' is not associative on **N**.

(*ii*) $a * b = \dfrac{a+b}{2}$, $a, b \in \mathbf{Q}$ is a binary operation.

(I) Commutativity.

For $a, b \in \mathbf{Q}$

$$a * b = \frac{a+b}{2} = \frac{b+a}{2}$$

[$\because$ *Rational numbers are commutative under addition*]

$$= b*a.$$

Hence, '*' is commutative on **Q**.

(II) Associativity.

As in part (*i*), '*' is not associative on **Q**.

Hence, '*' is not associative.

Example 13. Discuss the commutativity and associativity of binary operation '*' defined on A = Q – {1} by the rule :

a * b = a – b + ab for a, b ∈ A.

Also find the identity element of '*' in A and hence find the invertible elements of A. ***(C.B.S.E. 2017)***

Solution. We have : $a * b = a - b + ab \;\; \forall \; a, b \in A$, where $A = \mathbf{Q} - \{1\}$.

(*i*) **Commutativity :** Let $a, b \in \mathbf{Q} - \{1\}$.

Now $a * b = a - b + ab \neq b - a + ab = b * a$

Hence, '*' is not commutative.

(*ii*) **Associativity :** Let $a, b, c \in \mathbf{Q} - \{1\}$.

Now
$$\begin{aligned} a * (b * c) &= a * (b - c + bc) \\ &= a - (b - c + bc) + a(b - c + bc) \\ &= a - b + c - bc + ab - ac + abc. \end{aligned}$$

And
$$\begin{aligned} (a * b) * c &= (a - b + ab) * c \\ &= a - b + ab - c + (a - b + ab)c \\ &= a - b + ab - c + ac - bc + abc. \end{aligned}$$

Thus $a * (b * c) \neq (a * b) * c$.

Hence, '*' is not associative.

(*iii*) Let 'e' be the identity element in A.

$\therefore \quad a * e = a = e * a$

$\Rightarrow \quad a - e + ae = e - a + ea$

$\Rightarrow \quad a - e = e - a \quad [\because ae = ea]$

$\Rightarrow \quad 2e = 2a$

$\Rightarrow \quad e = a,$

which is not possible. [$\because$ *identity is unique*]

Hence, inverse element does not exist.

Example 14. Let 'X' be a non-empty set and P (X) be its power set. Let '*' be an operation defined on elements of P (X), by :

$$\mathbf{A * B = A \cap B \;\; \forall \; A, B \in P(X).}$$

Then,

(i) Prove that '*' is a binary operation on P (X).

(ii) Is '*' commutative ?

(iii) Is '*' associative ?

(iv) Find the identity element in P (X) w.r.t. '*'.

(v) Find all invertible elements of P (X).

(vi) If 'o' is another binary operation defined on P (X) as AoB = A ∪ B, then verify that 'o' distributes itself over '*'.

Solution. We have : $A * B = A \cap B \;\; \forall \; A, B \in P(X)$.

(*i*) Since '*' associates any two elements $A, B \in P(X)$ to a unique element $A \cap B$ in P (X),

$\therefore$ 'X' is a binary operation in P (X).

(*ii*) For all $A, B \in P(X)$, $A * B = A \cap B$

$$= B \cap A = B * A.$$

$\therefore$ '*' is commutative.

(*iii*) For all $A, B, C \in P(X)$

$$\begin{aligned} (A * B) * C &= (A \cap B) * C = (A \cap B) \cap C \\ &= A \cap (B \cap C) = A \cap (B * C) \\ &= A * (B * C). \end{aligned}$$

$\therefore$ '*' is associative.

(*iv*) Let E be the identity in P (X) w.r.t. '*'.

Then $A * E = A = E * A$ for all $A \in P(X)$

$\Rightarrow \quad A \cap E = A = E \cap A$ for all $A \subset X$

$\Rightarrow \quad E = X.$

Hence, X is the identity element w.r.t. '*' on P (X).

(*v*) Let A be an invertible element of P (X).

Let S be its inverse.

Then $A * S = X = S * A$

$\Rightarrow \quad A \cap S = X = S \cap A$

$\Rightarrow \quad A = S = X.$

Hence, X is the only invertible element of P (X) w.r.t. '*' and it is the inverse of itself.

(*vi*) We have $A * B = A \cap B \ \forall\ A, B \in P(X)$

and $A\ o\ B = A \cup B \ \forall\ A, B \in P(X)$.

For all A, B, C ∈ P (X),

$$\begin{aligned} A\ o\ (B * C) &= A\ o\ (B \cap C) \\ &= A \cup (B \cap C) \\ &= (A \cup B) \cap (A \cup C) \\ &= (A\ o\ B) * (A\ o\ C). \end{aligned}$$

Hence, '*o*' distributes itself over '*'.

Example 15. Let A = R × R and '*' be a binary operation on A defined by :

$$(a, b) * (c, d) = (a + c, b + d).$$

Show that '*' is commutative and associative. Find the identity element for '*' on A. Also find the inverse of every element (*a*, *b*) ∈ A. **(A.I.C.B.S.E. 2016)**

Solution. (*i*) (I) $(a, b) * (c, d) = (a + c, b + d)$

$= (c + a, d + b)$

[∵ *Commutative law holds in* **R** *under addition*]

$= (c, d) * (a, b).$

Thus '*' *is commutative* on A.

(*ii*) $[(a, b) * (c, d)] * (e, f)$

$$\begin{aligned} &= (a + c, b + d) * (e, f) \\ &= ((a + c) + e, (b + d) + f) \\ &= (a + (c + e), b + (d + f)) \end{aligned}$$

[∵ *Associative law holds in R under addition*]

$= (a, b) * [(c, d) * (e, f)].$

Thus '*' *is associative* on A.

(*iii*) Let (x, y) be the identity element of A.

Then $(a, b) * (x, y) = (a, b)$ for all $a, b \in \mathbf{R}$

⇒ $(a + x, b + y) = (a, b)$ for all $a, b \in \mathbf{R}$

⇒ $a + x = a$ and $b + y = b$

⇒ $x = 0$ and $y = 0$.

Hence, (0, 0) is the identity element.

(*iv*) Let (x, y) be the inverse of $(a, b) \in A$.

Then $(a, b) * (x, y) = (0, 0)$

⇒ $(a + x, b + y) = (0, 0)$

⇒ $a + x = 0, b + y = 0$

⇒ $x = -a, y = -b$.

Hence, $(-a, -b)$ is the inverse of (a, b).

Example 16. Let A = Z × Z and '*' be a binary operation on A defined by :

(a, b) * (c, d) = (ad + bc, bd).

Find the identity element for '*' in A.

(C.B.S.E. Sample Paper 2019)

Solution. Let (x, y) be the identity element.

Then, $(a, b) * (x, y) = (a, b) = (x, y) * (a, b)$ $\forall\ (a, b) \in A$

⇒ $(ay + bx, by) = (a, b) = (xb + ya, yb)$

⇒ $ay + bx = a = xb + ya$ and $by = b = yb$ ...(1)

⇒ $y = 1$ and $x = 0$.

Hence, (0, 1) is the identity element.

Example 17. Consider the binary operation '*' on the set {1, 2, 3, 4, 5} defined by a * b = min. {a, b}. Write the operation table of the operation '*'. **(C.B.S.E. 2011)**

Solution. The binary operation '*' on the set {1, 2, 3, 4, 5} defined by $a * b = \min.\{a, b\}$.

Thus we have the operation table :

Operation Table

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

EXERCISE 1 (e)

Fast Track Answer Type Questions

FTATQ

1. (*i*) Let $a * b = 2a + b - 3$, find $3 * 4$. **(H.B. 2015)**
   (*ii*) Let $a * b = a + 2b - 5$, find $3 * 5$. **(H.B. 2015)**
2. Let '*' be a binary operation on **N** given by $a * b$ = H.C.F. (a, b), $a, b \in \mathbf{N}$. Write the value of $22 * 4$.
3. (*a*) (*i*) Show that $+ : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\times : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are commutative binary operations. **(N.C.E.R.T.)**
   (*ii*) Show that $- : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\div : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are not commutative binary operations. **(N.C.E.R.T.)**
   (*b*) (*i*) Show that addition and multiplication are associative binary operations on **R.** **(N.C.E.R.T.)**
   (*ii*) Show that subtraction and division are not associative binary operations on **R.** **(N.C.E.R.T.)**
4. (*i*) Show that $-a$ is not the inverse of $a \in \mathbf{N}$ for addition operation '+' on **N.** **(N.C.E.R.T.)**
   (*ii*) Show that $\frac{1}{a}$ is not the inverse of $a \in \mathbf{N}$ for multiplication operation '×' on **N** for $a \neq 1$. **(N.C.E.R.T.)**

Very Short Answer Type Questions

VSATQ

5. If the binary operation * on the set **Z** of integers is defined by $a * b = a + b - 5$, then write the identity element for the operation '*' in **Z**. *(C.B.S.E. (F) 2012)*
6. Show that $* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by :
   $a * b \to a + 4b^2$ is a binary operation. *(N.C.E.R.T.)*
7. Let P be the set of all subsets of a given set X. Show that $\cup : P \times P \to P$ given by :
   $(A, B) \to A \cup B$
   and $\cap : P \times P \to P$ given by :
   $(A, B) \to A \cap B$
   are binary operations on the set P. *(N.C.E.R.T.)*
8. Determine whether or not each of the definition of '*' given below gives a binary operation. In the event that '*' is not a binary operation, give justification for this :
   (*i*) On $\mathbf{Z}^+$, define '*' by $a * b = a - b$
   (*ii*) On $\mathbf{Z}^+$, define '*' by $a * b = ab$
   (*iii*) On **R**, define '*' by $a * b = ab^2$
   (*iv*) On $\mathbf{Z}^+$, define '*' by $a * b = |a - b|$
   (*v*) On $\mathbf{Z}^+$, define '*' by $a * b = a$. *(N.C.E.R.T.)*
9. Show that the binary operation '*' defined from $\mathbf{N} \times \mathbf{N} \to \mathbf{N}$ and given by $a * b = 2a + 3b$ is not commutative. *(H.B. 2016)*
10. For each binary operation '*'defined below, determine whether '*' is commutative and whether '*' is associative :
    (*i*) On **Z**, define '*' by $a * b = a - b$
    (*ii*) On **Q**, define '*' by $a * b = ab + 1$ *(Meghalaya B. 2018, 17)*
    (*iii*) On **Q**, define '*' by $a * b = \frac{ab}{2}$ *(Karnataka B. 2014)*
    (*iv*) On **Q,** define '*' by $a * b = \frac{ab}{3}$ *(Kerala B. 2014)*
    (*v*) On $\mathbf{Z}^+$, define '*' by $a * b = 2^{ab}$
    (*vi*) On $\mathbf{Z}^+$, define '*' by $a * b = a^b$
    (*vii*) On $\mathbf{R} - \{-1\}$, define '*' by $a * b = \frac{a}{b+1}$. *(N.C.E.R.T.)*
11. Is '*' defined on the set {1, 2, 3, 4, 5} by :
    $a * b =$ *l.c.m.* of a and b
    a binary operation ? Justify your answer. *(N.C.E.R.T.)*
12. Let '*' be the binary operation on **N** given by :
    $a * b =$ *l.c.m.* of a and b.
    Find :
    (*i*) 5 * 7, 20 * 16.
    (*ii*) Is '*' commutative ?
    (*iii*) Is '*' associative ?
    (*iv*) Find the identity of '*' in **N.**
    (*v*) Which elements of **N** are invertible for the operation '*' ? *(N.C.E.R.T.)*
13. Let '*' be a binary operation on **N** defined by :
    $a * b =$ H.C.F. of a and b.
    Is '*' commutative ? Is '*' associative ? *(Kerala B. 2013)*
    Does there exist identity for this binary operation on **N** ? *(N.C.E.R.T.)*
14. (*a*) Let '*' be a binary operation defined on **Q**, the set of rational numbers, as follows :
    (*i*) $a * b = a - b$, for $a, b \in \mathbf{Q}$
    (*ii*) $a * b = a^2 + b^2$, for $a, b \in \mathbf{Q}$
    (*iii*) $a * b = a + ab$, for $a, b \in \mathbf{Q}$
    (*iv*) $a * b = (a - b)^2$, for $a, b \in \mathbf{Q}$
    (*v*) $a * b = \frac{ab}{4}$, for $a, b \in \mathbf{Q}$
    (*vi*) $a * b = ab^2$, for $a, b \in \mathbf{Q}$. *(H.B. 2016)*
    Find which of the binary operations are commutative and which are associative. *(N.C.E.R.T.)*
    (*b*) Show that none of the operations given in part (*a*) has identity. *(N.C.E.R.T.)*
15. In the binary operation $* : \mathbf{Q} \times \mathbf{Q} \to \mathbf{Q}$ is defined as :
    (*i*) $a * b = a + b - ab;$ $a, b \in \mathbf{Q}$ *(H.B. 2015)*
    (*ii*) $a * b = a + b + ab;$ $a, b \in \mathbf{Q}$
    (*iii*) $a * b = \frac{ab}{4};$ $a, b \in \mathbf{Q}$. *(H.B. 2016)*
    Show that '*' is commutative but is associative in both in (*ii*) – (*iii*) *(H.B. 2011)*
16. Show that the operation '*' on $\mathbf{Q} - \{1\}$ defined by $a * b = a + b - ab$ for all $a, b \in \mathbf{Q} - \{1\}$, satisfies the commutative law. *(Meghalaya B. 2013)*

