# Session 3

## Two Important Theorems, Divisibility Problems

### Two Important Theorems

**Theorem 1 If $(\sqrt{P}+Q)^n = I + f$, where $I$ and $n$ are positive integers, $n$ being odd and $0 \le f < 1$, then show that $(I+f)\,f = k^n$, where $P - Q^2 = k > 0$ and $\sqrt{P} - Q < 1$.**

**Proof** Given, $\sqrt{P} - Q < 1$ $\quad\therefore\quad 0 < (\sqrt{P} - Q)^n < 1$

Now, let $(\sqrt{P} - Q)^n = f'$, where $0 < f' < 1$

Also $\quad I + f = (\sqrt{P} + Q)^n \quad$ ...(i)

$0 \le f < 1 \quad$ ...(ii)

$f' = (\sqrt{P} - Q)^n \quad$ ...(iii)

and $\quad 0 < f' < 1 \quad$ ...(iv)

On subtracting Eq. (iii) from Eq. (i), we get

$$I + f - f' = (\sqrt{P} + Q)^n - (\sqrt{P} - Q)^n$$

$$= 2\,[{}^nC_1\,(\sqrt{P})^{n-1} \cdot Q + {}^nC_3\,(\sqrt{P})^{n-3} \cdot Q^3 + \dots]$$

$= 2$ (integer) $=$ Even integer $\quad$ ...(v)

[Since, $n$ is odd, RHS contains even powers of $\sqrt{P}$, so RHS is an even integer]

$\therefore$ LHS is also an integer.

$\because$ $I$ is an integer.

$\therefore$ $(f - f')$ is also an integer.

$\Rightarrow \quad f - f' = 0 \quad [\because -1 < (f - f') < 1]$

or $\quad f = f'$

From Eq. (v), $I$ is an even integer and

$$(I+f)\,f = (I+f)\,f' = (\sqrt{P}+Q)^n\,(\sqrt{P}-Q)^n$$

$$= (P - Q^2)^n = k^n$$

#### Remark

If $n$ is even integer, then $(\sqrt{P}+Q)^n + (\sqrt{P}-Q)^n = I + f + f'$

Since, LHS and $I$ are integers.

$\therefore \quad (f + f')$ is also an integer.

$\Rightarrow \quad f + f' = 1 \quad [\because 0 < (f + f') < 2]$

$\therefore \quad f' = 1 - f$

Hence, $\quad (I+f)(1-f) = (I+f)\,f' = (\sqrt{P}+Q)^n\,(\sqrt{P}-Q)^n$

$= (P - Q^2)^n = k^n$

**Theorem 2 If $(P+\sqrt{Q})^n = I + f$, where $I$ and $n$ are positive integers and $0 \le f < 1$, show that $(I+f)(1-f) = k^n$, where $P^2 - Q = k > 0$ and $P - \sqrt{Q} < 1$.**

**Proof** Given, $\quad P - \sqrt{Q} < 1$

$\therefore \quad 0 < (P - \sqrt{Q})^n < 1$

Now, let $(P - \sqrt{Q})^n = f'$, where $0 < f' < 1$

Also, $\quad I + f = (P + \sqrt{Q})^n \quad$ ...(i)

$0 \le f < 1 \quad$ ...(ii)

$f' = (P - \sqrt{Q})^n \quad$ ...(iii)

and $\quad 0 < f' < 1 \quad$ ...(iv)

On adding Eqs. (i) and (iii), we get

$$I + f + f' = (P + \sqrt{Q})^n + (P - \sqrt{Q})^n$$

$$= 2[{}^nC_0\,P^n + {}^nC_2\,P^{n-2}\,(\sqrt{Q})^2 + {}^nC_4\,P^{n-4}\,(\sqrt{Q})^4 + \dots]$$

$= 2$ (integer) $=$ Even integer $\quad$ ...(v)

[Since, RHS contains even power of $\sqrt{Q}$, so RHS is an even integer]

$\therefore$ LHS is also an integer.

$\because$ $I$ is an integer.

$\Rightarrow$ $f + f'$ is also an integer.

