Session 6

Arithmetico-Geometric Series (AGS), Sigma (Σ) Notation, Natural Numbers

Arithmetico-Geometric Series (AGS)

Definition

A series formed by multiplying the corresponding terms of an AP and a GP is called **Arithmetico - Geometric Series** (or shortly written as AGS)

For example, $1 + 4 + 7 + 10 + \dots$ is an AP and $1 + x + x^2 + x^3 + \dots$ is a GP.

Multiplying together the corresponding terms of these series, we get

$1 + 4x + 7x^2 + 10x^3 + \dots$ which is an Arithmetico-Geometric Series.

Again, $a + (a + d) + (a + 2d) + \dots + [a + (n - 1)d]$ is a typical AP

and $1 + r + r^2 + \dots + r^{n-1}$ is a typical GP.

Multiplying together the corresponding terms of these series, we get

$$a + (a + d) r + (a + 2d) r^2 + \dots + [a + (n - 1)d] r^{n-1}$$

which is called a standard Arithmetico-Geometric series.

Sum of *n* Terms of an Arithmetico-Geometric Series

Let the series be $a + (a + d) r + (a + 2d) r^2 + \dots + [a + (n - 1)d] r^{n-1}$

Let S_n denotes the sum to n terms, then

$$S_n = a + (a + d) r + (a + 2d) r^2 + \dots + [a + (n - 2) d] r^{n-2} + [a + (n - 1) d] r^{n-1} \quad \dots\text{(i)}$$

Multiplying both sides of Eq. (i) by r, we get

$$rS_n = ar + (a + d) r^2 + (a + 2d) r^3 + \dots + [a + (n - 2) d] r^{n-1} + [a + (n - 1) d] r^n \quad \dots\text{(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$(1 - r) S_n = a + (dr + dr^2 + \dots + dr^{n-1}) - [a + (n - 1) d] r^n$$

$$= a + \frac{dr(1 - r^{n-1})}{(1 - r)} - [a + (n - 1) d] r^n$$

$$\therefore \quad \boldsymbol{S_n = \frac{a}{(1 - r)} + \frac{dr(1 - r^{n-1})}{(1 - r)^2} - \frac{[a + (n - 1) d] r^n}{(1 - r)}} \quad \dots\text{(iii)}$$

Remark

The above result (iii) is not used as standard formula in any examination. You should follow all steps as shown above.

To Deduce the Sum upto Infinity from the Sum upto *n* Terms of an Arithmetico - Geometric Series, when $|r| < 1$

From Eq. (iii), we have

$$S_n = \frac{a}{1 - r} + \frac{dr}{(1 - r)^2} - \frac{dr^n}{(1 - r)^2} - \frac{[a + (n - 1) d] r^n}{(1 - r)}$$

If $|r| < 1$, when $n \to \infty, r^n \to 0$

and $\dfrac{dr^n}{(1 - r)^2}$ and $\dfrac{[a + (n - 1) d] r^n}{(1 - r)}$ both $\to 0$

$$\therefore \quad S_\infty = \frac{a}{(1 - r)} + \frac{dr}{(1 - r)^2}$$

Independent method Let S_∞ denotes the sum to infinity, then

$$S_\infty = a + (a + d) r + (a + 2d) r^2 + (a + 3d) r^3 + \dots \text{ upto } \infty \quad \dots\text{(iv)}$$

Multiplying both sides of Eq. (iv) by r, we get

$$rS_\infty = ar + (a + d) r^2 + (a + 2d) r^3 + \dots \text{ upto } \infty \quad \dots\text{(v)}$$

Subtracting Eq. (v) from Eq. (iv), we get

$$(1 - r) S_\infty = a + (dr + dr^2 + dr^3 + \dots \text{ upto } \infty)$$

$$= a + \frac{dr}{(1 - r)}$$

$$\therefore \quad S_\infty = \frac{a}{(1 - r)} + \frac{dr}{(1 - r)^2}$$

Example 82. Find the sum of the series $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\dots$

(i) to n terms. (ii) to infinity.

Sol. The given series can be written as

$$1+4\left(\frac{1}{5}\right)+7\left(\frac{1}{5}\right)^2+10\left(\frac{1}{5}\right)^3+\dots$$

The series is an Arithmetico-Geometric series, since each term is formed by multiplying corresponding terms of the series 1, 4, 7, ... which are in AP and

$1, \frac{1}{5}, \frac{1}{5^2}, \dots$ which are in GP.

$$\therefore \quad T_n = [n\text{ th term of } 1, 4, 7, \dots]\left[n\text{ th term of } 1, \frac{1}{5}, \left(\frac{1}{5}\right)^2, \dots\right]$$

$$= [1+(n-1)3]\times 1\cdot\left(\frac{1}{5}\right)^{n-1} = (3n-2)\left(\frac{1}{5}\right)^{n-1}$$

$$\therefore T_{n-1} = (3n-5)\left(\frac{1}{5}\right)^{n-2}$$

(i) Let sum of n terms of the series is denoted by S_n.

Then, $S_n = 1+4\left(\frac{1}{5}\right)+7\left(\frac{1}{5}\right)^2+\dots$

$$+(3n-5)\left(\frac{1}{5}\right)^{n-2}+(3n-2)\left(\frac{1}{5}\right)^{n-1} \quad \dots\text{(i)}$$

Multiplying both the sides of Eq. (i) by $\frac{1}{5}$, we get

$$\therefore \quad \frac{1}{5}S_n = \frac{1}{5}+4\left(\frac{1}{5}\right)^2+7\left(\frac{1}{5}\right)^3+\dots+(3n-5)\left(\frac{1}{5}\right)^{n-1}$$

$$+(3n-2)\left(\frac{1}{5}\right)^n \quad \dots\text{(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$\left(1-\frac{1}{5}\right)S_n = 1+3\left[\frac{1}{5}+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\dots+\left(\frac{1}{5}\right)^{n-1}\right]$$

$$-(3n-2)\left(\frac{1}{5}\right)^n$$

or $\quad \frac{4}{5}S_n = 1+3\left[\left(\frac{1}{5}\right)+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\dots+(n-1)\text{ terms}\right]$

$$-(3n-2)\left(\frac{1}{5}\right)^n$$

$$= 1+3\left\{\frac{\frac{1}{5}\left[1-\left(\frac{1}{5}\right)^{n-1}\right]}{1-\frac{1}{5}}\right\}-(3n-2)\left(\frac{1}{5}\right)^n$$

$$= 1+\frac{3}{4}\left\{1-\left(\frac{1}{5}\right)^{n-1}\right\}-(3n-2)\left(\frac{1}{5}\right)^n$$

$$\therefore \quad S_n = \frac{5}{4}+\frac{15}{16}\left[1-\left(\frac{1}{5}\right)^{n-1}\right]-\frac{(3n-2)}{4}\left(\frac{1}{5}\right)^{n-1}$$

$$= \frac{35}{16}-\frac{(12n+7)}{16}\left(\frac{1}{5}\right)^{n-1}$$

(ii) $S_\infty = 1+4\left(\frac{1}{5}\right)+7\left(\frac{1}{5}\right)^2+10\left(\frac{1}{5}\right)^3+\dots$ upto ∞ ...(iii)

Multiplying both sides of Eq. (i) by $\frac{1}{5}$, we get

$$\frac{1}{5}S_\infty = \left(\frac{1}{5}\right)+4\left(\frac{1}{5}\right)^2+7\left(\frac{1}{5}\right)^3+\dots\text{ upto }\infty \quad \dots\text{(iv)}$$

Subtracting Eq. (iv) from Eq. (iii), we get

$$\left(1-\frac{1}{5}\right)S_\infty = 1+3\left[\left(\frac{1}{5}\right)+\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\dots\text{upto }\infty\right]$$

$$= 1+3\left(\frac{\frac{1}{5}}{1-\frac{1}{5}}\right) = 1+\frac{3}{4}$$

$$\Rightarrow \quad \frac{4}{5}S_\infty = \frac{7}{4}$$

$$\therefore \quad S_\infty = \frac{35}{16}$$

Example 83. If the sum to infinity of the series $1+4x+7x^2+10x^3+\dots$ is $\frac{35}{16}$, find x.