Short Answer Type Questions

SATQ

17. Consider the infimum operation '∧' on the set :
    {1, 2, 3, 4, 5} defined by :
    $a \wedge b =$ minimum of a and b.
    Write the operation table of the operation '∧'. *(N.C.E.R.T.)*
18. State whether the following statements are true or false. Justify.
    (*i*) For an arbitrary binary operation '*' on a set **N**, $a * a = a, \forall a \in \mathbf{N}$.
    (*ii*) If '*' is a commutative binary operation on **N,** then
    $a * (b * c) = (c * b) * a$. *(N.C.E.R.T.)*

Long Answer Type Questions

LATQ

19. Let '*' be a binary operation defined on **N** × **N** by :
$(a, b) * (c, d) = (a + c, b + d)$.
(*i*) Find (1, 2) * (2, 3). *(Kerala B. 2018)*
(*ii*) Prove that '*' is commutative and associative. *(Kerala B. 2018)*
(*iii*) Find the identity element for '*', if it exists. *(Kerala B. 2017)*

20. Let A = **Q** × **Q** and let '*' be a binary operation on A defined by :
$(a, b) * (c, d) = (ac, b + ad)$ for $(a, b), (c, d) \in A$.
Determine, whether '*' is commutative or associative. Then, with respect to '*' on A :
(*i*) Find the identity element in A
(*ii*) Find the invertible elements of A. *(A.I.C.B.S.E. 2017)*

21. Let A = **Q** × **Q**, where **Q** is the set of all rational numbers and '*' be the binary operation on A defined by :
$(a, b) * (c, d) = (ac, b + ad)$ for $(a, b), (c, d) \in A$.
Then, find : (*i*) The identity element of '*' in A
(*ii*) Invertible elements of A and hence write the inverse of elements (5, 3) and $\left(\frac{1}{2}, 4\right)$. *(A.I.C.B.S.E. 2015)*

22. A binary operation '*' is defined on the set **R** of real numbers by :

$$a * b = \begin{cases} a, & \text{if } b = 0 \\ |a| + b, & \text{if } b \neq 0 \end{cases}$$

If at least one of a and b is 0, then prove that $a * b = b * a$. Check whether '*' is commutative. Find the identity element for '*', if it exists. *(C.B.S.E. Sample Paper 2018)*

Answers

1. (*i*) 7 (*ii*) 8. **2.** 2. **5.** 5.

8. (*i*) No (*ii*) Yes (*iii*) Yes (*iv*) Yes (*v*) Yes.

10. (*i*) Neither commutative nor associative
(*ii*) Commutative but not associative
(*iii*)-(*iv*) Both commutative and associative
(*v*) Commutative but not associative
(*vi*) – (*vii*) Neither commutative nor associative.

11. No.

12. (*i*) 5 * 7 = 35, 20 * 16 = 80
(*ii*) Yes (*iii*) Yes (*iv*) 1 (*v*) 1.

13. Both commutative and associative ; does not have any identity in **N**.

14. (*a*) (*ii*), (*iv*), (*v*) are commutative; (*v*) is associative.

17.

^	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

18. (*i*) False (*ii*) True. **19.** Yes, Yes, Yes.

19. (*i*) (3, 5), (*iii*) No identity element.

20. Not commutative but associative (*i*) (1, 0) (*ii*) $\left(\frac{1}{a}, \frac{-b}{a}\right)$.

21. (*i*) (1, 0) (*ii*) (I) $\left(\frac{1}{5}, -\frac{3}{5}\right)$ (II) (2, – 8).

22. Commutative; 0.

Hints to Selected Questions

3. (*a*) (*i*) For all $a, b \in$ **R**,
$a + b = b + a$ and $a \times b = b \times a$.
Hence, '+' and '×' are commutative binary operations.

(*b*) (*i*) For all $a, b, c \in$ **R**,
$(a + b) + c = a + (b + c)$
and $(a \times b) \times c = a \times (b \times c)$.
Hence, '+' and '×' are associative binary operations.

4. (*i*) Since $-a \notin$ **N**, when $a \in$ **N**,
∴ $-a$ is not the inverse of a under addition.

6. For $a, b \in$ **R**, $a + 4b^2 \in$ **R**.

8. (*i*) For $a, b \in$ **Z**, $a - b$ may not belong to $\mathbf{Z}^+$
⇒ '*' is not a binary operation.

10. (*i*) For $a, b \in$ **Z**, $a - b \neq b - a$, in general
⇒ '*' is not commutative.
For $a, b, c \in$ **Z**, $(a * b) * c \neq a * (b * c)$
⇒ '*' is not associative.

11. Since $a * b$ = l.c.m. of a, b.
∴ 2 * 3 = l.c.m. of 2, 3 = 6 ∉ {1, 2, 3, 4, 5}.
Hence, '*' is not a binary operation.

17.

^	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

21. (*ii*) Inverse of (a, b) will be $\left(\frac{1}{a}, \frac{-b}{a}\right)$.

(I) Inverse of (5, 3) = $\left(\frac{1}{5}, \frac{-3}{5}\right)$.

(II) Inverse of $\left(\frac{1}{2}, 4\right) = \left(2, \frac{-4}{1/2}\right) = (2, -8)$.

NCERT-FILE

Questions from NCERT Book

(For each Unsolved question, refer : "Solution of Modern's abc of Mathematics")

Exercise 1.1

1. Determine whether each of the following relations are reflexive, symmetric and transitive :

(*i*) Relation R in the set A = {1, 2, 3, ..., 13, 14}] defined as :

R $= \{(x, y) : 3x - y = 0\}$

(*ii*) Relation R in the set **N** of natural numbers defined as :

R $= \{(x, y) : y = x + 5 \text{ and } x < 4\}$

(*iii*) Relation R in the set A = {1, 2, 3, 4, 5, 6} as :

R $= \{(x, y) : y \text{ is divisible by } x\}$

(*iv*) Relation R in the set **Z** of all integers defined as :

R $= \{(x, y) : x - y \text{ is an integer}\}$

(*v*) Relation R in the set A of human beings in a town at a particular time given by :

(*a*) R = {(*x*, *y*) : *x* and *y* work at the same place}

(*b*) R = {(*x*, *y*) : *x* and *y* live in the same locality}

(*c*) R = {(*x*, *y*) : *x* is exactly 7 cm taller than *y*}

(*d*) R = {(*x*, *y*) : *x* is wife of *y*}

(*e*) R = {(*x*, *y*) : *x* is father of *y*}.

[**Solution :** Refer Q.7 ; Ex. 1(*a*)]

2. Show that the relation R in the set **R** of real numbers, defined as :

$R = \{(a, b) : a \leq b^2\}$ is neither reflexive nor symmetric nor transitive.

Solution :

R is not reflexive. [$\because$ *$a \leq a^2$ is not true $\forall$ $a \in$* **R**

e.g. $\frac{1}{3}$ *is not less than* $\frac{1}{9}$]

R is not symmetric.

[$\because$ *If $a \leq b^2$, then $b \leq a^2$ is not true e.g. $2 \leq 6^2$ but 6 is not less than 2^2*]

R is not transitive.

[$\because$ *If $a \leq b^2$, $b \leq c^2$, then $a \leq c^2$ is not true e.g. $2 < (-2)^2$, $-2 < (-1)^2$ but 2 is not less than $(-1)^2$*]

3. Check whether the relation R defined in the set.

{1, 2, 3, 4, 5, 6} as :

$R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive.

[**Solution :** Refer Q. 11(*a*) ; Ex. 1(*a*)]

4. Show that the relation R in **R** defined as $R = \{(a, b) : a \leq b\}$, is reflexive and transitive but not symmetric.

[**Solution :** Refer Q. 8(*a*) ; Ex. 1(*a*)]

5. Check whether the relation R in **R** defined by $R = \{(a, b) : a \leq b^3\}$ is reflexive, symmetric or transitive.

[**Solution :** Refer Q. 10 ; Ex. 1(*a*)]

6. Show that the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} is symmetric but neither reflexive nor transitive.

[**Solution :** Refer Q. 12(*a*) ; Ex. 1(*a*)]

7. Show that the relation R in the set A of all the books in a library of a college, given by R = {(*x*, *y*) : *x* and *y* have same number of pages} is an equivalence relation.

Solution :

R = {(*x*, *y*) : *x* and *y* have same number of pages}.

R is reflexive.

[$\because$ *(x, x) $\in$ R as x and x have same number of pages $\forall$ $x \in A$*]

R is symmetric. [$\because$ *(x, y) $\Rightarrow$ x and y have same number of pages $\Rightarrow$ y and x have same number of pages $\Rightarrow$ (y, x) $\in$ R*]

R is transitive. [$\because$ *(x, y) $\in$ R and (y, z) $\in$ R $\Rightarrow$ x and y have same number of pages & y and z have same number of pages $\Rightarrow$ x and z have same number of pages $\Rightarrow$ (x, z) $\in$ R*]

Hence, R is an equivalence relation.

8. Show that the relation R in the set A = {1, 2, 3, 4, 5} given by :

$R = \{(a, b) : |a - b| \text{ is even}\}$, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of {2, 4}.

[**Solution :** Refer Q. 16 ; Ex. 1(*a*)]

9. Show that each of the relation R in the set $A = \{x \in \mathbf{Z} : 0 \leq x \leq 12\}$, given by :

(*i*) $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$

(*ii*) $R = \{(a, b) : a = b\}$

is an equivalence relation. Find the set of all elements related to 1 in each case.

Solution :

Here A = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}.

(*i*) $R = \{(a, b) : |a - b| \text{ is a multiple of } 4\}$

R is reflexive.[$\because$ *$|a - a| = 0$, which is a multiple of 4 $\Rightarrow$ (a, a) $\in$ R $\forall$ $a \in A$*]

R is symmetric.

[$\because$ *(a, b) $\in$ R $\Rightarrow$ $|a - b|$ is a multiple of 4 $\Rightarrow$ $|b - a|$ is a multiple of 4 $\Rightarrow$ (b, a) $\in$ R*]

R is transitive.

[$\because$ *(a, b) $\in$ R and (b, c) $\in$ R $\Rightarrow$ $|a - b|$ is a multiple of 4 and $|b - c|$ is a multiple of 4*

$\Rightarrow$ (a – b) is a multiple of 4 and b – c is a multiple of 4

$\Rightarrow$ (a – b) + (b – c) i.e., (a – c) is a multiple of 4

⇒ $|a-c|$ *is a multiple of 4* ⇒ $(a, c) \in R$]

Hence, R is an equivalence relation.

Set of elements, which are related to 1

$= \{a \in A; (a, 1) \in R\}$

$= \{a \in A; |a-1| \text{ is multiple of } 4\}$

$= \{1, 5, 9\}$.

[$\because$ $|1-1| = 0, |5-1| = 4, |9-1| = 8$, *which are multiples of 4*]

(*ii*) $R = \{(a, b) : a = b\}$.

R is reflexive. [$\because a = a \ \forall \ a \in A$]

R is symmetric.

[$\because (a, b) \in R \Rightarrow a = b \Rightarrow b = a \Rightarrow (b, a) \in R$]

R is transitive.

[$\because (a, b) \in R$ *and* $(b, c) \in R \Rightarrow a = b$ *and* $b = c \Rightarrow a = c \Rightarrow (a, c) \in R$]

Hence, R is an equivalence relation.

Set of elements which are related to 1

$= \{a \in A ; (a, 1) \in R\}$

$= \{a \in A; a = 1\} = \{1\}$.

10. Give an example of a relation, which is :

(*i*) Symmetric but neither reflexive nor transitive

(*ii*) Transitive but neither reflexive nor symmetric

(*iii*) Reflexive and symmetric but not transitive

(*iv*) Reflexive and transitive but not symmetric

(*v*) Symmetric and transitive but not reflexive

[**Solution :** Refer Q. 4 ; Ex. 1(*a*)]

11. Show that the relation R in the set A of points in a plane given by R = {(P, Q) : distance of the point P from the origin is same as the distance of the point Q from the origin}, is an equivalence relation. Further, show that the set of all points related to a point P ≠ (0, 0) is the circle passing through P with origin as centre.

Solution :

Here R = {(P, Q) : | OP | = | OQ |, where O is the origin}

R is reflexive. [$\because |OP| = |OP|$,

$\therefore (P, P) \in R \ \forall \ P \in A$]

R is symmetric. [$\because (P, Q) \in R \Rightarrow |OP| = |OQ|$]

⇒ [$|OQ| = |OP| \Rightarrow (Q, P) \in R$]

R is transitive. [$\because (P, Q) \in R$ *and* $(Q, S) \in R$

⇒ $|OP| = |OQ|$ *and* $|OQ| = |OS|$

⇒ $|OP| = |OS| \Rightarrow (P, S) \in R$]

Hence, R is an equivalence relation.

Set of points related to P ≠ (0, 0)

= {Q ∈ A : (Q, P) ∈ R}

= {Q ∈ A : | OQ | = | OP |}

= {Q ∈ A : Q lies on the circle through P with centre O}.

12. Show that the relation R defined in the set A of all triangles as $R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}$, is equivalence relation. Consider three right angle triangles T_1 with sides 3, 4, 5, T_2 with sides 5, 12, 13 and T_3 with sides 6, 8, 10. Which triangles among T_1, T_2 and T_3 are related ?