$\therefore \quad f + f' = 1 \quad [\because 0 < (f + f') < 2]$

or $\quad f' = 1 - f$

From Eq. (v), $I =$ even integer $- 1 =$ odd integer and

$$(I+f)(1-f) = (I+f)\,f'$$

$$= (P+\sqrt{Q})^n\,(P-\sqrt{Q})^n = (P^2 - Q)^n = k^n$$

**Example 34.** Show that the integral part of $(5+2\sqrt{6})^n$ is odd, where $n$ is natural number.

**Sol.** $(5+2\sqrt{6})^n$ can be written as $(5+\sqrt{24})^n$

Now, let $\quad I + f = (5+\sqrt{24})^n \quad$ ...(i)

$0 \le f < 1 \quad$ ...(ii)

and let $\quad f' = (5 - \sqrt{24})^n \quad$ ...(iii)

$0 < f' < 1 \quad$ ...(iv)

On adding Eqs. (i) and (iii), we get

$$I + f + f' = (5+\sqrt{24})^n + (5-\sqrt{24})^n$$

$I + 1 = 2p$,

$\forall\ p \in N =$ Even integer $\quad$ [from theorem 2]

$\therefore \quad I = 2p - 1 =$ Odd integer

**Example 35.** Show that the integral part of $(5\sqrt{5}+11)^{2n+1}$ is even, where $n \in N$.

**Sol.** $(5\sqrt{5}+11)^{2n+1}$ can be written as $(\sqrt{125}+11)^{2n+1}$

Now, let $\quad I + f = (\sqrt{125}+11)^{2n+1} \quad$ ...(i)

$0 \le f < 1 \quad$ ...(ii)

and let $\quad f' = (\sqrt{125} - 11)^{2n+1} \quad$ ...(iii)

$0 < f' < 1 \quad$ ...(iv)

On subtracting Eq. (iii) from Eq. (i), we get

$$I + f - f' = (\sqrt{125} + 11)^{2n+1} - (\sqrt{125} - 11)^{2n+1}$$

$$I + 0 = 2p, \forall\ p \in N = \text{Even integer}$$

[from theorem 1]

$$\therefore \quad I = 2p = \text{Even integer}$$

**Example 36.** Let $R = (6\sqrt{6} + 14)^{2n+1}$ and $f = R - [R]$, where $[\cdot]$ denotes the greatest integer function. Find the value of $Rf, n \in N$.

***Sol.*** $(6\sqrt{6} + 14)^{2n+1}$ can be written as $(\sqrt{216} + 14)^{2n+1}$ and given that $f = R - [R]$

and $\quad R = (6\sqrt{6} + 14)^{2n+1} = (\sqrt{216} + 14)^{2n+1}$

$\therefore \quad [R] + f = (\sqrt{216} + 14)^{2n+1}$ ...(i)

$0 \le f < 1$ ...(ii)

Let $\quad f' = (\sqrt{216} - 14)^{2n+1}$ ...(iii)

$0 < f' < 1$ ...(iv)

On subtracting Eq. (iii) from Eq. (i), we get

$$[R] + f - f' = (\sqrt{216} + 14)^{2n+1} - (\sqrt{216} - 14)^{2n+1}$$

$$[R] + 0 = 2p, \forall\ p \in N = \text{Even integer [from theorem 1]}$$

$\therefore \quad f - f' = 0$ or $f = f'$

Now, $\quad Rf = Rf' = (\sqrt{216} + 14)^{2n+1}(\sqrt{216} - 14)^{2n+1}$

$$= (216 - 196)^{2n+1} = (20)^{2n+1}$$

**Example 37.** If $(7 + 4\sqrt{3})^n = s + t$, where $n$ and $s$ are positive integers and $t$ is a proper fraction, show that $(1 - t)(s + t) = 1$.

