

2

INVERSE-TRIGONOMETRIC FUNCTIONS



—Felix Klein

Mathematics, in general, is fundamentally the science of self-evident things

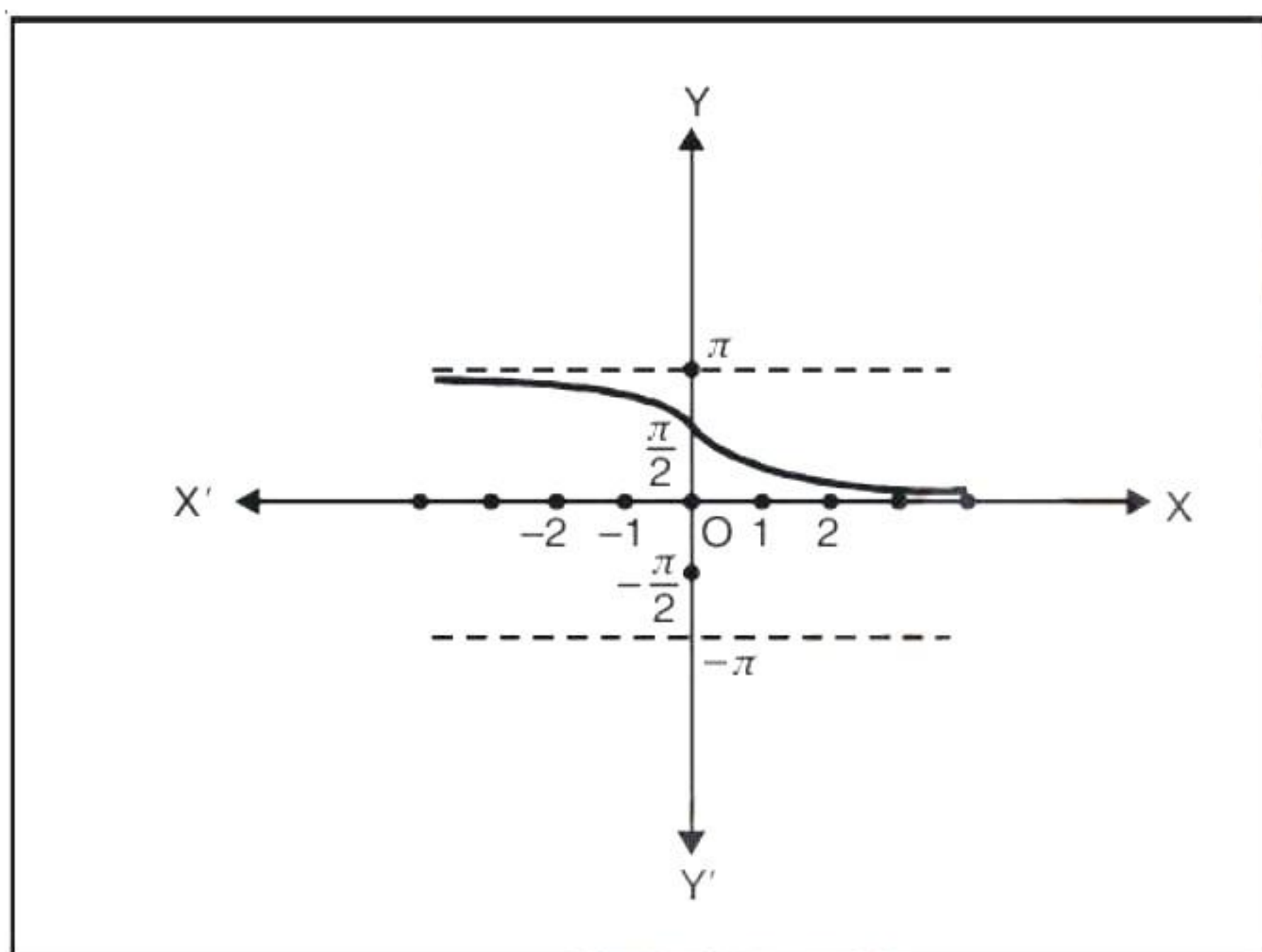
Objectives

After studying the material of this chapter you should be able to :

- Understand the definitions of inverse-trigonometric functions.
- Understand the domains and ranges of inverse-trigonometric functions with the help of their graphs.
- Understand the principal values of inverse-trigonometric functions.
- Understand the properties of inverse trigonometric functions.
- Understand the application of inverse trigonometric functions.



2.0. INTRODUCTION



Inverse trigonometric functions play a vital role in Calculus and help us to define several integrals. This concept is widely used in Engineering, Geometry and Physics. The restrictions on domains of trigonometric functions ensure the existence of their inverses.

Inverse trigonometric functions are the inverse functions of the trigonometric functions. They are not the reciprocals of the trigonometric functions.

In this chapter we will study the following concepts :

- Inverse trigonometric functions and their graphical representations
- Domain and Range of Inverse trigonometric functions
- Properties of inverse trigonometric functions

Chapter at Glance

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2.1. INVERSE-TRIGONOMETRIC FUNCTIONS

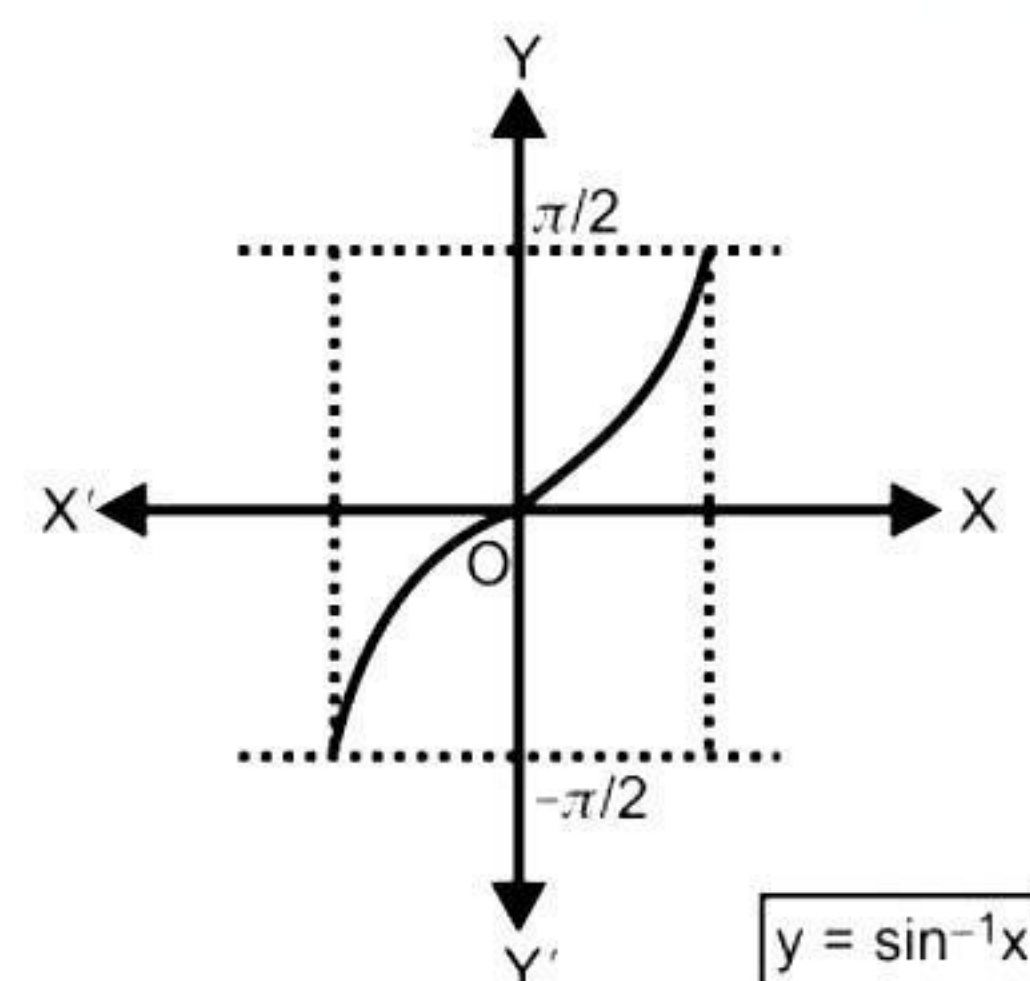
(i) Definition

The inverse-sine function is defined as :

$$y = f^{-1}(x) = \sin^{-1} x$$

$$\text{iff } x = \sin y \text{ and } y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

The graph of $\sin^{-1} x$ is as shown.



KEY POINT

Domain of $\sin^{-1} x = [-1, 1]$ and Range of $\sin^{-1} x = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

PREPARATION OF IMAGE

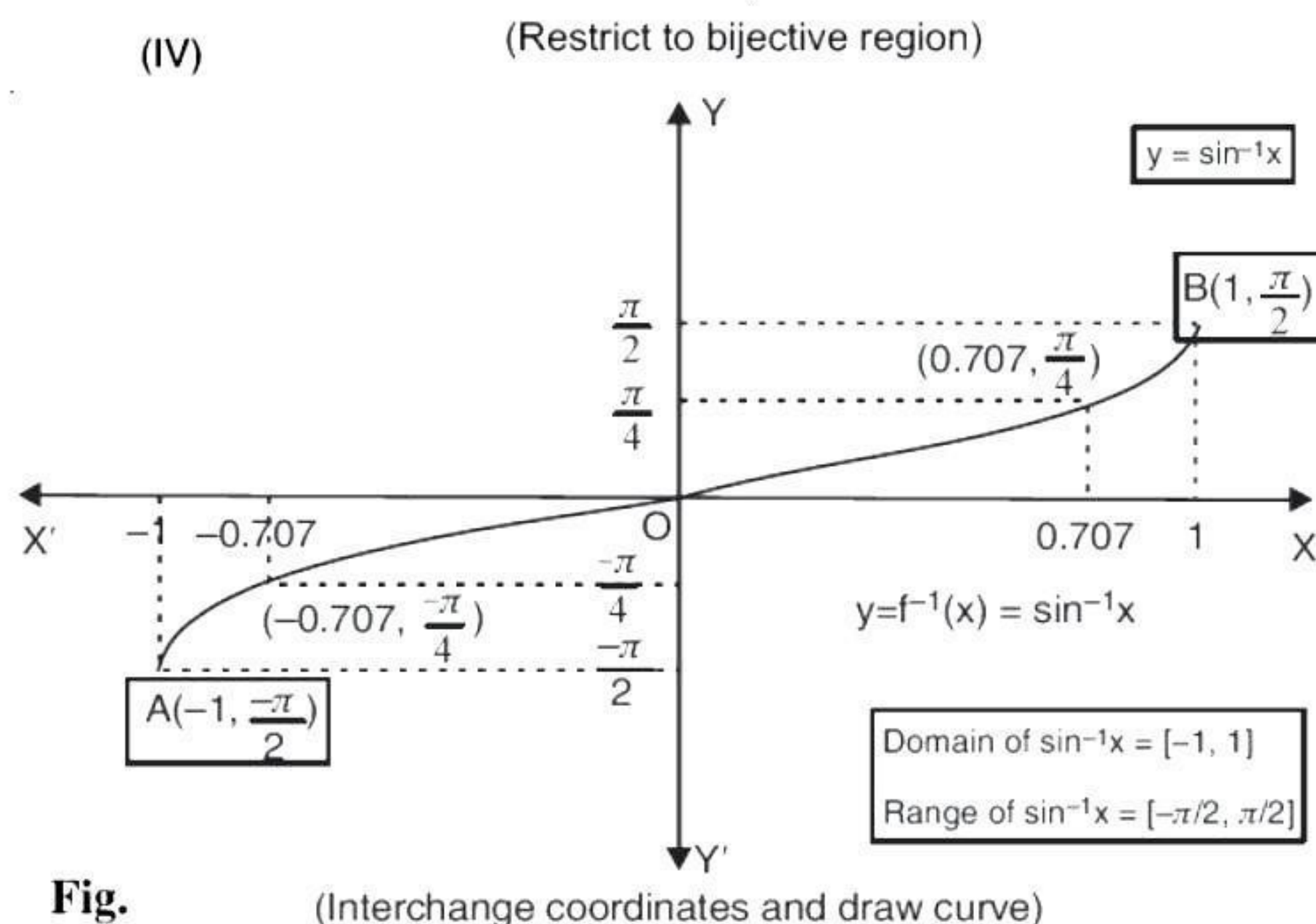
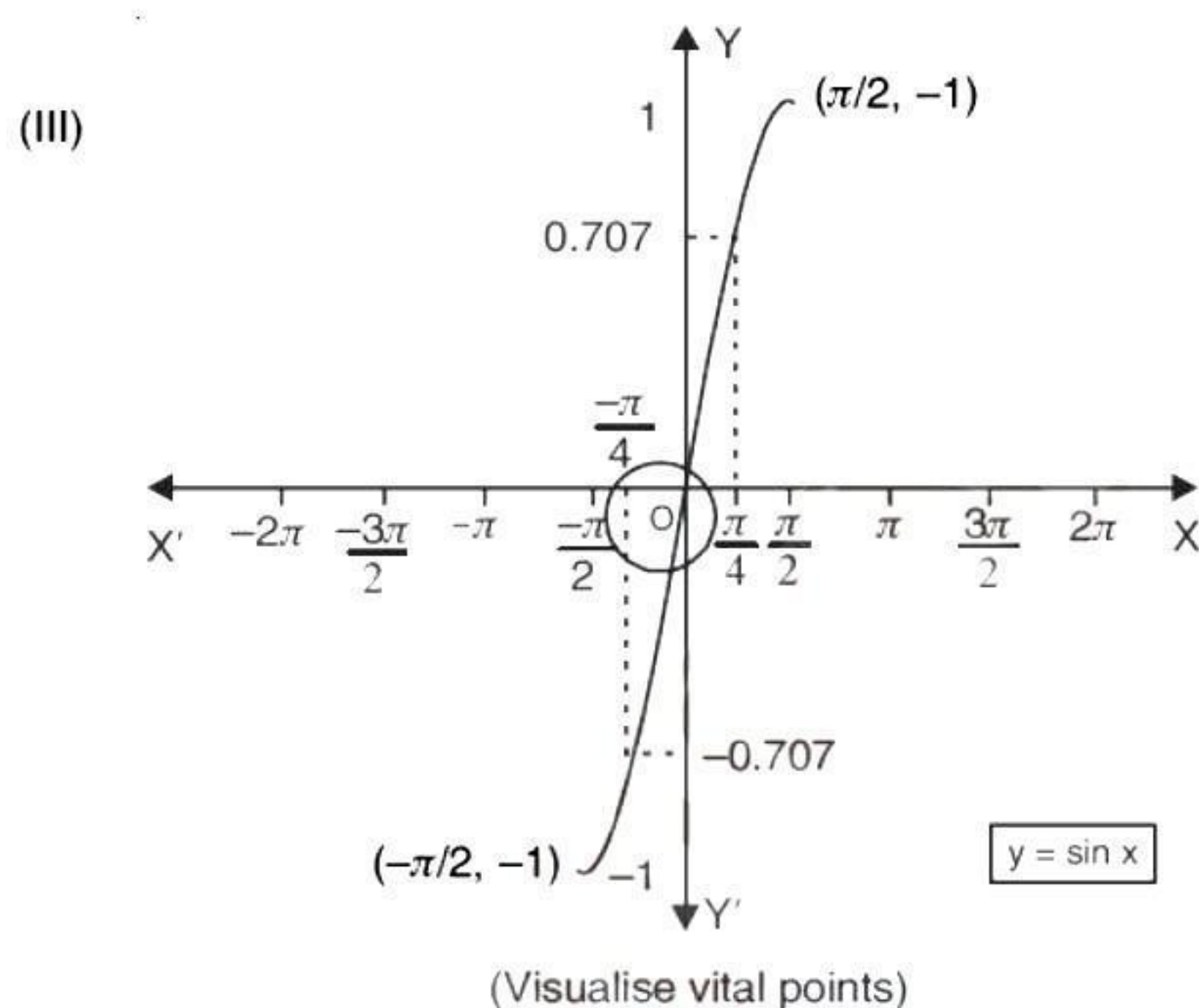
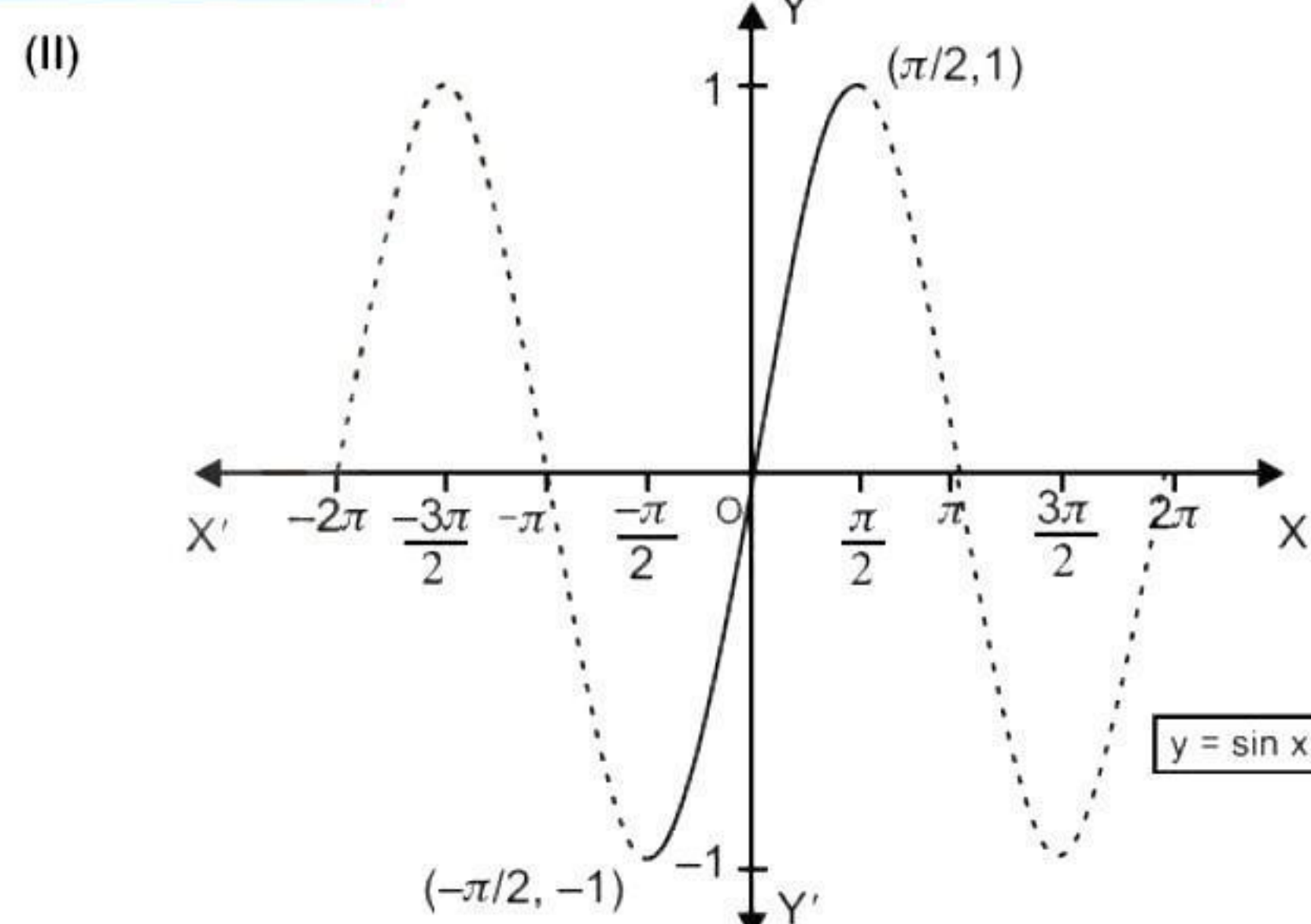
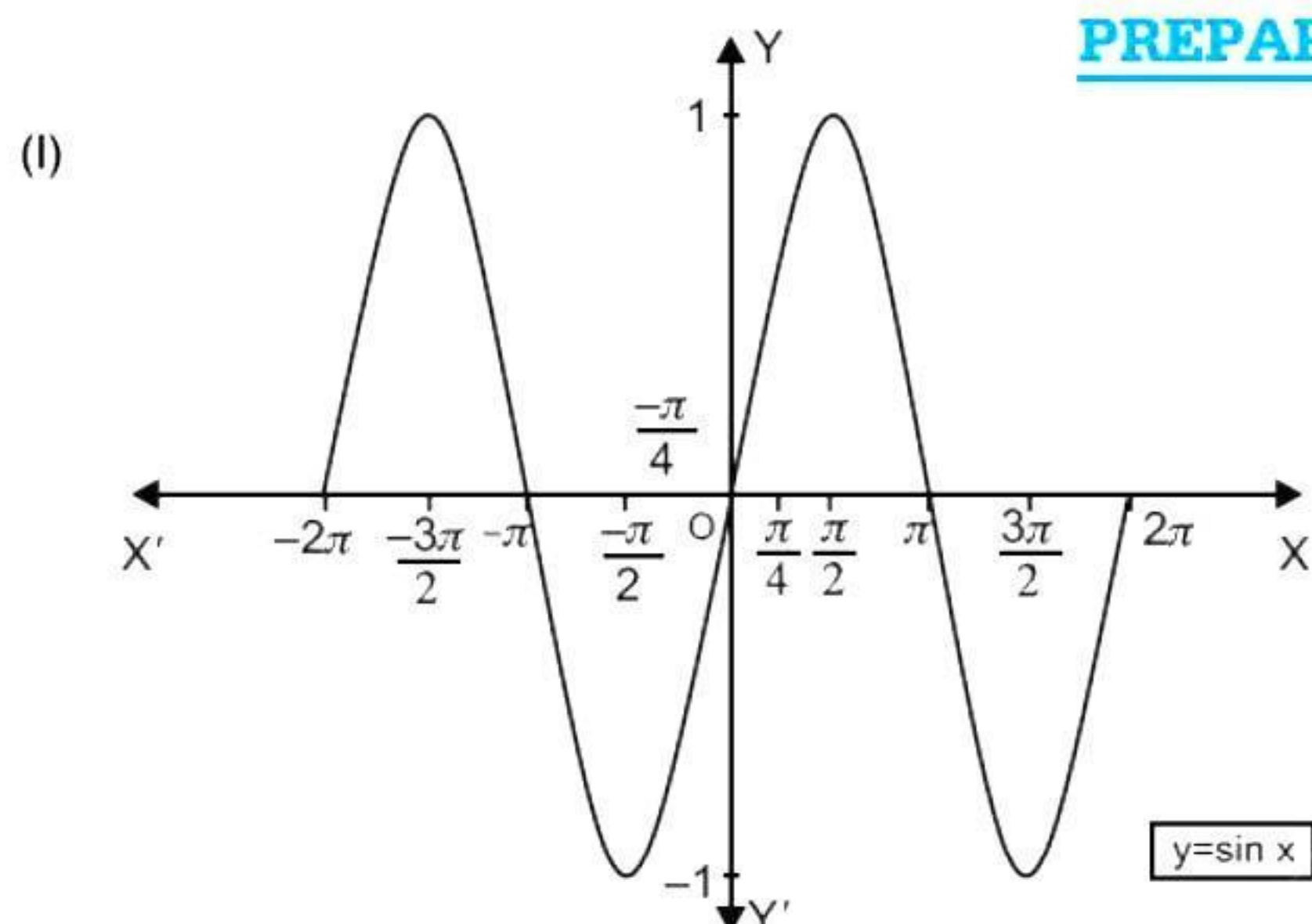


Fig.

(Interchange coordinates and draw curve)

(ii) Definition

The inverse-cosine function is defined as :

$$y = f^{-1}(x) = \cos^{-1} x$$

$$\text{iff } x = \cos y \text{ and } y \in [0, \pi].$$

The graph of $\cos^{-1} x$ is as shown.