[**Solution :** Refer Q. 22 ; Ex. 1(*a*)]

13. Show that the relation R defined in the set A of all polygons as $R = \{(P_1, P_2) : P_1 \text{ and } P_2 \text{ have same number of sides}\}$, is an equivalence relation. What is the set of all elements in A related to the right angle triangle T with sides 3, 4 and 5 ? ***(P.B. 2016)***

Solution :

We have : $R = \{(P_1, P_2), P_1 \text{ and } P_2$ have same number of sides}.

R is reflexive. [$\because$ *P and P have same number of sides* ⇒ $(P, P) \in R \ \forall \ P \in A$]

R is symmetric. [$\because (P_1, P_2) \in R \Rightarrow P_1$ *and* P_2 *have same number of sides*

⇒ P_2 *and* P_1 *have same number of sides*

⇒ $(P_2, P_1) \in R$]

R is transitive. [$\because (P_1, P_2) \in R$ *and* $(P_2, P_3) \in R$ ⇒ P_1 *and* P_2 *have same number of sides &* P_2 *and* P_3 *have same number of sides* ⇒ P_1 *and* P_3 *have same number of sides* ⇒ $(P_1, P_3) \in R$]

Hence, R is an equivalence relation.

Here T is a triangle.

∴ P ∈ A is related to T iff P and T have same number of sides

⇒ (P, T) ∈ R iff P and T have same number of sides

⇒ (P, T) ∈ R iff P is a triangle.

Hence, the set of all elements, which are related to T is the set of all triangles in A.

14. Let L be the set of all lines in XY plane and R be the relation in L defined as $R = \{(L_1, L_2) : L_1 \text{ is parallel to } L_2\}$. Show that R is an equivalence relation. Find the set of all lines related to the line $y = 2x + 4$.

[**Solution :** Refer Q. 23. ; Ex. 1(*a*)]

15. Let R be the relation in the set {1, 2, 3, 4} given by : R = {(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)}. Choose the correct answer :

(A) R is reflexive and symmetric but not transitive

(B) R is reflexive and transitive but not symmetric

(C) R is symmetric and transitive but not reflexive

(D) R is an equivalence relation. [**Ans.** (B)]

16. Let R be the relation in the set N given by : $R = \{(a, b) : a = b - 2, b > 6\}$. Choose the correct answer :

(A) (2, 4) ∈ R

(B) (3, 8) ∈ R

(C) (6, 8) ∈ R

(D) (8, 7) ∈ R. [**Ans.** (C)]

Exercise 1.2

1. Show that the function $f: \mathbf{R}_* \to \mathbf{R}_*$ defined by $f(x) = \frac{1}{x}$ is one-one and onto, where $\mathbf{R}_*$ is the set of all non-zero real numbers. Is the result true, if the domain $\mathbf{R}_*$ is replaced by **N** with co-domain being same as $\mathbf{R}_*$?

Solution :

(*i*) Let $x_1, x_2 \in \mathbf{R}_*$.

Now $f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$

$\Rightarrow f$ is one-one.

Let $y \in \mathbf{R}_*$.

Then $f(x) = y \Rightarrow \frac{1}{x} = y$

$\Rightarrow x = \frac{1}{y},\ y \neq 0$ as $y \in \mathbf{R}_* \Rightarrow f\left(\frac{1}{y}\right) = y$.

$\therefore$ Corresponding to each $y \in \mathbf{R}_*$, there exists $\frac{1}{y} \in \mathbf{R}_*$ s.t.

$$f\left(\frac{1}{y}\right) = y \Rightarrow f \text{ is onto.}$$

(*ii*) $f: \mathbf{N} \to \mathbf{R}_*$, then again f is one-one but not onto.

$$\left[\because \textit{Range of } f = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\right\} \neq \mathbf{R}_*\right]$$

2. Check the injectivity and surjectivity of the following functions :

(*i*) $f: \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^2$

(*ii*) $f: \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^2$

(*iii*) $f: \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^2$

(*iv*) $f: \mathbf{N} \to \mathbf{N}$ given by $f(x) = x^3$

(*v*) $f: \mathbf{Z} \to \mathbf{Z}$ given by $f(x) = x^3$.

[**Solution :** Refer Q. 14 ; Ex. 1(*b*)]

3. Prove that the Greatest Integer Function $f: \mathbf{R} \to \mathbf{R}$, given by $f(x) = [x]$, is neither one-one nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

Solution :

Since $f(x) = 0\ \forall\ x \in [0, 1)$,

$\therefore$ f is not one-one.

And $f: \mathbf{R} \to \mathbf{R}$ does not attain non-integral values

$\Rightarrow$ non-integral pair of **R** do not have their pre-images in the domain.

Thus f is not onto.

Hence, f is neither one-one nor onto.

4. Show that the Modulus Function $f: \mathbf{R} \to \mathbf{R}$, given by : $f(x) = |x|$, is neither one-one nor onto, where $|x|$ is x, if x is positive or 0 and $|x|$ is $-x$, if x is negative.

[**Solution :** Refer Q. 16 ; Ex. 1(*b*)]

5. Show that the Signum Function $f: \mathbf{R} \to \mathbf{R}$, given by :

$$f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$$

is neither one-one nor onto.

[**Solution :** Refer Q. 17 ; Ex. 1(*b*)]

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one.

[**Solution :** We have : f = {(1, 4), (2, 5), (3, 6)}.

Here $f(1) = 4$, $f(2) = 5(i)$ and $f(3) = 6$.

Since different elements have different images,

$\therefore$ 'f' is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(*i*) $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 3 - 4x$.

Solution : Let $x_1, x_2 \in \mathbf{R}$.

Now $f(x_1) = f(x_2) \Rightarrow 3 - 4x_1 = 3 - 4x_2$

$\Rightarrow x_1 = x_2$

$\Rightarrow$ 'f' is one–one.

Let $y \in \mathbf{R}$. Let $y = f(x_0)$.

Then $3 - 4x_0 = y \Rightarrow x_0 = \frac{3-y}{4}$.

Now $y \in \mathbf{R}$

$\Rightarrow \frac{3-y}{4} \in \mathbf{R} \Rightarrow x_0 \in \mathbf{R}$.

$$f(x_0) = 3 - 4x_0 = 3 - 4\,\frac{3-y}{4} = 3 - 3 + y = y.$$

$\therefore$ For each $y \in \mathbf{R}$, there exists $x_0 \in \mathbf{R}$ such that $f(x_0) = y$.

$\therefore$ 'f' is onto.

Hence, 'f' is one-one and onto *i.e.* bijective.

(*ii*) $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 1 + x^2$.

[**Solution :** Refer Q. 9 ; Ex. 1(*b*)]

8. Let A and B sets. Show that $f: A \times B \to B \times A$ such that $f(a, b) = (b, a)$ is bijective function.

[**Solution :** Refer Q. 15 ; Ex. 1(*b*)]

9. Let $f: \mathbf{N} \to \mathbf{N}$ be defined by $f(n) = \begin{cases} \frac{n+1}{2}, \text{ if } n \text{ is odd} \\ \frac{n}{2}, \text{ if } n \text{ is even} \end{cases}$

for all $n \in \mathbf{N}$.

State whether the function f is bijective. Justify your answer.

Solution : Here $f(1) = \frac{1+1}{2} = 1$,

$f(2) = \frac{2}{2} = 1$

$f(3) = \frac{3+1}{2} = 2$, $f(4) = \frac{4}{2} = 2$.

Thus $f(2k-1) = \dfrac{(2k-1)+1}{2} = k$

and $f(2k) = \dfrac{2k}{2} = k$

$\Rightarrow f(2k-1) = f(2k)$, where $k \in \mathbf{N}$

$\Rightarrow$ f is not one-one.

But f is onto because $R_f = \mathbf{N}$

[$\because$ *For any* $x \in N$, $2x \in N$ *such that* $f(2x) = \dfrac{2x}{2} = x$]

$\Rightarrow$ f is onto.

Hence, 'f' is not bijective.

10. Let $A = \mathbf{R} - \{3\}$ and $B = \mathbf{R} - \{1\}$. Consider the function :

$f : A \to B$ defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$. Is f one-one and onto ? Justify your answer.

Solution : Let $x_1, x_2 \in \mathbf{R} - \{3\}$.

Now $f(x_1) = f(x_2)$

$\Rightarrow \dfrac{x_1 - 2}{x_1 - 3} = \dfrac{x_2 - 2}{x_2 - 3}$

$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$

$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_2x_1 - 3x_2 - 2x_1 + 6$

$\Rightarrow -3x_1 - 2x_2 = -3x_2 - 2x_1$

$\Rightarrow x_1 = x_2$

$\Rightarrow$ f is one-one.

Let $y \in \mathbf{R} - \{1\}$.

Then $f(x) = y$.

When $\dfrac{x-2}{x-3} = y$, $x \neq 3$

$\Rightarrow x - 2 = yx - 3y$

$\Rightarrow x - xy = 2 - 3y$

$\Rightarrow x = \dfrac{2-3y}{1-y} \in A$

$\left[\because \dfrac{2-3y}{1-y} = 3 - \dfrac{1}{1-y} \neq 3\right]$

$\therefore$ Corresponding to each $y \in B$, there exists $\dfrac{2-3y}{1-y} \in A$ such that $f\left(\dfrac{2-3y}{1-y}\right) = y$

$\Rightarrow$ f is onto.

Hence, 'f' is one-one and onto.

11. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = x^4$.

Choose the correct answer :

(A) f is one-one onto

(B) f is many-one onto

(C) f is one-one but not onto

(D) f is neither one-one nor onto. [**Ans.** (D)]

12. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 3x$.

Choose the correct answer :

(A) f is one-one onto

(B) f is many-one onto

(C) f is one-one but not onto

(D) f is neither one-one nor onto. [**Ans.** (A)]

Exercise 1.3

1. Let $f : \{1, 3, 4\} \to \{1, 2, 5\}$ and $g : \{1, 2, 5\} \to \{1, 3\}$ be given by :

$f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down gof.

Solution : We have : $f : \{1, 3, 4\} \to \{1, 2, 5\}$

and $g : \{1, 2, 5\} \to \{1, 3\}$.

Here $f(1) = 2$, $f(3) = 5$ and $f(4) = 1$

and $g(1) = 3$, $g(2) = 3$ and $g(5) = 1$.

Here $R_f = \{1, 2, 5\} = D_g$.

$\therefore D_{gof} = D_f = \{1, 3, 4\}$.

Now $(gof)(1) = g(f(1)) = g(2) = 3$

$(gof)(3) = g(f(3)) = g(5) = 1$

and $(gof)(4) = g(f(4)) = g(1) = 3$.

Hence, $(gof) : \{(1, 3), (3, 1), (4, 3)\}$.

2. Let f, g and h be functions from $\mathbf{R}$ to $\mathbf{R}$. Show that :

$(f + g)oh = foh + goh$

$(f \cdot g)oh = (foh) \cdot (goh)$.

[**Solution :** Refer Q. 12 ; Ex. 1(c)]

3. Find gof and fog, if :

(i) $f(x) = |x|$ and $g(x) = |5x - 2|$

(ii) $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$.

(H.B. 2016, 14; Kerala B. 2016)

Solution :

(i) (a) $gof(x) = g(f(x)) = g(|x|) = |5|x| - 2|$.

(b) $fog(x) = f(g(x)) = f(|5x-2|)$
$= \big||5x - 2|\big| = |5x - 2|$.

(ii) (a) $gof(x) = g(f(x)) = g(8x^3)$
$= (8x^3)^{1/3} = 2x$.

(b) $fog(x) = f(g(x))$
$= f(x^{1/3}) = 8(x^{1/3})^3 = 8x$.

4. If $f(x) = \dfrac{(4x+3)}{(6x-4)}$, $x \neq \dfrac{2}{3}$, show that $fof(x) = x$, for all $x \neq \dfrac{2}{3}$. What is the inverse of f ?

Solution :

We have : $f(x) = \dfrac{(4x+3)}{(6x-4)},\ x \neq \dfrac{2}{3}.$

(*a*) $fof(x) = f(f(x)) = f\left(\dfrac{4x+3}{6x-4}\right) = \dfrac{4\left(\dfrac{4x+3}{6x-4}\right)+3}{6\left(\dfrac{4x+3}{6x-4}\right)-4}$

$= \dfrac{16x+12+18x-12}{24x+18-24x+16} = \dfrac{34x}{34} = x.$

(*b*) Let $y = \dfrac{4x+3}{6x-4}$

$\Rightarrow\ 6xy - 4y = 4x + 3 \Rightarrow (6y-4)x = 4y + 3$

$\Rightarrow\ x = \dfrac{4y+3}{6y-4}$

$\Rightarrow\ g(y) = f^{-1}(y) = \dfrac{4y+3}{6y-4}.$

$\therefore\ f^{-1}(x) = \dfrac{4x+3}{6x-4} = f(x).$

Hence, $f^{-1} = f.$

5. State with reason whether following functions have inverse :

(*i*) $f : \{1, 2, 3, 4\} \to \{10\}$ with $f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$

(*ii*) $g : \{5, 6, 7, 8\} \to \{1, 2, 3, 4\}$ with $g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$

(*iii*) $h : \{2, 3, 4, 5\} \to \{7, 9, 11, 13\}$ with $h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$.

Solution : (*i*) Here $f(1) = f(2) = f(3) = f(4) = 10$,

$\therefore$ f is not one-one [$\because$ *1, 2, 3, 4 have same image 10*]

$\Rightarrow$ f is many-one

$\Rightarrow$ f has no inverse.

(*ii*) Here $f(5) = f(7) = 4$.

