Session 6

Arrangement in Groups, Multinomial Theorem, Multiplying Synthetically

Arrangement in Groups

(a) The number of ways in which n different things can be arranged into r different groups is $r(r+1)(r+2)\dots(r+n-1)$ or $n!\cdot {}^{n-1}C_{r-1}$
according as blank groups are or are not admissible.

Proof

(i) Let n letters $a_1, a_2, a_3, \dots, a_n$ be written in a row in any order. All the arrangements of the letters in r, groups, blank groups being admissible, can be obtained thus, place among the letters $(r-1)$ marks of partition and arrange the $(n+r-1)$ things (consisting of letters and marks) in all possible orders. Since, $(r-1)$ of the things are alike, the number of different arrangements is $\dfrac{(n+r-1)!}{(r-1)!} = r(r+1)(r-2)\dots(r+n-1)$.

(ii) All the arrangements of the letters in r groups, none of the groups being blank, can be obtained as follows:

(I) Arrange the letters in all possible orders. This can be done in $n!$ ways.

(II) In every such arrangement, place $(r-1)$ marks of partition in $(r-1)$ out of the $(n-1)$ spaces between the letters. This can be done in ${}^{n-1}C_{r-1}$ ways.

Hence, the required number is $n!\cdot {}^{n-1}C_{r-1}$.

Example 77. In how many ways 5 different balls can be arranged into 3 different boxes so that no box remains empty?

Sol. The required number of ways $= 5!\cdot {}^{5-1}C_{3-1} = 5!\cdot {}^4C_2$

$$= (120)\cdot\left(\frac{4\cdot 3}{1\cdot 2}\right) = 720$$

Aliter

Each box must contain atleast one ball, since no box remains empty. Boxes can have balls in the following systems

Box	I	II	III
Number of balls	1	1	3

Or

Box	I	II	III
Number of balls	1	2	2

All 5 balls can be arranged by 5! ways and boxes can be arranged in each system by $\dfrac{3!}{2!}$.

Hence, required number of ways $= 5! \times \dfrac{3!}{2!} + 5! \times \dfrac{3!}{2!}$

$$= 120 \times 3 + 120 \times 3 = 720$$

(b) The number of ways in which n different things can be distributed into r different groups is

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \dots + (-1)^{r-1}\cdot {}^rC_{r-1}$$

Or

$$\sum_{p=0}^{r} (-1)^p \cdot {}^rC_p \cdot (r-p)^n$$

Or

Coefficient of x^n in $n!(e^x - 1)^r$.

Here, blank groups are not allowed.

Proof In any distribution, denote the groups by $g_1, g_2, g_3, \dots, g_r$ and consider the distributions in which blanks are allowed.

The total number of these is r^n.

The number in which g_1 is blank, is $(r-1)^n$.

Therefore, the number in which g_1 is not blank, is

$$r^n - (r-1)^n$$

of these last, the number in which g_2 is blank, is

$$(r-1)^n - (r-2)^n$$

Therefore, the number in which g_1, g_2 are not blank, is

$$r^n - 2(r-1)^n + (r-2)^n$$

of these last, the number in which g_3 is blank, is

$$(r-1)^n - 2(r-2)^n + (r-3)^n$$

Therefore, the number in which g_1, g_2, g_3 are not blank, is

$$r^n - 3(r-1)^n + 3(r-2)^n - (r-3)^n$$

This process can be continued as far as we like and it is obvious that the coefficients are formed as in a binomial expansion.

Hence, the number of distributions in which no one of x assigned groups is blank, is

$$r^n - {}^xC_1(r-1)^n + {}^xC_2(r-2)^n - \dots + (-1)^x(r-x)^n$$

when $x = r$, then

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \ldots + (-1)^{r-1} \cdot {}^rC_{r-1}$$
$$(r-(r-1))^n + (-1)^r \cdot {}^rC_r(r-r)^n$$

Or

$$r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \ldots + (-1)^{r-1} \cdot {}^rC_{r-1}$$

Aliter

By Principle of Inclusion and Exclusion

Let A_i denotes the set of distribution of things, if ith group gets nothing. Then, $n(A_i) = (r-1)^n$