***Sol.*** $(7 + 4\sqrt{3})^n$ can be written as $(7 + \sqrt{48})^n$

$\therefore \quad s + t = (7 + \sqrt{48})^n$ ...(i)

$0 < t < 1$ ...(ii)

Now, let $\quad t' = (7 - \sqrt{48})^n$ ...(iii)

$0 < t' < 1$ ...(iv)

On adding Eqs. (i) and (iii), we get

$$s + t + t' = (7 + \sqrt{48})^n + (7 - \sqrt{48})^n$$

$s + 1 = 2p, \forall\ p \in N = \text{Even integer}$ [from theorem 2]

$\therefore \quad t + t' = 1$ or $1 - t = t'$

Then, $\quad (1 - t)(s + t) = t'(s + t) = (7 - \sqrt{48})^n (7 + \sqrt{48})^n$

[from Eqs. (i) and (iii)]

$$= (49 - 48)^n = (1)^n = 1$$

**Example 38.** If $x = (8 + 3\sqrt{7})^n$, where $n$ is a natural number, prove that the integral part of $x$ is an odd integer and also show that $x - x^2 + x[x] = 1$, where $[\cdot]$ denotes the greatest integer function.

***Sol.*** $(8 + 3\sqrt{7})^n$ can be written as $(8 + \sqrt{63})^n$

$\therefore \quad x = [x] + f$

or $\quad [x] + f = (8 + \sqrt{63})^n$ ...(i)

$0 \le f < 1$ ...(ii)

Now, let $\quad f' = (8 - \sqrt{63})^n$ ...(iii)

$0 < f' < 1$ ...(iv)

On adding Eqs. (i) and (iii), we get

$$[x] + f + f' = (8 + \sqrt{63})^n + (8 - \sqrt{63})^n$$

$[x] + 1 = 2p, \forall\ p \in N = \text{Even integer}$

[from theorem 2]

$\therefore \quad [x] = 2p - 1 = \text{Odd integer}$

i.e., Integral part of $x$ = Odd integer

$\because \quad f + f' = 1 \Rightarrow 1 - f = f'$ ...(v)

LHS $= x - x^2 + x[x] = x - x(x - [x]) = x - xf$

$[\because x = [x] + f]$

$= x(1 - f) = x f'$ [from Eq.(v)]

$= (8 + \sqrt{63})^n (8 - \sqrt{63})^n$ [from Eqs.(i) and (iii)]

$= (64 - 63)^n = (1)^n = 1 = \text{RHS}$

***Remark***

Sometimes, students find it difficult to decide whether a problem is on addition or subtraction. Now, if $x = [x] + f$ and $0 < f' < 1$ and if $[x] + f + f' =$ Integer. Then, addition and if $[x] + f - f' =$ Integer, the subtraction and values of $(f + f')$ and $(f - f')$ are 1 and 0, respectively.

# Divisibility Problems

## Type I

**(i) $(x^n - a^n)$ is divisible by $(x - a)$, $\forall\ n \in N$.**

**(ii) $(x^n + a^n)$ is divisible by $(x + a)$, $\forall\ n \in$ Only odd natural numbers.**

**Example 39.** Show that $1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$ is divisible by 1998.

***Sol.*** Here, $n = 1998$ (Even)

$\therefore$ Only result (i) applicable.

Let $\quad P = 1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$

$= (1992^{1998} - 1955^{1998}) - (1938^{1998} - 1901^{1998})$

divisible by (1992 − 1955) i.e. 37 ; divisible by (1938 − 1901) i.e. 37

$\therefore$ $P$ is divisible by 37.

Also, $P = (1992^{1998} - 1938^{1998}) - (1955^{1998} - 1901^{1998})$

divisible by (1992 − 1938) i.e., 54 ; divisible by (1955 − 1901) i.e., 54

$\therefore$ $P$ is also divisible by 54.

Hence, $P$ is divisible by $37 \times 54$, i.e., 1998.

**Example 40.** Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

***Sol.*** We have, $2222^{5555} + 5555^{2222}$

$= (2222^{5555} + 4^{5555}) + (5555^{2222} - 4^{2222}) - (4^{5555} - 4^{2222})$ ...(i)

The number $(2222^{5555} + 4^{5555})$ is divisible by $2222 + 4 = 2226 = 7 \times 318$, which is divisible by 7 and the number $(5555^{2222} - 4^{2222})$ is divisible by

$5555 - 4 = 5551 = 7 \times 793$, which is divisible by 7 and the number $(4^{5555} - 4^{2222}) = 4^{2222}(4^{3333} - 1) = 4^{2222}(64^{1111} - 1^{1111})$ is divisible by $64 - 1 = 63 = 7 \times 9$, which is divisible by 7.