Sol. Let $S_\infty = 1+4x+7x^2+10x^3+\dots$ upto ∞ ...(i)

Multiplying both sides of Eq. (i) by x we get

$$x\,S_\infty = x+4x^2+7x^3+10x^4+\dots\text{ upto }\infty \quad \dots\text{(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$(1-x)\,S_\infty = 1+3x+3x^2+3x^3+\dots\text{ upto }\infty$$

$$= 1+3(x+x^2+x^3+\dots\text{upto }\infty) = 1+3\left(\frac{x}{1-x}\right) = \frac{(1+2x)}{(1-x)}$$

$$\therefore \quad S_\infty = \frac{(1+2x)}{(1-x)^2} = \frac{35}{16} \quad \left[\because S_\infty = \frac{35}{16}\right]$$

$$\Rightarrow \quad 16+32x = 35-70x+35x^2$$

$$\Rightarrow \quad 35x^2-102x+19 = 0$$

$$\Rightarrow \quad (7x-19)(5x-1) = 0$$

$$x \neq \frac{19}{7}$$

[$\because$ for infinity series common ratio $-1 < x < 1$]

Hence, $\quad x = \frac{1}{5}$

Example 84. ***Find the sum of the series*** $1+2^2x+3^2x^2+4^2x^3+\dots$ ***up to*** ∞, $|x|<1$.

Sol. Here, the numbers $1^2, 2^2, 3^2, 4^2, \dots$ i.e. 1, 4, 9, 16, … are not in AP but 1, $4-1=3$, $9-4=5$, $16-9=7$, … are in AP.

Let $S_\infty = 1 + 2^2x + 3^2x^2 + 4^2x^3 + \dots$ upto ∞

$$= 1 + 4x + 9x^2 + 16x^3 + \dots \text{ upto } \infty \quad \dots\text{(i)}$$

Multiplying both sides of Eq. (i) by x, we get

$$xS_\infty = x + 4x^2 + 9x^3 + 16x^4 + \dots \text{ upto } \infty \quad \dots\text{(ii)}$$

Subtracting Eq. (ii) from Eq. (i), we get

$$(1-x)S_\infty = 1 + 3x + 5x^2 + 7x^3 + \dots \text{ upto } \infty \quad \dots\text{(iii)}$$

Again, multiplying both sides of Eq. (iii) by x, we get

$$x(1-x)S_\infty = x + 3x^2 + 5x^3 + 7x^4 + \dots \text{ upto } \infty \quad \dots\text{(iv)}$$

Subtracting Eq. (iv) from Eq. (iii), we get

$$(1-x)(1-x)S_\infty = 1 + 2x + 2x^2 + 2x^3 + \dots \text{ upto } \infty$$

$$= 1 + 2(x + x^2 + x^3 + \dots \text{ upto } \infty)$$

$$= 1 + 2\left(\frac{x}{1-x}\right) = \frac{(1+x)}{(1-x)}$$

$$\therefore \quad S_\infty = \frac{(1+x)}{(1-x)^3}$$

Sigma (Σ) Notation

Σ is a letter of greek alphabets and it is called 'sigma'. The symbol sigma (Σ) represents the sum of similar terms. Usually sum of n terms of any series is represented by placing Σ the nth term of the series. But if we have to find the sum of k terms of a series whose nth term is u_n, this will be represented by $\sum_{n=1}^{k} u_n$.

For example, $\sum_{n=1}^{n=9} n$, i.e. $\sum_{1}^{9} n$ only means the sum of n similar terms when n varies from 1 to 9.

Thus, $\sum_{1}^{9} n = 1+2+3+4+5+6+7+8+9$

Remark

Shortly Σ is written in place of $\sum_{1}^{n}$.

Properties of Sigma Notation

1. $\sum_{r=1}^{n} T_r = T_1 + T_2 + T_3 + \dots + T_n$, when T_n is the general term of the series.

2. $\sum_{r=1}^{n} (T_r \pm T_r') = \sum_{r=1}^{n} T_r \pm \sum_{r=1}^{n} T_r'$

[sigma operator is distributive over addition and subtraction]

3. $\sum_{r=1}^{n} T_r T_r' \neq \left(\sum_{r=1}^{n} T_r\right)\left(\sum_{r=1}^{n} T_r'\right)$

[sigma operator is not distributive over multiplication]

4. $\sum_{r=1}^{n} \left(\frac{T_r}{T_r'}\right) \neq \frac{\left(\sum_{r=1}^{n} T_r\right)}{\left(\sum_{r=1}^{n} T_r'\right)}$

[sigma operator is not distributive over division]

5. $\sum_{r=1}^{n} aT_r = a\sum_{r=1}^{n} T_r$ [where a is constant]

6. $\sum_{j=1}^{n} \sum_{i=1}^{n} T_i T_j = \left(\sum_{i=1}^{n} T_i\right)\left(\sum_{j=1}^{n} T_j\right)$

[where i and j are independent]

Examples on Sigma Notation

(i) $\sum_{i=1}^{m} a = a + a + a + \dots$ upto m times $= am$

(ii) $\sum a = a + a + a + \dots$ upto n times $= an$

i.e. $\sum 5 = 5n$, $\sum 3 = 3n$

(iii) $\sum_{i=1}^{5} (i^2 - 3i) = \sum_{i=1}^{5} i^2 - 3\sum_{i=1}^{5} i$

$$= (1^2 + 2^2 + 3^2 + 4^2 + 5^2) - 3(1+2+3+4+5)$$

$$= 55 - 45 = 10$$

(iv) $\sum_{r=1}^{3} \left(\frac{r+1}{2r+4}\right) = \left(\frac{1+1}{2\cdot 1+4}\right) + \left(\frac{2+1}{2\cdot 2+4}\right) + \left(\frac{3+1}{2\cdot 3+4}\right)$

$$= \frac{2}{6} + \frac{3}{8} + \frac{4}{10} = \frac{40+45+48}{120} = \frac{133}{120} = 1\frac{13}{120}$$

Important Theorems on Σ (Sigma) Operator

Theorem 1 $\sum_{r=1}^{n} f(r+1) - f(r) = f(n+1) - f(1)$

Theorem 2

$$\sum_{r=1}^{n} f(r+2) - f(r) = f(n+2) + f(n+1) - f(2) - f(1)$$

Proof (Theorem 1) $\sum_{r=1}^{n} f(r+1) - f(r)$

$$= [f(2) - f(1)] + [f(3) - f(2)] + [f(4) - f(3)] + \ldots + [f(n+1) - f(n)]$$

$$= f(n+1) - f(1)$$

Proof (Theorem 2)

$$\sum_{r=1}^{n} f(r+2) - f(r) = \sum_{r=1}^{n} [f(r+2) - f(r+1)] + [f(r+1) - f(r)]$$

$$= \sum_{r=1}^{n} f(r+2) - f(r+1) + \sum_{r=1}^{n} f(r+1) - f(r)$$

$$= [f(n+2) - f(2)] + [f(n+1) - f(1)] \quad \text{[from Theorem 1]}$$

$$= f(n+2) + f(n+1) - f(2) - f(1)$$

Remark

1. $\sum_{r=1}^{n} f(r+k) - f(r) = \sum_{m=1}^{k} f(n+m) - \sum_{m=1}^{k} f(m), \forall\, k \in N$
2. $\sum_{r=1}^{n} f(2r+1) - f(2r-1) = f(2n+1) - f(1)$
3. $\sum_{r=1}^{n} f(2r) - f(2r-2) = f(2n) - f(0)$

Natural Numbers

The positive integers 1, 2, 3, ... are called natural numbers. These form an AP with first term and common difference, each equal to unity.