$\therefore$ f is not one-one [$\because$ *5 and 7 have same image 4*]

$\Rightarrow$ f is many-one

$\Rightarrow$ f has no inverse.

(*iii*) Here each element of {2, 3, 4, 5} has a unique element in {7, 9, 11, 13}.

Similarly each element of {7, 9, 11, 13} has a unique pre-image in {2, 3, 4, 5}.

$\therefore$ h is one-one onto $\Rightarrow$ f is invertible.

Hence, 'h' has the inverse.

6. Show that $f : [-1, 1] \to \mathbf{R}$, given by $f(x) = \dfrac{x}{(x+2)}$ is one-one. Find the inverse of the function $f : [-1, 1] \to$ Range f.

$\left(\right.$**Hint :** For $y \in$ Range f, $y = f(x) = \dfrac{x}{x+2}$, for some x in $[-1, 1]$, *i.e.*, $x = \dfrac{2y}{(1-y)}\left.\right)$

[Solution : Refer Q. 9 ; Ex. 1(*d*)]

7. Consider $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.

[Solution : Refer Q. 5(*a*) ; Ex. 1(*d*)]

8. Consider $f : \mathbf{R}_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where $\mathbf{R}_+$ is the set of non-negative real numbers. ***(Jammu B. 2017)***

Solution : $f(x_1) = f(x_2)$

$\Rightarrow\ x_1^2 + 4 = x_2^2 + 4$

$\Rightarrow\ x_1^2 = x_2^2$

$\Rightarrow\ |x_1| = |x_2|$

$\Rightarrow\ x_1 = x_2$ [$\because x_1, x_2 \geq 0$]

$\Rightarrow f$ is one-one $\Rightarrow$ f is invertible.

Let $y = f(x) \Rightarrow y = x^2 + 4$

$\Rightarrow\ x = \sqrt{y-4}.$ [$\because x \geq 0$]

Hence, $f^{-1}(y) = \sqrt{y-4}.$ [$\because$ *x is real*]

9. Consider $f : \mathbf{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\dfrac{\left(\sqrt{y+6}\right)-1}{3}\right)$.

Solution : In order to prove that 'f' is invertible, it is sufficient to prove that 'f' is bijective.

f is one-one :

Let $x_1, x_2 \in \mathbf{R}^+$.

Then $f(x_1) = f(x_2)$

$\Rightarrow 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$

$\Rightarrow 9\left(x_1^2 - x_2^2\right) + 6\left(x_1 - x_2\right) = 0$

$\Rightarrow (x_1 - x_2)\,[9(x_1 + x_2) + 6] = 0$

$\Rightarrow\ x_1 - x_2 = 0$

$\Rightarrow\ x_1 = x_2.$

Thus 'f' is one-one.

f is onto.

Obviously $f : \mathbf{R}^+ \to$ Range (f) is onto.

Thus 'f' is onto.

Hence, 'f' is one-one and onto function

$\Rightarrow$ 'f' is invertible.

Inverse. Let f^{-1} denote the inverse of f.

Then $f(x) = y$

$\Rightarrow\ 9x^2 + 6x - 5 = y$

$\Rightarrow 9x^2 + 6x - (5 + y) = 0.$

$$\therefore \quad x = \frac{-6 \pm \sqrt{36 + 36(5+y)}}{18}$$

$$= \frac{-6 \pm 6\sqrt{6+y}}{18}$$

$$= \frac{-1 \pm \sqrt{6+y}}{3}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{-1+\sqrt{6+y}}{3}. \quad [\textit{Taking +ve value}]$$

$$\text{Hence,} \quad f^{-1}(x) = \frac{-1+\sqrt{6+x}}{3}.$$

10. Let $f: X \to Y$ be an invertible function. Show that f has unique inverse.

(**Hint :** Suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$,

$fog_1(y) = 1_Y(y) = fog_2(y)$. Use one-one ness of f)

[**Solution :** Refer Th. III ; Page 1/23]

11. Consider $f: \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a, f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

[**Solution :** Refer Q. 10 ; Ex. 1(*d*)]

12. Let $f: X \to Y$ be an invertible function. Show that the inverse of f^{-1} is f *i.e.* $(f^{-1})^{-1} = f$.

Solution : $f: X \to Y$ is invertible

$\Rightarrow f$ is one-one and onto.

$f^{-1}: Y \to X$, defined as $f^{-1}(y) = x$.

Now $\quad f^{-1}(y_1) = f^{-1}(y_2)$

$\Rightarrow \quad f(f^{-1}(y_1)) = f(f^{-1}(y_2))$

$\Rightarrow \quad (fof^{-1})(y_1) = (fof^{-1})(y_2)$

$\Rightarrow \quad I_Y(y_1) = I_Y(y_2)$

$\Rightarrow \quad y_1 = y_2$

$\Rightarrow \quad f^{-1}$ is one-one $\Rightarrow f^{-1}$ is invertible.

Let $\quad g = (f^{-1})^{-1}$.

Then $\quad gof^{-1} = I_Y$

and $\quad f^{-1}og = I_X$.

For all $x \in X$,

$I_X(x) = x \Rightarrow (f^{-1}og)(x) = x$

$\Rightarrow \quad f^{-1}(g(x)) = x$

$\Rightarrow \quad f\{f^{-1}(g(x))\} = f(x)$

$\Rightarrow (fof^{-1})\, g(x) = f(x) \Rightarrow g(x) = f(x)$

$[\because fog^{-1} = I_Y]$

$\Rightarrow \quad g = x.$

13. If $f: \mathbf{R} \to \mathbf{R}$ be given by $f(x) = (3 - x^3)^{1/3}$, then $fof(x)$ is :

(A) $x^{1/3}$ (B) x^3

(C) x (D) $(3 - x^3)$. [**Ans.** (C)]

14. Let $f: \mathbf{R} - \left\{-\frac{4}{3}\right\} \to \mathbf{R}$ be a function defined as

$f(x) = \frac{4x}{3x+4}$. The inverse of f is the map :

g : Range $f \to \mathbf{R} - \left\{-\frac{4}{3}\right\}$ given by :

(A) $g(y) = \frac{3y}{3-4y}$ (B) $g(y) = \frac{4y}{4-3y}$

(C) $g(y) = \frac{4y}{3-4y}$ (D) $g(y) = \frac{3y}{4-3y}$.

[**Ans.** (B)]

Exercise 1.4

1. Determine whether or not each of the definition of * given below gives a binary operation. In the event that * is not a binary operation, give justification for this.

(*i*) On $\mathbf{Z}^+$, define * by $a * b = a - b$

(*ii*) On $\mathbf{Z}^+$, define * by $a * b = ab$

(*iii*) On $\mathbf{R}$, define * by $a * b = ab^2$

(*iv*) On $\mathbf{Z}^+$, define * by $a * b = |a - b|$

(*v*) On $\mathbf{Z}^+$, define * by $a * b = a$.

[**Solution :** Refer Q. 8 ; Ex. 1(*e*)]

2. For each operation * defined below, determine whether * is binary, commutative or associative.

(*i*) On $\mathbf{Z}$, define $a * b = a - b$

(*ii*) On $\mathbf{Q}$, define $a * b = ab + 1$

(*iii*) On $\mathbf{Q}$, define $a * b = \frac{ab}{2}$

(*iv*) On $\mathbf{Z}^+$, define $a * b = 2^{ab}$

(*v*) On $\mathbf{Z}^+$, define $a * b = a^b$

(*vi*) On $\mathbf{R} - \{-1\}$, define $a * b = \frac{a}{b+1}$.

[**Solution :** Refer Q. 10 ; Ex. 1(*e*)]

3. Consider the binary operation ^ on the set {1, 2, 3, 4, 5} defined by $a \wedge b = \min\{a, b\}$. Write the operation table of the operation ^.

[**Solution :** Refer Q. 17 ; Ex. 1(*e*)]

4. Consider a binary operation * on the set {1, 2, 3, 4, 5} given by the following multiplication table (Table : Below).

(*i*) Compute (2 * 3) * 4 and 2 * (3 * 4)

(*ii*) Is * commutative ?

(*iii*) Compute (2 * 3) * (4 * 5).

(**Hint :** use the following table)

Table

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Solution : (*i*) (2 * 3) * 4 = 1 * 4 = 1.
2 * (3 * 4) = 2 * 1 = 1.

(*ii*) Since the given composition table is symmetrical about the main diagonal,

∴ the binary operation is commutative.

(*iii*) (2 * 3) * (4 * 5) = 1 * 1 = 1.

5. Let *′ be the binary operation on the set {1, 2, 3, 4, 5} defined by $a *' b$ = H.C.F. of a and b. Is the operation *′ same as the operation * defined in Q. 4 above ? Justify your answer.

Solution :

We have : $a * b$ = H.C.F. of a and b.

The composition table is as below :

*′	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Clearly the composition table is same as that of **Q. 4**

6. Let * be the binary operation on **N** given by $a * b$ = L.C.M. of a and b. Find :

(*i*) 5 * 7, 20 * 16

(*ii*) Is * commutative ?

(*iii*) Is * associative ?

(*iv*) Find the identity of * in **N**

(*v*) Which elements of **N** are invertible for the operation * ?

[**Solution :** Refer Q. 12 ; Ex. 1(*e*)]

7. Is * defined on the set {1, 2, 3, 4, 5} by $a * b$ = L.C.M. of a and b a binary operation ? Justify your answer.

[**Solution :** Refer Q. 11 ; Ex. 1(*e*)]

8. Let * be the binary operation on **N** defined by $a * b$ = H.C.F. of a and b. Is * commutative ? Is * associative ? Does there exist identity for this binary operation on **N** ?

[**Solution :** Refer Q. 13 ; Ex. 1(*e*)]

9. Let * be a binary operation on the set **Q** of rational numbers as follows :

(*i*) $a * b = a - b$ (*ii*) $a * b = a^2 + b^2$

(*iii*) $a * b = a + ab$ (*iv*) $a * b = (a - b)^2$

(*v*) $a * b = \dfrac{ab}{4}$ (*vi*) $a * b = ab^2$.

Find which of the binary operations are commutative and which are associative.

[**Solution :** Refer Q. 14(*a*) ; Ex. 1 (*e*)]

10. Find which of the operations given above has identity.

[**Solution :** Refer Q. 14(*b*), Ex. 1(*e*)]

11. Let A = **N** × **N** and * be the binary operation on A defined by :

$(a, b) * (c, d) = (a + c, b + d)$.

Show that * is commutative and associative. Find the identity element for * on A, if any.

Solution : (i) Commutativity.

$$(a, b) * (c, d) = (a + c, b + d)$$
$$= (c + a, d + b)$$
[∵ *Addition is commutative in* **N**]
$$= (c, d) * (a, b).$$

Hence, '*' is commutative.

(ii) Associativity.

$$[(a * b) * (c, d)] * (e, f)] = (a + c, b + d) * (e, f)$$
$$= [(a + c) + e, (b + d) + f)$$
$$= (a + (c + e), b + (d + f))$$
[∵ *Addition is associative in* **N**]
$$= (a, b) * (c + e, d + f)$$
$$= (a, b) * [(c, d) * (e, f)].$$

Hence, '*' is associative.

(*iii*) Let (x, y) be the identity element.

Then $(a, b) * (x, y) = (a, b)$

⇒ $(a + x, b + y) = (a, b)$

⇒ $a + x = a, b + y = b$

⇒ $x = 0, y = 0.$

But $0 \notin$ **N.**

Hence, identity element does not exist.

12. State whether the following statements are true or false. Justify.

(*i*) For an arbitrary binary operation * on a set **N**,
$a * a = a \ \forall \ a \in$ **N**.

(*ii*) If * is a commutative binary operation on **N,** then
$a * (b * c) = (c * b) * a$.

[**Ans.** (*i*) False (*ii*) True]

13. Consider a binary operation * on **N** defined as :
$a * b = a^3 + b^3$. Choose the correct answer :

(A) Is * both associative and commutative ?

(B) Is * commutative but not associative ?

(C) Is * associative but not commutative ?

(D) Is * neither commutative nor associative ?

[**Ans.** (B)]

Miscellaneous Exercise on Chapter 1

1. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = 10x + 7$. Find the function $g : \mathbf{R} \to \mathbf{R}$ such that $g \circ f = f \circ g = 1_R$.

Solution : We have : $gof = I_R$, where $f(x) = 10x + 7$.

Now $gof(x) = I_R(x)$ for all $x \in \mathbf{R}$

$\Rightarrow g(f(x)) = x$ for all $x \in \mathbf{R}$ $[\because I_R(x) = x]$

$\Rightarrow g(10x + 7) = x$ for all $x \in \mathbf{R}$ $[\because f(x) = 10x + 7]$

$\Rightarrow g(y) = \dfrac{y-7}{10}$ for all $y \in \mathbf{R}$, where $10x + 7 = y$

$\Rightarrow g(x) = \dfrac{x-7}{10}$ for all $x \in \mathbf{R}$.

Hence, $g : \mathbf{R} \to \mathbf{R}$ is defined by $g(x) = \dfrac{x-7}{10}$.

2. Let $f : \mathbf{W} \to \mathbf{W}$ be defined as $f(n) = n - 1$, if n is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f. Here, $\mathbf{W}$ is the set of all whole numbers.

Solution : Let n_1 and n_2 be two distinct elements of $\mathbf{W}$.

To Prove : f is one-one.

Case I. *When n_1 and n_2 are both odd.*

Here $n_1 \neq n_2 \Rightarrow n_1 - 1 \neq n_2 - 1$

$\Rightarrow f(n_1) \neq f(n_2)$.