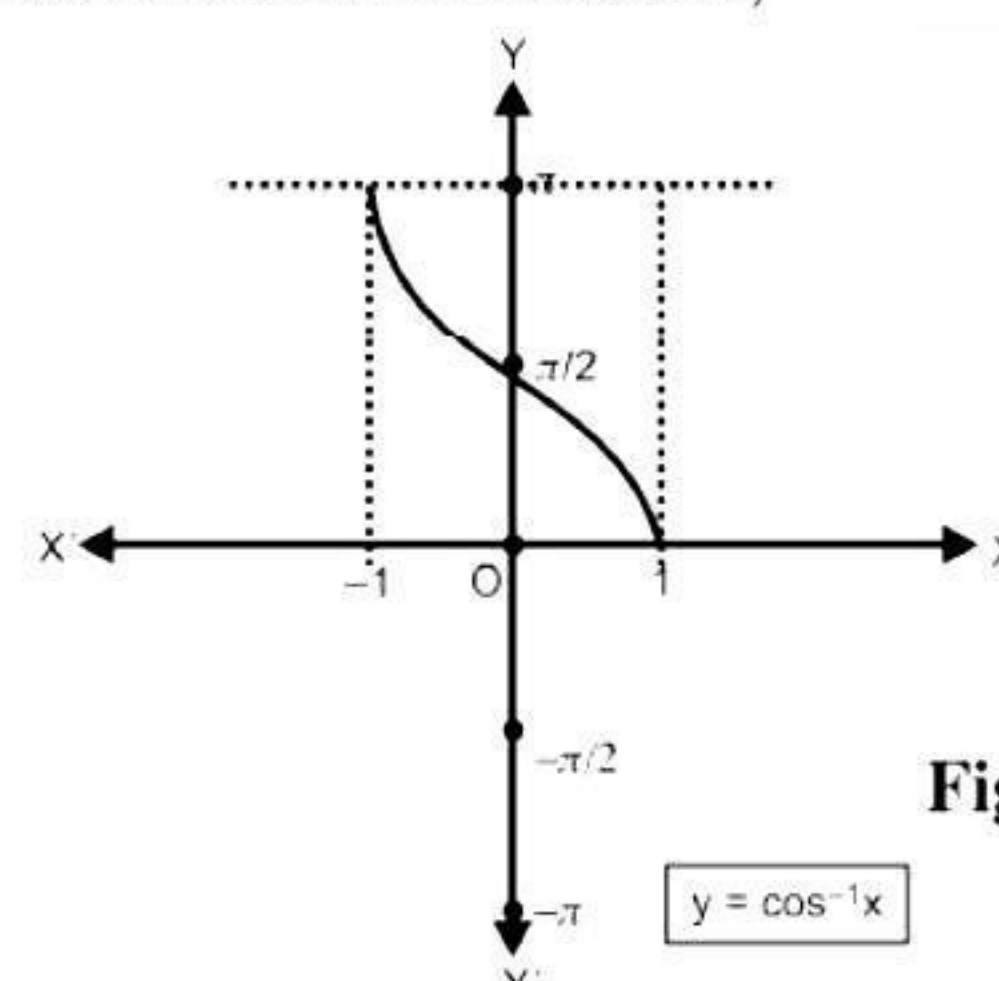
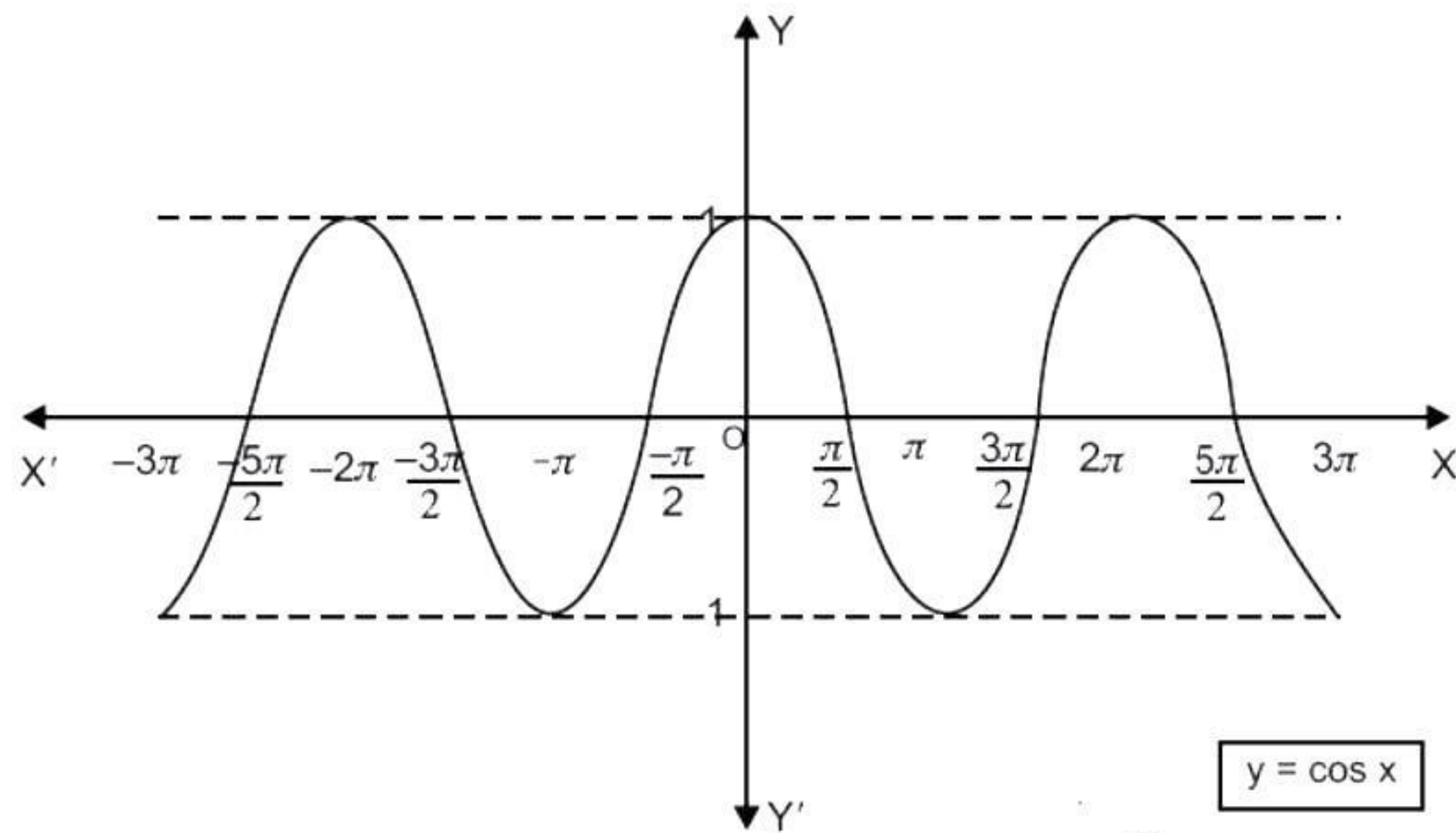


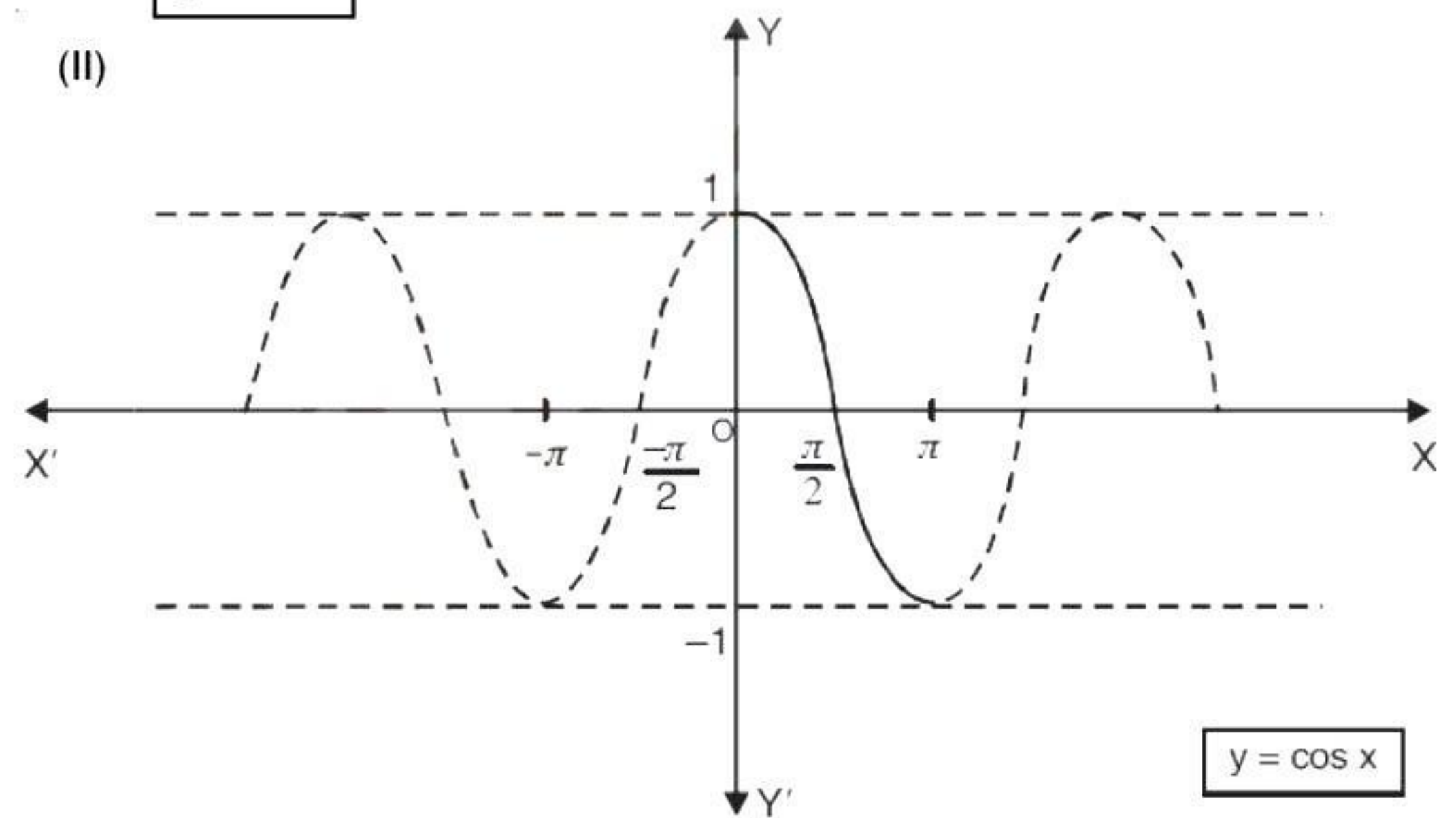
Fig.

PREPARATION OF IMAGE

(I)

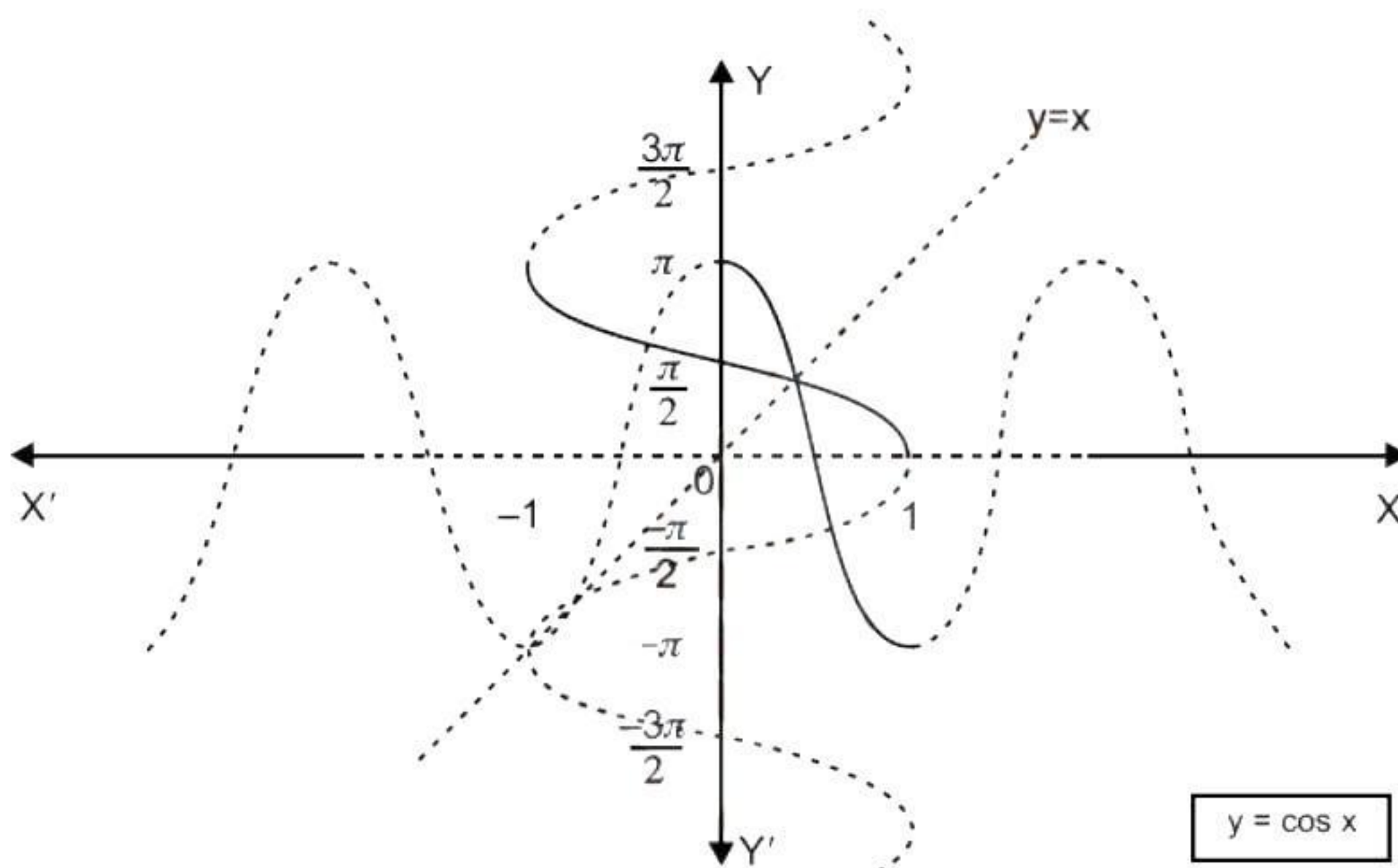


(II)

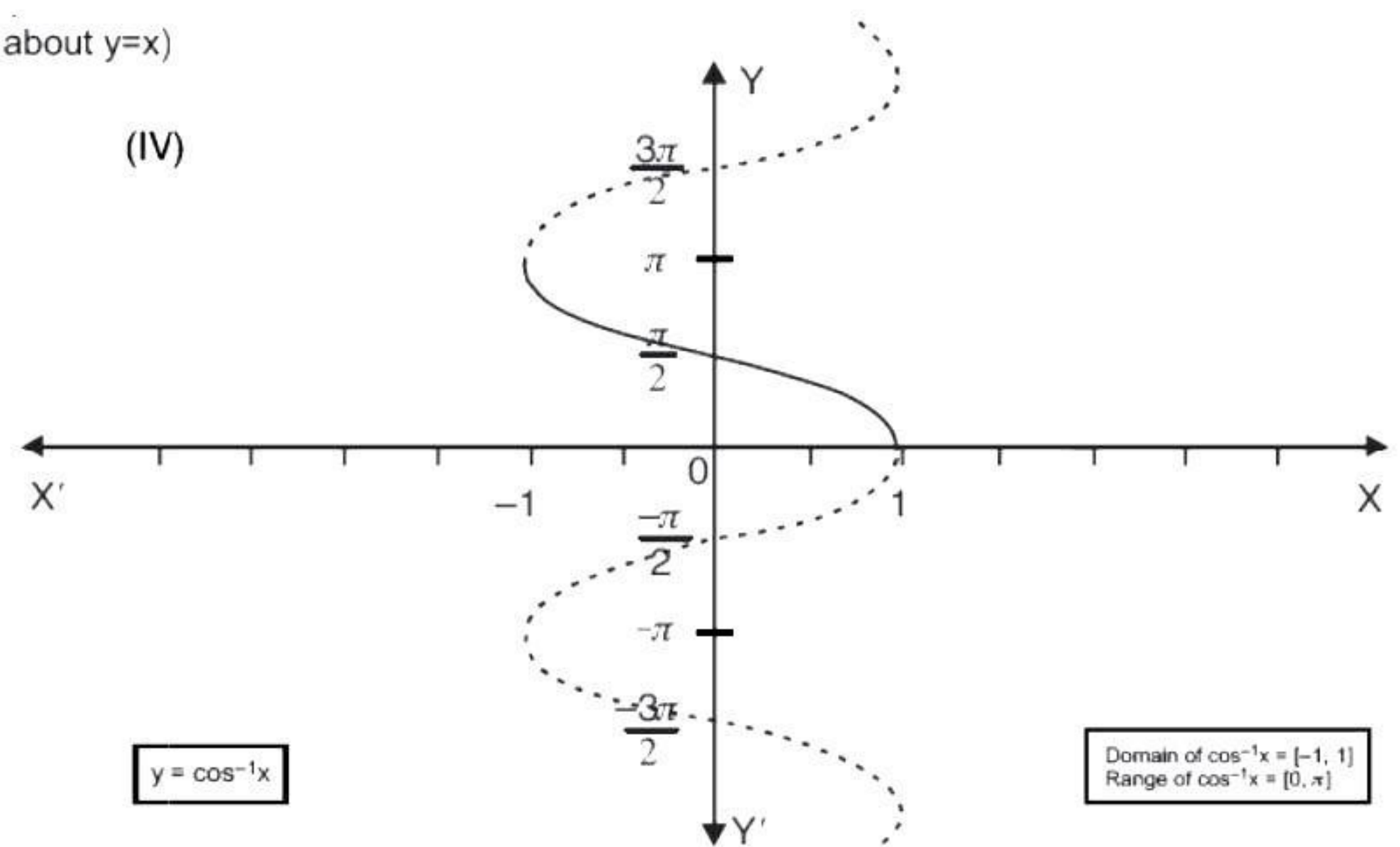


(Restrict to bijective region)

(III)

(Take mirror image of curve in 2nd step about $y=x$)

(IV)



KEY POINT

Domain of $\cos^{-1} x = [-1, 1]$ and Range of $\cos^{-1} x = [0, \pi]$.

(iii) Definition

The inverse-tangent function is defined as :

$$y = f^{-1}(x) = \tan^{-1} x$$

$$\text{iff } x = \tan y \text{ and } y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