(i) Sum of the First n Natural Numbers

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} = \Sigma n$$

$$\Rightarrow \quad \Sigma n = \frac{n(n+1)}{2} \quad \text{[Remember]}$$

(ii) Sum of the First n Odd Natural Numbers

$$1 + 3 + 5 + \ldots \text{ upto } n \text{ terms} = \frac{n}{2}[2 \cdot 1 + (n-1) \cdot 2] = n^2$$

$$\Rightarrow \quad \sum (2n-1) = n^2 \quad \text{[Remember]}$$

(iii) Sum of the Squares of the First n Natural Numbers

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof We know that, $r^3 - (r-1)^3 = 3r^2 - 3r + 1$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^3 - (r-1)^3 = 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$\Rightarrow \quad n^3 - 0^3 = 3\sum n^2 - 3\sum n + n \quad \ldots\text{(i)}$$

[from important Theorem 1]

Substituting the value of $\sum n$ in Eq. (i), we get

$$\Rightarrow \quad n^3 = 3\sum n^2 - \frac{3 \cdot n(n+1)}{2} + n$$

$$\Rightarrow \quad 3\sum n^2 = n^3 + \frac{3n(n+1)}{2} - n = \frac{n}{2}(2n^2 + 3n + 1)$$

$$= \frac{n(n+1)(2n+1)}{2}$$

$$\Rightarrow \quad \sum n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{[Remember]}$$

Independent Proof We know that,

$$(2r+1)^3 - (2r-1)^3 = 24r^2 + 2$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} (2r+1)^3 - (2r-1)^3 = \sum_{r=1}^{n} (24r^2 + 2)$$

$$\Rightarrow \quad (2n+1)^3 - 1^3 = 24\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} 1$$

[from points to consider-2]

$$\Rightarrow \quad (2n+1)^3 - 1 = 24\sum n^2 + 2n$$

$$\Rightarrow \quad (2n+1)^3 - (2n+1) = 24\sum n^2$$

$$\Rightarrow \quad (2n+1)[(2n+1)^2 - 1] = 24\sum n^2$$

$$\Rightarrow \quad (2n+1)(2n+1+1)(2n+1-1) = 24\sum n^2$$

$$\Rightarrow \quad \sum n^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{[Remember]}$$

(iv) Sum of the Cubes of the First n Natural Numbers

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \sum n^3 = \left(\sum n\right)^2 = \left\{\frac{n(n+1)}{2}\right\}^2$$

Proof We know that,

$$r^4 - (r-1)^4 = 4r^3 - 6r^2 + 4r - 1$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^4 - (r-1)^4 = 4\sum_{r=1}^{n} r^3 - 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1$$

$$\Rightarrow \quad n^4 - 0^4 = 4\sum n^3 - 6\sum n^2 + 4\sum n - n \quad \text{...(i)}$$

[from important theorem 1]

Substituting the values of $\sum n^2$ and $\sum n$ in Eq. (i), we get

$$\Rightarrow \quad n^4 = 4\sum n^3 - \frac{6\,n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} - n$$

$$\Rightarrow 4\sum n^3 = n^4 + n(n+1)(2n+1) - 2n(n+1) + n$$

$$= n[n^3 + (n+1)(2n+1) - 2(n+1) + 1]$$

$$= n(n^3 + 2n^2 + n)$$

$$= n^2(n+1)^2$$

$$\therefore \quad \sum n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = (\sum n)^2 \quad \text{[Remember]}$$

Independent Proof We know that,

$$r^2(r+1)^2 - r^2(r-1)^2 = 4r^3$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^2(r+1)^2 - r^2(r-1)^2 = 4\sum_{r=1}^{n} r^3$$

$$\Rightarrow \quad n^2(n+1)^2 - 1^2 \cdot 0^2 = 4\sum n^3$$

[from important Theorem 1]

$$\Rightarrow \quad \sum n^3 = \left\{\frac{n(n+1)}{2}\right\}^2 = (\sum n)^2 \quad \text{[Remember]}$$

Corollary $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2$

(v) Sum of the Powers Four of the First *n* Natural Numbers

$$1^4 + 2^4 + 3^4 + \ldots + n^4 = \sum n^4$$

$$= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Proof We know that,

$$r^5 - (r-1)^5 = 5r^4 - 10r^3 + 10r^2 - 5r + 1$$

Taking $\sum_{r=1}^{n}$ on both sides, we get

$$\sum_{r=1}^{n} r^5 - (r-1)^5 = 5\sum_{r=1}^{n} r^4 - 10\sum_{r=1}^{n} r^3 + 10\sum_{r=1}^{n} r^2 - 5\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$$

$$\Rightarrow \quad n^5 - 0^5 = 5\sum n^4 - 10\sum n^3 + 10\sum n^2 - 5\sum n + n \quad \text{...(i)}$$

[from important Theorem 1]

Substituting the values of $\sum n, \sum n^2, \sum n^3$ in Eq. (i), we get

$$\Rightarrow \quad n^5 = 5\sum n^4 - \frac{10\,n^2(n+1)^2}{4} + \frac{10\,n(n+1)(2n+1)}{6} - \frac{5n(n+1)}{2} + n$$

$$\therefore \quad 5\sum n^4 = n\left\{n^4 + \frac{5n(n+1)^2}{2} - \frac{5(n+1)(2n+1)}{3} + \frac{5(n+1)}{2} - 1\right\}$$

$$= \frac{n}{6}\{6n^4 + 15n(n^2 + 2n + 1) - 10(2n^2 + 3n + 1) + 15n + 15 - 6\}$$

$$\Rightarrow \quad \sum n^4 = \frac{n}{30}(6n^4 + 15n^3 + 10n^2 - 1)$$

$$= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Remark

If nth term of a sequence is given by $T_n = an^3 + bn^2 + cn + d$, where a, b, c, d are constants.

Then, sum of n terms, $S_n = \Sigma T_n = a\Sigma n^3 + b\,\Sigma n^2 + c\,\Sigma n + d\,\Sigma 1$

This can be evaluated using the above results.

Example 85. ***Find the sum to n terms of the series*** $1^2 + 3^2 + 5^2 + \ldots$ ***upto n terms.***

Sol. Let T_n be the nth term of this series, then

$$T_n = [1 + (n-1)2]^2 = (2n-1)^2 = 4n^2 - 4n + 1$$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = 4\Sigma n^2 - 4\Sigma n + \Sigma 1$

$$= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$= \frac{n}{3}(4n^2 + 6n + 2 - 6n - 6 + 3)$$

$$= \frac{n(4n^2 - 1)}{3}$$

Example 86. ***Find the sum to n terms of the series*** $1\cdot 2^2 + 2\cdot 3^2 + 3\cdot 4^2 + \ldots$.

Sol. Let T_n be the nth term of this series, then

T_n = (nth term of 1, 2, 3, ...) (nth term of $2^2, 3^2, 4^2, \ldots$)

$$= n(n+1)^2 = n^3 + 2n^2 + n$$

$\therefore$ Sum of n terms $S_n = \Sigma T_n$

$$2 = \Sigma n^3 + 2\Sigma n^2 + \Sigma n$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right\}$$

$$= \frac{n(n+1)}{12}(3n^2 + 3n + 8n + 4 + 6)$$

$$= \frac{n(n+1)(3n^2 + 11n + 10)}{12} = \frac{n(n+1)(n+2)(3n+5)}{12}$$

Example 87. *Find the sum of n terms of the series whose nth terms is (i) $n(n-1)(n+1)$ (ii) $n^2 + 3^n$.*

Sol. (i) We have, $T_n = n(n-1)(n+1) = n^3 - n$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \Sigma n^3 - \Sigma n$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 - \left\{\frac{n(n+1)}{2}\right\}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} - 1\right\}$$

$$= \frac{n(n+1)(n-1)(n+2)}{4}$$

(ii) We have, $T_n = n^2 + 3^n$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \Sigma n^2 + \Sigma 3^n$

$$= \Sigma n^2 + (3^1 + 3^2 + 3^3 + \ldots + 3^n)$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3(3^n - 1)}{(3-1)}$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3}{2}(3^n - 1)$$