[as n things can be distributed among $(r-1)$ groups in $(r-1)^n$ ways]

Then, $n(A_i \cap A_j)$ represents number of distribution ways in which groups i and j get no object. Then,

$$n(A_i \cap A_j) = (r-2)^n$$

Also, $\quad n(A_i \cap A_j \cap A_k) = (r-3)^n$

This process can be continued, then the required number is

$$n(A_1' A_2' \cap \ldots \cap A_r')$$
$$= n(U) - n(A_1 \cup A_2 \cup \ldots \cup A_r)$$
$$= r^n - \left\{ \sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) \ldots + (-1)^n \sum n(A_1 \cap A_2 \cap \ldots \cap A_r) \right\}$$
$$= r^n - \{{}^rC_1(r-1)^n - {}^rC_2(r-2)^n + {}^rC_3(r-3)^n - \ldots + (-1)^r \cdot {}^rC_{r-1}\}$$
$$= r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - {}^rC_3(r-3)^n + \ldots + (-1)^{r-1} \cdot {}^rC_{r-1} \cdot 1$$

Note Coefficient of x^r in $e^{px} = \dfrac{p^r}{r!}$.

Example 78. In how many ways 5 different balls can be distributed into 3 boxes so that no box remains empty?

Sol. The required number of ways

$$= 3^5 - {}^3C_1(3-1)^5 + {}^3C_2(3-2)^5 - {}^3C_3(3-3)^5$$
$$= 243 - 96 + 3 - 0 = 150$$

Or

Coefficient of x^5 in $5!(e^x - 1)^3$

$= $ Coefficient of x^5 in $5!(e^{3x} - 3e^{2x} + 3e^x - 1)$

$$= 5!\left(\frac{3^5}{5!} - 3 \times \frac{2^5}{5!} + 3 \times \frac{1}{5!}\right) = 3^5 - 3 \cdot 2^5 + 3 = 243 - 96 + 3 = 150$$

Aliter

Each box must contain atleast one ball, since number box remains empty. Boxes can have balls in the following systems

Box	I	II	III
Number of balls	1	1	3

Or

Box	I	II	III
Number of balls	1	2	2

The number of ways to distribute the balls in I system

$$= {}^5C_1 \times {}^4C_1 \times {}^3C_3$$

$\therefore$ The total number of ways to distribute 1, 1, 3 balls to the boxes

$$= {}^5C_1 \times {}^4C_1 \times {}^3C_3 \times \frac{3!}{2!} = 5 \times 4 \times 1 \times 3 = 60$$

and the number of ways to distribute the balls in II system

$$= {}^5C_1 \times {}^4C_2 \times {}^2C_2$$

$\therefore$ The total number of ways to distribute 1, 2, 2 balls to the boxes

$$= {}^5C_1 \times {}^4C_2 \times {}^2C_2 \times \frac{3!}{2!}$$
$$= 5 \times 6 \times 1 \times 3 = 90$$

$\therefore$ The required number of ways = 60 + 90 = 150

Example 79. In how many ways can 5 different books be tied up in three bundles?

Sol. The required number of ways $= \dfrac{1}{3!}(3^5 - {}^3C_1 \cdot 2^5 + {}^3C_2 \cdot 1^5)$

$$= \frac{150}{6} = 25$$

Example 80. *If $n(A) = 5$ and $n(B) = 3$, find number of onto mappings from A to B.*

Sol. We know that in onto mapping, each image must be assigned atleast one pre-image.

This is equivalent to number of ways in which 5 different balls (pre-images) can be distributed in 3 different boxes (images), if no box remains empty. The total number of onto mappings from A to B

$$= 3^5 - {}^3C_1(3-1)^5 + {}^3C_2(3-2)^5$$
$$= 243 - 96 + 3 = 150$$

(c) The number of ways in which n identical things can be distributed into r different groups is

$$^{n+r-1}C_{r-1} \text{ or } ^{n-1}C_{r-1}$$

According, as blank groups are or are not admissible.

Proof

If blank groups are not allowed Any such distribution can be effected as follows: place the n things in a row and put marks of partition in a selection of $(r-1)$ out of the $(n-1)$ spaces between them. This can be done in $^{n-1}C_{r-1}$.