Case II. *When n_1 and n_2 are both even.*

Here $n_1 \neq n_2 \Rightarrow n_1 + 1 \neq n_2 + 1$

$\Rightarrow f(n_1) \neq f(n_2)$.

Case III. *When n_1 is odd and n_2 is even.*

Here $f(n_1) = n_1 - 1$ is even

$f(n_2) = n_1 + 1$ is odd

$\Rightarrow f(n_1) \neq f(n_2)$.

Case IV. *When n_1 is even and n_2 is odd.*

Here $f(n_1) = n_1 + 1$ is odd

$f(n_2) = n_1 - 1$ is even

$\Rightarrow f(n_1) \neq f(n_2)$.

Thus in all cases, $n_1 \neq n_2 \Rightarrow f(n_1) \neq f(n_2)$

$\Rightarrow f$ is one-one.

To Prove : *f is onto.*

If $n \in \mathbf{W}$ is any element, then :

$f(n - 1) = n$, if n is odd $[\because n - 1$ *is even*$]$

and $f(n + 1) = n$, if n is even. $[\because n + 1$ *is odd*$]$

$\therefore$ Each element of $\mathbf{W}$ is f-image of some element of $\mathbf{W}$.

Thus f is onto.

Hence, f is invertible.

Now $f(n - 1) = n$, if n is odd

$f(n + 1) = n$, if n is even

$\Rightarrow n - 1 = f^{-1}(n)$ if n is odd

$n + 1 = f^{-1}(n)$ if n is even.

$$\therefore \quad f^{-1}(n) = \begin{cases} n - 1, \text{ if } n \text{ is odd} \\ n + 1, \text{ if } n \text{ is even}. \end{cases}$$

Hence, $f^{-1} = f$.

3. If $f : \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Solution :

$$\begin{aligned} f(f(x)) &= (f(x))^2 - 3f(x) + 2 \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 \\ &= (x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2) \\ &\qquad - 3(x^2 - 3x + 2) + 2 \\ &= x^4 - 6x^3 + 10x^2 - 3x. \end{aligned}$$

4. Show that the function $f : \mathbf{R} \to \{x \in \mathbf{R} : -1 < x < 1\}$ defined by $f(x) = \dfrac{x}{1+|x|}$, $x \in \mathbf{R}$ is one one and onto function.

[**Solution :** Refer Q. 13 ; Ex. 1(*b*)]

5. Show that the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$ is injective.

[**Solution :** Refer Q. 10 (*a*) ; Ex. 1(*b*)]

6. Give examples of two functions $f : \mathbf{N} \to \mathbf{Z}$ and $g : \mathbf{Z} \to \mathbf{Z}$ such that $g \circ f$ is injective but g is not injective.

(**Hint :** Consider $f(x) = x$ and $g(x) = |x|$).

[**Solution :** Refer Q. 4 ; Ex. 1(*c*)]

7. Give examples of two functions $f : \mathbf{N} \to \mathbf{N}$ and $g : \mathbf{N} \to \mathbf{N}$ such that $g \circ f$ is onto but f is not onto.

(**Hint :** Consider $f(x) = x + 1$ and $g(x) = \begin{cases} x - 1 \text{ if } x > 1 \\ 1 \text{ if } x = 1 \end{cases}$.

[**Solution :** Refer Q. 5 ; Ex. 1(*c*)]

8. Given a non empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows :

For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X) ? Justify your answer.

Solution :

Since $A \subset A \ \forall \ A \in P(X)$.

$\therefore$ R is reflexive.

For A, B, C $\in$ P (X),

ARB and BRC $\Rightarrow A \subset B$ and $B \subset C \Rightarrow A \subset C \Rightarrow$ ARC.

$\therefore$ R is transitive.

However, R is not symmetric $[\because A \subset B \not\Rightarrow B \subset A]$

$\Rightarrow$ ARB $\not\Rightarrow$ BRA.

Hence, R is not an equivalence relation.

9. Given a non-empty set X, consider the binary operation $* : P(X) \times P(X) \to P(X)$ given by $A * B = A \cap B \ \forall$ A, B in P(X), where P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in P(X) with respect to the operation *.

Solution :

(*i*) Let E $\in$ P (X) be the identity element.

Then $A * E = E * A = A \ \forall \ A \in P(X)$

$\Rightarrow A \cap E = E \cap A = A \ \forall \ A \in P(X)$

$\Rightarrow X \cap E = X$ because $X \in P(X)$

$\Rightarrow X \subset E$.

Also $E \subset X$. $[\because E \in P(X)]$

Thus $E = X$.

Hence, X is the identity element.

(*ii*) Let $A \in P(X)$ be invertible.

Then there exists $B \in P(X)$ such that :

$$A * B = B * A = X,$$

where X is the identity element

$\Rightarrow \quad A \cap B = B \cap A = X$

$\Rightarrow X \subset A, X \subset B.$

Also $A, B \subset X$ $\quad [\because A, B \in P(X)]$

$\therefore \quad A = X = B.$

Hence, X is the only invertible element and $X^{-1} = B = X$.

10. Find the number of all onto functions from the set $\{1, 2, 3, ..., n\}$ to itself.

[**Solution :** Refer Q. 7 ; Rev. Ex.]

11. Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. Find F^{-1} of the following functions F from S to T, if it exists.

(*i*) $F = \{(a, 3), (b, 2), (c, 1)\}$

(*ii*) $F = \{(a, 2), (b, 1), (c, 1)\}$.

Solution : (*i*) We have :

$F : \{(a, 3), (b, 2), (c, 1)\}$.

$\therefore \quad R_F = \{1, 2, 3\} = T$

$\Rightarrow$ F is onto.

Also F is one-one.

[$\because$ *Different elements of S have different F-images*]

Thus F is one-one onto

$\Rightarrow F^{-1}$ exists.

And $F^{-1} = \{(3, a)\ (2, b), (1, c)\}$.

(*ii*) We have : $F = \{(a, 2), (b, 1), (c, 1)\}$.

F is not one-one. $\quad [\because F(b) = F(c) = 1]$

Hence, F is not invertible.

12. Consider the binary operations $* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $o : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined as $a * b = |a - b|$ and $a \ o \ b = a$, $\forall a, b \in \mathbf{R}$. Show that * is commutative but not associative, o is associative but not commutative. Further, show that $\forall a, b, c \in \mathbf{R}$, $a * (b \ o \ c) = (a * b) \ o \ (a * c)$. [If it is so, we say that the operation * distributes over the operation o]. Does o distribute over * ? Justify your answer.

Solution : We have :

$$a * b = |a - b| \text{ and } a \ o \ b = a.$$

(*i*) (*a*) $a * b = |a - b|$ and

$b * a = |b - a| = |a - b|$

Thus $a * b = b * a$

$\Rightarrow$ '*' is commutative.

(*b*) $a * (b * c) = a * |b - c|$

$= |a - |b - c||$

$(a * b) * c = |a - b| * c$

$= ||a - b| - c|.$

Thus $a * (b * c) \neq (a * b) * c$

$\Rightarrow$ '*' is not assocative.

(*ii*) (*a*) $a \ o \ b = a$ and $b \ o \ a = b$.

Thus $a \ o \ b \neq b \ o \ a$

$\Rightarrow$ 'o' is not commutative.

(*b*) $a \ o \ (b \ o \ c) = a \ o \ b = a$

$(a \ o \ b) \ o \ c = a \ o \ c = a.$

Thus $a \ o \ (b \ o \ c) = (a \ o \ b) \ o \ c$

$\Rightarrow$ 'o' is associative.

(*iii*) (*a*) $a * (b \ o \ c) = a * b = |a - b|$

$(a * b) \ o \ (a * c) = |a - b| \ o \ |a - c| = |a - b|.$

Hence, $a * (b \ o \ c) = (a * b) \ o \ (a * c)$.

(*b*) $a \ o \ (b * c) = a \ o \ |b - c| = a$

$(a \ o \ b) * (a \ o \ c) = a * a = |a - a| = 0.$

Thus $a \ o \ (b * c) \neq (a \ o \ b) * (a \ o \ c)$.

Hence, 'o' is not distributive over '*'.

13. Given a non-empty set X, let $* : P(X) \times P(X) \to P(X)$ be defined as $A * B = (A - B) \cup (B - A)$, $\forall A, B \in P(X)$. Show that the empty set ϕ is the identity for the operation * and all the elements A of P(X) are invertible with $A^{-1} = A$. (**Hint :** $(A - \phi) \cup (\phi - A) = A$ and $(A - A) \cup (A - A) = A * A = \phi$).

Solution : (*i*) Let E be the identity element.

Then $A * E = E * A = A \ \forall \ A \in P(X)$

$\Rightarrow (A - E) \cup (E - A) = A \ \forall \ A \in P(X).$

Let us take A as ϕ.

Then $(\phi - E) \cup (E - \phi) = \phi$

$\Rightarrow \quad \phi \cup E = \phi \Rightarrow E = \phi.$

Thus $A * \phi = \phi * A = (A - \phi) \cup (\phi - A)$

$= A \ \forall \ A \in P(X).$

Thus ϕ is the identity element.

(*ii*) Let $A \in P(X)$ be invertible.

Then there is $B \in P(X)$ such that

$$A * B = B * A = \phi$$

$\Rightarrow (A - B) \cup (B - A) = \phi$

$\Rightarrow \quad A - B = \phi$ and $B - A = \phi$

$\Rightarrow \quad A \subset B$ and $B \subset A$

$\Rightarrow \quad A = B.$

$\therefore$ For all $A \in P(X)$, $A * A = \phi$.

Hence, A is invertible and $A^{-1} = A$.

14. Define a binary operation * on the set $\{0, 1, 2, 3, 4, 5\}$ as

$$a * b = \begin{cases} a + b, & \text{if } a + b < 6 \\ a + b - 6 & \text{if } a + b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element $a \neq 0$ of the set is invertible with $6 - a$ being the inverse of a.

Solution : (*i*) If '*e*' be the identity element, then :

$$a * e = e * a = a.$$

Now $a * 0 = a + 0,\ 0 * a = 0 + a = a.$

Thus $a * 0 = 0 * a = a.$

Hence, '0' is the identity of the operation.

(*ii*) If '*b*' be the inverse of '*a*', then :

$$a * b = b * a = e.$$

Now $a * (6 - a) = a + (6 - a) - 6 = 0$

$(6 - a) * a = (6 - a) + a - 6 = 0.$

Hence, each element *a* of the set is invertible with inverse $6 - a$.

15. Let $A = \{-1, 0, 1, 2\}$, $B = \{-4, -2, 0, 2\}$ and $f, g : A \to B$ be functions defined by $f(x) = x^2 - x$, $x \in A$ and $g(x) = 2\left|x - \frac{1}{2}\right| - 1$, $x \in A$. Are f and g equal ? Justify your answer.

(**Hint :** One may note that two functions $f : A \to B$ and $g : A \to B$ such that $f(a) = g(a)\ \forall\ a \in A$, are called equal functions).

Solution : Here $f(-1) = (-1)^2 - (-1) = 1 + 1 = 2,$

$$g(-1) = 2\left|-1 - \frac{1}{2}\right| - 1$$

$$= 2\left|-\frac{3}{2}\right| - 1 = 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2.$$

$$f(0) = 0^2 - 0 = 0,$$

$$g(0) = 2\left|0 - \frac{1}{2}\right| - 1 = 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0.$$

$$f(1) = 1^2 - 1 = 1 - 1 = 0,\ g(1) = 2\left|1 - \frac{1}{2}\right| - 1$$

$$= 2\left(\frac{1}{2}\right) - 1 = 1 - 1 = 0.$$

$$f(2) = 2^2 - 2 = 4 - 2 = 2,$$

$$g(2) = 2\left|2 - \frac{1}{2}\right| - 1$$

$$= 2\left(\frac{3}{2}\right) - 1 = 3 - 1 = 2.$$

From above, $f(-1) = g(-1)$; etc.

Hence, $f = g$.

16. Let $A = \{1, 2, 3\}$. Then number of relations containing (1, 2) and (1, 3), which are reflexive and symmetric but not transitive is :

(A) 1 (B) 2

(C) 3 (D) 4. [**Ans.** (A)]

17. Let $A = \{1, 2, 3\}$. Then number of equivalence relations containing (1, 2) is :

(A) 1 (B) 2

(C) 3 (D) 4. [**Ans.** (B)]

18. Let $f : \mathbf{R} \to \mathbf{R}$ be the Signum Function defined as :

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

and $g : \mathbf{R} \to \mathbf{R}$ be the Greatest Integer Function given by : $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x. Then, does *fog* and *gof* coincide in (0, 1] ?

[**Solution :** Refer Q. 18 ; Ex. 1(*c*)]

19. Number of binary operations on the set $\{a, b\}$ is :

(A) 10 (B) 16

(C) 20 (D) 8. [**Ans.** (B)]

Questions From NCERT Exemplar

Example 1. For the set A = {1, 2, 3}, define a relation *R* in the set A as follows :

***R* = {(1, 1), (2, 2) (3, 3), (1, 3)}.**

Write the ordered pairs to be added to R to make it the smallest equivalence relation.

Solution. (3, 1) is the single ordered pair, which needs to be added to R in order to make the smallest equivalence relation.

Example 2. Let *R* be the equivalence relation in the set Z of integers given by :

R = {(a, b) : 2 divides a – b}.

Write the equivalence class [0].