Example 88. *Find the sum of the series*

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots \text{ upto } n \text{ terms.}$$

Sol. Let T_n be the nth term of the given series. Then,

$$T_n = \frac{(1^3 + 2^3 + 3^3 + \ldots + n^3)}{(1 + 3 + 5 + \ldots + (2n-1))} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}(1 + 2n - 1)}$$

$$= \frac{(n+1)^2}{4} = \frac{1}{4}(n^2 + 2n + 1)$$

Let S_n denotes the sum of n terms of the given series. Then,

$$S_n = \Sigma T_n = \frac{1}{4}\Sigma(n^2 + 2n + 1)$$

$$= \frac{1}{4}(\Sigma n^2 + 2\Sigma n + \Sigma 1)$$

$$= \frac{1}{4}\left\{\frac{n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} + n\right\}$$

$$= \frac{n}{24}\{2n^2 + 3n + 1 + 6n + 6 + 6\}$$

Hence, $S_n = \dfrac{n(2n^2 + 9n + 13)}{24}$

Example 89. Show that

$$\frac{1\cdot 2^2 + 2\cdot 3^2 + \ldots + n\cdot(n+1)^2}{1^2\cdot 2 + 2^2\cdot 3 + \ldots + n^2\cdot(n+1)} = \frac{3n+5}{3n+1}.$$

Sol. Let T_n and T_n' be the nth terms of the series in numerator and denominator of LHS. Then,

$\therefore \quad T_n = n(n+1)^2$ and $T_n' = n^2(n+1)$

$$\therefore \quad \text{LHS} = \frac{\Sigma T_n}{\Sigma T_n'} = \frac{\Sigma n(n+1)^2}{\Sigma n^2(n+1)} = \frac{\Sigma(n^3 + 2n^2 + n)}{\Sigma(n^3 + n^2)}$$

$$= \frac{\Sigma n^3 + 2\Sigma n^2 + \Sigma n}{\Sigma n^3 + \Sigma n^2}$$

$$= \frac{\left\{\frac{n(n+1)}{2}\right\}^2 + 2\left\{\frac{n(n+1)(2n+1)}{6}\right\} + \left\{\frac{n(n+1)}{2}\right\}}{\left\{\frac{n(n+1)}{2}\right\}^2 + \left\{\frac{n(n+1)(2n+1)}{6}\right\}}$$

$$= \frac{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right\}}{\frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + \frac{(2n+1)}{3}\right\}}$$

$$= \frac{\frac{1}{6}(3n^2 + 3n + 8n + 4 + 6)}{\frac{1}{6}(3n^2 + 3n + 4n + 2)}$$

$$= \frac{(3n^2 + 11n + 10)}{(3n^2 + 7n + 2)} = \frac{(3n+5)(n+2)}{(3n+1)(n+2)} = \frac{(3n+5)}{(3n+1)} = \text{RHS}$$

Example 90. Find the sum of the series

$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + 3\cdot 4\cdot 5 + \ldots$ upto n terms.

Sol. Here, T_n = {nth term of 1, 2, 3, ...}

× {nth term of 2, 3, 4, ...} × {nth term of 3, 4, 5, ...}

$\therefore \quad T_n = n(n+1)(n+2) = n^3 + 3n^2 + 2n$

$\therefore \quad S_n$ = Sum of n terms of the series

$$= \Sigma T_n = \Sigma n^3 + 3\Sigma n^2 + 2\Sigma n$$

$$= \left\{\frac{n(n+1)}{2}\right\}^2 + 3\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 2\left\{\frac{n(n+1)}{2}\right\}$$

$$= \frac{n(n+1)}{2}\left\{\frac{n(n+1)}{2} + (2n+1) + 2\right\}$$

$$= \frac{n(n+1)}{4}(n^2 + n + 4n + 2 + 4)$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

Example 91. Find sum to n terms of the series

$$1 + (2+3) + (4+5+6) + \ldots .$$

Sol. Now, number of terms in first bracket is 1, in the second bracket is 2, in the third bracket is 3, etc. Therefore, the number of terms in the nth bracket will be n.

Let the sum of the given series of n terms = S

$\therefore$ Number of terms in $S = 1 + 2 + 3 + \dots + n = \dfrac{n(n+1)}{2}$

Also, the first term of S is 1 and common difference is also 1.

$$\therefore \quad S = \frac{\left\{\frac{n(n+1)}{2}\right\}}{2}\left[2\cdot 1 + \left(\frac{n(n+1)}{2} - 1\right)\cdot 1\right]$$

$$= \frac{n(n+1)}{8}(4 + n^2 + n - 2)$$

$$= \frac{n(n+1)(n^2+n+2)}{8}$$

Example 92. *Find the sum of the series*

$1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + 4\cdot(n-3) + \dots + (n-1)\cdot 2 + n\cdot 1$ also, find the coefficient of x^{n-1} in the expansion of $(1 + 2x + 3x^2 + \dots + nx^{n-1})^2$.

Sol. The rth term of the given series is

$$T_r = r\cdot(n - r + 1) = (n+1)\,r - r^2$$

$\therefore$ Sum of the series

$$S_n = \sum_{r=1}^{n} T_r = (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2 = (n+1)\,\Sigma n - \Sigma n^2$$

$$= (n+1)\frac{n\,(n+1)}{2} - \frac{n\,(n+1)(2n+1)}{6}$$

$$= \frac{n\,(n+1)}{6}(3n + 3 - 2n - 1) = \frac{n\,(n+1)(n+2)}{6}$$

Now,

$$(1 + 2x + 3x^2 + \dots + nx^{n-1})^2 = (1 + 2x + 3x^2 + \dots + nx^{n-1}) \times (1 + 2x + 3x^2 + \dots + nx^{n-1})$$

$\therefore$ Coefficient of x^{n-1} in $(1 + 2x + 3x^2 + \dots + nx^{n-1})^2$

$$= 1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \dots + n\cdot 1$$

$$= S_n = \frac{n\,(n+1)(n+2)}{6}$$

Method of Differences

If the differences of the successive terms of a series are in AP or GP, we can find the nth term of the series by the following steps.

***Step* I** Denote the nth term and the sum of the series upto n terms of the series by T_n and S_n, respectively.

***Step* II** Rewrite the given series with each term shifted by one place to the right.

***Step* III** Then, subtract the second expression of S_n from the first expression to obtain T_n.

Example 93. Find the nth term and sum of n terms of the series, $1 + 5 + 12 + 22 + 35 + \dots$.

Sol. The sequence of differences between successive terms is 4, 7, 10, 13,... . Clearly, it is an AP with common difference 3. So, let the nth term of the given series be T_n and sum of n terms be S_n.

Then, $S_n = 1 + 5 + 12 + 22 + 35 + \dots + T_{n-1} + T_n$ ...(i)

$S_n = 1 + 5 + 12 + 22 + \dots + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$$0 = 1 + 4 + 7 + 10 + 13 + \dots + (T_n - T_{n-1}) - T_n$$

$$\Rightarrow \quad T_n = 1 + 4 + 7 + 10 + 13 + \dots n \text{ terms}$$

$$= \frac{n}{2}\{2\cdot 1 + (n-1)3\} = \frac{1}{2}(3n^2 - n)$$

Hence, $T_n = \dfrac{3}{2}n^2 - \dfrac{1}{2}n$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \dfrac{3}{2}\Sigma n^2 - \dfrac{1}{2}\Sigma n$

$$= \frac{3}{2}\left(\frac{n\,(n+1)(2n+1)}{6}\right) - \frac{1}{2}\left(\frac{n\,(n+1)}{2}\right)$$

$$= \frac{n\,(n+1)}{4}(2n + 1 - 1)$$

$$= \frac{1}{2}n^2(n+1) = \frac{1}{2}(n^3 + n^2)$$

Example 94. *Find the nth term and sum of n terms of the series,* $1 + 3 + 7 + 15 + 31 + \dots$.