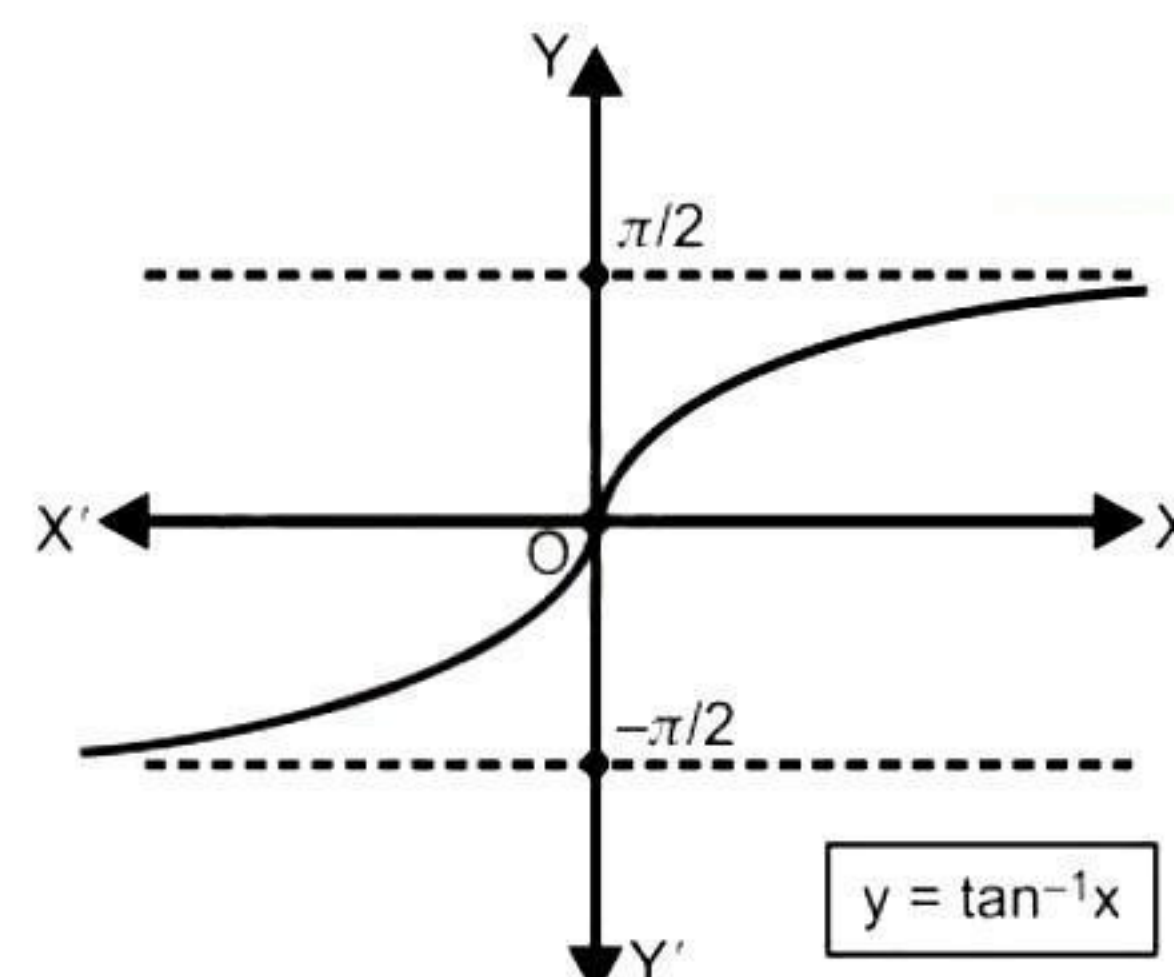


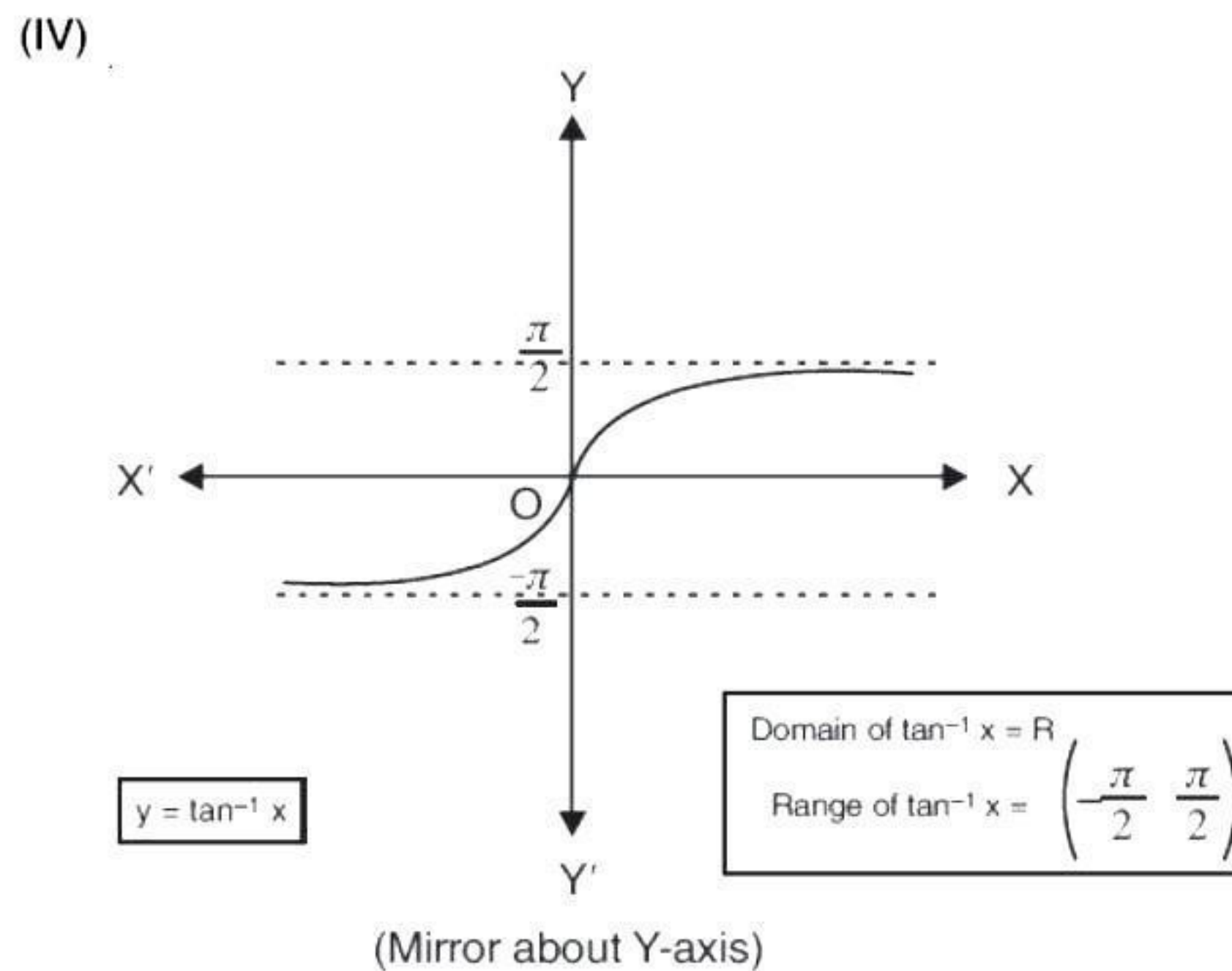
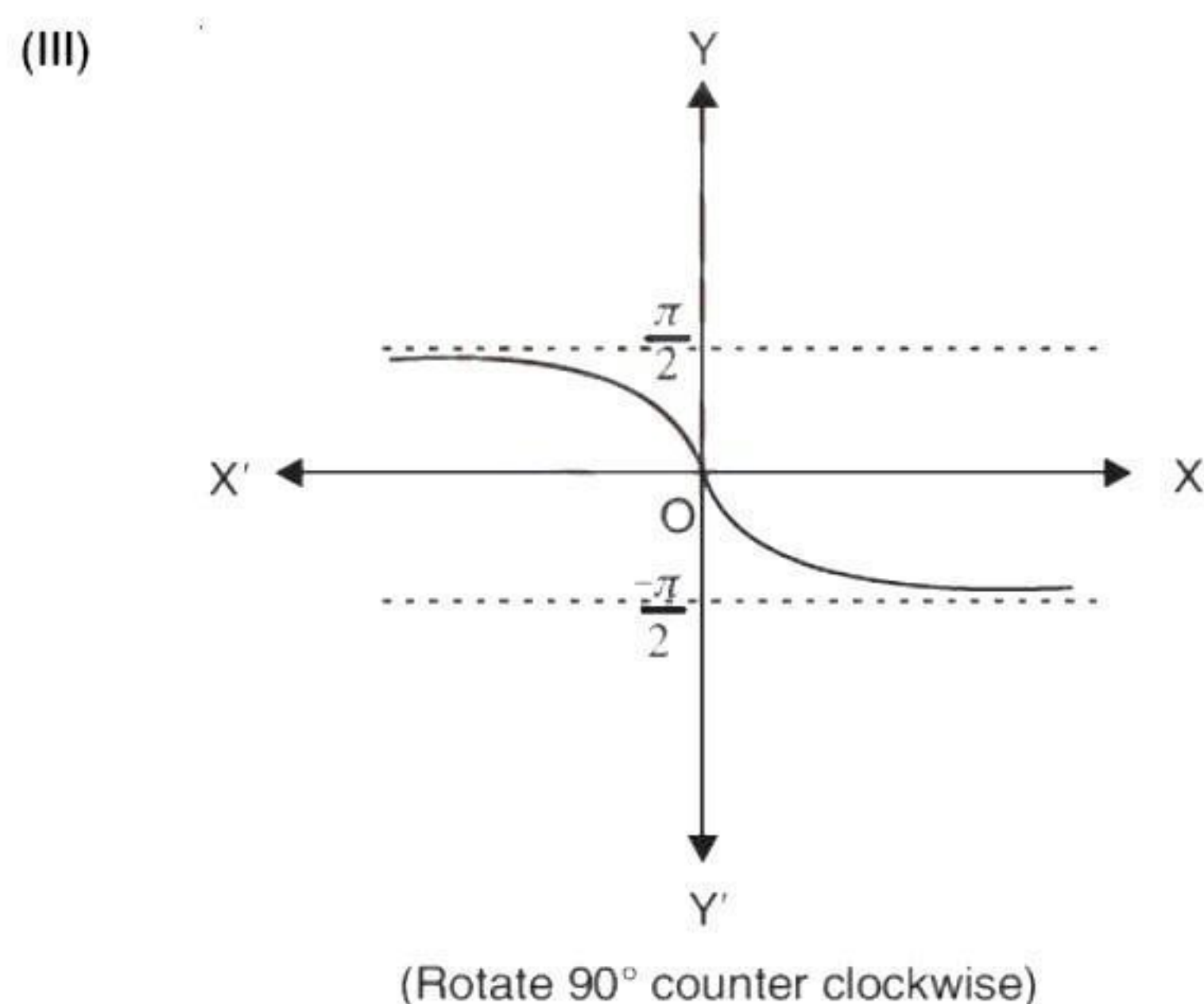
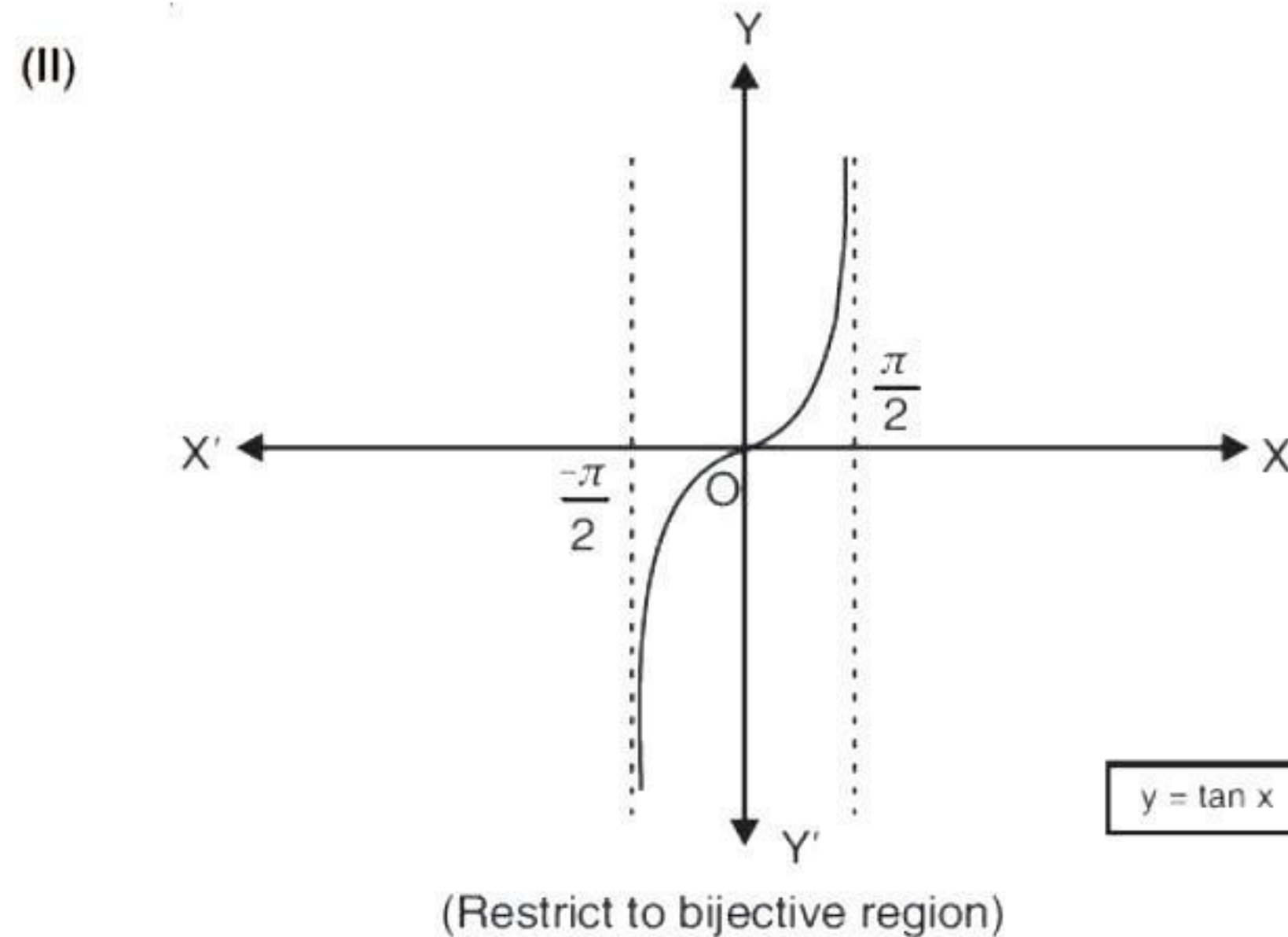
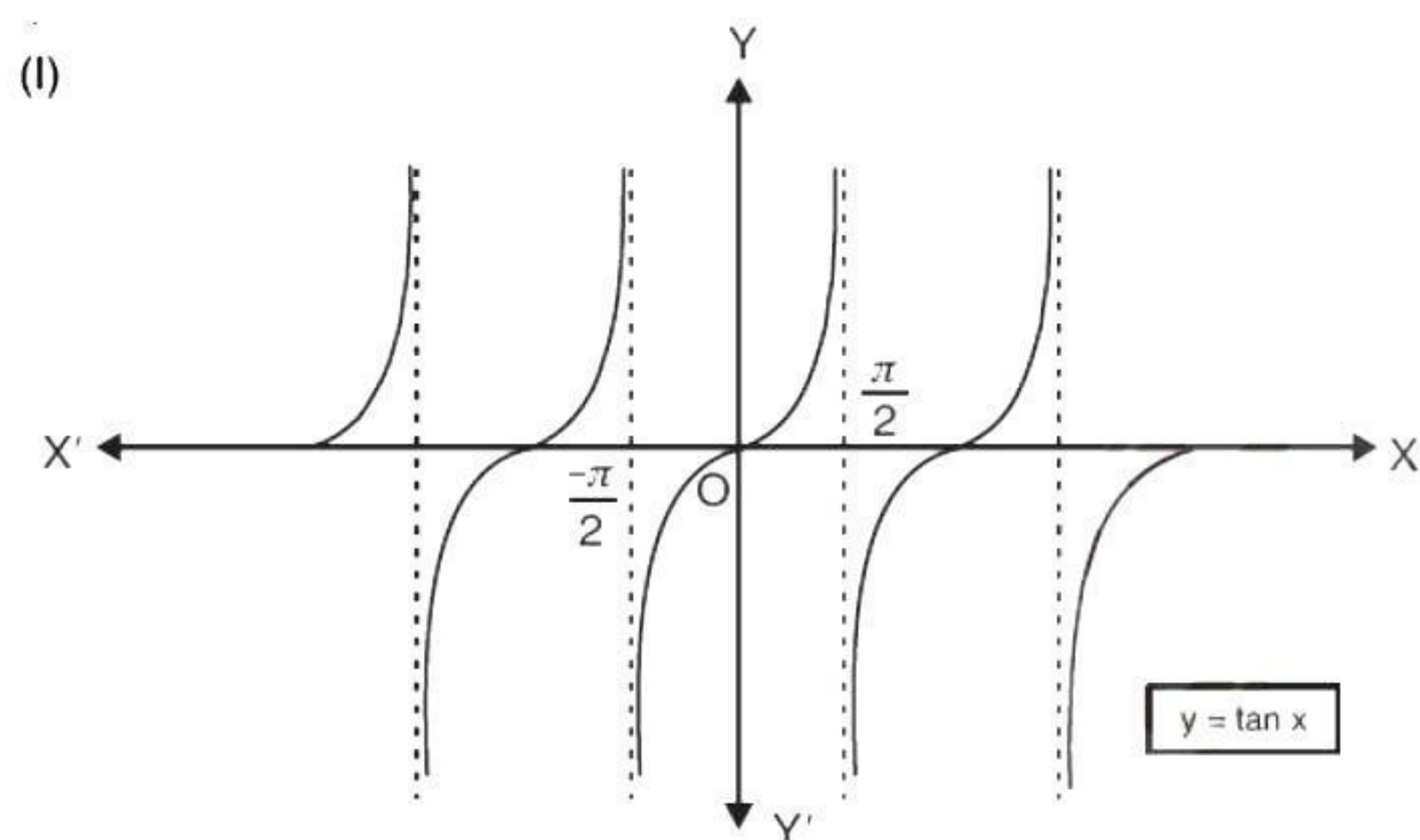
Fig.

The graph of $\tan^{-1} x$ is as shown.

KEY POINT

Domain of $\tan^{-1} x = \mathbf{R}$ and Range of $\tan^{-1} x = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

PREPARATION OF IMAGE



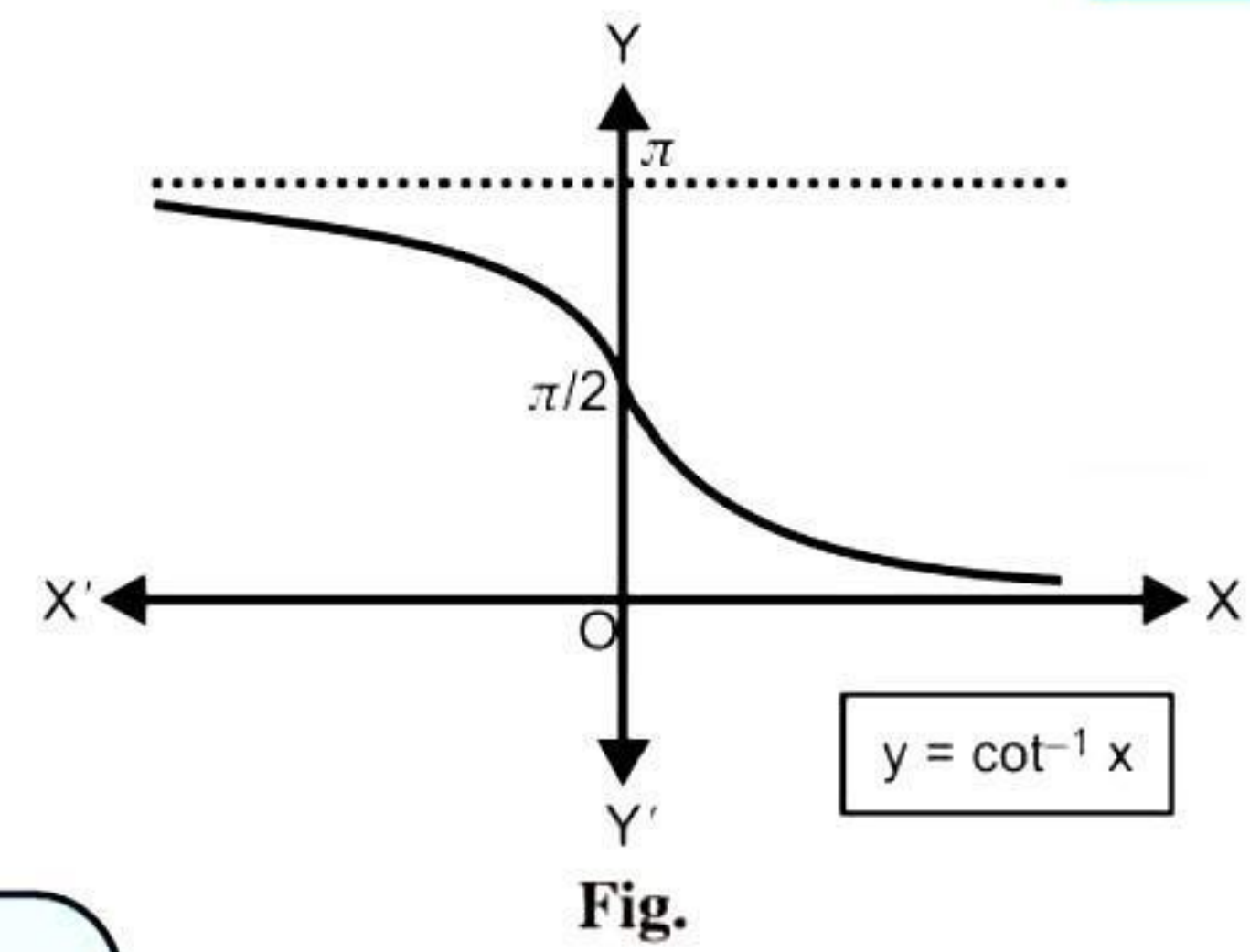
(iv) **Definition**

The inverse-cotangent function is defined as :

$$y = f^{-1}(x) = \cot^{-1} x$$

iff $x = \cot y$ and $y \in (0, \pi)$.

The graph of $\cot^{-1} x$ is as shown.

**KEY POINT**

Domain of $\cot^{-1} x = \mathbf{R}$ and Range of $\cot^{-1} x = (0, \pi)$.

(v) **Definition**

The inverse-secant function is defined as :

$$y = f^{-1}(x) = \sec^{-1} x$$

iff $x = \sec y$

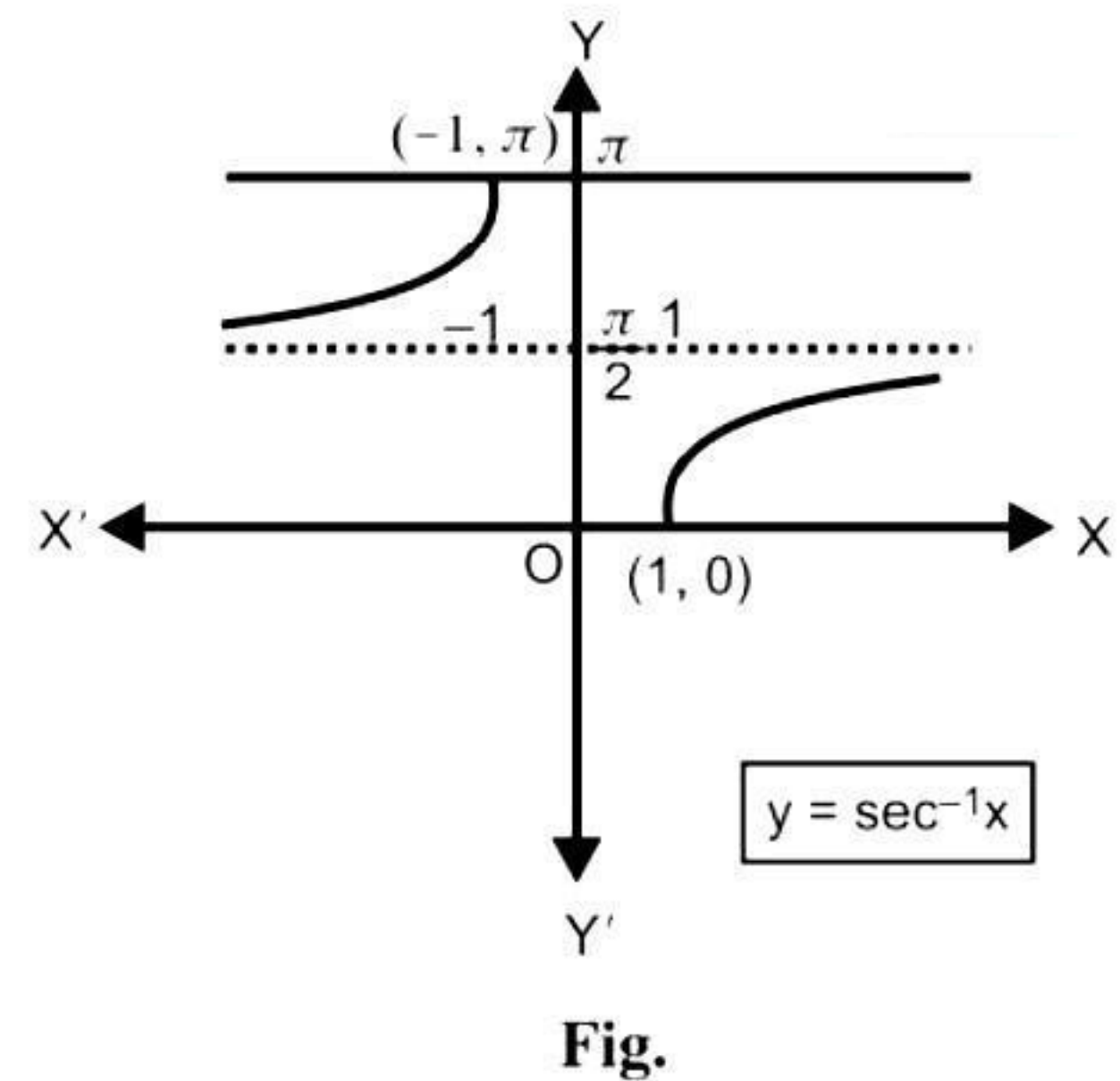
and $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

The graph of $\sec^{-1} x$ is as shown.

KEY POINT

Domain of $\sec^{-1} x = (-\infty, -1] \cup [1, \infty)$.

Range of $\sec^{-1} x = \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$. (Assam B. 2017)

(vi) **Definition**

The inverse-cosecant function is defined as :

$$y = f^{-1}(x) = \operatorname{cosec}^{-1} x$$

iff $x = \operatorname{cosec} y$

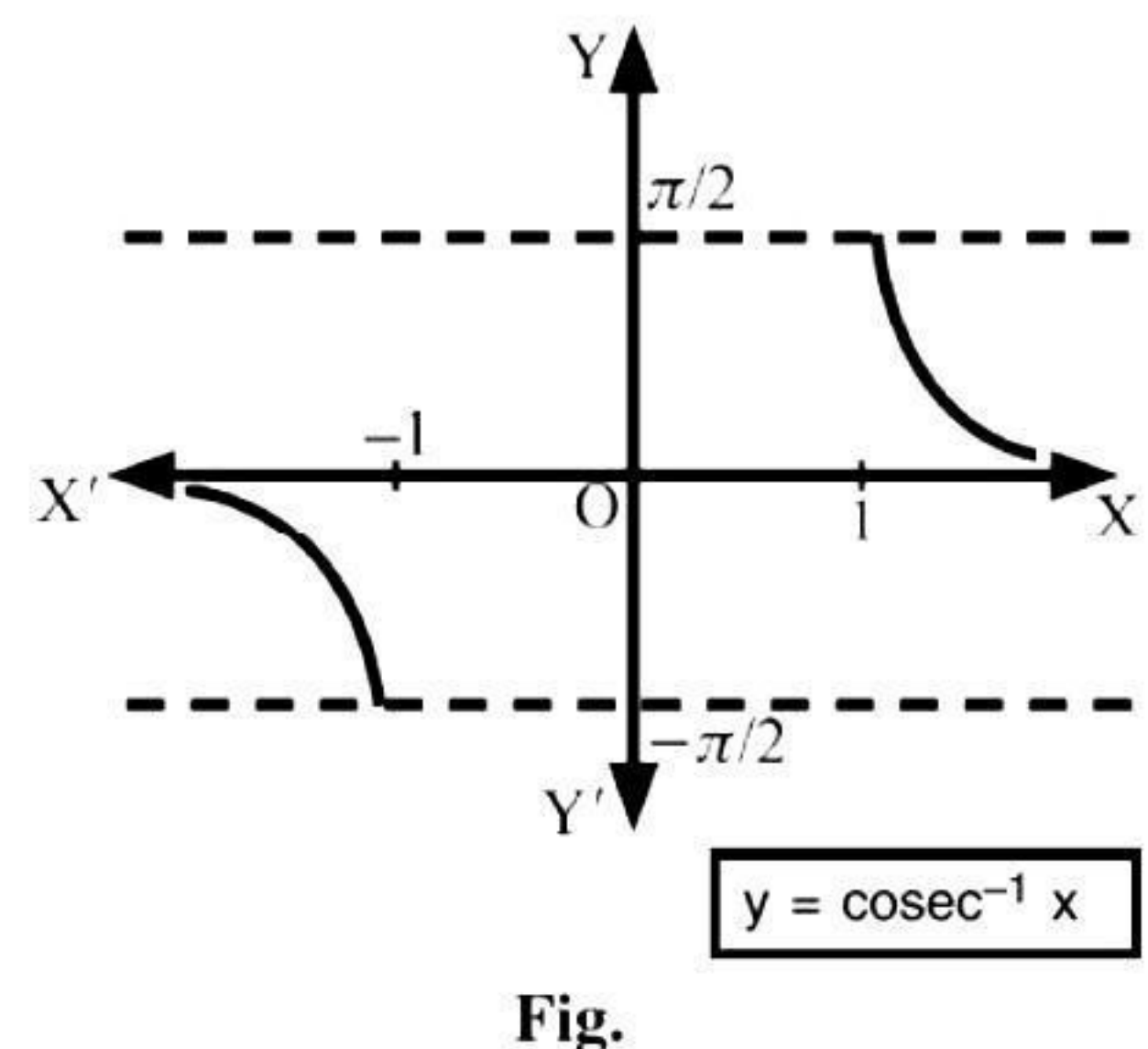
and $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$.