Therefore, each brackets of Eq. (i) are divisible by 7. Hence, $2222^{5555} + 5555^{2222}$ is divisible by 7.

## Type II To show that an Expression is Divisible by An Integer

### Solution Process

**(i)** If $a, p, n$ and $r$ are positive integers, first of all write $a^{pn+r} = a^{pn} \cdot a^r = (a^p)^n \cdot a^r$

**(ii)** If we will show that the given expression is divisible by $c$. Then, expression $a^p = \{1 + (a^p - 1)\}$, if some power of $(a^p - 1)$ has $c$ as a factor.

or $a^p = \{2 + (a^p - 2)\}$, if some power of $(a^p - 2)$ has $c$ as a factor.

or $a^p = \{3 + (a^p - 3)\}$, if some power of $(a^p - 3)$ has $c$ as a factor.

⋮ ⋮ ⋮ ⋮ ⋮ ⋮

or $a^p = \{k + (a^p - k)\}$, if some power of $(a^p - k)$ has $c$ as a factor.

**Example 41.** ***If $n$ is any positive integer, show that $2^{3n+3} - 7n - 8$ is divisible by 49.***

***Sol.*** Given expression

$$= 2^{3n+3} - 7n - 8 = 2^{3n} \cdot 2^3 - 7n - 8$$
$$= 8^n \cdot 8 - 7n - 8 = 8(1+7)^n - 7n - 8$$
$$= 8(1 + {}^nC_1 \cdot 7 + {}^nC_2 \cdot 7^2 + \ldots + {}^nC_n \cdot 7^n) - 7n - 8$$
$$= 8 + 56n + 8({}^nC_2 \cdot 7^2 + \ldots + {}^nC_n \cdot 7^n) - 7n - 8$$
$$= 49n + 8({}^nC_2 \cdot 7^2 + \ldots + {}^nC_n \cdot 7^n)$$
$$= 49\{n + 8({}^nC_2 + \ldots + {}^nC_n \cdot 7^{n-2})\}$$

Hence, $2^{3n+3} - 7n - 8$ is divisible by 49.

**Example 42.** If $10^n$ divides the number $101^{100} - 1$, find the greatest value of $n$.

***Sol.*** We have, $101^{100} - 1 = (1 + 100)^{100} - 1$

$$= 1 + {}^{100}C_1 \cdot 100 + {}^{100}C_2 \cdot 100^2 + \ldots + {}^{100}C_{100}\, 100^{100} - 1$$
$$= {}^{100}C_1 \cdot 100 + {}^{100}C_2 \cdot 100^2 + \ldots + {}^{100}C_{100} \cdot 10^{100}$$
$$= (100)(100) + {}^{100}C_2 \cdot 100^2 + \ldots + {}^{100}C_{100} \cdot 100^{100}$$
$$= (100)^2 [1 + {}^{100}C_2 + \ldots + 100^{98}]$$
$$= 100^2 k, \text{ where } k \text{ is a positive integer}$$

Therefore, $101^{100} - 1$ is divisible by $100^2$ i.e., $10^4$.

$\therefore \quad n = 4$

## How to Find Remainder by Using Binomial Theorem

If $a, p, n$ and $r$ are positive integers, then to find the remainder when $a^{pn+r}$ is divided by $b$, we adjust power of $a$ to $a^{pn+r}$ which is very close to $b$, say with difference 1 i.e., $b \pm 1$. Also, the remainder is always positive. When number of the type $5n - 2$ is divided by 5, then we have

$$5)\ 5n - 2\ (n$$
$$\underline{-5n}$$
$$\underline{-2}$$

We can write $-2 = -2 - 3 + 3 = -5 + 3$

or $\quad \dfrac{5n-2}{5} = \dfrac{5n-5+3}{5} = n - 1 + \dfrac{3}{5}$

Hence, the remainder is 3.