Sol. The sequence of differences between successive terms is 2, 4, 8, 16, ... Clearly, it is a GP with common ratio 2. So, let the nth term and sum of the series upto n terms of the series be T_n and S_n, respectively. Then,

$S_n = 1 + 3 + 7 + 15 + 31 + \dots + T_{n-1} + T_n$ ...(i)

$S_n = 1 + 3 + 7 + 15 + \dots + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$$0 = 1 + 2 + 4 + 8 + 16 + \dots + (T_n - T_{n-1}) - T_n$$

$$\Rightarrow \quad T_n = 1 + 2 + 4 + 8 + 16 + \dots \text{ upto } n \text{ terms}$$

$$= \frac{1\cdot(2^n - 1)}{2 - 1}$$

Hence, $T_n = (2^n - 1)$

$\therefore$ Sum of n terms $S_n = \Sigma T_n = \Sigma(2^n - 1) = \Sigma 2^n - \Sigma 1$

$$= (2 + 2^2 + 2^3 + \dots + 2^n) - n$$

$$= \frac{2\cdot(2^n - 1)}{(2-1)} - n = 2^{n+1} - 2 - n$$

Example 95. *Find the nth term of the series* $1 + 4 + 10 + 20 + 35 + \dots$

Sol. The sequence of first consecutive differences is 3, 6, 10, 15, ... and second consecutive differences is 3, 4, 5, ... Clearly, it is an AP with common difference 1. So, let the nth term and sum of the series upto n terms of the series be T_n and S_n, respectively.

Then,

$S_n = 1 + 4 + 10 + 20 + 35 + \dots + T_{n-1} + T_n$ ...(i)

$S_n = \quad 1 + 4 + 10 + 20 + \dots + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$$0 = 1 + 3 + 6 + 10 + 15 + \dots + (T_n - T_{n-1}) - T_n$$

$\Rightarrow \quad T_n = 1 + 3 + 6 + 10 + 15 + \dots$ upto n terms

or $\quad T_n = 1 + 3 + 6 + 10 + 15 + \dots + t_{n-1} + t_n$ ...(iii)

$\quad T_n = 1 + 3 + 6 + 10 + \dots + t_{n-1} + t_n$ ...(iv)

Now, subtracting Eq. (iv) from Eq. (iii), we get

$$0 = 1 + 2 + 3 + 4 + 5 + \dots + (t_n - t_{n-1}) - t_n$$

or $\quad t_n = 1 + 2 + 3 + 4 + 5 + \dots$ upto n terms

$$= \Sigma n = \frac{n(n+1)}{2}$$

$$\therefore \quad T_n = \Sigma t_n = \frac{1}{2}(\Sigma n^2 + \Sigma n)$$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{n(n+1)}{6}(2n+1+3) = \frac{1}{6}n(n+1)(n+2)$$

Example 96. ***Find the nth term of the series*** $1 + 5 + 18 + 58 + 179 + \dots$

Sol. The sequence of first consecutive differences is 4, 13, 40, 121, ... and second consecutive differences is 9, 27, 81, ... Clearly, it is a GP with common ratio 3. So, let the nth term and sum of the series upto n terms of the series be T_n and S_n, respectively. Then,

$S_n = 1 + 5 + 18 + 58 + 179 + \dots + T_{n-1} + T_n$ ...(i)

$S_n = 1 + 5 + 18 + 58 + \dots + T_{n-1} + T_n$ ...(ii)

Subtracting Eq. (ii) from Eq. (i), we get

$$0 = 1 + 4 + 13 + 40 + 121 + \dots + (T_n - T_{n-1}) - T_n$$

$\Rightarrow \quad T_n = 1 + 4 + 13 + 40 + 121 + \dots$ upto n terms

or $\quad T_n = 1 + 4 + 13 + 40 + 121 + \dots + t_{n-1} + t_n$ ...(iii)

$\quad T_n = \quad 1 + 4 + 13 + 40 + \dots + t_{n-1} + t_n$ ...(iv)

Now, subtracting Eq. (iv) from Eq. (iii), we get

$$0 = 1 + 3 + 9 + 27 + 81 + \dots + (t_n - t_{n-1}) - t_n$$

or $\quad t_n = 1 + 3 + 9 + 27 + 81 + \dots$ upto n terms

$$= \frac{1 \cdot (3^n - 1)}{(3-1)} = \frac{1}{2}(3^n - 1)$$

$$\therefore \quad T_n = \Sigma t_n = \frac{1}{2}(\Sigma 3^n - \Sigma 1)$$

$$= \frac{1}{2}\{(3 + 3^2 + 3^3 + \dots + 3^n) - n\}$$

$$= \frac{1}{2}\left\{\frac{3(3^n - 1)}{(3-1)} - n\right\}$$

$$= \frac{3}{4}(3^n - 1) - \frac{1}{2}n$$

Method of Differences (Shortcut) to find nth term of a Series

The nth term of the series can be written directly on the basis of successively differences, we use the following steps to find the nth term T_n of the given sequence.

***Step* I** If the first consecutive differences of the given sequence are in AP, then take
$T_n = a(n-1)(n-2) + b(n-1) + c$, where a, b, c are constants. Determine a, b, c by putting $n = 1, 2, 3$ and putting the values of T_1, T_2, T_3.

***Step* II** If the first consecutive differences of the given sequence are in GP, then take
$T_n = ar^{n-1} + bn + c$, where a, b, c are constants and r is the common ratio of GP. Determine a, b, c by putting $n = 1, 2, 3$ and putting the values of T_1, T_2, T_3.

***Step* III** If the differences of the differences computed in Step I are in AP, then take
$T_n = a(n-1)(n-2)(n-3) + b(n-1)(n-2) + c(n-1) + d$, where a, b, c, d are constants.Determine by putting $n = 1, 2, 3, 4$ and putting the values of T_1, T_2, T_3, T_4.

***Step* IV** If the differences of the differences computed in Step I are in GP with common ratio r, then take
$T_n = ar^{n-1} + bn^2 + cn + d$, where a, b, c, d are constants. Determine by putting $n = 1, 2, 3, 4$ and putting the values of T_1, T_2, T_3, T_4.

Example 97. Find the nth term and sum of n terms of the series $2 + 4 + 7 + 11 + 16 + \dots$.

Sol. The sequence of first consecutive differences is 2, 3, 4, 5, ... Clearly, it is an AP.

Then, nth term of the given series be

$$T_n = a(n-1)(n-2) + b(n-1) + c \quad \text{...(i)}$$

Putting $n = 1, 2, 3$, we get

$$2 = c \Rightarrow 4 = b + c \Rightarrow 7 = 2a + 2b + c$$

After solving, we get $a = \frac{1}{2}, b = 2, c = 2$

Putting the values of a, b, c in Eq. (i), we get

$$T_n = \frac{1}{2}(n-1)(n-2) + 2(n-1) + 2 = \frac{1}{2}(n^2 + n + 2)$$

Hence, sum of series $S_n = \Sigma T_n = \frac{1}{2}(\Sigma n^2 + \Sigma n + 2\Sigma 1)$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + 2n\right)$$

$$= \frac{1}{6}n(n^2 + 3n + 8)$$

Example 98. ***Find the nth term and sum of n terms of the series*** $5+7+13+31+85+\ldots$.

Sol. The sequence of first consecutive differences is 2, 6, 18, 54, ... Clearly, it is a GP with common ratio 3. Then, nth term of the given series be

$$T_n = a(3)^{n-1} + bn + c \quad \ldots(i)$$

Putting $n = 1, 2, 3$, we get

$$5 = a + b + c \quad \ldots(ii)$$
$$7 = 3a + 2b + c \quad \ldots(iii)$$
$$13 = 9a + 3b + c \quad \ldots(iv)$$

Solving these equations, we get

$$a = 1, b = 0, c = 4$$

Putting the values of a, b, c in Eq. (i), we get

$$T_n = 3^{n-1} + 4$$

Hence, sum of the series

$$S_n = \Sigma T_n = \Sigma(3^{n-1} + 4) = \Sigma(3^{n-1}) + 4\Sigma 1$$
$$= (1 + 3 + 3^2 + \ldots + 3^{n-1}) + 4n$$
$$= 1\cdot\frac{(3^n - 1)}{(3-1)} + 4n = \frac{1}{2}(3^n + 8n - 1)$$

Example 99. ***Find the nth term of the series*** $1+2+5+12+25+46+\ldots$.