The graph of $\operatorname{cosec}^{-1} x$ is as shown.

KEY POINT

Domain of $\operatorname{cosec}^{-1} x = (-\infty, -1] \cup [1, \infty)$. (Assam B. 2018)

Range of $\operatorname{cosec}^{-1} x = \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$. (Assam B. 2017)



Note : On similar lines, the readers and try the images for (iv) to (vi)

2.2. TABLE OF INVERSE-TRIGONOMETRIC FUNCTIONS

We give the table giving the inverse-trigonometric functions and their principal value branches :

TABLE

Function	Principal Value Branch	
	Domain	Range
(i) $y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
(ii) $y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
(iii) $y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv) $y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$
(v) $y = \sec^{-1} x$	$1 \leq x < \infty$	$0 \leq y < \frac{\pi}{2}$
	$-\infty < x \leq -1$	$\frac{\pi}{2} < y \leq \pi$
(vi) $y = \operatorname{cosec}^{-1} x$	$1 \leq x < \infty$	$0 < y \leq \frac{\pi}{2}$
	$-\infty < x \leq -1$	$-\frac{\pi}{2} \leq y < 0$

Note : When y is +ve ($0 \leq y \leq 1$), there are two angles : one between 0 and $\pi/2$ and the other between $-\pi/2$ and 0 having their cosine equal to y ($\cos x$ is an even function of x).

Here we take smallest positive angle as the principal value of $\cos^{-1} y$.

For Example : $\cos^{-1} \left(\frac{\sqrt{3}}{2} \right) = 30^\circ$ and not -30° even though $\cos(-30^\circ) = \frac{\sqrt{3}}{2}$.

ILLUSTRATIVE EXAMPLES

Example 1. Find the value of

$$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3}). \quad (\text{C.B.S.E. 2018})$$

Solution. $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$

$$= \frac{\pi}{3} - \left(\pi - \frac{\pi}{6} \right)$$

$$= -\frac{\pi}{2}.$$

Example 2. Find the principal value of $\sin^{-1} \left(\frac{1}{2} \right)$.

(Jammu B. 2017 ; Kerala B. 2013; H.B. 2012)

Solution. $\sin^{-1} \left(\frac{1}{2} \right) = \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$.

Hence, the principal value of $\sin^{-1} \left(\frac{1}{2} \right)$ is $\frac{\pi}{6}$.

Example 3. What is the principal value of :

$$\cos^{-1} \left(\cos \frac{2\pi}{3} \right) + \sin^{-1} \left(\sin \frac{2\pi}{3} \right)? \quad (\text{A.I.C.B.S.E. 2011})$$

Solution. $\pi.$ $\left[\because \cos^{-1} \left(\cos \frac{2\pi}{3} \right) + \sin^{-1} \left(\sin \frac{2\pi}{3} \right) \right]$

$$= \cos^{-1} \left(-\frac{1}{2} \right) + \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) = \frac{2\pi}{3} + \frac{\pi}{3} = \pi.$$

Example 4. Find the principal value of :

$$\tan^{-1}(\sqrt{3}) - \sec^{-1}(-2). \quad (\text{A.I.C.B.S.E. 2012})$$

Solution. $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$

$$\sec^{-1}(-2) = \frac{2\pi}{3} \in \left[0, \frac{\pi}{2} \right] - \left\{ \frac{\pi}{2} \right\}.$$

$$\begin{aligned} \therefore \tan^{-1}(\sqrt{3}) - \sec^{-1}(-2) \\ = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}. \end{aligned}$$

Example 5. Evaluate : $\tan^{-1} \left(2 \cos \left(2 \sin^{-1} \left(\frac{1}{2} \right) \right) \right)$.

(N.C.E.R.T. ; H.P.B. 2013 ; Jammu B. 2012)

Solution. $\tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right]$

$$\begin{aligned} &= \tan^{-1} \left[2 \cos \left(2 \cdot \frac{\pi}{6} \right) \right] \\ &= \tan^{-1} \left[2 \cos \frac{\pi}{3} \right] = \tan^{-1} \left(2 \cdot \frac{1}{2} \right) \\ &= \tan^{-1} (1) = \frac{\pi}{4}. \end{aligned}$$

EXERCISE 2 (a)

Fast Track Answer Type Questions



1. (a) Fill in the blanks :

(i) Principal value of $\cot^{-1}(\sqrt{3})$ is

(Jammu B. 2018)

(ii) Principal value of $\sin^{-1} \left(-\frac{1}{2} \right)$ is

(Jammu B. 2017)

(iii) Principal value of $\tan^{-1}(-1)$ is.....

(Jammu B. 2017)

(iv) Principal value of $\cos^{-1} \left(-\frac{1}{2} \right)$ is.....

(Kashmir B. 2016)

(v) Principal value of $\tan^{-1}(-\sqrt{3})$ is.....

(Meghalaya B. 2018; Kashmir B. 2016)

(vi) Principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$ is.....

(Kashmir B. 2016)

(b) Write the domain of $f(x) = \cos^{-1}(x)$.

(Karnataka B. 2014)

Find the principal values of the following (2–8) :

2. (i) $\sin^{-1} \left(-\frac{\sqrt{3}}{2} \right)$

(Jammu B. 2012; C.B.S.E. 2010)

(ii) $\sin^{-1} \left(\frac{1}{\sqrt{2}} \right)$. (N.C.E.R.T. ; Uttarakhand B. 2013)

3. (i) $\cos^{-1} \left(-\frac{1}{2} \right)$ (Kerala B. 2014; H.P.B. 2010 S)

(ii) $\cos^{-1} \left(-\frac{\sqrt{3}}{2} \right)$ (C.B.S.E. 2010)

(iii) $\cos^{-1} \left(-\frac{1}{\sqrt{2}} \right)$. (N.C.E.R.T.; Kashmir B. 2013)

4. (i) $\tan^{-1}(-\sqrt{3})$ (ii) $\tan^{-1}(-1)$.
(N.C.E.R.T.; C.B.S.E. (F) 2011)

5. (i) $\cot^{-1} \left(-\frac{1}{\sqrt{3}} \right)$ (N.C.E.R.T.)

(ii) $\cot^{-1}(-\sqrt{3})$. (A.I.C.B.S.E. 2010)

6. (i) $\sec^{-1} \left(\frac{2}{\sqrt{3}} \right)$ (N.C.E.R.T.)

(ii) $\sec^{-1}(-2)$. (A.I.C.B.S.E. 2010)

7. (i) $\operatorname{cosec}^{-1}(2)$ (ii) $\operatorname{cosec}^{-1}(-\sqrt{2})$.
(H.P.B. 2010 S) (N.C.E.R.T. Karnataka B. 2017)

8. (i) $\cos^{-1} \left(\cos \frac{13\pi}{6} \right)$ (Kashmir B. 2017)

(ii) $\cos^{-1} \left(\cos \frac{7\pi}{6} \right)$
(Tripura B. 2016; C.B.S.E. 2011)

(iii) $\tan^{-1} \left(\tan \frac{3\pi}{4} \right)$ (Jammu B. 2012)

(iv) $\sin^{-1} \left(\sin \frac{3\pi}{5} \right)$ (Assam B. 2015)

(v) $\sin \left(\cos^{-1} \frac{1}{2} \right)$ (H.B. 2015, 12)

(vi) $\cos \left(\sin^{-1} \frac{5}{13} \right)$. (H.B. 2014)

Short Answer Type Questions

9. Find the value of $\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$.
(C.B.S.E. 2010)
10. Write the principal value of :
(i) $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right)$ (C.B.S.E. 2013)
(ii) $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ (A.I.C.B.S.E. 2013)
(iii) $\cos^{-1}\left(\frac{1}{2}\right) - 2 \sin^{-1}\left(-\frac{1}{2}\right)$ (C.B.S.E. 2012)

SATQ

11. Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2 \sin^{-1}\left(\frac{1}{2}\right)$.
(N.C.E.R.T.)
12. Find the value of :
 $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) + \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) + \sin^{-1}\left(\frac{1}{2}\right)$. (H.B. 2013)
13. Write the value of $\tan^{-1}\left[2 \sin\left(2 \cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$.
(A.I.C.B.S.E. 2013)

Answers

1. (a) (i) $\frac{\pi}{6}$ (ii) $-\frac{\pi}{6}$ (iii) $-\frac{\pi}{4}$ (iv) $\frac{2\pi}{3}$ (v) $-\frac{\pi}{3}$
(vi) $-\frac{\pi}{4}$
(b) $[-1, 1]$.
2. (i) $-\frac{\pi}{3}$ (ii) $\frac{\pi}{4}$.
3. (i) $\frac{2\pi}{3}$ (ii) $\frac{5\pi}{6}$ (iii) $\frac{3\pi}{4}$.
4. (i) $-\frac{\pi}{3}$ (ii) $-\frac{\pi}{4}$.
5. (i) $\frac{2\pi}{3}$ (ii) $\frac{5\pi}{6}$.

6. (i) $\frac{\pi}{6}$ (ii) $\frac{2\pi}{3}$.
7. (i) $\frac{\pi}{6}$ (ii) $-\frac{\pi}{4}$.
8. (i) $\frac{\pi}{6}$ (ii) $\frac{5\pi}{6}$ (iii) $-\frac{\pi}{4}$ (iv) $\frac{2\pi}{5}$ (v) $\frac{\sqrt{3}}{2}$ (vi) $\frac{12}{13}$.
9. $\frac{\pi}{2}$. 10. (i) $\frac{11\pi}{12}$ (ii) $-\frac{\pi}{2}$ (iii) $\frac{2\pi}{3}$.
11. $\frac{2\pi}{3}$. 12. $\frac{5\pi}{6}$.
13. $\frac{\pi}{3}$.

Hints to Selected Questions

9. $\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right)$
 $= \sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right) + \cos^{-1}\left(\cos\frac{2\pi}{3}\right)$
 $= -\frac{\pi}{6} + \frac{2\pi}{3} = \frac{\pi}{2}$.
10. (i) $\tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{4} + \cos^{-1}\left(\cos\frac{2\pi}{3}\right)$
 $= \frac{\pi}{4} + \frac{2\pi}{3} = \frac{11\pi}{12}$.
- (ii) $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$
 $= \tan^{-1}\left(\tan\frac{\pi}{3}\right) - \cot^{-1}\left(-\cot\frac{\pi}{6}\right)$
 $= \frac{\pi}{3} - \cot^{-1}\left[\cot\left(-\frac{\pi}{6}\right)\right]$

$$= \frac{\pi}{3} - \cot^{-1}\left[\cot\left(\pi - \frac{\pi}{6}\right)\right]$$

$$= \frac{\pi}{3} - \cot^{-1}\left(\cot\frac{5\pi}{6}\right) = \frac{\pi}{3} - \frac{5\pi}{6} = -\frac{\pi}{2}$$

11. $\cos^{-1}\left(\frac{1}{2}\right) + 2 \sin^{-1}\left(\frac{1}{2}\right)$
 $= \cos^{-1}\left(\cos\frac{\pi}{3}\right) + 2 \sin^{-1}\left(\sin\frac{\pi}{6}\right)$
 $= \frac{\pi}{3} + 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$.
13. $\tan^{-1}\left[2 \sin\left(2 \cos^{-1}\frac{\sqrt{3}}{2}\right)\right] = \tan^{-1}\left[2 \sin\left(2 \cdot \frac{\pi}{6}\right)\right]$
 $= \tan^{-1}\left[2 \sin\frac{\pi}{3}\right] = \tan^{-1}\left[2 \cdot \frac{\sqrt{3}}{2}\right]$
 $= \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$.

2.3. PROPERTIES OF INVERSE-TRIGONOMETRIC FUNCTIONS

$$(a) \quad x = \sin^{-1}(\sin x) = \cos^{-1}(\cos x) = \tan^{-1}(\tan x) \\ = \cot^{-1}(\cot x) = \sec^{-1}(\sec x) = \operatorname{cosec}^{-1}(\operatorname{cosec} x).$$

Proof. Let $\sin x = y$.

$$\text{Then } x = \sin^{-1} y = \sin^{-1}(\sin x).$$

Similarly other parts.

$$(b) \text{ (i) } \operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x}, x \geq 1 \text{ or } x \leq -1 \quad \text{(ii) } \sec^{-1} x = \cos^{-1} \frac{1}{x}, x \geq 1 \text{ or } x \leq -1 \quad \text{(iii) } \cot^{-1} x = \tan^{-1} \frac{1}{x}, x > 0.$$

Proof. (i) Let $\operatorname{cosec}^{-1} x = y$. Then $x = \operatorname{cosec} y$

$$\Rightarrow \frac{1}{x} = \sin y \quad \Rightarrow y = \sin^{-1} \left(\frac{1}{x} \right).$$

$$\text{Hence, } \operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x}.$$

(ii) and (iii) can be proved similarly.

$$(c) \text{ (i) } \sin^{-1}(-x) = -\sin^{-1} x, x \in [-1, 1]$$

$$\text{(iii) } \tan^{-1}(-x) = -\tan^{-1} x, x \in \mathbf{R}$$

$$\text{(v) } \sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \geq 1$$

$$\text{(ii) } \cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1, 1]$$

$$\text{(iv) } \cot^{-1}(-x) = \pi - \cot^{-1} x, x \in \mathbf{R}$$

$$\text{(vi) } \operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, |x| \geq 1.$$

Proof. (i) Let $\sin^{-1}(-x) = y$.

$$\therefore -x = \sin y$$

$$\Rightarrow x = -\sin y = \sin(-y) \quad \Rightarrow -y = \sin^{-1} x$$

$$\Rightarrow y = -\sin^{-1} x.$$

$$\text{Hence, } \sin^{-1}(-x) = -\sin^{-1} x.$$

(ii) Let $\cos^{-1}(-x) = y$.

$$\therefore -x = \cos y$$

$$\Rightarrow x = -\cos y = \cos(\pi - y)$$

$$\Rightarrow \cos^{-1} x = \pi - y = \pi - \cos^{-1}(-x).$$

$$\text{Hence, } \cos^{-1}(-x) = \pi - \cos^{-1} x.$$

(iii)–(vi) can be proved similarly.

$$(d) \text{ (i) } \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, x \in [-1, 1]$$

(H.B. 2018; Jammu B. 2017; Jharkhand B. 2013)

$$\text{(ii) } \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in \mathbf{R}$$

(H.B. 2018)

$$\text{(iii) } \sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}, |x| \geq 1$$

(H.B. 2018)

$$\text{(iv) } \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}, xy < 1$$

(H.B. 2018; Kerala B. 2016)

$$\text{(v) } \tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}, xy > -1$$

(Jammu B. 2017)

$$\text{(vi) (I) } 2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2}, |x| \leq 1$$

$$\text{(II) } 2 \tan^{-1} x = \cos^{-1} \frac{1-x^2}{1+x^2}, x \geq 0$$

$$\text{(III) } 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}, -1 < x < 1.$$

Proof. (i) Let $\sin^{-1} x = y$.

$$\therefore x = \sin y \quad \Rightarrow \quad x = \cos \left(\frac{\pi}{2} - y \right)$$

$$\Rightarrow \cos^{-1}x = \frac{\pi}{2} - y \quad \Rightarrow \cos^{-1}x + y = \frac{\pi}{2}$$

$$\Rightarrow \cos^{-1}x + \sin^{-1}x = \frac{\pi}{2} \quad \text{Hence, } \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}.$$

$$(ii) \text{ Let } \tan^{-1}x = y.$$

$$\therefore x = \tan y \quad \Rightarrow x = \cot\left(\frac{\pi}{2} - y\right)$$

$$\Rightarrow \cot^{-1}x = \frac{\pi}{2} - y \quad \Rightarrow \cot^{-1}x + y = \frac{\pi}{2}$$

$$\Rightarrow \cot^{-1}x + \tan^{-1}x = \frac{\pi}{2} \quad \text{Hence, } \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}.$$

(iii) can be proved similarly.

$$(iv) \text{ Let } \tan^{-1}x = \theta_1 \quad \text{and} \quad \tan^{-1}y = \theta_2.$$

$$\therefore x = \tan \theta_1 \quad \text{and} \quad y = \tan \theta_2.$$

$$\text{Now } \tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2} = \frac{x + y}{1 - xy} \quad \Rightarrow \theta_1 + \theta_2 = \tan^{-1} \frac{x + y}{1 - xy}.$$

$$\text{Hence, } \tan^{-1}x + \tan^{-1}y = \tan^{-1} \frac{x + y}{1 - xy}, \quad xy < 1^* \quad \dots (1)$$

(v) can be proved similarly.

$$(vi) \text{ Let } \tan^{-1}x = y.$$

$$\therefore x = \tan y.$$

$$(I) \quad \frac{2x}{1+x^2} = \frac{2 \tan y}{1 + \tan^2 y} = \frac{2 \sin y \cos y}{\cos^2 y + \sin^2 y} = \frac{\sin 2y}{1} = \sin 2y.$$

$$\text{Thus } 2y = \sin^{-1} \frac{2x}{1+x^2}.$$

$$\text{Hence, } 2 \tan^{-1}x = \sin^{-1} \frac{2x}{1+x^2}.$$

$$(II) \quad \frac{1-x^2}{1+x^2} = \frac{1 - \tan^2 y}{1 + \tan^2 y} = \cos 2y.$$

$$\text{Thus } 2y = \cos^{-1} \frac{1-x^2}{1+x^2}.$$

$$\text{Hence, } 2 \tan^{-1}x = \cos^{-1} \frac{1-x^2}{1+x^2}.$$

$$(III) \quad \frac{2x}{1-x^2} = \frac{2 \tan y}{1 - \tan^2 y} = \tan 2y.$$

$$\text{Thus } 2y = \tan^{-1} \frac{2x}{1-x^2}.$$

$$\text{Hence, } 2 \tan^{-1}x = \tan^{-1} \frac{2x}{1-x^2}.$$

* Because the above result does not hold if $xy \geq 1$.

If $xy = 1$, then RHS of (1) is not defined.

If $xy > 1$ and $y < 0$, then $\frac{x+y}{1-xy} > 0$ and RHS of (1) is +ve while LHS of (1) is -ve. As such (1) does not hold.

Hence, (iv) does not hold for $xy \geq 1$.

KEY POINT

$$\sin^{-1} \frac{2x}{1+x^2} = 2 \tan^{-1} x \quad \text{if } |x| \leq 1$$

$$\cos^{-1} \frac{1-x^2}{1+x^2} = 2 \tan^{-1} x \quad \text{if } x \geq 0$$

$$\tan^{-1} \frac{2x}{1-x^2} = 2 \tan^{-1} x \quad \text{if } -1 < x < 1.$$

$$(e) (i) \quad \sin^{-1} x + \sin^{-1} y = \sin^{-1} (x \sqrt{1-y^2} + y \sqrt{1-x^2})$$

$$(ii) \quad \sin^{-1} x - \sin^{-1} y = \sin^{-1} (x \sqrt{1-y^2} - y \sqrt{1-x^2})$$

$$(iii) \quad \cos^{-1} x + \cos^{-1} y = \cos^{-1} (xy - \sqrt{1-x^2} \sqrt{1-y^2})$$

$$(iv) \quad \cos^{-1} x - \cos^{-1} y = \cos^{-1} (xy + \sqrt{1-x^2} \sqrt{1-y^2}).$$

Proof. (i) Let $\sin^{-1} x = \theta_1$ and $\sin^{-1} y = \theta_2$.
 $\therefore \sin \theta_1 = x$ and $\sin \theta_2 = y$

so that $\cos \theta_1 = \sqrt{1-x^2}$ and $\cos \theta_2 = \sqrt{1-y^2}$.

$$\text{Now } \sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2 = x \sqrt{1-y^2} + \sqrt{1-x^2} \cdot y$$

$$\Rightarrow \theta_1 + \theta_2 = \sin^{-1} (x \sqrt{1-y^2} + y \sqrt{1-x^2}).$$

$$\text{Hence, } \sin^{-1} x + \sin^{-1} y = \sin^{-1} (x \sqrt{1-y^2} + y \sqrt{1-x^2}).$$

(ii) can be proved similarly.

(iii) Let $\cos^{-1} x = \theta_1$ and $\cos^{-1} y = \theta_2$.
 $\therefore \cos \theta_1 = x$ and $\cos \theta_2 = y$

so that $\sin \theta_1 = \sqrt{1-x^2}$ and $\sin \theta_2 = \sqrt{1-y^2}$.

$$\text{Now } \cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 = xy - \sqrt{1-x^2} \sqrt{1-y^2}$$

$$\Rightarrow \theta_1 + \theta_2 = \cos^{-1} (xy - \sqrt{1-x^2} \sqrt{1-y^2}).$$

$$\text{Hence, } \cos^{-1} x + \cos^{-1} y = \cos^{-1} (xy - \sqrt{1-x^2} \sqrt{1-y^2}).$$

(iv) can be proved similarly.

Frequently Asked Questions

Example 1. (i) If $\sin^{-1} \left(\frac{1}{3} \right) + \cos^{-1} x = \frac{\pi}{2}$, then find x .

(C.B.S.E. 2010 C)

(ii) If $\sec^{-1} (2) + \operatorname{cosec}^{-1} (y) = \frac{\pi}{2}$, then find y .

(C.B.S.E. 2010 C)

FAQs

Solution. (i) $\sin^{-1} \left(\frac{1}{3} \right) + \cos^{-1} x = \frac{\pi}{2}$

$$\Rightarrow x = \frac{1}{3}, \quad \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right]$$

(ii) $\sec^{-1} (2) + \operatorname{cosec}^{-1} (y) = \frac{\pi}{2}$

$$\Rightarrow y = 2, \quad \left[\because \sec^{-1} (x) + \operatorname{cosec}^{-1} (x) = \frac{\pi}{2} \right]$$

Example 2. Prove the following :

$$\cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}.$$

(A.I.C.B.S.E. 2012)

Solution.

$$\begin{aligned} \text{LHS} &= \cos\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right) \\ &= \cos\left(\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{3}{\sqrt{13}}\right) \\ &= \cos\left[\cos^{-1}\left(\frac{4}{5} \cdot \frac{3}{\sqrt{13}} - \sqrt{1 - \frac{16}{25}} \sqrt{1 - \frac{9}{13}}\right)\right] \\ &= \frac{12}{5\sqrt{13}} - \frac{3}{5} \cdot \frac{2}{\sqrt{13}} = \frac{6}{5\sqrt{13}} = \text{RHS.} \end{aligned}$$

Example 3. If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$, $xy < 1$, then

write the value of $x + y + xy$. (A.I.C.B.S.E. 2014)

Solution. We have : $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1} \frac{x+y}{1-xy} = \frac{\pi}{4} \Rightarrow \frac{x+y}{1-xy} = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{x+y}{1-xy} = 1 \Rightarrow x+y = 1-xy.$$

Hence, $x + y + xy = 1$.

Example 4. Prove that $3 \sin^{-1} x = \sin^{-1} (3x - 4x^3)$;

$$x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

(N.C.E.R.T.; C.B.S.E. 2018; Jammu B. 2013; H.B. 2012)

Solution. Put $\sin^{-1} x = \theta$ so that $x = \sin \theta$.

$$\begin{aligned} \text{RHS} &= \sin^{-1}(3 \sin \theta - 4 \sin^3 \theta) \\ &= \sin^{-1}(\sin 3\theta) \\ &= 3\theta \\ &= 3 \sin^{-1} x = \text{LHS.} \end{aligned}$$

Example 5. If $\sin^{-1} x = \tan^{-1} y$, then show that :

$$\frac{1}{x^2} - \frac{1}{y^2} = 1.$$

(W. Bengal B. 2018)

Solution. We have : $\sin^{-1} x = \tan^{-1} y$

$$\Rightarrow \sin^{-1} x = \sin^{-1} \frac{y}{\sqrt{1+y^2}}$$

$$\Rightarrow x = \frac{y}{\sqrt{1+y^2}} \Rightarrow x\sqrt{1+y^2} = y.$$

$$\text{Squaring, } x^2(1+y^2) = y^2 \Rightarrow y^2 - x^2 = x^2 y^2.$$

$$\text{Hence, } \frac{1}{x^2} - \frac{1}{y^2} = 1. \quad [\text{Dividing by } x^2 y^2]$$

Example 6. Prove that $\frac{1}{2} \leq x \leq 1$, then :

$$\cos^{-1} x + \cos^{-1} \left[\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2} \right] = \frac{\pi}{3}.$$

(C.B.S.E. Sample Paper 2018)

$$\text{Solution. LHS} = \cos^{-1} x + \cos^{-1} \left[\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2} \right]$$

$$= \cos^{-1} x + \cos^{-1} \left[\frac{1}{2} x + \frac{\sqrt{3}}{2} \sqrt{1-x^2} \right]$$

$$= \theta + \cos^{-1} \left[\cos \frac{\pi}{3} \cdot \cos \theta + \sin \frac{\pi}{3} \cdot \sin \theta \right]$$

[Putting $x = \cos \theta$ so that $\sqrt{1-x^2} = \sin \theta$]

$$= \theta + \cos^{-1} \left[\cos \left(\frac{\pi}{3} - \theta \right) \right]$$

$$= \theta + \left(\frac{\pi}{3} - \theta \right)$$

$$= \frac{\pi}{3} = \text{RHS.}$$

Example 7. Find the value of :

$$\sin^{-1} \left(2 \tan^{-1} \frac{1}{4} \right) + \cos (\tan^{-1} 2\sqrt{2}).$$

(C.B.S.E. Sample Paper 2019)

Solution. To evaluate : $\sin^{-1} \left(2 \tan^{-1} \frac{1}{4} \right)$.