If blank groups are allowed The number of distribution is the same as that of $(n+r)$ things of the same sort into r groups with no blank groups. For such a distribution can be effected thus, put one of the

$(n+r)$ things into each of the r groups and distribute the remaining n things into r groups, blank lots being allowed. Hence, the required number is ${}^{n+r-1}C_{r-1}$.

Aliter The number of distribution of n identical things into r different groups is the coefficient of x^n in $(1+x+x^2+\ldots+\infty)^r$ or in $(x+x^2+x^3+\ldots+\infty)^r$ according as blank groups are or are not allowed. These expressions are respectively equal to $(1-x)^{-r}$ and $x^r(1-x)^{-r}$

Hence, coefficient of x^n in two expressions are ${}^{n+r-1}C_{r-1}$ and ${}^{n-1}C_{r-1}$, respectively.

Example 81. In how many ways 5 identical balls can be distributed into 3 different boxes so that no box remains empty?

Sol. The required number of ways $= {}^{5-1}C_{3-1} = {}^4C_2 = \dfrac{4\cdot 3}{1\cdot 2} = 6$

Aliter Each box must contain atleast one ball, since no box remains empty. Boxes can have balls in the following systems.

Box	I	II	III
Number of balls	1	1	3

Or

Box	I	II	III
Number of balls	1	2	2

Here, balls are identical but boxes are different the number of combinations will be 1 in each systems.

$\therefore$ Required number of ways $= 1\times\dfrac{3!}{2!} + 1\times\dfrac{3!}{2!} = 3+3 = 6$

Example 82. Four boys picked up 30 mangoes. In how many ways can they divide them, if all mangoes be identical?

Sol. Clearly, 30 mangoes can be distributed among 4 boys such that each boy can receive any number of mangoes.

Hence, total number of ways $= {}^{30+4-1}C_{4-1}$

$$= {}^{33}C_3 = \frac{33\cdot 32\cdot 31}{1\cdot 2\cdot 3} = 5456$$

Example 83. Find the positive number of solutions of $x+y+z+w=20$ under the following conditions

(i) Zero value of x, y, z and w are included.

(ii) Zero values are excluded.

Sol. (i) Since, $x+y+z+w=20$

Here, $x\ge 0, y\ge 0, z\ge 0, w\ge 0$

The number of Sols of the given equation in this case is same as the number of ways of distributing 20 things among 4 different groups.

Hence, total number of Sols $= {}^{20+4-1}C_{4-1}$

$$= {}^{23}C_3 = \frac{23\cdot 22\cdot 21}{1\cdot 2\cdot 3} = 1771$$

(ii) Since, $x+y+z+w=20$ ...(i)

Here, $x\ge 1, y\ge 1, z\ge 1, w\ge 1$

or $x-1\ge 0, y-1\ge 0, z-1\ge 0, w-1\ge 0$

Let $x_1 = x-1 \Rightarrow x = x_1+1$

$y_1 = y-1 \Rightarrow y = y_1+1$

$z_1 = z-1 \Rightarrow z = z_1+1$

$w_1 = w-1 \Rightarrow w = w_1+1$

Then, from Eq. (i), we get

$$x_1+1+y_1+1+z_1+1+w_1+1 = 20$$

$\Rightarrow$ $x_1+y_1+z_1+w_1 = 16$

and $x_1\ge 0, y_1\ge 0, z_1\ge 0, w_1\ge 0$

Hence, total number of Solutions $= {}^{16+4-1}C_{4-1}$

$$= {}^{19}C_3 = \frac{19\cdot 18\cdot 17}{1\cdot 2\cdot 3} = 57\cdot 17 = 969$$

Aliter

Part (ii) $\because$ $x+y+z+w=20$

$x\ge 1, y\ge 1, z\ge 1, w\ge 1$

Hence, total number of solutions

$$= {}^{20-1}C_{4-1} = {}^{19}C_3 = 969$$

Example 84. How many integral solutions are there to $x+y+z+t=29$, when $x\ge 1$, $y>1$, $z\ge 3$ and $t\ge 0$?