Sol. The sequence of first consecutive differences is 1, 3, 7, 13, 21, ... The sequence of the second consecutive differences is 2, 4, 6, 8, ... Clearly, it is an AP. Then, nth term of the given series be

$$T_n = a(n-1)(n-2)(n-3) + b(n-1)(n-2) + c(n-1) + d \quad \ldots(i)$$

Putting $n = 1, 2, 3, 4$, we get

$$1 = d \quad \ldots(ii)$$
$$2 = c + d \quad \ldots(iii)$$
$$5 = 2b + 2c + d \quad \ldots(iv)$$
$$12 = 6a + 6b + 3c + d \quad \ldots(v)$$

After, solving these equations, we get

$$a = \frac{1}{3}, b = 1, c = 1, d = 1$$

Putting the values of a, b, c, d in Eq. (i), we get

$$T_n = \frac{1}{3}(n^3 - 6n^2 + 11n - 6) + (n^2 - 3n + 2) + (n-1) + 1$$
$$= \frac{1}{3}(n^3 - 3n^2 + 5n) = \frac{n}{3}(n^2 - 3n + 5)$$

Example 100. Find the nth term of the series $2+5+12+31+86+\ldots$.

Sol. The sequence of first consecutive differences is 3, 7, 19, 55, ... The sequence of the second consecutive differences is 4, 12, 36, ... Clearly, it is a GP with common ratio 3. Then, nth term of the given series be

$$T_n = a(3)^{n-1} + bn^2 + cn + d \quad \ldots(i)$$

Putting $n = 1, 2, 3, 4$, we get

$$2 = a + b + c + d \quad \ldots(ii)$$
$$5 = 3a + 4b + 2c + d \quad \ldots(iii)$$
$$12 = 9a + 9b + 3c + d \quad \ldots(iv)$$
$$31 = 27a + 16b + 4c + d \quad \ldots(v)$$

After, solving these equations, we get

$$a = 1, b = 0, c = 1, d = 0$$

Putting the values of a, b, c, d in Eq. (i), we get

$$T_n = 3^{n-1} + n$$

Method of Differences

(Maha Shortcut)

To find $t_1 + t_2 + t_3 + \ldots + t_{n-1} + t_n$

Let $S_n = t_1 + t_2 + t_3 + \ldots + t_{n-1} + t_n$

Then, $\Delta t_1, \Delta t_2, \Delta t_3, \ldots, \Delta t_{n-1}$ [1st order differences]

$\Delta^2 t_1, \Delta^2 t_2, \Delta^2 t_3, \ldots, \Delta^2 t_{n-1}$ [2nd order differences]

$\vdots \quad \vdots \quad \vdots$

$$\therefore\ t_n = {}^{n-1}C_0\, t_1 + {}^{n-1}C_1\, \Delta t_1 + {}^{n-1}C_2\, \Delta^2 t_1 + \ldots + {}^{n-1}C_{r-1}\, \Delta^{r-1} t_1$$

and $S_n = {}^nC_1\, t_1 + {}^nC_2\, \Delta t_1 + {}^nC_3\, \Delta^2 t_1 + \ldots + {}^nC_r\, \Delta^r t_1$

where, $\Delta t_1 = t_2 - t_1, \Delta t_2 = t_3 - t_2$, etc.

$\Delta^2 t_1 = \Delta t_2 - \Delta t_1, \Delta^3 t_1 = \Delta^2 t_2 - \Delta^2 t_1$, etc.

Example 101. Find the n th term and sum to n terms of the series $12+40+90+168+280+432+\ldots$.

Sol. Let $S_n = 12 + 40 + 90 + 168 + 280 + 432 + \ldots$, then

1st order differences are 28, 50, 78, 112, 152, ... (i.e. $\Delta t_1, \Delta t_2, \Delta t_3, \ldots$)

and 2nd order differences are

22, 28, 34, 40, ... (i.e. $\Delta^2 t_1, \Delta^2 t_2, \Delta^2 t_3, \ldots$)

and 3rd order differences are

6, 6, 6, 6,... (i.e. $\Delta^3 t_1, \Delta^3 t_2, \Delta^3 t_3, \ldots$)

and 4th order differences are

0, 0, 0, 0, ... (i.e. $\Delta^4 t_1, \Delta^4 t_2, \Delta^4 t_3, \ldots$)

$$\therefore\quad t_n = 12\cdot{}^{n-1}C_0 + 28\cdot{}^{n-1}C_1 + 22\cdot{}^{n-1}C_2 + 6\cdot{}^{n-1}C_3$$
$$= 12 + 28(n-1) + \frac{22(n-1)(n-2)}{2} + \frac{6(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}$$
$$= n^3 + 5n^2 + 6n$$

and $S_n = 12\cdot{}^nC_1 + 28\cdot{}^nC_2 + 22\cdot{}^nC_3 + 6\cdot{}^nC_4$

$$= 12n + \frac{28n(n-1)}{2} + \frac{22n(n-1)(n-2)}{1\cdot 2\cdot 3} + \frac{6\cdot n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$$

$$= \frac{n}{12}(n+1)(3n^2+23n+46)$$

V_n Method

To find the sum of the series of the forms

I. $a_1a_2 \dots a_r + a_2a_3 \dots a_{r+1} + \dots + a_na_{n+1} \dots a_{n+r-1}$

II. $\dfrac{1}{a_1a_2\dots a_r} + \dfrac{1}{a_2a_3\dots a_{r+1}} + \dots + \dfrac{1}{a_na_{n+1}\dots a_{n+r-1}}$

where, $a_1, a_2, a_3, \dots, a_n, \dots$ are in AP.

***Solution of form* I** Let S_n be the sum and T_n be the nth term of the series, then

$$S_n = a_1a_2 \dots a_r + a_2a_3 \dots a_{r+1} + \dots + a_na_{n+1} + \dots + a_{n+r-1}$$

$\therefore \quad T_n = a_na_{n+1}a_{n+2} \dots a_{n+r-2}\, a_{n+r-1}$ ...(i)

Let $V_n = a_na_{n+1}a_{n+2} \dots a_{n+r-2}\, a_{n+r-1}\, a_{n+r}$

[taking one extra factor in T_n for V_n]

$\therefore \quad V_{n-1} = a_{n-1}a_n\, a_{n+1} \dots a_{n+r-3}\, a_{n+r-2}\, a_{n+r-1}$

$\Rightarrow \quad V_n - V_{n-1} = a_na_{n+1}a_{n+2} \dots a_{n+r-1}(a_{n+r} - a_{n-1})$

$= T_n(a_{n+r} - a_{n-1})$ [from Eq. (i)] ...(ii)

Let d be the common difference of AP, then

$$a_n = a_1 + (n-1)d$$

Then, from Eq. (ii)

$$V_n - V_{n-1} = T_n[\{a_1 + (n+r-1)d\} - \{a_1 + (n-2)d\}] = (r+1)d\,T_n$$

$$\Rightarrow \quad T_n = \frac{1}{(r+1)d}(V_n - V_{n-1})$$

$$\therefore \quad S_n = \Sigma T_n = \sum_{n=1}^{n} T_n = \frac{1}{(r+1)d}\sum_{n=1}^{n}(V_n - V_{n-1})$$

$$= \frac{1}{(r+1)d}(V_n - V_0)$$

[from important Theorem 1 of Σ]

$$= \frac{1}{(r+1)(a_2 - a_1)}(a_na_{n+1} \dots a_{n+r} - a_0a_1 \dots a_r)$$

***Corollary* I** If $a_1, a_2, a_3, \dots, a_n, \dots$ are in AP, then

(i) For $r = 2$, $a_1a_2 + a_2a_3 + \dots + a_na_{n+1} = \dfrac{1}{3(a_2 - a_1)}(a_na_{n+1}a_{n+2} - a_0a_1a_2)$