Solution. $[0] = \{0, \pm 2, \pm 4, \pm 6, ...\}$.

Example 3. If A = {1, 2, 3} and f, g are relations corresponding to the subset of A × A indicated against them, which of f, g is a function ? Why ?

f = {(1, 3) (2, 3), (3, 2)} ; g = {(1, 2), (1, 3), (3, 1)}.

Solution. (*i*) '*f*' is a function.

[$\because$ *each element of A in the first place in the ordered pair is related to only one element of A in the second place*]

(*ii*) '*g*' is not a function.

[$\because$ *1 is related to two elements of A namely 2 and 3*]

Example 4. In the set of natural numbers N, define a relation R as follows :

∀ n, m ∈ N, nRm if on division by 5 each of the integers n and m leaves the remainder less than 5 i.e. one of the numbers 0, 1, 2, 3 and 4. Show that R is equivalence relation. Also, obtain the pairwise disjoint subsets determined by R.

Solution. Partition the set **N** into pairwise disjoint subsets.

The equivalent classes are as given by :

$A_0 = \{5, 10, 15, 20, ...\}$

$A_1 = \{1, 6, 11, 16, 21, ...\}$

$A_2 = \{2, 7, 12, 17, 22, ...\}$

$A_3 = \{3, 8, 13, 18, 23, ...\}$

$A_4 = \{4, 9, 14, 19, 24, ...\}$.

Clearly the above five sets are pairwise disjoint and

$$A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 = \bigcup_{i=0}^{4} A_i = \mathbf{N}.$$

Example 5. Show that the function $f : \mathbf{R} \to \mathbf{R}$ defined by :

$$\mathbf{f(x) = \frac{x}{x^2+1} \forall x \in R,}$$

is neither one-one nor onto.

Solution. For $x_1, x_2 \in \mathbf{R}, f(x_1) = f(x_2)$

$$\Rightarrow \quad \frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$$

$$\Rightarrow \quad x_1x_2^2 + x_1 = x_2x_1^2 + x_2$$

$$\Rightarrow \quad x_1x_2(x_2 - x_1) = x_2 - x_1$$

$$\Rightarrow \quad x_1 = x_2 \text{ or } x_1x_2 = 1.$$

Here there are points x_1 and x_2 with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

[For Ex. *Take* $x_1 = 2,\ x_2 = \frac{1}{2}$,

then $f(x_1) = \frac{2}{4+1} = \frac{2}{5}$ *and* $f(x_2) = \frac{1/2}{1+1/4} = \frac{2}{5}$.

Thus $f(x_1) = f(x_2) = \frac{2}{5}$ *but* $x_1 \neq x_2$ *i.e.* $f(x) = \frac{1}{2}$]

Hence, 'f' is not one-one.

Also 'f' is not onto.

For if so, then for $1 \in \mathbf{R}$, there exists $x \in \mathbf{R}$ such that $f(x) = 1$,

which gives $\frac{x}{x^2+1} = 1$.

But there is no such x in the domain **R**.

[$\because x^2 - x + 1 = 0$ *does not give any real value of* x]

Exercise

1. Let A = {0, 1, 2, 3} and define a relation R on A as follows :

   R = {(0, 0), (0, 1), (0, 3), (1, 0), (1, 1), (2, 2). (3, 0), (3, 3)}.

   Is R Reflexive ? Symmetric ? Transitive ?
2. Let A = {1, 2, 3,......9} and R be the relation in A × A defined by (a, b) R (c, d) if $a + d = b + c$,

   for $(a, b), (c, d) \in A \times A$. Prove that R is an equivalence relation. Also, obtain the equivalnce class [(2, 5)].
3. If f = {(5, 2), (6, 3)}, g = {(2, 5), (3, 6)}, write fog.
4. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2 + 1$.

   Find the pre-image of (*i*) 17 (*ii*) – 3.
5. Let the function : f $\mathbf{R} \to \mathbf{R}$ to be defined by $f(x) = \cos x$ for all $x \in \mathbf{R}$. Show that 'f' is neither one-one nor onto.
6. Let A = **R** – {3}, B = **R** – {1}.

   Let f : A → B be defined by :

   $$f(x) = \frac{x-2}{x-3} \forall x \in A.$$

   Then show that 'f' is bijective.
7. If the mappings f and g are given by :

   f = {(1, 2), (3, 5), (4, 1)} and g = {(2, 3), (5, 1), (1, 3)}, write fog.
8. Let '*' be the binary operation on **Q.** Find which of the following binary operations are commutative :

   (*i*) $a * b = a - b \ \forall a, b \in \mathbf{Q}$

   (*ii*) $a * b = a^2 + b^2 \ \forall a, b \in \mathbf{Q}$

   (*iii*) $a * b = a + ab \ \forall a, b \in \mathbf{Q}$

   (*iv*) $a * b = (a - b)^2 \ \forall a, b \in \mathbf{Q}$.
9. Let '*' be the binary operation defined on **R** by $a * b = 1 + ab \ \forall a, b \in \mathbf{R}$.

   Then the operation '*' is :

   (*i*) commutative but not associative

   (*ii*) associative but not commutative

   (*iii*) neither commutative nor associative

   (*iv*) both commutative and associative.
10. Is the binary operation '*' defined on **Z** (set of integers) by $m * n = m - n + mn$ for all $m, n \in \mathbf{Z}$ commutative ?

Answers

1. R is reflexive and symmetric but not transitive.
2. [(2, 5)] = [(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)].
3. fog = {(2, 2), (3, 3)}.
4. (*i*) {– 4,4} (*ii*) ϕ.
7. fog = {(2, 5), (5, 2), (1, 5)}.
8. (*ii*) and (*iv*)
9. (*i*).
10. No.

Revision Exercise

1. Let f : X → Y be a function. Define a relation *R* in X given by :

$$R = \{(a, b) : f(a) = f(b)\}.$$

Examine, if *R* is an equivalence relation. *(N.C.E.R.T.)*

Solution. For each $a \in X$, $(a, a) \in R$.

Thus R is *reflexive*. $[\because\ f(a) = f(a)]$

Now $(a, b) \in R \Rightarrow f(a) = f(b)$

$\Rightarrow f(b) = f(a)$

$\Rightarrow (b, a) \in R$.

Thus R is *symmetric*.

And $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow f(a) = f(b)$ and $f(b) = f(c)$

$\Rightarrow f(a) = f(c)$

$\Rightarrow (a, c) \in R$.

Thus R is *transitive*.

Hence, R is an equivalence relation.

2. If R_1 and R_2 are equivalence relations in a set A, show that $R_1 \cap R_2$ is also an equivalence relation. *(N.C.E.R.T.)*

Solution. Since R_1 and R_2 are equivalence relations, [*Given*]

$\therefore (a, a) \in R_1$ and $(a, a) \in R_2\ \forall\ a \in A$

$\Rightarrow (a, a) \in R_1 \cap R_2\ \forall\ a \in A$.

Thus $R_1 \cap R_2$ is *reflexive*.

Now $(a, b) \in R_1 \cap R_2$

$\Rightarrow (a, b) \in R_1$ and $(a, b) \in R_2$

$\Rightarrow (b, a) \in R_1$ and $(b, a) \in R_2$

$\Rightarrow (b, a) \in R_1 \cap R_2$.

Thus $R_1 \cap R_2$ is *symmetric*.

And $(a, b) \in R_1 \cap R_2$ and $(b, c) \in R_1 \cap R_2$

$\Rightarrow (a, c) \in R_1$ and $(a, c) \in R_2$

$\Rightarrow (a, c) \in R_1 \cap R_2$.

Thus $R_1 \cap R_2$ is *transitive*.

Hence, $R_1 \cap R_2$ is an equivalence relation.

3. Let X = {1, 2, 3, 4, 5, 6, 7, 8, 9}.

Let R_1 be a relation on X given by :

$R_1 = \{(x, y) : x - y$ is divisible by $3\}$

and R_2 be another relation on X given by :

$R_2 = \{(x, y) : \{x, y\} \subset \{1, 4, 7\}$ or $\{x, y\} \subset \{2, 5, 8\}$ or $\{x, y\} \subset \{3, 6, 9\}\}$.

Show that $R_1 = R_2$. *(N.C.E.R.T.)*

4. Show that the number of equivalence relations in the set {1, 2, 3} containing {1, 2} and {2, 1} is two. *(N.C.E.R.T.)*

5. Let A = {1, 2, 3}. Then show that the number of relations containing (1, 2) and (2, 3), which are reflexive and transitive but not symmetric, is four. *(N.C.E.R.T.)*

6. Find the number of all one–one functions from the set A = {1, 2, 3} to itself. *(N.C.E.R.T.)*

Solution. One–one function from {1, 2, 3} to itself is a permutation of 1, 2, 3.

But the total number of permutations of 1, 2, 3 = 3 ! = 6.

Hence, number of all one–one functions = 6.

7. Find the number of all onto functions from the set $\{1, 2, 3, ..., n\}$ to itself. *(N.C.E.R.T.)*

8. Give examples of two one–one functions f_1 and f_2 from **R** to **R** such that $f_1 + f_2 : \mathbf{R} \to \mathbf{R}$ defined by :

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

is not one–one. *(N.C.E.R.T.)*

9. Show that if f_1 and f_2 are one–one (respectively onto) maps from **R** to **R**, then the product :

$f_1 \times f_2 : \mathbf{R} \to \mathbf{R}$ defined by $(f_1 \times f_2)(x) = f_1(x) \times f_2(x)$

need not be one–one (respectively onto). *(N.C.E.R.T.)*

10. Let $f : X \to Y$ be such that $fof = f$. Show that f is onto if and only if f is one–one. *(N.C.E.R.T)*

11. Consider the identity function $I_N : N \to N$ defined as :

$$I_N(x) = x\ \forall\ x \in N.$$

Show that although I_N is onto but $I_N + I_N : N \to N$ defined as :

$$(I_N + I_N)(x) = I_N(x) + I_N(x) = x + x = 2x$$

is not onto. *(N.C.E.R.T.)*

Solution. Here I_N is onto. [*Given*]

But $I_N + I_N$ is not onto

[$\because$ *We can find 3 in the co-domain* **N**, *where 3 is not a multiple of 2*]

12. Consider a function $f : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by :

$f(x) = g(x)$ and $g : \left[0, \frac{\pi}{2}\right] \to \mathbf{R}$ given by :

$g(x) = \cos x$. Show that f and g are one–one, but $f + g$ is not one–one. *(N.C.E.R.T.)*

13. Find fof^{-1} and f^{-1} of for the function :

$f(x) = \frac{1}{x}, x \neq 0$. Also prove that $fof^{-1} = f^{-1}of$.

Solution. Here $f(x) = y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$.

$\therefore \quad f^{-1} = \{(y, x)\} = \left\{\left(y, \frac{1}{y}\right)\right\}$

$\Rightarrow \quad f^{-1}(x) = \frac{1}{x}$.

$\therefore \quad fof^{-1}(x) = f\left(\frac{1}{x}\right) = \frac{1}{1/x} = x.$

Also $f^{-1}of(x) = f^{-1}\left(\frac{1}{x}\right) = \frac{1}{1/x} = x.$

Thus $fof^{-1}(x) = f^{-1}of(x)$.

Hence, $fof^{-1} = f^{-1}of$.

14. Show that the number of binary operations on {1, 2} having 1 as identity and having 2 as the inverse of 2 is exactly one. *(N.C.E.R.T.)*

Solution. We have binary operation :

'*' on {1, 2} is a function from {1, 2} × {1, 2} to {1, 2}

i.e. {(1, 1), (1, 2), (2, 1), (2, 2)} → {1, 2}.

Since 1 is the identity, [*Given*]

∴ * (1, 1) = 1, * (1, 2) = 2, * (2, 1) = 2.

Since 2 is the inverse of 2 is 2, [*Given*]

∴ * (2, 2) = 1.

Hence, the number of desired binary operations is only one.

15. Determine whether the following binary operation on the set N is associative and commutative :

$$\mathbf{a * b = 1 \ \forall\ a,\ b \in N.}$$ *(N.C.E.R.T.)*

Solution. (*a*) (I) $a * b = 1 = b * a \ \ \forall \ (a, b) \in \mathbf{N}$.

Thus '*' is *commutative.*

(II) $(a * b) * c = 1 * c = 1$

$a * (b * c) = a * 1 = 1 \quad \forall\ a, b, c \in \mathbf{N}$

$\Rightarrow (a * b) * c = a * (b * c)$.

Thus '*' is *associative.*

Hence, '*' is commutative and associative.

16. Determine whether the following binary operation on the set **N** is associative and commutative :

$$a * b = \frac{a+b}{2} \quad \forall\ a, b \in \mathbf{N}.$$ *(N.C.E.R.T.)*

17. Consider the binary operations :

$* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $o : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined as :

$a * b = |a - b|$ and $a \ o \ b = a \quad \forall\ a, b \in \mathbf{R}$.

Show that * is commutative but not associative and *o* is associative but not commutative.

Also show that $a * (b \ o \ c) = (a * b) \ o \ (a * c) \ \forall\ a, b, c \in \mathbf{R}$. *(N.C.E.R.T.)*

18. Define a binary operation '*' on the set A = {0, 1, 2, 3, 4, 5}, given by a * b = (ab) mod 6. Show that for *, 1 and 5 are only invertible elements with $\mathbf{1^{-1} = 1}$ and $\mathbf{5^{-1} = 5}$.