Put $\tan^{-1} \frac{1}{4} = \theta$ so that $\tan \theta = \frac{1}{4}$.

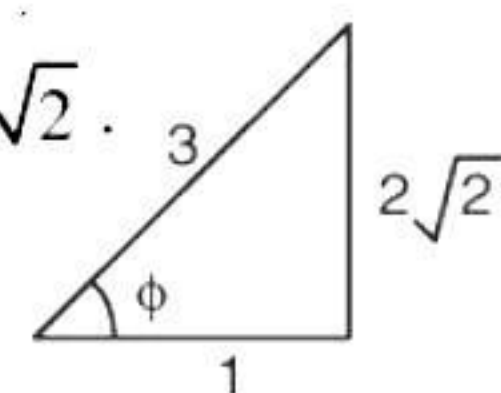
$$\text{Now, } \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2 \left(\frac{1}{4} \right)}{1 + \left(\frac{1}{4} \right)^2} = \frac{1/2}{1 + 1/16}$$

$$= \frac{1/2}{17/16} = \frac{8}{17} \quad \dots(1)$$

To evaluate : $\cos (\tan^{-1} 2\sqrt{2})$.

Put $\tan^{-1} 2\sqrt{2} = \phi$ so that $\tan \phi = 2\sqrt{2}$.

$$\therefore \cos \phi = \frac{1}{3} \quad \dots(2)$$



Hence, $\sin^{-1} \left(2 \tan^{-1} \frac{1}{4} \right) + \cos (\tan^{-1} 2\sqrt{2})$

$$= \frac{8}{17} + \frac{1}{3} \quad [\text{Using (1) \& (2)}]$$

$$= \frac{24+17}{51} = \frac{41}{51}.$$

Example 8. Prove that :

$$\tan^{-1} \left(\frac{1}{2} \right) + \tan^{-1} \left(\frac{1}{5} \right) + \tan^{-1} \left(\frac{1}{8} \right) = \frac{\pi}{4}.$$

(C.B.S.E. 2013; A.I.C.B.S.E. 2011; H.B. 2010)

Solution.

$$\begin{aligned}
 \text{LHS} &= \left(\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) \right) + \tan^{-1}\left(\frac{1}{8}\right) \\
 &= \tan^{-1} \frac{\frac{1}{2} + \frac{1}{5}}{1 - \frac{1}{2} \cdot \frac{1}{5}} + \tan^{-1}\left(\frac{1}{8}\right) \\
 &= \tan^{-1} \frac{5+2}{10-1} + \tan^{-1}\left(\frac{1}{8}\right) \\
 &= \tan^{-1}\left(\frac{7}{9}\right) + \tan^{-1}\left(\frac{1}{8}\right) \\
 &= \tan^{-1} \frac{\frac{7}{9} + \frac{1}{8}}{1 - \frac{7}{9} \cdot \frac{1}{8}} = \tan^{-1} \frac{56+9}{72-7} \\
 &= \tan^{-1}\left(\frac{65}{65}\right) = \tan^{-1}(1) = \frac{\pi}{4} = \text{RHS.}
 \end{aligned}$$

Example 9. Show that :

$$\sin^{-1} \frac{3}{5} - \sin^{-1} \frac{8}{17} = \cos^{-1} \frac{84}{85}.$$

(N.C.E.R.T.; H.P.B. 2011)

Solution. Let $\sin^{-1} \frac{3}{5} = x$ and $\sin^{-1} \frac{8}{17} = y$.

$$\therefore \sin x = \frac{3}{5} \quad \text{and} \quad \sin y = \frac{8}{17}$$

$$\text{so that } \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\text{and } \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \frac{64}{289}} = \sqrt{\frac{225}{289}} = \frac{15}{17}.$$

Now $\cos(x - y) = \cos x \cos y + \sin x \sin y$

$$\begin{aligned}
 &= \left(\frac{4}{5}\right) \left(\frac{15}{17}\right) + \left(\frac{3}{5}\right) \left(\frac{8}{17}\right) \\
 &= \frac{60+24}{85} = \frac{84}{85}
 \end{aligned}$$

$$\Rightarrow x - y = \cos^{-1} \left(\frac{84}{85} \right).$$

$$\text{Hence, } \sin^{-1} \frac{3}{5} - \sin^{-1} \frac{8}{17} = \cos^{-1} \frac{84}{85}.$$

Example 10. Prove that :

$$\sin^{-1} \left(\frac{5}{13} \right) + \cos^{-1} \left(\frac{4}{5} \right) = \frac{1}{2} \sin^{-1} \left(\frac{3696}{4225} \right).$$

(P.B. 2018)

$$\text{Solution. LHS} = \sin^{-1} \left(\frac{5}{13} \right) + \cos^{-1} \left(\frac{4}{5} \right)$$

$$= \sin^{-1} \left(\frac{5}{13} \right) + \sin^{-1} \left(\frac{3}{5} \right)$$

$$= \frac{1}{2} \left[2 \sin^{-1} \left(\frac{5}{13} \right) + 2 \sin^{-1} \left(\frac{3}{5} \right) \right]$$

$$= \frac{1}{2} \left[\sin^{-1} \left(\frac{5}{13} \right) + \sin^{-1} \left(\frac{5}{13} \right) \right] + \left[\sin^{-1} \left(\frac{3}{5} \right) + \sin^{-1} \left(\frac{3}{5} \right) \right]$$

$$= \frac{1}{2} \left(\sin^{-1} \left(2 \cdot \frac{5}{13} \sqrt{1 - \frac{25}{169}} \right) + \sin^{-1} \left(2 \cdot \frac{3}{5} \sqrt{1 - \frac{9}{25}} \right) \right)$$

$$= \frac{1}{2} \left(\sin^{-1} \frac{10}{13} \times \frac{12}{13} + \sin^{-1} \left(\frac{6}{5} \times \frac{4}{5} \right) \right)$$

$$= \frac{1}{2} \left[\sin^{-1} \left(\frac{120}{169} \right) + \sin^{-1} \left(\frac{24}{25} \right) \right]$$

$$= \frac{1}{2} \left[\sin^{-1} \left(\frac{120}{169} \sqrt{1 - \frac{576}{625}} + \frac{24}{25} \sqrt{1 - \frac{14400}{28561}} \right) \right]$$

$$= \frac{1}{2} \left[\sin^{-1} \left(\frac{120}{169} \times \frac{7}{25} + \frac{24}{25} \times \frac{119}{169} \right) \right]$$

$$= \frac{1}{2} \sin^{-1} \left(\frac{840 + 2856}{4225} \right)$$

$$= \frac{1}{2} \sin^{-1} \left(\frac{3696}{4225} \right) = \text{RHS.}$$

Example 11. Show that :

$$2 \sin^{-1} \left(\frac{3}{5} \right) - \tan^{-1} \left(\frac{17}{31} \right) = \frac{\pi}{4}.$$

(A.I.C.B.S.E. 2015)

$$\text{Solution. } 2 \sin^{-1} \left(\frac{3}{5} \right) = \sin^{-1} \left(2 \cdot \frac{3}{5} \sqrt{1 - \frac{9}{25}} \right)$$

$$\left[\because 2 \sin^{-1} x = \sin^{-1} (2x \sqrt{1 - x^2}) \right]$$

$$= \sin^{-1} \left(2 \times \frac{3}{5} \times \frac{4}{5} \right)$$

$$= \sin^{-1} \left(\frac{24}{25} \right) = \tan^{-1} \left(\frac{24}{7} \right) \quad \dots(1)$$

$$\text{Now LHS} = 2 \sin^{-1} \left(\frac{3}{5} \right) - \tan^{-1} \left(\frac{17}{31} \right)$$

$$= \tan^{-1} \left(\frac{24}{7} \right) - \tan^{-1} \left(\frac{17}{31} \right) \quad [\text{Using (1)}]$$

$$\begin{aligned}
 &= \tan^{-1} \frac{\frac{24}{7} - \frac{17}{31}}{1 + \frac{24}{7} \times \frac{17}{31}} = \tan^{-1} \frac{744 - 119}{217 + 408} \\
 &= \tan^{-1} \frac{625}{625} = \tan^{-1}(1) \\
 &= \frac{\pi}{4} = \text{RHS.}
 \end{aligned}$$

Example 12. Prove that :

$$\tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}.$$

(C.B.S.E. 2017)

Solution. Put $\cos^{-1}\frac{a}{b} = \theta$ so that $\cos \theta = \frac{a}{b}$.

$$\begin{aligned}
 \text{LHS} &= \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) \\
 &= \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} + \frac{1 - \tan \frac{\theta}{2}}{1 + \tan \frac{\theta}{2}} \\
 &= \frac{\left(1 + \tan \frac{\theta}{2}\right)^2 + \left(1 - \tan \frac{\theta}{2}\right)^2}{1 - \tan^2 \frac{\theta}{2}} \\
 &= \frac{2(1 + \tan^2 \theta)}{1 - \tan^2 \frac{\theta}{2}} = \frac{2}{\cos 2 \cdot \frac{\theta}{2}} = \frac{2}{\cos \theta} \\
 &= \frac{2}{a/b} = \frac{2b}{a} = \text{RHS.}
 \end{aligned}$$

Example 13. Prove the following :

$$\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right).$$

(Meghalaya B. 2016; H.P.B. 2015;
C.B.S.E. (F) 2011)

Solution. Putting $\sin^{-1}\left(\frac{1}{3}\right) = x$ and $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = y$,

we get :

$$\sin x = \frac{1}{3} \text{ and } \sin y = \frac{2\sqrt{2}}{3}.$$

$$\therefore \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{1}{9}}$$

$$= \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

$$\text{and } \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \frac{8}{9}} = \sqrt{\frac{1}{9}} = \frac{1}{3}.$$

$$\text{Now } \sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{2\sqrt{2}}{3}\right)\left(\frac{2\sqrt{2}}{3}\right)$$

$$= \frac{1}{9} + \frac{8}{9} = 1 = \sin \frac{\pi}{2}$$

$$\Rightarrow x + y = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{\pi}{2}$$

$$\Rightarrow \frac{9}{4}\left(\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right) = \frac{9}{4}\left(\frac{\pi}{2}\right)$$

[Multiplying by $\frac{9}{4}$]

$$\Rightarrow \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{9\pi}{8}.$$

$$\text{Hence, } \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right).$$

Example 14. Prove that :

$$\tan^{-1}\left(\frac{6x - 8x^3}{1 - 12x^2}\right) - \tan^{-1}\left(\frac{4x}{1 - 4x^2}\right) = \tan^{-1} 2x; |2x| < \frac{1}{\sqrt{3}}.$$

(A.I.C.B.S.E. 2016)

Solution.

$$\text{LHS} = \tan^{-1}\left(\frac{6x - 8x^3}{1 - 12x^2}\right) - \tan^{-1}\left(\frac{4x}{1 - 4x^2}\right)$$

$$= \tan^{-1} \frac{\frac{6x - 8x^3}{1 - 12x^2} - \frac{4x}{1 - 4x^2}}{1 + \frac{6x - 8x^3}{1 - 12x^2} \cdot \frac{4x}{1 - 4x^2}}$$

$$= \tan^{-1} \frac{(6x - 8x^3)(1 - 4x^2) - 4x(1 - 12x^2)}{(1 - 12x^2)(1 - 4x^2) + 4x(6x - 8x^3)}$$

$$= \tan^{-1} \frac{6x - 8x^3 - 24x^3 + 32x^5 - 4x + 48x^3}{1 - 12x^2 - 4x^2 + 48x^4 + 24x^2 - 32x^4}$$

$$= \tan^{-1} \frac{2x + 16x^3 + 32x^5}{1 + 8x^2 + 16x^4} = \tan^{-1} \frac{2x(1 + 8x^2 + 16x^4)}{1 + 8x^2 + 16x^4}$$

$$= \tan^{-1} 2x = \text{RHS.}$$

Example 15. Prove that :

$$\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}.$$

(C.B.S.E. 2016)

Solution.

$$\text{LHS} = \left(\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} \right) + \left(\tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} \right)$$

$$= \tan^{-1} \frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \times \frac{1}{7}} + \tan^{-1} \frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \times \frac{1}{8}}$$

$$= \tan^{-1} \frac{12}{34} + \tan^{-1} \frac{11}{23} = \tan^{-1} \frac{6}{17} + \tan^{-1} \frac{11}{23}$$

$$= \tan^{-1} \frac{\frac{6}{17} + \frac{11}{23}}{1 - \frac{6}{17} \times \frac{11}{23}} = \tan^{-1} \frac{138 + 187}{391 - 66} = \tan^{-1} \frac{325}{325}$$

$$= \tan^{-1} (1) = \frac{\pi}{4} = \text{RHS.}$$

Example 16. Solve for x :

$$2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x). \quad (\text{C.B.S.E. 2016})$$

Solution. Here $2 \tan^{-1} (\cos x) = \tan^{-1} (\cos x) + \tan^{-1} (\cos x)$

$$= \tan^{-1} \frac{\cos x + \cos x}{1 - \cos x \cdot \cos x} = \tan^{-1} \frac{2 \cos x}{1 - \cos^2 x}$$

$$= \tan^{-1} \frac{2 \cos x}{\sin^2 x} = \tan^{-1} (2 \cot x \operatorname{cosec} x) \quad \dots (1)$$

$$\text{Now } 2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1} (2 \cot x \operatorname{cosec} x) = \tan^{-1} (2 \operatorname{cosec} x)$$

[Using (1)]

$$\Rightarrow 2 \cot x \operatorname{cosec} x = 2 \operatorname{cosec} x$$

$$\Rightarrow \cot x \operatorname{cosec} x = \operatorname{cosec} x$$

$$\Rightarrow \sin x = \tan x \sin x$$

$$\Rightarrow \text{either } \sin x = 0 \text{ or } \tan x = 1.$$

$$\text{Hence, } x = n\pi \quad \forall n \in \mathbb{Z} \text{ or } x = m\pi + \frac{\pi}{4} \quad \forall m \in \mathbb{Z}.$$

$$\text{Example 17. If } \tan^{-1} \left(\frac{x-2}{x-4} \right) + \tan^{-1} \left(\frac{x+2}{x+4} \right) = \frac{\pi}{4},$$

find the value of 'x'.

(A.I.C.B.S.E. 2014)

Solution. We have :

$$\tan^{-1} \left(\frac{x-2}{x-4} \right) + \tan^{-1} \left(\frac{x+2}{x+4} \right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{\left(\frac{x-2}{x-4} \right) + \left(\frac{x+2}{x+4} \right)}{1 - \left(\frac{x-2}{x-4} \right) \left(\frac{x+2}{x+4} \right)} = \frac{\pi}{4}$$

$$\Rightarrow \frac{(x-2)(x+4) + (x+2)(x-4)}{(x-4)(x+4) - (x-2)(x+2)} = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{(x^2 - 2x + 4x - 8) + (x^2 + 2x - 4x - 8)}{(x^2 - 16) - (x^2 - 4)} = 1$$

$$\Rightarrow \frac{2x^2 - 16}{-12} = 1 \quad \Rightarrow \quad 2x^2 - 16 = -12$$

$$\Rightarrow 2x^2 = 16 - 12 \quad \Rightarrow \quad 2x^2 = 4$$

$$\Rightarrow x^2 = 2.$$

Hence, $x = \pm\sqrt{2}.$

Example 18. If $\tan^{-1} \frac{x-3}{x-4} + \tan^{-1} \frac{x+3}{x+4} = \frac{\pi}{4}$, then find the value of x.

(A.I.C.B.S.E. 2017)

$$\text{Solution. We have : } \tan^{-1} \frac{x-3}{x-4} + \tan^{-1} \frac{x+3}{x+4} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{\frac{x-3}{x-4} + \frac{x+3}{x+4}}{1 - \frac{x-3}{x-4} \cdot \frac{x+3}{x+4}} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{(x-3)(x+4) + (x-4)(x+3)}{(x^2 - 16) - (x^2 - 9)} = \frac{\pi}{4}$$

$$\Rightarrow \frac{(x^2 + x - 12) + (x^2 - x - 12)}{-16 + 9} = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{2x^2 - 24}{-7} = 1$$

$$\Rightarrow 2x^2 - 24 = -7$$

$$\Rightarrow 2x^2 = 17 \quad \Rightarrow \quad x^2 = \frac{17}{2}.$$

$$\text{Hence, } x = \pm\sqrt{\frac{17}{2}}.$$

Example 19. Solve : (i) $\tan^{-1} \frac{x}{2} + \tan^{-1} \frac{x}{3} = \frac{\pi}{4}$;

$$\sqrt{6} > x > 0$$

(P.B. 2015; C.B.S.E. 2010 C)

(ii) $\sin \left(\sin^{-1} \frac{1}{5} + \cos^{-1} x \right) = 1.$

(N.C.E.R.T. ; H.B. 2017; Jammu B. 2013)

Solution. (i) We have :

$$\tan^{-1} \frac{x}{2} + \tan^{-1} \frac{x}{3} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{\frac{x}{2} + \frac{x}{3}}{1 - \left(\frac{x}{2}\right)\left(\frac{x}{3}\right)} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{3x + 2x}{6 - x^2} = \frac{\pi}{4} ; 6 > x^2 > 0 \text{ i.e. } \sqrt{6} > x > 0$$

$$\Rightarrow \frac{5x}{6 - x^2} = \tan \frac{\pi}{4} \Rightarrow \frac{5x}{6 - x^2} = 1$$

$$\Rightarrow 5x = 6 - x^2 \Rightarrow x^2 + 5x - 6 = 0$$

$$\Rightarrow (x + 6)(x - 1) = 0 \Rightarrow x = -6, 1.$$

Hence, $x = 1.$ [$\because x > 0$]

(ii) We have : $\sin \left(\sin^{-1} \frac{1}{5} + \cos^{-1} x \right) = 1 = \sin \frac{\pi}{2}$

$$\Rightarrow \sin^{-1} \frac{1}{5} + \cos^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} \frac{1}{5} = \cos^{-1} \frac{1}{5}.$$

Hence, $x = \frac{1}{5}.$

Example 20. If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$, then

find 'x'.

(C.B.S.E. 2015)

Solution. We have : $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$... (1)

Put $\tan^{-1} x = t$ so that $\cot^{-1} x = \frac{\pi}{2} - t.$

$\therefore$ (1) becomes : $t^2 + \left(\frac{\pi}{2} - t\right)^2 = \frac{5\pi^2}{8}$

$$\Rightarrow t^2 + \frac{\pi^2}{4} - \pi t + t^2 = \frac{5\pi^2}{8}$$

$$\Rightarrow 2t^2 - \pi t - \frac{3\pi^2}{8} = 0$$

$$\Rightarrow 16t^2 - 8\pi t - 3\pi^2 = 0.$$

Solving, $t = \frac{8\pi \pm \sqrt{64\pi^2 + 192\pi^2}}{32}$

$$= \frac{8\pi \pm 16\pi}{32} = \frac{3\pi}{4}, -\frac{\pi}{4}.$$

When $\tan^{-1} x = \frac{3\pi}{4}$, then $x = \tan \frac{3\pi}{4} = -1.$

When $\tan^{-1} x = -\frac{\pi}{4}$, then $x = \tan \left(-\frac{\pi}{4}\right) = -1.$

Hence, $x = -1.$

Example 21. Write : $\tan^{-1} \frac{1}{\sqrt{x^2 - 1}}$, $|x| > 1$

in the simplest form. (H.P.B. 2013 S, 10 S)

Solution. Put $x = \sec \theta.$

Then $\sqrt{x^2 - 1} = \sqrt{\sec^2 \theta - 1} = \tan \theta.$

$$\therefore \tan^{-1} \frac{1}{\sqrt{x^2 - 1}} = \tan^{-1} \left(\frac{1}{\tan \theta} \right) = \tan^{-1} (\cot \theta)$$

$$= \tan^{-1} \left(\tan \left(\frac{\pi}{2} - \theta \right) \right)$$

$$= \frac{\pi}{2} - \theta.$$

Hence, $\tan^{-1} \frac{1}{\sqrt{x^2 - 1}} = \frac{\pi}{2} - \sec^{-1} x.$

Example 22. Prove that :

$$\tan^{-1} \left(\frac{\cos x}{1 + \sin x} \right) = \frac{\pi}{4} - \frac{x}{2}, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

(C.B.S.E. 2012)

Solution. Here

$$\begin{aligned} \frac{\cos x}{1 + \sin x} &= \frac{\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}} \\ &= \frac{\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right) \left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)}{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} \end{aligned}$$

$$= \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}}$$

$$= \frac{1 - \tan \frac{x}{2}}{1 + \tan \frac{x}{2}}$$

[Dividing Num. & Denom. by $\cos \frac{x}{2}$]

$$= \tan \left(\frac{\pi}{4} - \frac{x}{2} \right).$$

Hence, $\tan^{-1} \left(\frac{\cos x}{1 + \sin x} \right) = \frac{\pi}{4} - \frac{x}{2}.$

Example 23. Prove that :

$$\tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2} = \tan^{-1} \frac{3x-x^3}{1-3x^2}, \quad |x| < \frac{1}{\sqrt{3}}.$$

(N.C.E.R.T.; Jammu B. 2018; Kashmir B. 2012 ; A.I.C.B.S.E. 2010)

Solution. Put $x = \tan \theta$ so that $\theta = \tan^{-1} x$.

$$\begin{aligned} \text{RHS} &= \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right) = \tan^{-1} \left(\frac{3 \tan \theta - \tan^3 \theta}{1-3 \tan^2 \theta} \right) \\ &= \tan^{-1} (\tan 3\theta) = 3\theta = 3 \tan^{-1} x \\ &= \tan^{-1} x + 2 \tan^{-1} x = \tan^{-1} x + \tan^{-1} \frac{2x}{1-x^2} = \text{LHS.} \\ &\quad \text{[Property (d) (vi) (III)]} \end{aligned}$$

Example 24. Simplify : $\tan^{-1} \left[\frac{a \cos x - b \sin x}{b \cos x + a \sin x} \right],$
if $\frac{a}{b} \tan x > -1.$

(N.C.E.R.T. ; H.P.B. 2013 S)

Solution. $\tan^{-1} \left[\frac{a \cos x - b \sin x}{b \cos x + a \sin x} \right]$

$$= \tan^{-1} \left[\frac{\frac{a \cos x - b \sin x}{b \cos x}}{\frac{b \cos x + a \sin x}{b \cos x}} \right]$$

[Dividing Numerator and Denominator by $b \cos x$]

$$= \tan^{-1} \left[\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b} \tan x} \right]$$

$$= \tan^{-1} \left(\frac{a}{b} \right) - \tan^{-1} (\tan x) = \tan^{-1} \frac{a}{b} - x.$$

Example 25. Show that :

$$\tan^{-1} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x; -\frac{1}{\sqrt{2}} \leq x \leq 1.$$

(N.C.E.R.T.; Assam B. 2018; Jammu B. 2015W, 13; H.P.B. 2015, 13 S; A.I.C.B.S.E. 2014, 11 ; H.B. 2013; P.B. 2013; Assam. B. 2013)

Solution. LHS = $\tan^{-1} \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$
 $= \tan^{-1} \frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}$
 [Putting $x = \cos 2\theta$]

$$= \tan^{-1} \frac{\sqrt{2 \cos^2 \theta} - \sqrt{2 \sin^2 \theta}}{\sqrt{2 \cos^2 \theta} + \sqrt{2 \sin^2 \theta}}$$

$$= \tan^{-1} \left(\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right)$$

$$= \tan^{-1} \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right)$$

[Dividing Numerator and Denominator by $\cos \theta$]

$$= \tan^{-1} \left[\tan \left(\frac{\pi}{4} - \theta \right) \right]$$

$$= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x = \text{RHS.}$$

$$\left[\because \cos 2\theta = x \Rightarrow \theta = \frac{1}{2} \cos^{-1} x \right]$$

Example 26. Prove the following :

$$\cos \left[\tan^{-1} \left\{ \sin \left(\cot^{-1} x \right) \right\} \right] = \sqrt{\frac{1+x^2}{2+x^2}}.$$

(H.B. 2015 ; A.I.C.B.S.E. 2010)

Solution. Put $\cot^{-1} x = \theta$ so that $x = \cot \theta$.

$$\therefore \sin \theta = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \sin (\cot^{-1} x) = \frac{1}{\sqrt{1+x^2}} \quad \dots(1)$$

$$\therefore \tan^{-1} \{ \sin (\cot^{-1} x) \} = \tan^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right) \quad [\text{Using (1)}]$$

$$\text{Put } \tan^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right) = \phi \quad \dots(2)$$

$$\text{so that } \frac{1}{\sqrt{1+x^2}} = \tan \phi.$$

$$\therefore \cos \phi = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}$$

$$\Rightarrow \cos \left(\tan^{-1} \frac{1}{\sqrt{1+x^2}} \right) = \frac{\sqrt{1+x^2}}{\sqrt{2+x^2}}. \quad [\text{Using (2)}]$$

$$\text{Hence, } \cos \left[\tan^{-1} \{ \sin (\cot^{-1} x) \} \right] = \sqrt{\frac{1+x^2}{2+x^2}}. \quad [\text{Using (1)}]$$

EXERCISE 2 (b)

Fast Track Answer Type Questions

1. (i) $2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}$.