**Example 43.** If $7^{103}$ is divided by 25, find the remainder.

***Soln.*** We have, $7^{103} = 7 \cdot 7^{102} = 7 \cdot (7^2)^{51} = 7(49)^{51} = 7(50 - 1)^{51}$

$$= 7[(50)^{51} - {}^{51}C_1(50)^{50} + {}^{51}C_2(50)^{49} - \ldots - 1]$$
$$= 7[(50)^{51} - {}^{51}C_1(50)^{50} + {}^{51}C_2(50)^{49} - \ldots + {}^{51}C_{50}(50)] - 7 - 18 + 18$$
$$= 7[50((50)^{50} - {}^{51}C_1(50)^{49} + {}^{51}C_2(50)^{48} - \ldots + {}^{51}C_{50})] - 25 + 18$$
$$= 7[50k] - 25 + 18, \text{ where } k \text{ is an integer.}$$
$$= 25[14k - 1] + 18 = 25p + 18 \qquad [\text{where } p \text{ is an integer}]$$

Now, $\dfrac{7^{103}}{25} = p + \dfrac{18}{25}$. Hence, the remainder is 18.

**Example 44.** Find the remainder, when $5^{5^{5^{\cdots^5}}}$ (24 times 5) is divided by 24.

***Sol.*** Here, $5^{5^{5^{\cdots^5}}}$ (23 times 5) is an odd natural number.

Let $5^{5^{\cdots^5}}$ (23 times 5) $= 2m + 1$

Now, let $x = 5^{5^{5^{\cdots^5}}}$ (24 times 5) $= 5^{2m+1} = 5 \cdot 5^{2m}$, where $m$ is a natural number.

$$\therefore \quad x = 5 \cdot (5^2)^m = 5(24 + 1)^m$$
$$= 5[{}^mC_0(24)^m + {}^mC_1(24)^{m-1} + \ldots + {}^mC_{m-1}(24) + 1]$$
$$= 5(24k + 1) = 24(5k) + 5$$
$$\therefore \quad \frac{x}{24} = 5k + \frac{5}{24}$$

Hence, the remainder is 5.

**Example 45.** If 7 divides $32^{32^{32}}$, then find the remainder.

***Solution.*** We have, $32 = 2^5$

$$\therefore \quad 32^{32} = (2^5)^{32} = 2^{160} = (3 - 1)^{160}$$
$$= {}^{160}C_0(3)^{160} - {}^{160}C_1(3)^{159} + \ldots - {}^{160}C_{159}(3) + 1$$
$$= 3(3^{159} - {}^{160}C_1(3)^{158} + \ldots - {}^{160}C_{159}) + 1$$
$$= 3m + 1,\ m \in I^+$$

Now, $32^{32^{32}} = 32^{3m+1} = 2^{5(3m+1)} = 2^{15m+5}$

$= 2^2 \cdot 2^{3(5m+1)} = 4(8)^{5m+1} = 4(7+1)^{5m+1}$

$= 4[{}^{5m+1}C_0 (7)^{5m+1} + {}^{5m+1}C_1 (7)^{5m} + {}^{5m+1}C_2 (7)^{5m-1} + \dots + {}^{5m+1}C_{5m} (7) + 1]$

$= 4[7({}^{5m+1}C_0 (7)^{5m} + {}^{5m+1}C_1 (7)^{5m-1} + \dots + {}^{5m+1}C_{5m}) + 1]$

$= 4[7k+1]$, where $k$ is positive integer $= 28k + 4$

$\therefore \quad \dfrac{32^{32^{32}}}{7} = 4k + \dfrac{4}{7}$

Hence, the remainder is 4.

## How to Find Last Digit, Last Two Digits, Last Three Digits, ... and so on.

If $a, p, n$ and $r$ are positive integers, then $a^{pn+r}$ is adjust of the form $(10k \pm 1)^m$, where $k$ and $m$ are positive integers. For last digit, take 10 common. For last two digits, take 100 common, for last three digits, take 1000 common , ... and so on.

i.e. $(10k \pm 1)^m = (10k)^m + {}^mC_1 (10k)^{m-1} (\pm 1) + {}^mC_2 (10k)^{m-2} (\pm 1)^2 + \dots + {}^mC_{m-2} (10k)^2 (\pm 1)^{m-2} + {}^mC_{m-1} (10k)(\pm 1)^{m-1} + (\pm 1)^m$

For last digit $= 10\lambda + (\pm 1)^m$

For last two digits $= 100\mu + {}^mC_{m-1} (10k)(\pm 1)^{m-1} + (\pm 1)^m$

For last three digits $= 1000\nu + {}^mC_{m-2} (10k)^2 (\pm 1)^{m-2} + {}^mC_{m-1} (10k)(\pm 1)^{m-1} + (\pm 1)^m$ and so on where $\lambda, \mu, \nu \in I$.