(ii) For $r = 3$,

$a_1a_2a_3 + a_2a_3a_4 + \dots + a_na_{n+1}a_{n+2} = \dfrac{1}{4(a_2 - a_1)}(a_na_{n+1}a_{n+2}a_{n+3} - a_0a_1a_2a_3)$

***Corollary* II**

(i) $1\cdot 2 + 2\cdot 3 + \dots + n(n+1) = \dfrac{1}{3}\{n(n+1)(n+2) - 0\cdot 1\cdot 2\} = \dfrac{n(n+1)(n+2)}{3}$

(ii) $1\cdot 3\cdot 5\cdot 7 + 3\cdot 5\cdot 7\cdot 9 + \dots + (2n-1)\cdot(2n+1)(2n+5)\cdot(2n+3)\cdot$

$$= \frac{1}{11}\{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7) - (-1)\cdot 1\cdot 3\cdot 5\cdot 7\}$$

$$= \frac{1}{11}\{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7) + 105\}$$

***Solution of form* II** Let d be the common difference of AP, then $a_n = a_1 + (n-1)d$

Let sum of the series and nth term are denoted by S_n and T_n, respectively. Then,

$$S_n = \frac{1}{a_1a_2 \dots a_r} + \frac{1}{a_2a_3 \dots a_{r+1}} + \dots + \frac{1}{a_na_{n+1} \dots a_{n+r-1}}$$

$\therefore \quad T_n = \dfrac{1}{a_na_{n+1}a_{n+2} \dots a_{n+r-2}\, a_{n+r-1}}$ ...(i)

Let $\quad V_n = \dfrac{1}{a_{n+1}\, a_{n+2} \dots a_{n+r-2}\, a_{n+r-1}}$ ...(ii)

[leaving first factor from denominator of T_n]

So, $\quad V_{n-1} = \dfrac{1}{a_na_{n+1} \dots a_{n+r-3}\, a_{n+r-2}}$

$$\Rightarrow \quad V_n - V_{n-1} = \frac{1}{a_{n+1}\, a_{n+2} \dots a_{n+r-2}\, a_{n+r-1}} - \frac{1}{a_na_{n+1} \dots a_{n+r-3}\, a_{n+r-2}}$$

$$= \frac{a_n - a_{n+r-1}}{a_na_{n+1}\, a_{n+2} - a_{n+r-2}\, a_{n+r-1}}$$

$= T_n(a_n - a_{n+r-1})$ [from Eq. (i)]

$= T_n[\{a_1 + (n-1)d\} - \{a_1 + (n+r-2)d\}]$

$= d(1-r)T_n$

$$\therefore \quad T_n = \frac{(V_n - V_{n-1})}{d(1-r)}$$

$$\therefore \quad S_n = \Sigma T_n = \sum_{n=1}^{n}\frac{(V_n - V_{n-1})}{d(1-r)} = \frac{1}{d(1-r)}(V_n - V_0)$$

[from important Theorem 1 of Σ]

$$= \frac{1}{(a_2 - a_1)(1-r)}\left\{\frac{1}{a_{n+1}\, a_{n+2} \dots a_{n+r-2}\, a_{n+r-1}} - \frac{1}{a_1\, a_2 \dots a_{r-2}\, a_{r-1}}\right\}$$

Hence, the sum of n terms is $S_n = \dfrac{1}{(r-1)(a_2 - a_1)}$

$$\left\{\frac{1}{a_1 a_2 \dots a_{r-1}} - \frac{1}{a_{n+1} a_{n+2} \dots a_{n+r-1}}\right\}$$

***Corollary* I** If $a_1, a_2, a_3, \dots, a_n, \dots$ are in AP, then

(i) For $r = 2$,

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_n a_{n+1}} = \frac{1}{(a_2 - a_1)}$$

$$\left\{\frac{1}{a_1} - \frac{1}{a_{n+1}}\right\} = \frac{1}{d}\left(\frac{a_{n+1} - a_1}{a_1\, a_{n+1}}\right)$$

$$= \frac{1}{d}\left(\frac{a_1 + nd - a_1}{a_1 a_{n+1}}\right) = \frac{n}{a_1\, a_{n+1}}$$

(ii) For $r = 3$, $\dfrac{1}{a_1 a_2 a_3} + \dfrac{1}{a_2 a_3 a_4} + \dots + \dfrac{1}{a_n\, a_{n+1}\, a_{n+2}}$

$$= \frac{1}{2(a_2 - a_1)}\left\{\frac{1}{a_1 a_2} - \frac{1}{a_{n+1}\, a_{n+2}}\right\}$$

(iii) For $r = 4$,

$$\frac{1}{a_1 a_2 a_3 a_4} + \frac{1}{a_2 a_3 a_4 a_5} + \dots + \frac{1}{a_n a_{n+1} a_{n+2} a_{n+3}}$$

$$= \frac{1}{3(a_2 - a_1)}\left\{\frac{1}{a_1 a_2 a_3} - \frac{1}{a_{n+1}\, a_{n+2}\, a_{n+3}}\right\}$$

***Corollary* II**

(i) $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \dots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$

(ii) $\dfrac{1}{1\cdot 2\cdot 3} + \dfrac{1}{2\cdot 3\cdot 4} + \dfrac{1}{3\cdot 4\cdot 5} + \dots + \dfrac{1}{n(n+1)(n+2)}$

$$= \frac{1}{2}\left\{\frac{1}{1\cdot 2} - \frac{1}{(n+1)(n+2)}\right\} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

(iii) $\dfrac{1}{1\cdot 3\cdot 5\cdot 7} + \dfrac{1}{3\cdot 5\cdot 7\cdot 9}$

$$+ \dots + \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}$$

$$= \frac{1}{6}\left\{\frac{1}{1\cdot 3\cdot 5} - \frac{1}{(2n+1)(2n+3)(2n+5)}\right\}$$

$$= \frac{1}{90} - \frac{1}{6(2n+1)(2n+3)(2n+5)}$$

Example 102. Find the sum upto n terms of the series $1\cdot 4\cdot 7\cdot 10 + 4\cdot 7\cdot 10\cdot 13 + 7\cdot 10\cdot 13\cdot 16 + \dots$

Sol. Let T_n be the nth term of the given series.

$\therefore T_n =$ (nth term of 1, 4, 7, ...(nth term of 4, 7, 10, ...) (nth term of 7, 10, 13, ...) (nth term of 10, 13, 16, ...)

$$T_n = (3n-2)(3n+1)(3n+4)(3n+7) \quad \text{...(i)}$$

$$\therefore V_n = (3n-2)(3n+1)(3n+4)(3n+7)(3n+10)$$

$$V_{n-1} = (3n-5)(3n-2)(3n+1)(3n+4)(3n+7)$$

$\Rightarrow$ $V_n = (3n+10)\,T_n$ [from Eq. (i)]

and $V_{n-1} = (3n-5)\,T_n$

$\therefore$ $V_n - V_{n-1} = 15\,T_n$

$\therefore$ $T_n = \dfrac{1}{15}(V_n - V_{n-1}))$

$\therefore$ $S_n = \Sigma T_n = \displaystyle\sum_{n=1}^{n} \frac{1}{15}(V_n - V_{n-1})$

$$= \frac{1}{15}(V_n - V_0)$$

[from important Theorem 1 of Σ]

$$= \frac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) - (-2)(1)(4)(7)(10)\}$$

$$= \frac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) + 560\}$$

Shortcut Method

$$S_n = \frac{1}{(\text{last factor of III term} - \text{first factor of I term})}$$

(Taking one extra factor in T_n in last – Taking one extra factor in I term in start)

$$= \frac{1}{(16-1)}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) - (-2)\cdot 1\cdot 4\cdot 7\cdot 10\}$$

$$= \frac{1}{15}\{(3n-2)(3n+1)(3n+4)(3n+7)(3n+10) + 560\}$$

Example 103. Find the sum to n terms of the series $\dfrac{1}{1\cdot 3\cdot 5\cdot 7\cdot 9} + \dfrac{1}{3\cdot 5\cdot 7\cdot 9\cdot 11} + \dfrac{1}{5\cdot 7\cdot 9\cdot 11\cdot 13} + \dots$. Also, find the sum to infinity terms.