(True False) (Jammu B. 2018)

(ii) Is $\sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \geq 1$?

(True False) (Jammu B. 2014)

(iii) $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{4}$. (True/False)

(Kashmir B. 2016)

2. Find the value of the following :

(a) (i) $\sin^{-1} \left(\sin \frac{2\pi}{3} \right)$ (N.C.E.R.T.)

(ii) $\sin^{-1} \left(\sin \frac{3\pi}{5} \right)$ (N.C.E.R.T.)

(iii) $\sin^{-1} \left(\sin \frac{4\pi}{5} \right)$ (A.I.C.B.S.E. 2010)

(iv) $\tan^{-1} \left(\tan \frac{3\pi}{4} \right)$ (N.C.E.R.T.; C.B.S.E. 2011)

(v) $\tan^{-1} \left(\tan \frac{7\pi}{6} \right)$ (N.C.E.R.T.; Kashmir B. 2018)

(vi) $\tan \left(2 \tan^{-1} \frac{1}{5} \right)$ (C.B.S.E. 2013)

(vii) $\cos (\sec^{-1} x + \operatorname{cosec}^{-1} x), |x| > 1$ (N.C.E.R.T.; H.B. 2012)

(viii) $\cot (\tan^{-1} a + \cot^{-1} a)$. (N.C.E.R.T.)

(b) $\sin \left[\frac{\pi}{3} - \sin^{-1} \left(\frac{-1}{2} \right) \right]$. (Karnataka B. 2014; Bihar B. 2014; C.B.S.E. 2011)

3. (a) Write down the value of $\operatorname{cosec}^{-1} x + \sec^{-1} x$, where $|x| \geq 1$. (Uttarakhand B. 2015)

(b) If $4 \sin^{-1} x + \cos^{-1} x = \pi$, then find the value of x . (C.B.S.E. Sample Paper 2018)

Very Short Answer Type Questions

4. Evaluate : (i) $\tan^{-1} 1 + \cos^{-1} \frac{1}{3} + \sin^{-1} \frac{1}{3}$ (Jharkhand B. 2016)

(ii) $\sec^2 (\tan^{-1} 2) + \operatorname{cosec}^2 (\cot^{-1} 3)$. (W. Bengal B. 2017)

Prove that (5-9) :

5. (a) $2 \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{24}{7}$. (N.C.E.R.T. ; H.B. 2017, 12; Kashmir B. 2015)

(b) $\tan \left(\frac{1}{2} \sin^{-1} \frac{3}{4} \right) = \frac{4-\sqrt{7}}{3}$ (A.I.C.B.S.E. 2013)

(c) $\tan^{-1} \frac{3}{5} + \tan^{-1} \frac{1}{4} = \frac{\pi}{4}$. (Mizoram B. 2017)

6. (i) $\cos^{-1} (\cos^2 x - \sin^2 x) = 2x$ (H.B. 2012, 10)

VSATQ

(ii) $3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x), x \in \left[\frac{1}{2}, 1 \right]$.

(N.C.E.R.T.; Jammu B. 2015 W; Karnataka B. 2014; H.P.B. 2012, 10)

7. (i) $\sin^{-1} (2x \sqrt{1-x^2}) = 2 \sin^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$ (N.C.E.R.T.; Kashmir B. 2011 ; H.B. 2010)

(ii) $\sin^{-1} (2x \sqrt{1-x^2}) = 2 \cos^{-1} x, \frac{1}{\sqrt{2}} \leq x \leq 1$ (N.C.E.R.T.; Karnataka B. 2017; Uttarakhand B. 2015; H.B. 2012; H.P.B. 2012, 10)

(iii) $2 \tan^{-1} x = \sin^{-1} \left(\frac{2x}{1+x^2} \right); -1 \leq x \leq 1$ (H.B. 2015)

$$(iv) \tan^{-1} \left(\frac{2x\sqrt{1-x^2}}{1-2x^2} \right) = 2 \sin^{-1} x \text{ when :}$$

$$-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}. \quad (\text{Karnataka B. 2013})$$

Short Answer Type Questions

Prove that (10 – 12) :

$$10. (a) (i) \sin^{-1} \frac{4}{5} + \sin^{-1} \frac{5}{13} + \sin^{-1} \frac{16}{65} = \frac{\pi}{2}$$

(H.B. 2018; C.B.S.E. 2009)

$$(ii) \sin^{-1} \frac{12}{13} + \cos^{-1} \frac{4}{5} + \tan^{-1} \frac{63}{16} = \pi.$$

(N.C.E.R.T.; Assam B. 2018; H.B. 2017)

$$(b) (i) \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$$

(C.B.S.E. 2010)

$$(ii) \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{2}{11} = \tan^{-1} \frac{3}{4}$$

(N.C.E.R.T.; Kerala B. 2018; H.B. 2017, 12; H.P.B. 2011)

$$(iii) 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}$$

(N.C.E.R.T.; Kashmir B. 2017; H.B. 2017; Jharkhand B. 2016; Kerala B. 2016, 15; P.B. 2016; H.P.B. 2013, 11, 10; P.B. 2012; A.I.C.B.S.E. 2011)

$$(iv) 2 \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{4} = \tan^{-1} \frac{32}{43}$$

(Jharkhand B. 2013; P.B. 2012)

$$(v) 2 \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{6} = \tan^{-1} \frac{42}{67} \quad (\text{P.B. 2016})$$

$$(vi) 2 \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{9}{13} \quad (\text{P.B. 2016})$$

$$(vii) \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{5} = \frac{1}{2} \cos^{-1} \frac{16}{65} \quad (\text{P.B. 2017})$$

$$(viii) \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{2}{9} = \frac{1}{2} \cos^{-1} \frac{3}{5}.$$

(Meghalaya B. 2017)

$$11. (i) \tan^{-1} \frac{3}{4} + \tan^{-1} \frac{3}{5} - \tan^{-1} \frac{8}{19} = \frac{\pi}{4}$$

(A.I.C.B.S.E. 2009 C)

$$(ii) \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$$

(N.C.E.R.T.; H.B. 2017; Kashmir B. 2013)

$$(iii) \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}.$$

(A.I.C.B.S.E. 2010, 09 C)

$$8. \tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right), x \in [0, 1].$$

(N.C.E.R.T.; H.B. 2017, 12; Jammu B. 2016; Nagaland B. 2016; C.B.S.E. 2012, 10)

$$9. \tan^{-1} \left(\frac{1-\sqrt{x}}{1+\sqrt{x}} \right) = \frac{\pi}{4} - \tan^{-1} \sqrt{x}, \text{ where } x > 0.$$

(H.B. 2011)

SATQ

$$12. (i) \cos^{-1} \frac{4}{5} + \cos^{-1} \frac{12}{13} = \cos^{-1} \frac{33}{65}$$

(H.P.B. Model Paper 2018; H.P.B. 2017, 15; A.I.C.B.S.E. 2012; C.B.S.E. 2010 C; P.B. 2010)

$$(ii) \sin^{-1} \frac{63}{65} = \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5}$$

(C.B.S.E. (F) 2012)

$$(iii) \cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{56}{65}$$

(H.P.B. 2017, 15; A.I.C.B.S.E. 2012; C.B.S.E. 2010; P.B. 2010)

$$(iv) \tan^{-1} \frac{63}{16} = \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5}$$

(N.C.E.R.T.; H.P.B. 2016, 11)

$$(v) (I) \sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{77}{85}$$

(H.P.B. 2017, 15)

$$(II) \sin^{-1} \frac{8}{17} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{77}{36}.$$

(Meghalaya B. 2018; H.B. 2017)

$$(vi) \sin^{-1} \frac{3}{5} + \sin^{-1} \frac{8}{17} = \cos^{-1} \frac{36}{85}$$

(Meghalaya B. 2013; C.B.S.E. 2012, 10 C; P.B. 2012)

$$(vii) 2 \sin^{-1} \frac{3}{5} - \tan^{-1} \frac{17}{31} = \frac{\pi}{4}$$

(Mizoram B. 2018)

$$(viii) \sin^{-1} \frac{3}{5} + \cos^{-1} \frac{12}{13} = \sin^{-1} \frac{56}{65}$$

(C.B.S.E. 2010 C)

$$(ix) \sin^{-1} \frac{3}{5} - \sin^{-1} \frac{8}{17} = \cos^{-1} \frac{84}{85}$$

(H.P.B. 2016, 11; Kerala B. 2013; Kashmir B. 2013; Uttarakhand B. 2013, 11; P.B. 2010)

$$(x) \cos^{-1} \frac{4}{5} + \tan^{-1} \frac{3}{5} = \tan^{-1} \frac{27}{11}$$

(Meghalaya B. 2015)

$$(xi) \tan^{-1} \left(\frac{63}{16} \right) = \sin^{-1} \left(\frac{5}{13} \right) + \cos^{-1} \left(\frac{3}{5} \right).$$

(H.P.B. 2016; P.B. 2014)

13. (a) Find the value of :

(i) $4 \left(2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} \right)$ (W. Bengal B. 2016)

(ii) $2 \tan^{-1} \frac{1}{5} + \sec^{-1} \frac{5\sqrt{2}}{7} + 2 \tan^{-1} \frac{1}{8}$.
(H.B. 2015, 10 ; C.B.S.E. 2014)

(b) $\tan^{-1} \left(\frac{x}{y} \right) - \tan^{-1} \left(\frac{x-y}{x+y} \right)$. (C.B.S.E. 2011)

14. Prove that :

$$\cot^{-1} \frac{ab+1}{a-b} + \cot^{-1} \frac{bc+1}{b-c} + \cot^{-1} \frac{ca+1}{c-a} = 0.$$

15. Find the value of :

$$\tan \frac{1}{2} \left(\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right),$$

where $|x| < 1$, $y > 0$ and $xy < 1$.

(N.C.E.R.T. ; Jammu B. 2015, 13 ; C.B.S.E. 2013)

16. Prove that $\tan \left(\frac{\pi}{4} + \frac{1}{2} \cos^{-1} \frac{a}{b} \right)$
 $+ \tan \left(\frac{\pi}{4} - \frac{1}{2} \cos^{-1} \frac{a}{b} \right) = \frac{2b}{a}$.
(W. Bengal B. 2018)

17. Solve the following equations :

(i) $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$, $x > 0$

(N.C.E.R.T.; H.P.B. 2018; Kashmir B. 2016; Bihar B. 2014)

(ii) $\tan^{-1} \left(\frac{x+1}{x-1} \right) + \tan^{-1} \left(\frac{x-1}{x} \right) = \tan^{-1} (-7)$
(A.I.C.B.S.E. 2009 C)

(iii) $\tan^{-1} \left(\frac{x-1}{x-2} \right) + \tan^{-1} \left(\frac{x+1}{x+2} \right) = \frac{\pi}{4}$, $|x| < 1$
(N.C.E.R.T.; Kerala B. 2017; Kashmir B. 2016;
Tripura B. 2016; H.B. 2015, 12 ; H.P.B. 2013, 10 ;
Jammu B. 2013 ; A.I.C.B.S.E. 2010 ; C.B.S.E. 2010)

(iv) $\sin^{-1}(1-x) - 2 \sin^{-1} x = \frac{\pi}{2}$
(A.I.C.B.S.E. 2015)

(v) $2 \tan^{-1}(\sin x) = \tan^{-1}(2 \sec x)$, $x \neq \frac{\pi}{2}$
(C.B.S.E. (F) 2012)

(vi) $2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$
(N.C.E.R.T.; H.P.B. 2018)

(vii) $\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1} \left(\frac{8}{79} \right)$;
 $x > 0$ (C.B.S.E. 2010 C)

(viii) $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1} \left(\frac{8}{31} \right)$; $x > 0$
(W. Bengal B. 2017, P.B. 2015, 14 S)

(ix) $\tan^{-1}(x+2) + \tan^{-1}(x-2) = \frac{\pi}{4}$; $x > 0$
(P.B. 2015)

(x) $\tan^{-1} \left(\frac{1-x}{1+x} \right) = \frac{1}{2} \tan^{-1} x$; $x > 0$
(N.C.E.R.T.; Karnataka B. 2017; H.B. 2015, 12 ;
C.B.S.E. (F) 2011)

(xi) $\tan^{-1} \frac{2x}{1-x^2} + \cot^{-1} \frac{1-x^2}{2x} = \frac{\pi}{3}$; $x > 0$
(H.B. 2012)

(xii) $\cos(\tan^{-1} x) = \sin \left(\cot^{-1} \frac{3}{4} \right)$.
(C.B.S.E. 2017; A.I.C.B.S.E. 2013)

(xiii) $\sin[\cot^{-1}(x+1)] = \cos(\tan^{-1} x)$
(C.B.S.E. 2015)

(xiv) $2 \tan^{-1}(\sin x) = \tan^{-1}(2 \sec x)$; $0 < x < \frac{\pi}{2}$
(C.B.S.E. 2010 C)

(xv) $\cos(\sin^{-1} x) = \frac{1}{2}$. (Assam B. 2017)

18. Solve for x : $\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x$.
(A.I.C.B.S.E. 2016)

19. Solve for x :

$$3 \sin^{-1} \left(\frac{2x}{1+x^2} \right) - 4 \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) + 2 \tan^{-1} \left(\frac{2x}{1-x^2} \right) = \frac{\pi}{2}.$$

20. Write the following in the simplest form :

(i) $\tan^{-1} \left(\frac{\cos x}{1 - \sin x} \right)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(N.C.E.R.T.; H.P.B. 2017, 16, 14, 10 S; Meghalaya B. 2014 ; Jammu B. 2012)

(ii) $\tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right)$, $x < \pi$
(N.C.E.R.T.; H.P.B. 2017, 16, 10; H.B. 2012, 11 ;
Jammu B. 2012)

(iii) $\tan^{-1} \left(\frac{\cos x + \sin x}{\cos x - \sin x} \right)$, $0 < x < \pi$
(Kerala B. 2014)

(iv) $\tan^{-1} \left(\frac{x}{\sqrt{a^2 - x^2}} \right)$, $|x| < a$

(N.C.E.R.T. ; H.P.B. 2018, 17, 13S, 13, 10S, 10; H.B. 2014)

(v) $\tan^{-1} \left(\frac{\sqrt{1+x^2} - 1}{x} \right)$, $x \neq 0$

(N.C.E.R.T.; H.B. 2018; H.P.B. 2018, 17, 13S; Karnataka B. 2017; Kashmir B. 2015; H.B. 2014 ; Jammu B. 2014, 13; P.B. 2010)

$$(vi) \tan^{-1} \left(\frac{1}{\sqrt{x^2 - 1}} \right), |x| > 1. \quad (\text{N.C.E.R.T.})$$

21. Prove that :

$$(i) \tan^{-1} \left(\frac{\sqrt{1-x^2}}{1+x} \right) = \frac{1}{2} \cos^{-1} x \quad (\text{P.B. 2010})$$

$$(ii) \frac{1}{2} \tan^{-1} x = \cos^{-1} \left\{ \frac{1 + \sqrt{1+x^2}}{2\sqrt{1+x^2}} \right\}^{\frac{1}{2}}. \quad (\text{Rajasthan B. 2013})$$

22. (a) Prove that :

$$(i) \tan^{-1} \left[\frac{\sqrt{1+z} + \sqrt{1-z}}{\sqrt{1+z} - \sqrt{1-z}} \right] = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} z \quad (\text{P.B. 2013})$$

$$(ii) \tan^{-1} \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x^2 \quad (\text{H.B. 2018, 15, 13; P.B. 2013})$$

$$(iii) \cot^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x.$$

(H.B. 2013)

(b) Prove that :

$$\cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4} \right).$$

(H.B. 2018; Assam B. 2017; Jammu B. 2015; H.P.B. 2015, 13, S; C.B.S.E. 2014, 11)

23. If $\tan^{-1} x + \tan^{-1} y - \tan^{-1} z = 0$, then prove that :
 $x + y + xyz = z.$ (H.B. 2011)

24. (i) If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, prove that :
 $x^2 + y^2 + z^2 + 2xyz = 1.$ (H.B. 2010)

(ii) If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, prove that :

$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz.$$

(Assam B. 2016; W. Bengal B. 2016)

25. Show that $\sin \left[\cot^{-1} \left\{ \cos (\tan^{-1} x) \right\} \right] = \sqrt{\frac{x^2 + 1}{x^2 + 2}}.$

Long Answer Type Questions

26. If $\tan^{-1} \frac{yz}{xr} + \tan^{-1} \frac{zx}{yr} + \tan^{-1} \frac{xy}{zr} = \frac{\pi}{4}$, prove that :

$$x^2 + y^2 + z^2 = r^2. \quad (\text{H.B. 2010})$$

27. (i) If $\cos^{-1} \frac{x}{a} + \cos^{-1} \frac{y}{b} = \alpha$, prove that :

$$\frac{x^2}{a^2} - \frac{2xy}{ab} \cos \alpha + \frac{y^2}{b^2} = \sin^2 \alpha.$$

(ii) If $\cos^{-1} \frac{x}{2} + \cos^{-1} \frac{y}{3} = \theta$,
 prove that $9x^2 - 12xy \cos \theta + 4y^2 = 36 \sin^2 \theta.$

Answers

1. (i) False (ii) True (iii) False.

$$2. (a) (i) \frac{\pi}{3} \quad (ii) \frac{2\pi}{5} \quad (iii) \frac{\pi}{5} \\ (iv) -\frac{\pi}{4} \quad (v) \frac{\pi}{6} \quad (vi) \frac{5}{12} \\ (vii) - (viii) 0 \quad (b) 1.$$

$$3. (a) \frac{\pi}{2} \quad (b) \frac{1}{2}. \quad 4. (i) \frac{3\pi}{4} \quad (ii) 15.$$

$$13. (a) (i) \pi \quad (ii) \frac{\pi}{4} \quad (b) \frac{\pi}{4}. \quad 15. \frac{x+y}{1-xy}.$$

$$17. (i) \frac{1}{6} \quad (ii) 2 \quad (iii) \pm \frac{1}{\sqrt{2}} \quad (iv) \frac{1}{2} \\ (v) \frac{\pi}{4} \quad (vi) n\pi \text{ or } m\pi + \frac{\pi}{4}; m, n \in \mathbb{I}$$

$$(vii) - (viii) \frac{1}{4} \quad (ix) \sqrt{6} - 1 \quad (x) \frac{1}{\sqrt{3}}$$

$$(xi) 2 - \sqrt{3} \quad (xii) \frac{3}{4} \quad (xiii) -\frac{1}{2}$$

$$(xiv) \frac{\pi}{4} \quad (xv) \frac{\sqrt{3}}{2}.$$

$$18. x = \pm \frac{1}{\sqrt{2}}. \quad 19. x = 1.$$

$$20. (i) \frac{\pi}{4} + \frac{x}{2} \quad (ii) \frac{\pi}{4} - x \quad (iii) \frac{\pi}{4} + x$$

$$(iv) \sin^{-1} \frac{x}{a} \quad (v) \frac{1}{2} \tan^{-1} x \quad (vi) \frac{x}{2} - \sec^{-1} x.$$

LATQ

Hints to Selected Questions

$$5. (a) 2 \sin^{-1} \frac{3}{5} = \sin^{-1} \left(2 \cdot \frac{3}{5} \sqrt{1 - \frac{9}{25}} \right)$$

$$\left[\because \sin^{-1} (2x\sqrt{1-x^2}) = 2 \sin^{-1} x \right]$$

$$= \sin^{-1} \left(\frac{6}{5} \times \frac{4}{5} \right) = \sin^{-1} \left(\frac{24}{25} \right) = \tan^{-1} \left(\frac{24}{7} \right).$$

$$(b) \text{ Put } \sin^{-1} \frac{3}{4} = x \text{ so that } \sin x = \frac{3}{4}.$$

$$\therefore \tan \left(\frac{1}{2} \sin^{-1} \frac{3}{4} \right) = \tan \frac{x}{2} = \frac{\sin x/2}{\cos x/2}$$

$$= \frac{2 \sin \frac{x}{2} \sin \frac{x}{2}}{2 \cos \frac{x}{2} \sin \frac{x}{2}} = \frac{2 \sin^2 \frac{x}{2}}{\sin x} = \frac{1 - \cos x}{\sin x}$$

$$= \frac{1 - \sqrt{7}/4}{3/4} = \frac{4 - \sqrt{7}}{3}.$$

$$6. (i) \cos^{-1} (\cos^2 x - \sin^2 x) = \cos^{-1} (\cos 2x) = 2x$$

$$(ii) \text{ Put } \cos^{-1} x = \theta.$$

$$7. (iii) \text{ Put } \tan^{-1} x = \theta \text{ so that } x = \tan \theta.$$

$$8. \text{ Put } \tan^{-1} \sqrt{x} = \theta.$$

$$10. (a) (i) \text{ Put } \sin^{-1} \frac{4}{5} = \theta \text{ and } \sin^{-1} \frac{5}{13} = \phi$$

$$\text{so that } \sin \theta = \frac{4}{5} \text{ and } \sin \phi = \frac{5}{13}.$$

$$\therefore \cos \theta = \frac{3}{5} \text{ and } \cos \phi = \frac{12}{13}.$$

$$\text{Now } \cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$$

$$= \frac{3}{5} \times \frac{12}{13} - \frac{4}{5} \times \frac{5}{13} = \frac{16}{65}.$$

$$\therefore \theta + \phi = \cos^{-1} \frac{16}{65}$$

$$\Rightarrow \sin^{-1} \frac{4}{5} + \sin^{-1} \frac{5}{13} = \frac{\pi}{2} - \sin^{-1} \frac{16}{65}.$$

$$11. (i) \text{ LHS} = \left(\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{3}{5} \right) - \tan^{-1} \left(\frac{8}{19} \right)$$

$$= \tan^{-1} \frac{\frac{3}{4} + \frac{3}{5}}{1 - \frac{3}{4} \times \frac{3}{5}} - \tan^{-1} \left(\frac{8}{19} \right)$$

$$= \tan^{-1} \left(\frac{27}{11} \right) - \tan^{-1} \left(\frac{8}{19} \right).$$

$$14. \text{ LHS} = \tan^{-1} \frac{a-b}{1+ab} + \tan^{-1} \frac{b-c}{1+bc} + \tan^{-1} \frac{c-a}{1+ca}$$

$$= (\tan^{-1} a - \tan^{-1} b) + (\tan^{-1} b - \tan^{-1} c)$$

$$+ (\tan^{-1} c - \tan^{-1} a) = 0.$$

$$17. (ii) \tan^{-1} \left(\frac{x+1}{x-1} \right) + \tan^{-1} \left(\frac{x-1}{x} \right) = \tan^{-1} (-7)$$

$$\Rightarrow \tan^{-1} \left[\frac{\frac{x+1}{x-1} + \frac{x-1}{x}}{1 - \frac{x+1}{x-1} \times \frac{x-1}{x}} \right] = \tan^{-1} (-7)$$

$$\Rightarrow \frac{x(x+1) + (x-1)^2}{x(x-1) - (x^2-1)} = -7 \Rightarrow 2x^2 - 8x + 8 = 0; \text{ etc.}$$

$$(vi) 2 \tan^{-1} (\cos x) = \tan^{-1} \left(\frac{2 \cos x}{1 - \cos^2 x} \right)$$

$$= \tan^{-1} \left(\frac{2 \cos x}{\sin^2 x} \right)$$

$$= \tan^{-1} (2 \cot x \operatorname{cosec} x).$$

$$(x) \tan^{-1} \left(\frac{1-x}{1+x} \right) = \frac{1}{2} \tan^{-1} x$$

$$\Rightarrow \tan^{-1} x = 2 \tan^{-1} \left(\frac{1-x}{1+x} \right)$$

$$= \tan^{-1} \frac{2 \left(\frac{1-x}{1+x} \right)}{1 - \left(\frac{1-x}{1+x} \right)^2}; \text{ etc.}$$

$$18. \tan^{-1} (x-1) + \tan^{-1} x + \tan^{-1} (x+1) = \tan^{-1} 3x$$

$$\Rightarrow \tan^{-1} (x-1) + \tan^{-1} (x+1) = \tan^{-1} 3x - \tan^{-1} x$$

$$\Rightarrow \tan^{-1} \frac{(x-1) + (x+1)}{1 - (x-1)(x+1)} = \tan^{-1} \frac{3x - x}{1 + 3x(x)}$$

$$\Rightarrow \tan^{-1} \frac{2x}{2-x^2} = \tan^{-1} \frac{2x}{1+3x^2}$$

$$\Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2}; \text{ etc.}$$

$$\begin{aligned}
 20. (i) \quad \frac{\cos x}{1 - \sin x} &= \frac{\sin\left(\frac{\pi}{2} + x\right)}{1 + \cos\left(\frac{\pi}{2} + x\right)} \\
 &= \frac{2 \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) \cos\left(\frac{\pi}{4} + \frac{x}{2}\right)}{2 \cos^2\left(\frac{\pi}{4} + \frac{x}{2}\right)} \\
 &= \frac{\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)}{\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right).
 \end{aligned}$$