**Example 46.** Find the last two digits of $3^{400}$.

***Sol.*** We have, $3^{400} = (3^2)^{200} = (9)^{200} = (10-1)^{200}$

$= (10)^{200} - {}^{200}C_1 (10)^{199} + {}^{200}C_2 (10)^{198} - {}^{200}C_3 (10)^{197} + \dots + {}^{200}C_{198} (10)^2 - {}^{200}C_{199} (10) + 1$

$= 100\mu - {}^{200}C_{199} (10) + 1$, where $\mu \in I$

$= 100\mu - {}^{200}C_1 (10) + 1 = 100\mu - 2000 + 1$

$= 100(\mu - 20) + 1 = 100p + 1$, where $p$ is an integer.

Hence, the last two digits of $3^{400}$ is $00 + 1 = 01$.

**Example 47.** If the number is $17^{256}$, find the

(i) last digit. (ii) last two digits.

(iii) last three digits of $17^{256}$.

***Sol.*** Since, $17^{256} = (17^2)^{128} = (289)^{128} = (290-1)^{128}$

$\therefore 17^{256} = {}^{128}C_0 (290)^{128} - {}^{128}C_1 (290)^{127} + {}^{128}C_2 (290)^{126} - {}^{128}C_3 (290)^{125} + \dots - {}^{128}C_{125} (290)^3 + {}^{128}C_{126} (290)^2 - {}^{128}C_{127} (290) + 1$

(i) **For last digit**

$17^{256} = 290[{}^{128}C_0 (290)^{127} - {}^{128}C_1 (290)^{126} + {}^{128}C_2 (290)^{125} - \dots - {}^{128}C_{127} (1)] + 1$

$= 290(k) + 1$, where $k$ is an integer.

$\therefore$ Last digit $= 0 + 1 = 1$

(ii) **For last two digits,**

$17^{256} = (290)^2 [{}^{128}C_0 (290)^{126} - {}^{128}C_1 (290)^{125} + {}^{128}C_2 (290)^{124} - \dots + {}^{128}C_{126} (1)] - {}^{128}C_{127} (290) + 1$

$= 100m - {}^{128}C_{127} (290) + 1$, where $m$ is an integer.

$= 100m - {}^{128}C_1 (290) + 1 = 100m - 128 \times 290 + 1$

$= 100m - 128 \times (300 - 10) + 1$

$= 100(m - 384) + 1281$

$= 100n + 1281$, where $n$ is an integer.

$\therefore$ Last two digits $= 00 + 81 = 81$

(iii) **For last three digits,**

$17^{256} = (290)^3 [{}^{128}C_0 (290)^{125} - {}^{128}C_1 (290)^{124} + {}^{128}C_2 (290)^{123} - \dots - {}^{128}C_{125} (1)] + {}^{128}C_{126} (290)^2 - {}^{128}C_{127} (290) + 1$

$= 1000m + {}^{128}C_{126} (290)^2 - {}^{128}C_{127} (290) + 1$

where, $m$ is an integer

$= 1000m + {}^{128}C_2 (290)^2 - {}^{128}C_1 (290) + 1$

$= 1000m + \dfrac{(128)(127)}{2} (290)^2 - 128 \times 290 + 1$

$= 1000m + (128)(127)(290)(145) - (128)(290) + 1$

$= 1000m + (128)(290)(127 \times 145 - 1) + 1$

$= 1000m + (128)(290)(18414) + 1$

$= 1000m + 683527680 + 1$

$= 1000m + 683527000 + 680 + 1$

$= 1000(m + 683527) + 681$

$\therefore$ Last three digits $= 000 + 681 = 681$

## Two Important Results

(i) $2 \le \left(1 + \dfrac{1}{n}\right)^n < 3, n \ge 1, n \in N$

(ii) If $n > 6$, then $\left(\dfrac{n}{3}\right)^n < n! < \left(\dfrac{n}{2}\right)^n$