[*Here (ab) mod 6, we mean the remainder after dividing ab by 6*] *(N.C.E.R.T.)*

Solution. Here $a * 1 = a \bmod 6 = a$

and $1 * a = a \bmod 6 = a \ \forall\ a \in A$.

Thus 1 is the identity operation for '*'.

Further $1 * 1 = 1 \bmod 6 = 1$

and $5 * 5 = 25 \bmod 6 = 1$

⇒ 1 is the inverse of 1 and 5 is the inverse of 5.

Also, $\forall\ a \in A$, other than 1 and 5, we cannot find *b* such that $a * b = a\,b \bmod 6 = 1$.

Hence, no element other than 1 and 5 is invertible.

Answers

7. *n* !.

16. Commutative but not associative.

CHECK YOUR UNDERSTANDING

1. Given set A = {1, 2, 3}, then the relation :
   R = {(1, 1) ; (2, 2) ; (3, 3)} is reflexive. **(True/False)** *(Jammu B. 2016)*

   Ans. True.
2. Give an example of a relation, which is symmetric and transitive but not reflexive.

   Ans. R = {(1, 2), (2, 1), (1, 1), (2, 2)}.
3. A bijective function is both one-one and onto. (True/False)

   Ans. True.
4. What is the domain of the function $f(x) = \dfrac{1}{x-2}$. *(Assam B. 2016)*

   Ans. **R** – {2}.
5. If $f(x) = \begin{cases} x-2 \ ; & x < 2 \\ 3 \ ; & x = 2 \\ x+2 \ ; & x > 3 \end{cases}$, then find $f(8)$. *(Jharkhand B. 2016)*

   Ans. 5.
6. If $a * b = 3a + 4b$, then the value of 3 * 4 is ....

   Ans. 25.
7. If $a * b = \dfrac{a}{2} + \dfrac{b}{3}$, then the value of 2 * 3 is ....

   Ans. 2.
8. Let A = {1, 2, 3}. For $x, y \in A$, let xRy if and only if $x > y$. Write down R as subset of A × A. *(Assam B. 2016)*

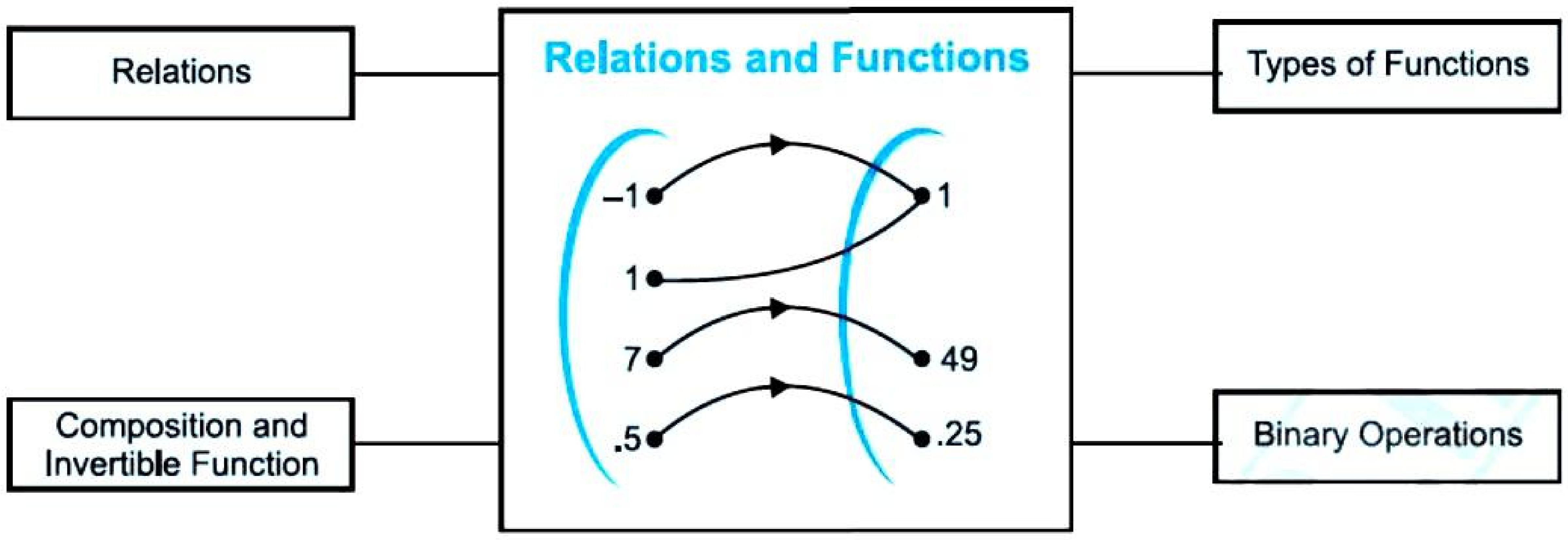
   Ans. {(2, 1), (3, 1), (3, 2)}.
9. Is – *a* the inverse of $a \in \mathbf{N}$ for addition operation '+' or **N** ?

   Ans. No.
10. Is $\dfrac{1}{a}$ the inverse of $a \in \mathbf{N}$ for multiplication operation '×' on **N** for $a \neq 1$?

    Ans. No.

SUMMARY

RELATIONS AND FUNCTIONS



RELATIONS

1. DEFINITIONS

(*i*) **Relation.** A relation R from a set A to a set B is subset of A × B.

(*ii*) **Classification of Relations :**

(*a*) **Reflexive Relation.** A relation R in a set E is said to be reflexive if $x R x \ \forall \ x \in E$.

(*b*) **Symmetric Relation.** A relation R in a set E is said to be symmetric if :

$$xRy = yRx \ \forall \ x, y \in E.$$

(*c*) **Transitive Relation.** A relation R in a set E is said to be transitive if :

$$xRy \text{ and } yRz \Rightarrow xRz \ \forall x, y, z \in E.$$

(*d*) **Equivalence Relation.** A relation R in a set E is said to be an equivalence relation if it is :

(I) reflexive (II) symmetric and (III) transitive.

FUNCTIONS

2. FUNCTIONS

(*a*) Let X and Y be two non-empty sets. Then 'f' is a rule, which associates to each element x in X a unique element y in Y.

(*i*) The unique element y of Y is called the **value** of f at x.

(*ii*) The element x of X is called **pre-image** of y.

(*iii*) The set X is called the **domain** of f.

(*iv*) The set of images of elements of X under f is called the **range** of f.

(*b*) (*i*) $D_f = \{x : x \in \mathbf{R}, f(x) \in \mathbf{R}\}$

(*ii*) $R_f = \{f(x) : x \in D_f\}$

(*iii*) f is **one-one** iff $x_1 = x_2$
$\Rightarrow f(x_1) = f(x_2)$ for $x_1, x_2 \in D_f$

or iff $x_1 \neq x_2 \ \Rightarrow f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in D_f$

(*iv*) f is **invertible** iff f is one-one onto and

$$D_{f^{-1}} = R_f, R_{f^{-1}} = D_f.$$

3. ALGEBRA OF FUNCTIONS

Let f and g be two functions.

Then (*i*) $(f + g)(x) = f(x) + g(x)$; $D_{f+g} = D_f \cap D_g$

(*ii*) $(f - g)(x) = f(x) - g(x)$; $D_{f-g} = D_f \cap D_g$

(*iii*) $(fg)(x) = f(x)\, g(x)$; $D_{fg} = D_f \cap D_g$

(*iv*) $\left(\dfrac{f}{g}\right) x = \dfrac{f(x)}{g(x)}$;

$$D_{f/g} = D_f \cap D_g - \{x : x \in D_g, g(x) = 0\}.$$

BINARY OPERATIONS

4. DEFINITIONS

(*a*) Let A be non-empty set. Then the rule denoted by 'o' is called **binary operation** on A if to each ordered pair (a, b) of the elements of A, it associates a unique element, denoted by $a \ o \ b$ of A.

(*b*) **Properties :**

(*i*) **Commutative.** If $aob = boa$ for $a, b \in A$, then 'o' is commutative.

(*ii*) **Associative.** If $ao(boc) = (aob)oc$ for $a, b, c \in A$, then 'o' is associative.

(*iii*) **Existence of Identity Element.** If 'o' is binary operation on A and there is $e \in A$ such that $aoe = a = eoa$, where e is the identity element of the operation.

(*iv*) **Existence of Inverse Element.** An element b in a set A is said to be inverse element of an element $a \in A$ *w.r.t.* binary operation 'o' if $aob = e = boa$.

(*c*) **Algebraic Structure.** A non-empty set with one or more binary operations defined on it, is called an algebraic structure.

MULTIPLE CHOICE QUESTIONS

For Board Examinations

1. If '*' is a binary operation such that $a * b = a^2 + b^2$, then $3 * 5$ is :

(A) 34 (B) 9
(C) 8 (D) 25. *(P.B. 2018)*

2. If $f(x) = \log(1 + x)$ and $g(x) = e^x$, then the value of $(gof)(x)$ is :

(A) e^{1+x} (B) $1 + x$
(C) $\log x$ (D) None of these. *(H.B. 2018)*

3. Let $A = \{(a, b)\}\ \forall\ a, b \in \mathbf{N}$. Then the relation R is :

(A) Reflexive (B) Symmetric
(C) Transitive (D) None of these. *(Mizoram B. 2018)*

4. The domain of the function defined by $f(x) = \sqrt{9 - x^2}$ is :

(A) $]-3, 3[$ (B) $[-3, 3]$
(C) $[0, 3]$ (D) $[-3, 0]$. *(Mizoram B. 2018)*

5. The domain of the function $f(x) = \dfrac{x}{|x|}$ is :

(A) $\mathbf{R} - \{0\}$ (B) $\mathbf{R}$
(C) $\mathbf{Z}$ (D) $\mathbf{W}$. *(Nagaland B. 2018)*

6. If a binary operation is defined by $a * b = a^b$, then $3 * 2$ is equal to :

(A) 4 (B) 2
(C) 9 (D) 8. *(P.B. 2017)*

7. Let $f : \mathbf{R} \to \mathbf{R}$ be defined as $f(x) = x^4$, then :

(A) f is one-one onto
(B) f is many-one onto
(C) f is one-one but not onto
(D) f is neither one-one nor onto. *(H.B. 2017)*

8. If the binary operation '*' on **N** defined as :
$a * b = a^3 + b^3$, then '*' is :

(A) both associative and commutative
(B) commutative but not associative
(C) associative but not commutative
(D) Neither commutative nor associative. *(H.B. 2017)*

9. Consider the set $A = \{1, 2, 3, 4\}$. Which of the following relations R form a reflexive relation ?

(A) $R = \{(1, 1), (1, 2), (2, 2), (3, 4)\}$
(B) $R = \{(1, 1), (2, 2), (2, 3), (3, 3), (3, 4)$
(C) $R = \{(1, 1), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4)\}$
(D) $R = \{(1, 1), (2, 1), (2, 3), (3, 3), (3, 4), (4, 4)\}$. *(Nagaland B. 2017)*

10. Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 2), (1, 2)\}$. Then R is :

(A) Reflexive and symmetric but not transitive
(B) Reflexive and transitive but not symmetric
(C) Symmetric and transitive but not reflexive
(D) An equivalence relation. *(Mizoram B. 2017)*

11. Let R be a relation defined on $A = \{1, 2, 3\}$ by : $R = \{(1, 3), (3, 1), (2, 2)\}$. R is :

(A) Reflexive
(B) Symmetric
(C) Transitive
(D) Reflexive but not Transitive. *(Kerala B. 2017)*

12. If function $f(x) = \dfrac{3x}{4x+3}$ is defined on :

$f : \mathbf{R} - \left\{-\dfrac{3}{4}\right\} \to \mathbf{R}$, then its inverse function defined on

$g : \text{Range } f \to \mathbf{R} - \left\{-\dfrac{3}{4}\right\}$ in which of the following ?