(iv) Put $x = a \sin \theta$.(v) Put $x = \tan \theta$.(vi) Put $x = \sec \theta$.

$$22. (b) \quad \sqrt{1 + \sin x} = \cos \frac{x}{2} + \sin \frac{x}{2} \text{ and}$$

$$\sqrt{1 - \sin x} = \cos \frac{x}{2} - \sin \frac{x}{2}.$$

$$26. \text{ Put } \tan^{-1} \frac{yz}{xr} = A, \tan^{-1} \frac{zx}{yr} = B, \tan^{-1} \frac{xy}{zr} = C.$$



NCERT-FILE

Questions from NCERT Book

(For each unsolved question, refer : "Solution of Modern's abc of Mathematics")

Exercise 2.1

Find the principal values of the following :

$$1. \sin^{-1}\left(-\frac{1}{2}\right).$$

$$\text{Solution : Let } \sin^{-1}(-1) = y, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}.$$

$$\therefore \sin y = -1 \Rightarrow y = -\frac{\pi}{2}.$$

$$\text{Hence, the reqd. principal value} = -\frac{\pi}{2}.$$

$$2. \cos^{-1}\left(\frac{\sqrt{3}}{2}\right).$$

$$\text{Solution : Let } \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = y, 0 \leq y \leq \pi.$$

$$\therefore \cos y = \frac{\sqrt{3}}{2} \Rightarrow \cos y = \cos \frac{\pi}{6}$$

$$\Rightarrow y = \frac{\pi}{6}.$$

$$\text{Hence, the reqd. principal value} = \frac{\pi}{6}.$$

$$3. \operatorname{cosec}^{-1}(2).$$

[Solution : Refer Q. 7(i) ; Ex. 2(a)]

$$4. \tan^{-1}(-\sqrt{3}).$$

[Solution : Refer Q. 4(i) ; Ex. 2(a)]

$$5. \cos^{-1}\left(-\frac{1}{2}\right).$$

[Solution : Refer Q. 3(i) ; Ex. 2(a)]

$$6. \tan^{-1}(-1).$$

[Solution : Refer Q. 4(ii) ; Ex. 2(a)]

$$7. \sec^{-1}\left(\frac{2}{\sqrt{3}}\right).$$

[Solution : Refer Q. 6(i) ; Ex. 2(a)]

$$8. \cot^{-1}(\sqrt{3}).$$

Solution : Let $\cot^{-1}(\sqrt{3}) = y, 0 < y < \pi$.

$$\therefore \cot y = \sqrt{3}$$

$$\Rightarrow y = \frac{\pi}{6}.$$

$$\text{Hence, the reqd. principal value} = \frac{\pi}{6}.$$

$$9. \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right).$$

[Solution : Refer Q. 3(iii) ; Ex. 2(a)]

$$10. \operatorname{cosec}^{-1}(-\sqrt{2})$$

[Solution : Refer Q. 7(ii) ; Ex. 2(a)]

Find the values of the following :

$$11. \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right).$$

$$\text{Solution : (I) Let } \tan^{-1}(1) = x, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\therefore \tan x = 1 \Rightarrow x = \frac{\pi}{4} \quad \dots(1)$$

$$(II) \text{ Let } \cos^{-1}\left(-\frac{1}{2}\right) = y, 0 \leq y \leq \pi$$

$$\begin{aligned} \therefore \cos y &= -\frac{1}{2} = -\cos \frac{\pi}{3} \\ &= \cos\left(\pi - \frac{\pi}{3}\right) = \cos \frac{2\pi}{3} \end{aligned}$$

$$\Rightarrow y = \frac{2\pi}{3} \quad \dots(2)$$

$$(III) \text{ Let } \sin^{-1}\left(-\frac{1}{2}\right) = z, -\frac{\pi}{2} \leq z \leq \frac{\pi}{2}$$

$$\begin{aligned} \therefore \sin z &= -\frac{1}{2} = -\sin \frac{\pi}{6} = \sin\left(-\frac{\pi}{6}\right) \\ \Rightarrow z &= -\frac{\pi}{6} \quad \dots(3) \end{aligned}$$

$$\therefore \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$$

$$\begin{aligned} &= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} \quad [\text{Using (1), (2) \& (3)}] \\ &= \frac{1}{12}(3\pi + 8\pi - 2\pi) = \frac{9\pi}{12} = \frac{3\pi}{4}. \end{aligned}$$

$$12. \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right).$$

[Solution : Refer Q. 11 ; Ex. 2(a)].

$$13. \text{ If } \sin^{-1} x = y, \text{ then :}$$

$$(A) 0 \leq y \leq \pi \quad (B) -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$(C) 0 < y < \pi \quad (D) -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

[Ans. (B)]

$$14. \tan^{-1}\sqrt{3} - \sec^{-1}(-2) \text{ is equal to :}$$

$$(A) \pi \quad (B) -\frac{\pi}{3}$$

$$(C) \frac{\pi}{3} \quad (D) \frac{2\pi}{3}. \quad [\text{Ans. (B)}]$$

Exercise 2.2

Prove the following :

$$1. 3\sin^{-1} x = \sin^{-1}(3x - 4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

[Solution : Refer Ex. 4 ; Page 2/12]

$$2. 3\cos^{-1} x = \cos^{-1}(4x^3 - 3x), x \in \left[\frac{1}{2}, 1\right]$$

[Solution : Refer Q. 6(ii) ; Ex. 2(b)]

$$3. \tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24} = \tan^{-1} \frac{1}{2}. \quad (\text{H.P.B. 2016, 14})$$

$$\text{Solution : LHS} = \tan^{-1} \frac{2}{11} + \tan^{-1} \frac{7}{24}$$

$$\begin{aligned} &= \tan^{-1} \frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11} \times \frac{7}{24}} \\ &\quad \left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \end{aligned}$$

$$= \tan^{-1} \frac{48+77}{264-14}$$

$$= \tan^{-1} \frac{125}{250} = \tan^{-1} \frac{1}{2} = \text{RHS.}$$

$$4. 2\tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{31}{17}.$$

[Solution : Refer Q. 10 (b)(iii) ; Ex. 2(b)]

Write the following functions in the simplest form :

$$5. \tan^{-1} \frac{\sqrt{1+x^2}-1}{x}, x \neq 0.$$

[Solution : Refer Q. 20 (v) ; Ex. 2(b)]

$$6. \tan^{-1} \frac{1}{\sqrt{x^2-1}}, |x| > 1.$$

[Solution : Refer Q. 20 (vi) ; Ex. 2(b)]

$$7. \tan^{-1} \left(\sqrt{\frac{1-\cos x}{1+\cos x}} \right), x < \pi.$$

$$\text{Solution : LHS} = \tan^{-1} \left(\sqrt{\frac{1-\cos x}{1+\cos x}} \right)$$

$$= \tan^{-1} \left(\sqrt{\frac{2\sin^2 \frac{x}{2}}{2\cos^2 \frac{x}{2}}} \right)$$

$$\left[\because 1 - \cos 2\theta = 2\sin^2 \theta; 1 + \cos 2\theta = 2\cos^2 \theta \right]$$

$$= \tan^{-1} \left(\sqrt{\tan^2 \frac{x}{2}} \right) = \tan^{-1} \left(\tan \frac{x}{2} \right) = \frac{x}{2}.$$

$$8. \tan^{-1} \left(\frac{\cos x - \sin x}{\cos x + \sin x} \right), 0 < x < \pi.$$

[Solution : Refer Q. 20 (ii) ; Ex. 2(b)]

$$9. \tan^{-1} \frac{x}{\sqrt{a^2-x^2}}, |x| < a.$$

[Solution : Refer Q. 20 (iv) ; Ex. 2(b)]

10. $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right), a > 0; \frac{-a}{\sqrt{3}} \leq x \leq \frac{a}{\sqrt{3}}.$

(H.P.B. 2016, 13)

Solution : $\tan^{-1}\left(\frac{3a^2x - x^3}{a^3 - 3ax^2}\right) = \tan^{-1}\left(\frac{3\frac{x}{a} - \frac{x^3}{a^3}}{1 - \frac{3x^2}{a^2}}\right).$

Put $\frac{x}{a} = \tan \theta$ so that $\theta = \tan^{-1} \frac{x}{a}.$

$\therefore$ Given expression

$$= \tan^{-1}\left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}\right) = \tan^{-1}(\tan 3\theta)$$

$$= 3\theta = 3 \tan^{-1} \frac{x}{a}.$$

Find the values of each of the following :

11. $\tan^{-1}\left[2 \cos\left(2 \sin^{-1} \frac{1}{2}\right)\right].$

Solution : $\tan^{-1}\left[2 \cos\left(2 \sin^{-1} \frac{1}{2}\right)\right]$

$$= \tan^{-1}\left[2 \cos\left(2 \cdot \frac{\pi}{6}\right)\right] = \tan^{-1}\left(2 \cos \frac{\pi}{3}\right)$$

$$= \tan^{-1}\left(2 \cdot \frac{1}{2}\right) = \tan^{-1}(1) = \frac{\pi}{4}.$$

12. $\cot(\tan^{-1} a + \cot^{-1} a).$

[**Solution :** Refer Q. 2 (viii) ; Ex. 2(b)]

13. $\tan \frac{1}{2} \left[\sin^{-1} \frac{2x}{1+x^2} + \cos^{-1} \frac{1-y^2}{1+y^2} \right], |x| < 1, y > 0$ and $xy < 1.$

[**Solution.** Refer Q. 15; Ex. 2(b)]

14. If $\sin\left(\sin^{-1} \frac{1}{5} + \cos^{-1} x\right) = 1$, then find the value of 'x'.

Solution : $\sin\left(\sin^{-1} \frac{1}{5} + \cos^{-1} x\right) = 1 = \sin \frac{\pi}{2}$

$$\Rightarrow \sin^{-1} \frac{1}{5} + \cos^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} \frac{1}{5}$$

$$\Rightarrow \cos^{-1} x = \cos^{-1} \frac{1}{5}$$

Hence, $x = \frac{1}{5}.$

15. If $\tan^{-1} \frac{x-1}{x-2} + \tan^{-1} \frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of 'x'.

[**Solution :** Refer Q. 17(iii); Ex. 2(b)]

Find the values of each of the expressions in Exercise 16 to 18 :

16. $\sin^{-1}\left(\sin \frac{2\pi}{3}\right).$

[**Solution :** Refer Q. 2(i); Ex. 2(b)]

17. $\tan^{-1}\left(\tan \frac{3\pi}{4}\right).$

[**Solution :** Refer Q. 2(iv); Ex. 2(b)]

18. $\tan\left(\sin^{-1} \frac{3}{5} + \cot^{-1} \frac{3}{2}\right).$

Solution : $\tan\left(\sin^{-1} \frac{3}{5} + \cot^{-1} \frac{3}{2}\right)$

$$= \tan\left(\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{2}{3}\right)$$

$$= \tan\left(\tan^{-1} \frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \times \frac{2}{3}}\right) = \tan\left(\tan^{-1} \frac{17}{6}\right) = \frac{17}{6}.$$

19. $\cos^{-1}\left(\cos \frac{7\pi}{6}\right)$ is equal to :

(A) $\frac{7\pi}{6}$

(B) $\frac{5\pi}{6}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{6}.$

[**Ans. (B)**]

20. $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to :

(A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{1}{4}$

(D) 1.

[**Ans. (D)**]

21. $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ is equal to :

(A) π

(B) $-\frac{\pi}{2}$

(C) 0

(D) $2\sqrt{3}.$

[**Ans. (B)**]

Miscellaneous Exercise on Chapter 2

Find the value of the following :

1. $\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

Solution : $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi + \frac{\pi}{6}\right)\right)$
 $= \cos^{-1}\left(\cos\frac{\pi}{6}\right) = \frac{\pi}{6}$.

2. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$.

[Solution : Refer Q. 2(v); Ex. 2(b)]

Prove that :

3. $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$.

[Solution : Refer Q. 5(a); Ex. 2(b)]

4. $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{36}$.

Solution : Let LHS = θ .

Then $\sin \theta = \sin\left[\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right)\right]$
 $= \sin\left(\sin^{-1}\left(\frac{8}{17}\right)\right)\cos\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$
 $+ \cos\left(\sin^{-1}\left(\frac{8}{17}\right)\right)\sin\left(\sin^{-1}\left(\frac{3}{5}\right)\right)$
 $= \frac{8}{17}\sqrt{1-\frac{9}{25}} + \sqrt{1-\frac{64}{289}}\left(\frac{3}{5}\right)$
 $= \frac{8}{17} \times \frac{4}{5} + \frac{15}{17} \times \frac{3}{5} = \frac{77}{85}$
 $\Rightarrow \theta = \sin^{-1}\frac{77}{85}$.

Hence, $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{77}{85}$.

5. $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$.

[Solution : Refer Q. 12(i); Ex. 2(b)]

6. $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$.

[Solution : Refer Q. 12(iii); Ex. 2(b)]

7. $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$.

[Solution : Refer Q. 12(iv); Ex. 2(b)]

8. $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$.

[Solution : Refer Q. 11(ii); Ex. 2(b)]

Prove that :

9. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in [0, 1]$.

[Solution : Refer Q. 8; Ex. 2(b)]

10. $\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$.

[Solution : Refer Q. 22 (b); Ex. 2(b)]

11. $\tan^{-1}\left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x, -\frac{1}{\sqrt{2}} \leq x \leq 1$.

[Hint : Put $x = \cos 2\theta$]

[Solution : Refer Ex. 25; Page 2/17]

12. $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$.

Solution. Putting

$\sin^{-1}\left(\frac{1}{3}\right) = x$ and $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = y$, we get :

$\sin x = \frac{1}{3}$ and $\sin y = \frac{2\sqrt{2}}{3}$.

$\therefore \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - \frac{1}{9}}$
 $= \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

and $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \frac{8}{9}} = \sqrt{\frac{1}{9}} = \frac{1}{3}$.

Now $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{2\sqrt{2}}{3}\right)\left(\frac{2\sqrt{2}}{3}\right)$
 $= \frac{1}{9} + \frac{8}{9} = 1 = \sin \frac{\pi}{2}$

$\Rightarrow x + y = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{\pi}{2}$

$\Rightarrow \frac{9}{4}\left(\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\right) = \frac{9}{4}\left(\frac{\pi}{2}\right)$

[Multiplying by $\frac{9}{4}$]

$\Rightarrow \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) = \frac{9\pi}{8}$.

Hence, $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) = \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$.

Solve the following equations :

13. $2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$.

[Solution : Refer Q. 17 (vi) ; Ex. 2(b)]

14. $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2} \tan^{-1} x, (x > 0)$.

[Solution : Refer Q. Ex. 17(x); Ex. 2(b)]

15. $\sin(\tan^{-1} x), |x| < 1$ is equal to :

(A) $\frac{x}{\sqrt{1-x^2}}$

(B) $\frac{1}{\sqrt{1-x^2}}$

(C) $\frac{1}{\sqrt{1+x^2}}$

(D) $\frac{x}{\sqrt{1+x^2}}$

[Ans. (D)]

16. $\sin(1-x) - 2 \sin^{-1} x = \frac{\pi}{2}$, then x is equal to :

(A) $0, \frac{1}{2}$

(B) $1, \frac{1}{2}$

(C) 0

(D) $\frac{1}{2}$

[Ans. (C)]

17. $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\frac{x-y}{x+y}$ is equal to :

(A) $\frac{\pi}{2}$

(B) $\frac{\pi}{3}$

(C) $\frac{\pi}{4}$

(D) $-\frac{3\pi}{4}$

[Ans. (C)]

Questions From NCERT Exemplar

Example 1. Prove that $\tan(\cot^{-1} x) = \cot(\tan^{-1} x)$.

State with reason whether the equality is valid for all values of x .

Solution. Let $\cot^{-1} x = \theta$. Then $\cot \theta = x$

$$\Rightarrow \tan\left(\frac{\pi}{2} - \theta\right) = x \Rightarrow \tan^{-1} x = \frac{\pi}{2} - \theta.$$

$$\therefore \tan(\cot^{-1} x) = \tan \theta$$

$$= \cot\left(\frac{\pi}{2} - \theta\right) = \cot\left(\frac{\pi}{2} - \cot^{-1} x\right)$$

$$= \tan(\cot^{-1} x) = \cot(\tan^{-1} x).$$

The equality is valid for all values of x .

[$\because \tan^{-1} x$ and $\cot^{-1} x$ are true for $x \in \mathbb{R}$]

Example 2. Find the value of $\tan(\cos^{-1} x)$ and hence evaluate $\tan\left(\cos^{-1} \frac{8}{17}\right)$.

Solution. Let $\cos^{-1} x = \theta$ so that $\cos \theta = x$, where $\theta \in [0, \pi]$.

$$\therefore \tan(\cos^{-1} x) = \tan \theta = \frac{\sqrt{1-\cos^2 \theta}}{\cos \theta} = \frac{\sqrt{1-x^2}}{x}.$$

$$\text{Hence, } \tan\left(\cos^{-1} \frac{8}{17}\right) = \frac{\sqrt{1-\left(\frac{8}{17}\right)^2}}{\frac{8}{17}} = \frac{\frac{15}{17}}{\frac{8}{17}} = \frac{15}{8}.$$

Example 3. Prove that : $\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18 = \cot^{-1} 3$.

Solution. LHS = $\cot^{-1} 7 + \cot^{-1} 8 + \cot^{-1} 18$

$$= \left(\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8}\right) + \tan^{-1} \frac{1}{18}$$

$$= \tan^{-1} \left(\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \times \frac{1}{8}} \right) + \tan^{-1} \left(\frac{1}{18} \right)$$

$$= \tan^{-1} \frac{3}{11} + \tan^{-1} \left(\frac{1}{18} \right)$$

$$= \tan^{-1} \left(\frac{\frac{3}{11} + \frac{1}{18}}{1 - \frac{3}{11} \times \frac{1}{18}} \right)$$

[$\because xy < 1$]

$$= \tan^{-1} \frac{65}{195} = \tan^{-1} \left(\frac{1}{3} \right) = \cot^{-1} 3 = \text{RHS.}$$

Example 4. Find the value of :

$$\sin\left(2 \tan^{-1} \frac{2}{3}\right) + \cos\left(\tan^{-1} \sqrt{3}\right).$$

Solution. Let $\tan^{-1} \frac{2}{3} = x$ and $\tan^{-1} \sqrt{3} = y$

so that $\tan x = \frac{2}{3}$ and $\tan y = \sqrt{3}$.

$$\text{Now } \sin\left(2 \tan^{-1} \frac{2}{3}\right) + \cos\left(\tan^{-1} \sqrt{3}\right)$$

$$= \sin 2x + \cos y = \frac{2 \tan x}{1 + \tan^2 x} + \frac{1}{\sqrt{1 + \tan^2 y}}$$

$$= \frac{2 \cdot \frac{2}{3}}{1 + \frac{4}{9}} + \frac{1}{\sqrt{1 + (\sqrt{3})^2}} = \frac{12}{13} + \frac{1}{2} = \frac{37}{26}.$$

Exercise

- Find the value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$.
- Evaluate : $\tan^{-1}\left\{\sin\left(\frac{-\pi}{2}\right)\right\}$.
- Evaluate : $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$.
- Find the value of $\sec\left(\tan^{-1}\frac{y}{2}\right)$ in terms of y .
- Find the value of $\sin\left[2\cot^{-1}\left(-\frac{5}{12}\right)\right]$.
- Which is greater $\tan 1$ or $\tan^{-1} 1$?
- Evaluate : $\cos\left[\sin^{-1}\frac{1}{4} + \sec^{-1}\frac{4}{3}\right]$.
- Find the value of the expression :
 $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos\left(\tan^{-1}2\sqrt{2}\right)$.
- Solve for x :
(i) $\sin^{-1}x + \sin^{-1}(1-x) = \cos^{-1}x$
(ii) $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$.
- Find the solution of the equation :
 $\tan^{-1}x - \cot^{-1}x = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Answers

- $\frac{\pi}{2}$.
- $-\frac{\pi}{4}$.
- $\frac{\pi}{6}$.
- $\frac{1}{2}\sqrt{4+y^2}$.
- $-\frac{120}{169}$.
- $\tan 1$.
- $\frac{3\sqrt{15}-\sqrt{7}}{16}$.
- $\frac{14}{15}$.
- (i) 0 , $\frac{1}{2}$ (ii) $\frac{3}{4}$.
- $\sqrt{3}$.

Revision Exercise

1. Show that :

$$(i) \sec(\operatorname{cosec}^{-1}x) = \frac{|x|}{\sqrt{x^2-1}}, \text{ for } |x| > 1$$

$$(ii) \cos(2\tan^{-1}x) = \frac{1-x^2}{1+x^2}.$$

2. Prove that :

$$(i) \sin(\tan^{-1}1) = \frac{1}{\sqrt{2}}$$

$$(ii) \operatorname{cosec}\left[\tan^{-1}(-\sqrt{3})\right] = -\frac{2}{\sqrt{3}}.$$

$$3. \text{ Solve : } 3\tan^{-1}\left(\frac{1}{2+\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{x}\right) = \tan^{-1}\left(\frac{1}{3}\right).$$

$$4. \text{ Solve : } \sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2}.$$

Solution. The given equation is :

$$\sin^{-1}\left(\frac{5}{x}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2} \quad \dots(1)$$

$$\Rightarrow \sin^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{5}{x}\right)$$

$$\Rightarrow \frac{12}{x} = \sin\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{5}{x}\right)\right]$$

$$\Rightarrow \frac{12}{x} = \cos\left(\sin^{-1}\left(\frac{5}{x}\right)\right) = \sqrt{1 - \left(\frac{5}{x}\right)^2}$$

$$\left[\because \cos(\sin^{-1}x) = \sqrt{1-x^2}, |x| < 1\right]$$

$$\Rightarrow \left(\frac{12}{x}\right)^2 = 1 - \left(\frac{5}{x}\right)^2 \Rightarrow 144 = x^2 - 25$$

$$\Rightarrow x^2 = 169.$$

Hence, $x = 13$. [$\because x = -13$ does not satisfy (1)]

$$5. \text{ Show that : } (i) \sin^{-1}\left[\sin\frac{3\pi}{4}\right] \neq \frac{3\pi}{4}.$$

What is its value ?

$$(ii) \tan^{-1}\left[\tan\frac{5\pi}{6}\right] \neq \frac{5\pi}{6}.$$

What is its value ?