**Example 48.** ***Find the positive integer just greater than*** $(1 + 0.0001)^{10000}$.

***Sol.*** $(1 + 0.0001)^{10000} = \left(1 + \dfrac{1}{10000}\right)^{10000}$

We know that, $2 \le \left(1 + \frac{1}{n}\right)^n < 3,\ n \ge 1,\ n \in N$ [Result (i)]

Hence, positive integer just greater than $(1 + 0.0001)^{10000}$ is 3.

**Example 49.** Find the greater number is $100^{100}$ and $(300)!$ .

***Sol.*** Using Result (ii), We know that, $\left(\frac{n}{3}\right)^n < n!$

Putting $n = 300$, we get

$$(100)^{300} < (300)! \quad \text{...(i)}$$

But $$(100)^{100} < (100)^{300} \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$(100)^{100} < (100)^{300} < (300)!$$

$$\Rightarrow \quad (100)^{100} < (300)!$$

Hence, the greater number is $(300)!$ .

**Example 50.** Find the greater number in 300! and $\sqrt{300^{300}}$.

***Sol.*** Since, $(100)^{150} > 3^{150}$

$$\Rightarrow \quad (100)^{150} \cdot (100)^{150} > 3^{150} \cdot (100)^{150}$$

$$\Rightarrow \quad (100)^{300} > (300)^{150}$$

or $$(100)^{300} > \sqrt{300^{300}} \quad \text{...(i)}$$

Using result (ii), $\left(\frac{n}{3}\right)^n < n!$

Putting $n = 300$, we get $(100)^{300} < 300!$ ...(ii)

From Eqs. (i) and (ii), we get

$$\sqrt{300^{300}} < (100)^{300} < 300!$$

$$\Rightarrow \quad \sqrt{300^{300}} < 300!$$

Hence, the greater number is 300 !.

# *Exercise for Session 3*

**1.** If $x = (7 + 4\sqrt{3})^{2n} = [x] + f$, where $n \in N$ and $0 \le f < 1$, then $x(1 - f)$ is equal to

(a) 1 (b) 0 (c) – 1 (d) even integer

**2.** If $(5 + 2\sqrt{6})^n = I + f$; $n, I \in N$ and $0 \le f < 1$, then $I$ equals

(a) $\frac{1}{f} - f$ (b) $\frac{1}{1+f} - f$ (c) $\frac{1}{1-f} - f$ (d) $\frac{1}{1+f} + f$

**3.** If $n > 0$ is an odd integer and $x = (\sqrt{2} + 1)^n$, $f = x - [x]$, then $\frac{1 - f^2}{f}$ is

(a) an irrational number (b) a non-integer rational number (c) an odd number (d) an even number

**4.** Integral part of $(\sqrt{2} + 1)^6$ is

(a) 196 (b) 197 (c) 198 (d) 199

**5.** $(103)^{86} - (86)^{103}$ is divisible by

(a) 7 (b) 13 (c) 17 (d) 23

**6.** Fractional part of $\frac{2^{78}}{31}$ is

(a) $\frac{2}{31}$ (b) $\frac{4}{31}$ (c) $\frac{8}{31}$ (d) $\frac{16}{31}$

**7.** The unit digit of $17^{1983} + 11^{1983} - 7^{1983}$ is

(a) 1 (b) 2 (c) 3 (d) 0

**8.** The last two digits of the number $(23)^{14}$ are

(a) 01 (b) 03 (c) 09 (d) 27

**9.** The last four digits of the number $3^{100}$ are

(a) 2001 (b) 3211 (c) 1231 (d) 0001

**10.** The remainder when $23^{23}$ is divided by 53 is

(a) 17 (b) 21 (c) 30 (d) 47

# Answers

## Exercise for Session 3

**1.** (a) **2.** (c) **3.** (d) **4.** (b) **5.** (c) **6.** (c)
**7.** (a) **8.** (c) **9.** (a) **10.** (c)