(A) $g(y) = \dfrac{4y+3}{3y}$

(B) $g(y) = \dfrac{3y}{3-4y},\ y \neq \dfrac{3}{4}$

(C) $g(y) = \dfrac{3y}{4y-3},\ y \neq \dfrac{3}{4}$

(D) None of these. *(H.B. 2016)*

13. The function $f : \mathbf{N} \to \mathbf{N}$, given by $f(2x) = 2x$ is :

(A) one-one and onto
(B) one-one but not onto
(C) not one-one and not onto
(D) onto, but not one-one. *(Kerala B. 2016)*

14. Let S be the set of all real numbers and let R be the relation in S defined by $R = \{(a, b) : a \leq b^2\}$, then :

(A) R is reflexive (B) R is symmetric
(C) R is transitive (D) None of these. *(Mizoram B. 2016)*

15. The operation '*' on $\mathbf{Q} - \{1\}$ is defined by
$a * b = a + b - ab$ for all $a, b \in \mathbf{Q} - \{1\}$.
Then the identity element in $\mathbf{Q} - \{1\}$ is :

(A) 0 (B) 1
(C) -1 (D) None of these. *(Mizoram B. 2016)*

RCQ Pocket

(Single Correct Answer Type)

(JEE-Main and Advanced)

16. Consider the following relations :
$R = \{(x, y) \mid x, y$ are real numbers and $x = wy$ for some rational number $w\}$;

$S = \left\{\left(\dfrac{m}{n}, \dfrac{p}{q}\right);\ m, n, p\right.$ and q are integers such that $n, q \neq 0$ and $qm = pn\}$. Then :

(A) R is an equivalence relation but S is not an equivalence relation

(B) neither R nor S is an equivalence relation
(C) S is an equivalence relation but R is not an equivalence relation
(D) R and S both are equivalence relations.
(A.I.E.E.E. 2010)

17. The domain of the function $f(x) = \dfrac{1}{\sqrt{|x| - x}}$ is :

(A) $(-\infty, \infty)$ (B) $(0, \infty)$
(C) $(-\infty, 0)$ (D) $(-\infty, \infty) - \{0\}$.
(A.I.E.E.E. 2011)

18. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in \mathbf{R}$.
Then the set of all x satisfying :
$(fogogof)(x) = (gogof)(x)$, where $(fog)(x) = f(g(x))$, is :

(A) $\pm\sqrt{n\pi}, n \in \{0, 1, 2, ...\}$

(B) $\pm\sqrt{n\pi}, n \in \{1, 2, ...\}$

(C) $\dfrac{\pi}{2} + 2n\pi, n \in \{...... -2, -1, 0, 1, 2, ...\}$

(D) $2n\pi, n \in \{...... -2, -1, 0, 1, 2, ...\}$.
(I.I.T. 2011)

19. The function $f : [0, 3] \to [1, 29]$ given by :
$f(x) = 2x^3 - 15x^2 + 36x + 1$, is :
(A) one-one and onto
(B) onto but not one-one
(C) one-one but not onto
(D) neither one-one nor onto. ***(I.I.T. 2012)***

20. If $a \in \mathbf{R}$ and the equation
$$-3(x - [x])^2 + 2(x - [x]) + a^2 = 0,$$
where $[x]$ denotes the greatest integer ($\le x$) has no integral solution, then all possible values of a lie in the interval :
(A) $(1, 2)$ (B) $(-2, -1)$
(C) $(-\infty, -2) \cup (2, \infty)$ (D) $(-1, 0) \cup (0, 1)$.
(J.E.E. (Main) 2014)

21. The function $f : \mathbf{R} \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as :

$$f(x) = \frac{x}{1+x^2}, \text{ is :}$$
(A) Surjective but not injective
(B) Neither injective nor surjective
(C) Invertible
(D) Injective but not surjective.
(J.E.E. (Main) 2017)

Answers

1. (A)	**2.** (B)	**3.** (D)	**4.** (B)	**5.** (A)	**6.** (C)	**7.** (D)	**8.** (B)	**9.** (C)	**10.** (B)
11. (B)	**12.** (B)	**13.** (B)	**14.** (D)	**15.** (A)	**16.** (C)	**17.** (C)	**18.** (A)	**19.** (B)	**20.** (D)
21. (A).									

Hints to Selected Questions

RCQ Pocket

16. (C) Here $x\mathrm{R}y \Rightarrow x = wy$
$y\mathrm{R}x \Rightarrow y = wx$
$\Rightarrow x\mathrm{R}y \ne y\mathrm{R}x$
$\Rightarrow$ R is not symmetric.
Thus R is not an equivalence relation.

Now $\dfrac{m}{n}\mathrm{S}\dfrac{m}{n} = mn = mn$, which is true for $m, n \in \mathbf{Z}$

$\Rightarrow$ S is reflexive.

$\dfrac{m}{n}\mathrm{S}\dfrac{p}{q} \Rightarrow mq = pn$

$\Rightarrow pn = mq \Rightarrow \dfrac{p}{q}\mathrm{S}\dfrac{m}{n}$

$\Rightarrow$ S is symmetric.

$\dfrac{m}{n}\mathrm{S}\dfrac{p}{q}, \dfrac{p}{q}\mathrm{S}\dfrac{r}{s} \Rightarrow qm = pn,\ ps = qr$

$\Rightarrow (qm)(ps) = (pn)(qr)$

$\Rightarrow ms = rn \Rightarrow \dfrac{m}{n}\mathrm{S}\dfrac{r}{s}$

$\Rightarrow$ S is transitive.

Thus S is an equivalence relation.
Hence, S is an equivalence relation but R is not an equivalence relation.

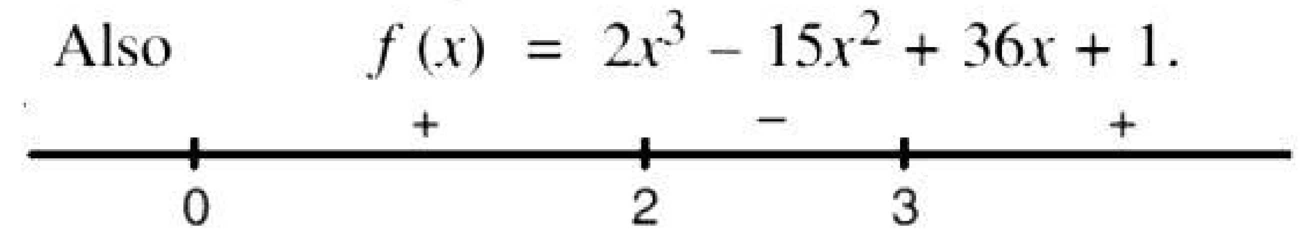
17. (C) $f(x) = \dfrac{1}{\sqrt{|x| - x}}$.

$f(x)$ is defined if $|x| - x > 0$
if $|x| > x$ if $x < 0$.
Hence, $D_f = (-\infty, 0)$.

18. (A) $(fogogof)(x) = (gogof)(x)$
$\Rightarrow (fogog)(f(x)) = (gog)(f(x))$
$\Rightarrow (fogog)(x^2) = (gog)(x^2)$
$\Rightarrow (fog)(g(x^2)) = g(g(x^2))$
$\Rightarrow (fog)(\sin x^2) = g(\sin x^2)$
$\Rightarrow f(g(\sin x^2)) = g(\sin x^2)$
$\Rightarrow f(\sin\sin x^2) = \sin\sin x^2$
$\Rightarrow (\sin\sin x^2)^2 = \sin\sin x^2$
$\Rightarrow \sin\sin x^2 = 0$ or 1

$\Rightarrow x = \pm\sqrt{n\pi}, n \in \{0, 1, 2, ...\}$.

19. (B) We have : $f : [0, 3] \to [1, 29]$.
Also $f(x) = 2x^3 - 15x^2 + 36x + 1$.



$\therefore \quad f'(x) = 6x^2 - 30x + 36$
$= 6(x^2 - 5x + 6) = 6(x-2)(x-3).$

Thus $f'(x)$ is +ve from 0 to 2, –ve from 2 to 3 and +ve from 3 onwards

$\Rightarrow$ $f(x)$ is maxima at $x = 2$

$\Rightarrow$ $f(x)$ is **many-one** and not one-one.

Now $f(0) = 1, f(2) = 16 - 60 + 72 + 1 = 29$

$f(3) = 54 - 135 + 108 + 1 = 28.$

Thus Range = [1, 29].

Hence, given function is onto.

20. (D) Put $x - [x] = t$.

Then $-3t^2 + 2t + a^2 = 0 \Rightarrow a^2 = 3t^2 - 2t.$

For non-integral solutions, $0 < a^2 < 1$.

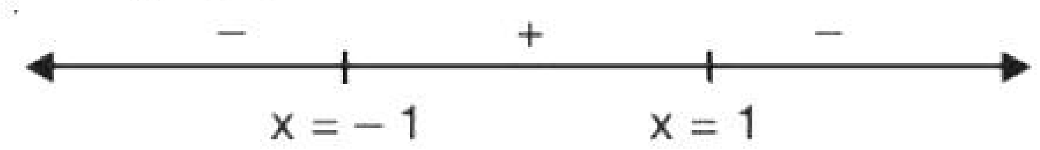
Hence, $a \in (-1, 0) \cup (0, 1)$.

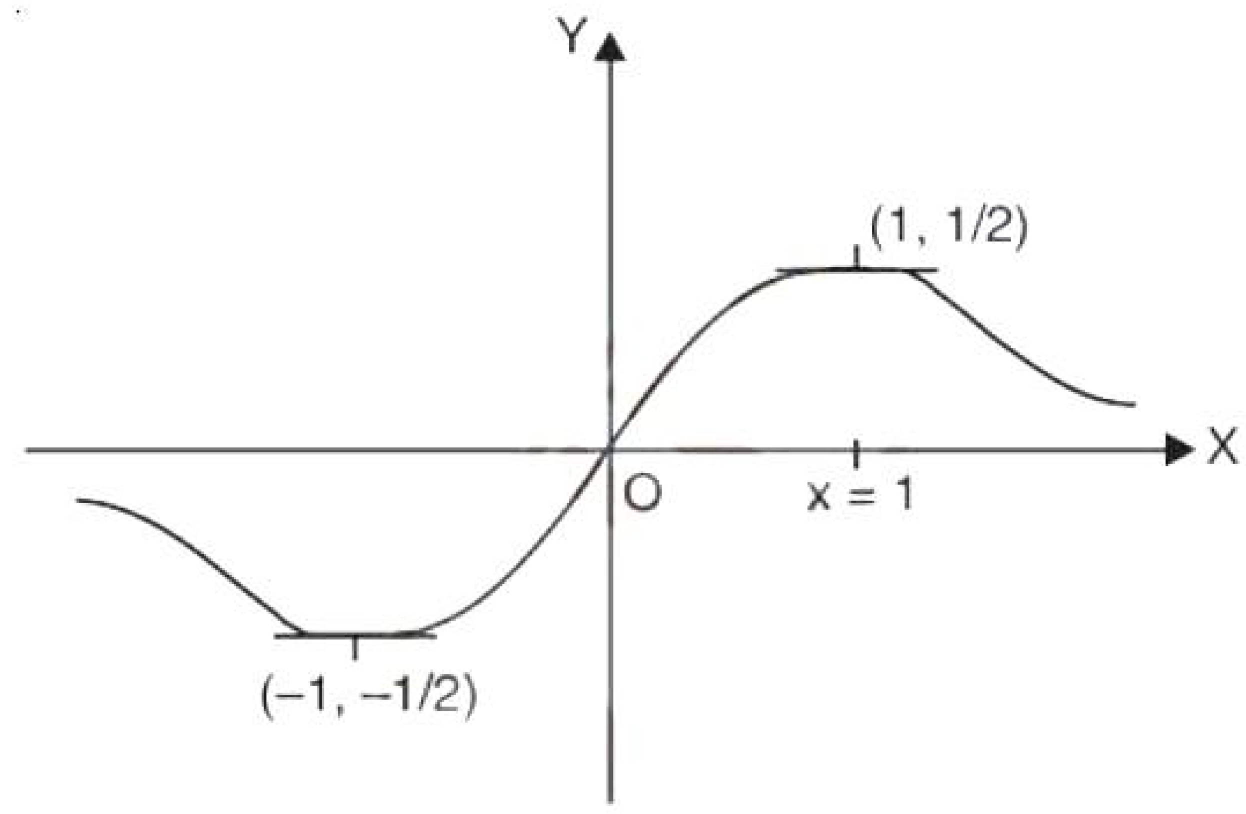
21. (A) $f: \mathbf{R} \to \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$f(x) = \frac{x}{1+x^2}, \quad \forall x \in \mathbf{R}.$$

$$\therefore \; f'(x) = \frac{(1+x^2)(1) - x(-2x)}{(1+x^2)^2} = \frac{-(x+1)(x-1)}{(1+x^2)^2}.$$

Signs of $f'(x)$





From the figure, $f(x)$ is surjective but not injective.

CHAPTER TEST 1

Time Allowed : 1 Hour **Max. Marks : 34**

Notes : **1.** *All questions are compulsory.*

2. *Marks have been indicated against each question.*

1. Give an example of a relation, which is symmetric but neither reflexive nor transitive ? **(1)**

2. Are f and g both necessarily onto, if gof is onto ? **(1)**

3. Show that $- : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\div \; \mathbf{R} \to \mathbf{R}$ are not commutative binary operations. **(2)**

4. Show that the relation R defined by :

$(a, b) \text{ R } (c, d) \Rightarrow a + d = b + c$ in the set **N** is an equivalence relation. **(2)**

5. Show that the Signum Function $f: \mathbf{R} \to \mathbf{R}$, given by :

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0; & \text{if } x = 0 \\ -1; & \text{if } x < 0 \end{cases}$$

is neither one-one nor onto. **(4)**

6. If $f(x) = \frac{x-1}{x+1}, x \neq -1,$ then show that :

$f(f(x)) = -\frac{1}{x}, x \neq 0.$ **(4)**

7. Let $Y = \{n^2, n \in \mathbf{N}\} \subset \mathbf{N}$. Consider $f: \mathbf{N} \to Y$ as $f(n) = n^2$.

Show that 'f' is invertible. Find the inverse of 'f'. **(4)**

8. Is '*' defined on the set {1, 2, 3, 4, 5] by : $a * b = l.$c.m of a and b a binary operation ? Justify your answer. **(4)**

9. If R_1 and R_2 are equivalence relations in a set A, show that $R_1 \cap R_2$ is also an equivalence relation. **(6)**

10. Consider the binary operations :

$* : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $o : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined as $a * b = |a - b|$ and $a \; o \; b = a \; \forall \; a, b \in \mathbf{R}.$

Show that '*' is commutative but not associative and 'o' is associative but not commutative.

Also show that $a * (b \; o \; c) = (a * b) \; o \; (a * c) \; \forall \; a, b, c \in \mathbf{R}.$ **(6)**

Answers

1. Let A = {1, 2, 3} and R = {(2, 3), (3, 2)}.

2. f is not onto.

7. $f^{-1} = g$, where $g(y) = \sqrt{y}$.

8. No.