6. Prove that :

$$(i) \tan^{-1} x + \cot^{-1} (x + 1) = \tan^{-1} (x^2 + x + 1)$$

$$(ii) \cot^{-1} 3 + \operatorname{cosec}^{-1} \sqrt{5} = \frac{\pi}{4}.$$

7. Prove that :

$$\sec^2 (\tan^{-1} 2) + \operatorname{cosec}^2 (\cot^{-1} 3) = 15.$$

8. Prove that

$$2 \tan^{-1} \left[\sqrt{\frac{a-b}{a+b}} \frac{\tan \theta}{2} \right]$$

$$= \cos^{-1} \left(\frac{a \cos \theta + b}{a + b \sin \theta} \right)$$

(Assam B. 2017)

Answers

3. $x = 2$.

5. (i) $\frac{\pi}{4}$ (ii) $-\frac{\pi}{6}$.



Hints to Selected Questions

1. (i) Put $\operatorname{cosec}^{-1} x = \theta$ (ii) $\tan^{-1} x = \theta$.

3. $\tan^{-1} \frac{1}{2+\sqrt{3}} = \tan^{-1} \frac{2-\sqrt{3}}{4-3} = \tan^{-1} (2-\sqrt{3}) = 15^\circ$.

7. Put $\tan^{-1} 2 = \theta$ and $\cot^{-1} 3 = \phi$
so that $\tan \theta = 2$ and $\cot \phi = 3$.

$$\text{LHS} = \sec^2 \theta + \operatorname{cosec}^2 \theta = (1 + \tan^2 \theta) + (1 + \cot^2 \phi) = (1 + 4) + (1 + 9) = 15.$$



CHECK YOUR UNDERSTANDING

1. Write the domain of $f(x) = \tan^{-1} x$.

(Jammu B. 2015 W)

Ans. $\mathbf{R}$.

2. Write the principal value of $\sin^{-1} \left(\frac{1}{2} \right)$.

(Jammu B. 2016)

Ans. $\frac{\pi}{6}$.

3. Write the principal value of $\tan^{-1}(\sqrt{3}) + \operatorname{cosec}^{-1}(-2)$.

Ans. $\frac{\pi}{6}$.

4. If $\sec^{-1}(x) + \operatorname{cosec}^{-1} \left(\frac{1}{3} \right) = \frac{\pi}{2}$, then find x .

Ans. $x = \frac{1}{3}$.

5. Is $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $x \in [-1, 1]$?

(State True/False)

Ans. True.

6. Find the value of $\sin (\sec^{-1} x + \operatorname{cosec}^{-1} x)$, $|x| > 1$.

Ans. 1.

7. Write down the value of $\tan^{-1} x + \cot^{-1} x$, $x \in \mathbf{R}$.

Ans. $\frac{\pi}{2}$.

8. Write down the value of $2 \sin^{-1} \frac{3}{5}$.

Ans. $\sin^{-1} \frac{24}{25}$.

9. Find the solution of $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$, $x > 0$.

Ans. $\frac{1}{6}$.

10. Is $\tan^{-1} \left(\sqrt{\frac{1-x^2}{1+x^2}} \right) = \frac{1}{2} \cos^{-1} x$ true?

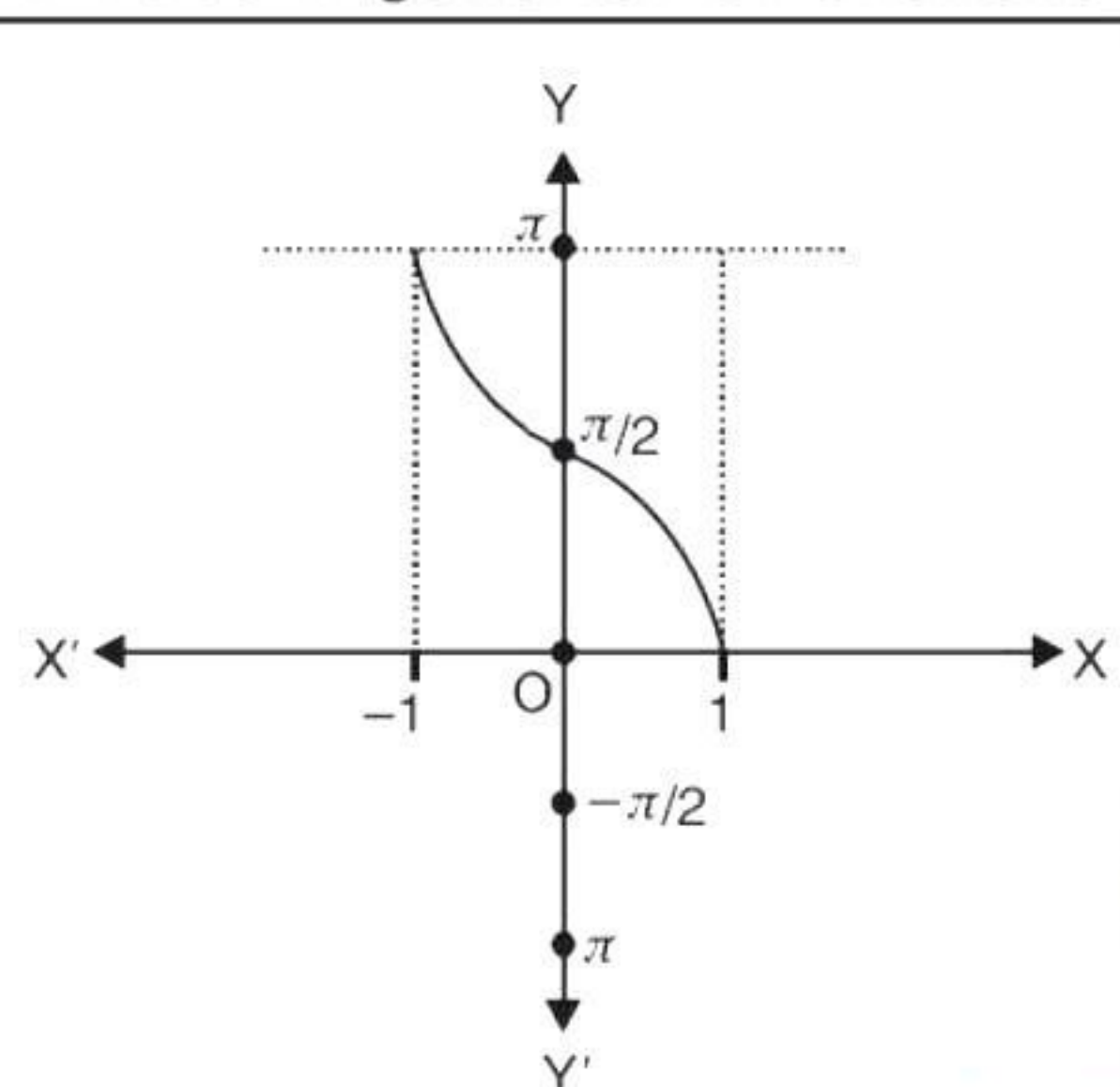
Ans. Yes.

SUMMARY

INVERSE-TRIGONOMETRIC FUNCTIONS

Inverse Trigonometric Functions

Inverse Trigonometric Functions



Properties of inverse trigonometric functions

1.

TABLE

Function	Principal Value Branch	
	Domain	Range
(i) $y = \sin^{-1} x$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
(ii) $y = \cos^{-1} x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
(iii) $y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
(iv) $y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$
(v) $y = \sec^{-1} x$	$1 \leq x < \infty$	$0 \leq y < \frac{\pi}{2}$
	$-\infty < x \leq -1$	$\frac{\pi}{2} < y \leq \pi$
(vi) $y = \operatorname{cosec}^{-1} x$	$1 \leq x < \infty$	$0 < y \leq \frac{\pi}{2}$
	$-\infty < x \leq -1$	$-\frac{\pi}{2} \leq y < 0$

2. Properties :

$$(a) \quad x = \sin^{-1}(\sin x) = \cos^{-1}(\cos x) = \tan^{-1}(\tan x); \text{ etc.}$$

$$(b) \quad (i) \quad \operatorname{cosec}^{-1} x = \sin^{-1} \frac{1}{x}, x \geq 1 \text{ or } x \leq -1$$

$$(ii) \quad \sec^{-1} x = \cos^{-1} \frac{1}{x}, x \geq 1 \text{ or } x \leq -1$$

$$(iii) \quad \cot^{-1} x = \tan^{-1} \frac{1}{x}, x > 0.$$

$$(c) \quad (i) \quad \sin^{-1}(-x) = -\sin^{-1} x, x \in [-1, 1]$$

$$(ii) \quad \cos^{-1}(-x) = \pi - \cos^{-1} x, x \in [-1, 1]$$

$$(iii) \quad \tan^{-1}(-x) = -\tan^{-1} x, x \in \mathbf{R}$$

$$(iv) \quad \cot^{-1}(-x) = \pi - \cot^{-1} x, x \in \mathbf{R}$$

$$(v) \quad \sec^{-1}(-x) = \pi - \sec^{-1} x, |x| \geq 1$$

$$(vi) \quad \operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, |x| \geq 1.$$

$$(d) \quad (i) \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, x \in [-1, 1]$$

$$(ii) \quad \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, x \in \mathbf{R}$$

$$(iii) \quad \sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}, |x| \geq 1$$

$$(iv) \quad \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}, xy < 1$$

$$(v) \quad \tan^{-1} x - \tan^{-1} y = \frac{x-y}{1+xy}, xy > -1$$

$$(vi) \quad 2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2} = \cos^{-1} \frac{1-x^2}{1+x^2} = \tan^{-1} \frac{2x}{1-x^2}$$

$$(e) \quad (i) \quad \sin^{-1} x \pm \sin^{-1} y = \sin^{-1} (x \sqrt{1-y^2} \pm y \sqrt{1-x^2})$$

$$(ii) \quad \cos^{-1} x \pm \cos^{-1} y = \cos^{-1} (xy \mp \sqrt{1-x^2} \sqrt{1-y^2}).$$



MULTIPLE CHOICE QUESTIONS

► For Board Examinations

1. If $\cos^{-1} x = y$, then :

- (A) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ (B) $-\pi \leq y \leq \pi$
 (C) $0 \leq y \leq \frac{\pi}{2}$ (D) $0 \leq y \leq \pi$. **(P.B. 2018)**

2. The principal value of $\sin^{-1} \left(\frac{1}{\sqrt{2}} \right)$ is :

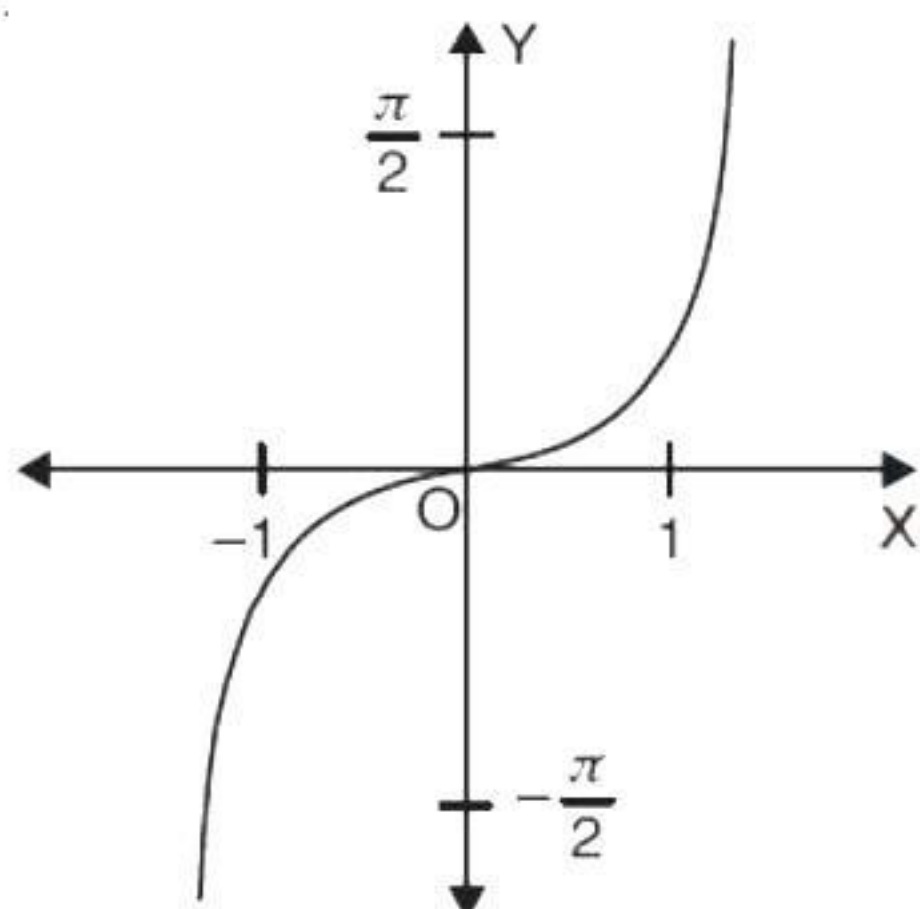
- (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{2}$
 (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$ **(H.P.B. 2018)**

3. The value of $\sin \left(\cos^{-1} \frac{3}{5} \right)$ is :

- (A) $\frac{4}{5}$ (B) $\frac{3}{5}$
 (C) $\frac{2}{5}$ (D) None of these.

(H.B. 2018)

4. Identify the function from the following figure :



- (A) $\tan^{-1} x$ (B) $\sin^{-1} x$
 (C) $\cos^{-1} x$ (D) $\operatorname{cosec}^{-1} x$.

(Kerala B. 2018)

5. $\sin^{-1} \left(\frac{1}{2} \right)$ is equal to :

- (A) 0 (B) $\frac{\pi}{6}$
 (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{3}$ **(P.B. 2017)**

6. (i) Principal value of $\cos^{-1} \left(-\frac{1}{2} \right)$ is :

- (A) $\frac{\pi}{3}$ (B) $\frac{2\pi}{3}$
 (C) $-\frac{\pi}{3}$ (D) $\frac{\pi}{6}$ **(H.P.B. 2017)**

7. (i) $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ is equal to :

- (A) $-\frac{\pi}{2}$ (B) $2\sqrt{3}$
 (C) 0 (D) π **(H.B. 2017)**

8. Principal value of $\cot^{-1} \left(-\frac{1}{\sqrt{3}} \right)$ is :

- (A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$
 (C) $\frac{2\pi}{3}$ (D) π **(H.B. 2017)**

9. The value of $\cos^{-1} \left(\cos \frac{4\pi}{3} \right)$ is :

- (A) $\frac{4\pi}{3}$ (B) $\frac{2\pi}{3}$
 (C) 0 (D) π . **(Nagaland B. 2017)**

10. The value of $\sin^{-1} \left\{ \sin \left(\frac{2\pi}{3} \right) \right\}$ is :

- (A) $\frac{2\pi}{3}$ (B) $\frac{\pi}{3}$
 (C) $-\frac{\pi}{3}$ (D) $-\frac{2\pi}{3}$. **(Mizoram B. 2017)**

11. If $\sin^{-1} x = y$, then :

- (A) $0 \leq y \leq \pi$ (B) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
 (C) $0 < y < \pi$ (D) $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

(H.P.B. 2016)

12. $\tan^{-1} \left(\tan \frac{3\pi}{4} \right)$ equals :

- (A) $\frac{3\pi}{4}$ (B) $-\frac{\pi}{4}$
 (C) $\frac{\pi}{4}$ (D) None of these.

(H.P.B. 2016)

13. Principal value of $\sin^{-1} \left(-\frac{1}{2} \right)$ is :

- (A) $\frac{5\pi}{6}$ (B) $\frac{\pi}{6}$
 (C) $-\frac{\pi}{6}$ (D) $-\frac{5\pi}{6}$. **(P.B. 2016)**

14. If $\sec^{-1} x = \operatorname{cosec}^{-1} y$, then the value of

$$\cos^{-1} \frac{1}{x} + \cos^{-1} \frac{1}{y} \text{ will be :}$$

- (A) π (B) $\frac{2\pi}{3}$
(C) $\frac{5\pi}{6}$ (D) $\frac{\pi}{2}$. (W. Bengal B. 2016)

15. The value of $\tan^{-1} \left\{ 2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right\}$ is :

- (A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$
(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$. (Mizoram B. 2016)

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(Single Correct Answer Type)

(JEE-Main and Advanced)

16. If $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$, then $4x^2 - 4xy \cos \alpha + y^2$

is equal to :

- (A) 4 (B) $2 \sin^2 \alpha$
(C) $-4 \sin^2 \alpha$ (D) $4 \sin^2 \alpha$.

(A.I.E.E.E. 2005)

17. If $\sin^{-1} \left(\frac{x}{5} \right) + \operatorname{cosec}^{-1} \left(\frac{5}{4} \right) = \frac{\pi}{2}$, then a value of x

is :

- (A) 3 (B) 4
(C) 5 (D) 1. (A.I.E.E.E. 2007)

18. The value of $\cot \left(\operatorname{cosec}^{-1} \frac{5}{3} + \tan^{-1} \frac{2}{3} \right)$ is :

- (A) $\frac{5}{17}$ (B) $\frac{6}{17}$
(C) $\frac{3}{17}$ (D) $\frac{4}{17}$.

(A.I.E.E.E. 2008)

19. If $0 < x < 1$, then :

$$\sqrt{1+x^2} \left[\left\{ x \cos(\cot^{-1} x) + \sin(\cot^{-1} x) \right\}^2 - 1 \right]^{1/2} =$$

- (A) $\frac{x}{\sqrt{1+x^2}}$ (B) x
(C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$.

(I.I.T. 2008)

20. The value of $\cot \left\{ \sum_{n=1}^{23} \cot^{-1} \left(1 + \sum_{k=1}^n 2k \right) \right\}$ is :

- (A) $\frac{23}{25}$ (B) $\frac{25}{23}$
(C) $\frac{23}{24}$ (D) $\frac{24}{23}$.

(I.I.T. (Advanced) 2013)

21. If $\tan^{-1} y = \tan^{-1} x + \tan^{-1} \left(\frac{2x}{1-x^2} \right)$, when $|x| < \frac{1}{\sqrt{3}}$,

then a value of y is :

- (A) $\frac{3x-x^3}{1-3x^2}$ (B) $\frac{3x+x^3}{1-3x^2}$
(C) $\frac{3x-x^3}{1+3x^2}$ (D) $\frac{3x+x^3}{1+3x^2}$.

(J.E.E. (Main) 2015)

Answers

1. (D) 2. (D) 3. (A) 4. (B) 5. (B) 6. (B) 7. (A) 8. (C) 9. (B) 10. (B)
11. (B) 12. (B) 13. (C) 14. (D) 15. (C) 16. (D) 17. (A) 18. (B) 19. (C) 20. (B)
21. (A).

Hints/Solutions

RCQ Pocket

16. (D) We have : $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$

$$\Rightarrow x = \cos \left(\cos^{-1} \frac{y}{2} + \alpha \right)$$

$$= \cos \left(\cos^{-1} \frac{y}{2} \right) \cos \alpha - \sin \left(\cos^{-1} \frac{y}{2} \right) \sin \alpha$$

$$= \frac{y}{2} \cos \alpha - \sqrt{1 - \frac{y^2}{4}} \sin \alpha$$

$$\Rightarrow 2x = y \cos \alpha - \sin \alpha \sqrt{4 - y^2}$$

$$\Rightarrow 2x - y \cos \alpha = -\sin \alpha \sqrt{4 - y^2}$$

Squaring, $4x^2 + y^2 \cos^2 \alpha - 4xy \cos \alpha$

$$= 4 \sin^2 \alpha - y^2 \sin^2 \alpha$$

$$\Rightarrow 4x^2 - 4xy \cos \alpha + y^2 = 4 \sin^2 \alpha$$

17. (A) $\sin^{-1} \frac{x}{5} + \operatorname{cosec}^{-1} \frac{5}{4} = \frac{\pi}{2}$

$$\Rightarrow \sin^{-1} \frac{x}{5} + \sin^{-1} \frac{4}{5} = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1} \frac{x}{5} = \frac{\pi}{2} - \sin^{-1} \frac{4}{5} = \cos^{-1} \frac{4}{5} = \sin^{-1} \frac{3}{5}$$

$$\Rightarrow x = 3$$

18. (B) $\cot \left(\operatorname{cosec}^{-1} \frac{5}{3} + \tan^{-1} \frac{2}{3} \right)$

$$= \cot \left(\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{2}{3} \right)$$

$$= \cot \left(\tan^{-1} \frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \cdot \frac{2}{3}} \right) = \cot \left(\tan^{-1} \frac{17}{6} \right)$$

$$= \cot \left(\cot^{-1} \frac{6}{17} \right) = \frac{6}{17}$$

19. (C)
Here

$$\sqrt{1+x^2} \left[\{x \cos(\cot^{-1} x) + \sin(\cot^{-1} x)\}^2 - 1 \right]^{1/2}$$

$$= \sqrt{1+x^2} \left[\left\{ x \cos \left(\cos^{-1} \frac{x}{\sqrt{1+x^2}} \right) + \sin \left(\sin^{-1} \frac{1}{\sqrt{1+x^2}} \right) \right\}^2 - 1 \right]^{1/2}$$

$$= \sqrt{1+x^2} \left[\left(\frac{x^2}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} \right)^2 - 1 \right]^{1/2}$$

$$= \sqrt{1+x^2} (x^2 + 1 - 1)^{1/2} = x\sqrt{1+x^2}$$

20. (B) Let $\cot \sum_{n=1}^{23} \left\{ \cot^{-1} \left(1 + \sum_{k=1}^n 2k \right) \right\}$

$$= \cot \sum_{n=1}^{23} \cot^{-1} (1 + 2(1 + 2 + \dots + n))$$

$$= \cot \sum_{n=1}^{23} \cot^{-1} \left(1 + 2 \frac{n(n+1)}{2} \right)$$

$$= \cot \sum_{n=1}^{23} \cot^{-1} (1 + n(n+1))$$

$$= \cot \sum_{n=1}^{23} \tan^{-1} \left(\frac{1}{1 + n(n+1)} \right)$$

$$= \cot \sum_{n=1}^{23} \tan^{-1} \left(\frac{(n+1) - n}{1 + (n+1)n} \right)$$

$$= \cot \sum_{n=1}^{23} (\tan^{-1}(n+1) - \tan^{-1} n)$$

$$= \cot [\{(\tan^{-1}(2) - \tan^{-1}(1)) + \{(\tan^{-1}(3) - \tan^{-1}(2))\} + \dots + \{(\tan^{-1}(24) - \tan^{-1}(23))\}$$

$$= \cot[\tan^{-1}(24) - \tan^{-1}(1)] = \cot \left[\tan^{-1} \frac{24-1}{1+(24)(1)} \right]$$

$$= \cot \left[\tan^{-1} \frac{23}{25} \right] = \cot \left[\cot^{-1} \frac{25}{23} \right] = \frac{25}{23}$$

21. (A) We have : $\tan^{-1} y = \tan^{-1} x + \tan^{-1} \left(\frac{2x}{1-x^2} \right)$

$$= \tan^{-1} \frac{x + \frac{2x}{1-x^2}}{1 - x \cdot \frac{2x}{1-x^2}} = \tan^{-1} \frac{3x - x^3}{1 - 3x^2}$$

Hence, $y = \frac{3x - x^3}{1 - 3x^2}$

CHAPTER TEST

2

Time Allowed : 1 Hour

Max. Marks : 34

Notes: 1. All question are compulsory.

2. Marks have been indicated against each questions.

1. Find the value of $\sin^{-1}\left(\sin\frac{4\pi}{5}\right)$. (1)
2. Prove that $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$. (1)
3. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$. (2)
4. Prove that $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$. (2)
5. If $\tan^{-1}\left(\frac{x-2}{x-4}\right) + \tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$, then find the value of 'x'. (4)
6. Prove that $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. (4)
7. Show that $2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$. (4)
8. Prove that $\tan^{-1}x + \tan^{-1}\frac{2x}{1-x^2} = \tan^{-1}\frac{3x-x^3}{1-3x^2}, |x| < \frac{1}{\sqrt{3}}$. (4)
9. Show that $\tan^{-1}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x, -\frac{1}{\sqrt{2}} \leq x \leq 1$. (6)
10. If $\tan^{-1}x + \tan^{-1}y - \tan^{-1}z = 0$, then prove that $x + y + xyz = z$. (6)

Answers

1. $\frac{\pi}{5}$. 3. $\frac{\pi}{3}$. 5. $x = \pm\sqrt{2}$.