Sol. Let T_n be the nth term of the given series.

Then, $T_n = \dfrac{1}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)}$ ...(i)

$\therefore$ $V_n = \dfrac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}$

[leaving first factor from denominator of T_n]

$$V_{n-1} = \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}$$

$$\Rightarrow V_n - V_{n-1} = \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)} - \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)}$$

$$= \frac{(2n-1)-(2n+7)}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)}$$

$= -8\,T_n$ [from Eq. (i)]

$\therefore \quad T_n = -\frac{1}{8}(V_n - V_{n-1})$

$\therefore \quad S_n = \Sigma T_n = \sum_{n=1}^{n} T_n = -\frac{1}{8}\sum_{n=1}^{n}(V_n - V_{n-1}) = -\frac{1}{8}(V_n - V_0)$

[from Important Theorem 1 of Σ]

$$= \frac{1}{8}(V_0 - V_n)$$

$$= \frac{1}{8}\left\{\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$$

$$= \frac{1}{840} - \frac{1}{8(2n+1)(2n+3)(2n+5)(2n+7)}$$

and $S_\infty = \frac{1}{840} - \frac{1}{\infty} = \frac{1}{840} - 0 = \frac{1}{840}$

Shortcut Method

$$\frac{1}{1\cdot3\cdot5\cdot7\cdot9} + \frac{1}{3\cdot5\cdot7\cdot9\cdot11} + \frac{1}{5\cdot7\cdot9\cdot11\cdot13} + \ldots + \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)} \quad \ldots\text{(i)}$$

Now, in each term in denominator

$$9 - 1 = 11 - 3 = 13 - 5 = \ldots = (2n+7) - (2n-1) = 8$$

Then, Eq. (i) can be written as

$$= \frac{1}{8}\left\{\frac{9-1}{1\cdot3\cdot5\cdot7\cdot9} + \frac{11-3}{3\cdot5\cdot7\cdot9\cdot11} + \frac{13-5}{5\cdot7\cdot9\cdot11\cdot13} + \ldots + \frac{(2n+7)-(2n-1)}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$$

$$= \frac{1}{8}\left\{\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{3\cdot5\cdot7\cdot9} + \frac{1}{3\cdot5\cdot7\cdot9} - \frac{1}{5\cdot7\cdot9\cdot11} + \frac{1}{5\cdot7\cdot9\cdot11} - \frac{1}{7\cdot9\cdot11\cdot13} + \ldots + \frac{1}{(2n-1)(2n+1)(2n+3)(2n+5)} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$$

$$= \frac{1}{8}\left\{\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right\}$$

[middle terms are cancelled out]

$$= \frac{1}{840} - \frac{1}{8(2n+1)(2n+3)(2n+5)(2n+7)} = S_n \quad \text{[say]}$$

$\therefore$ Sum to infinity terms $= S_\infty = \frac{1}{840} - 0 = \frac{1}{840}$

Maha Shortcut Method

Taking $\frac{1}{8}$ outside the bracket $\left(\text{i.e. } \frac{1}{9-1} = \frac{1}{11-3} = \frac{1}{13-5} = \ldots\right)$ and in bracket leaving last factor of denominator of first term – leaving first factor of denominator of last term

i.e., $S_n = \frac{1}{8}\left(\frac{1}{1\cdot3\cdot5\cdot7} - \frac{1}{(2n+1)(2n+3)(2n+5)(2n+7)}\right)$

$\therefore \quad S_\infty = \frac{1}{8}\left(\frac{1}{1\cdot3\cdot5\cdot7} - 0\right) = \frac{1}{840}$

Example 104. ***If*** $\sum_{r=1}^{n} T_r = \frac{n(n+1)(n+2)(n+3)}{12}$, ***where*** T_r ***denotes the*** r***th term of the series. Find*** $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{T_r}$.

Sol. We have, $T_n = \sum_{r=1}^{n} T_r - \sum_{r=1}^{n-1} T_r$

$$= \frac{n(n+1)(n+2)(n+3)}{12} - \frac{(n-1)n(n+1)(n+2)}{12}$$

$$= \frac{n(n+1)(n+2)}{12}[(n+3)-(n-1)]$$

$$= \frac{n(n+1)(n+2)}{3} \qquad \frac{1}{T_n} = \frac{3}{n(n+1)(n+2)}$$

$$\therefore \quad \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{T_r} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{3}{r(r+1)(r+2)}$$

$$= 3\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r(r+1)(r+2)}$$

$$= 3\lim_{n\to\infty}\left(\frac{1}{1\cdot2\cdot3} + \frac{1}{2\cdot3\cdot4} + \frac{1}{3\cdot4\cdot5} + \ldots + \frac{1}{n(n+1)(n+2)}\right)$$

Maha Shortcut Method

$$= 3\lim_{n\to\infty}\frac{1}{2}\left(\frac{1}{1\cdot2} - \frac{1}{(n+1)(n+2)}\right)$$

$$= \frac{3}{2}\left(\frac{1}{2} - 0\right) = \frac{3}{4}$$

Exercise for Session 6

1. The sum of the first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\dots$ is

(a) $2^n - n - 1$ (b) $1 - 2^{-n}$ (c) $n + 2^{-n} - 1$ (d) $2^n - 1$

2. $2^{1/4} \cdot 4^{1/8} \cdot 8^{1/16} \cdot 16^{1/32} \dots$ is equal to

(a) 1 (b) $\frac{3}{2}$ (c) 2 (d) $\frac{5}{2}$

3. $1 + 3 + 7 + 15 + 31 + \dots$ upto n terms equals

(a) $2^{n+1} - n$ (b) $2^{n+1} - n - 2$ (c) $2^n - n - 2$ (d) None of these

4. 99th term of the series $2 + 7 + 14 + 23 + 34 + \dots$ is

(a) 9998 (b) 9999 (c) 10000 (d) 100000

5. The sum of the series $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \dots$ upto n terms is

(a) $n(n+1)(n+2)$ (b) $(n+1)(n+2)(n+3)$

(c) $\frac{1}{4}n(n+1)(n+2)(n+3)$ (d) $\frac{1}{4}(n+1)(n+2)(n+3)$

6. $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\dots+\frac{1}{n(n+1)}$ equals

(a) $\frac{1}{n(n+1)}$ (b) $\frac{n}{n+1}$

(c) $\frac{2n}{n+1}$ (d) $\frac{2}{n(n+1)}$

7. Sum of the n terms of the series $\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^3}+\dots$ is

(a) $\frac{2n}{n+1}$ (b) $\frac{4n}{n+1}$

(c) $\frac{6n}{n+1}$ (d) $\frac{9n}{n+1}$

8. If $t_n = \frac{1}{4}(n+2)(n+3)$ for $n = 1, 2, 3, \dots$, then $\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\dots+\frac{1}{t_{2003}}$ equals

(a) $\frac{4006}{3006}$ (b) $\frac{4003}{3007}$

(c) $\frac{4006}{3008}$ (d) $\frac{4006}{3009}$

9. The value of $\frac{1}{(1+a)(2+a)}+\frac{1}{(2+a)(3+a)}+\frac{1}{(3+a)(4+a)}+\dots$ upto ∞ is

(where, a is constant)

(a) $\frac{1}{1+a}$ (b) $\frac{2}{1+a}$

(c) ∞ (d) None of these

10. If $f(x)$ is a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$. Then, the value of n is

(a) 4 (b) 5 (c) 6 (d) None of these