Sol. Since, $x+y+z+t=29$ ...(i)

and x, y, z, t are integers

$\therefore$ $x\ge 1, y\ge 2, z\ge 3, t\ge 0$

$\Rightarrow$ $x-1\ge 0, y-2\ge 0, z-3\ge 0, t\ge 0$

Let $x_1 = x-1, x_2 = y-2, x_3 = z-3$

or $x = x_1+1, y = x_2+2, z = x_3+3$ and then $x_1\ge 0, x_2\ge 0, x_3\ge 0, t\ge 0$

From Eq. (i), we get

$$x_1+1+x_2+2+x_3+3+t = 29$$

$\Rightarrow$ $x_1+x_2+x_3+t = 23$

Hence, total number of solutions $= {}^{23+4-1}C_{4-1}$

$$= {}^{26}C_3 = \frac{26\cdot 25\cdot 24}{1\cdot 2\cdot 3} = 2600$$

Aliter

$\because$ $x+y+z+t=29$ ...(i)

and $x\ge 1, y-1\ge 1, z-2\ge 1, t+1\ge 1$

Let $x_1 = x, y_1 = y-1, z_1 = z-2, t_1 = t+1$

or $x = x_1, y = y_1+1, z = z_1+2, t = t_1-1$

and then $x_1\ge 1, y_1\ge 1, z_1\ge 1, t_1\ge 1$

From Eq. (i), $x_1+y_1+1+z_1+2+t_1-1 = 29$

$\Rightarrow$ $x_1+y_1+z_1+t_1 = 27$

Hence, total number of solutions $= {}^{27-1}C_{4-1} = {}^{26}C_3$

$$= \frac{26\cdot 25\cdot 24}{1\cdot 2\cdot 3} = 2600$$

Example 85. How many integral Solutions are there to the system of equations $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ and $x_1 + x_2 = 15$, when $x_k \geq 0$?

Sol. We have, $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ ...(i)

and $x_1 + x_2 = 15$ ...(ii)

Then, from Eqs. (i) and (ii), we get two equations

$$x_3 + x_4 + x_5 = 5 \quad \text{...(iii)}$$
$$x_1 + x_2 = 15 \quad \text{...(iv)}$$

and given $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$

Then, number of solutions of Eq. (iii)

$$= {}^{5+3-1}C_{3-1} = {}^7C_2$$
$$= \frac{7 \cdot 6}{1 \cdot 2} = 21$$

and number of solutions of Eq. (iv)

$$= {}^{15+2-1}C_{2-1} = {}^{16}C_1 = 16$$

Hence, total number of solutions of the given system of equations

$$= 21 \times 16 = 336$$

Example 86. Find the number of non-negative integral solutions of $3x + y + z = 24$.

Sol. We have,

$$3x + y + z = 24 \text{ and given } x \geq 0, y \geq 0, z \geq 0$$

Let $x = k$

$\therefore$ $y + z = 24 - 3k$ ...(i)

Here, $24 \geq 24 - 3k \geq 0 [\because x \geq 0]$

Hence, $0 \leq k \leq 8$

The total number of integral solutions of Eq. (i) is

$${}^{24-3k+2-1}C_{2-1} = {}^{25-3k}C_1 = 25 - 3k$$

Hence, the total number of Sols of the original equation

$$= \sum_{k=0}^{8}(25 - 3k) = 25\sum_{k=0}^{8} 1 - 3\sum_{k=0}^{8} k$$
$$= 25 \cdot 9 - 3 \cdot \frac{8 \cdot 9}{2} = 225 - 108 = 117$$

(d) The number of ways in which n identical things can be distributed into r groups so that no group contains less than l things and more than m things $(l < m)$ is coefficient of x^{n-lr} in the expansion of $(1 - x^{m-l+1})^r (1 - x)^{-r}$.

Proof Required number of ways

= Coefficient of x^n in the expansion of

$$(x^l + x^{l+1} + x^{l+2} + \ldots + x^m)^r$$

[$\because$ no group contains less than l things and more than m things, here r groups]

= Coefficient of x^n in the expansion of $x^{lr}(1 + x + x^2 + \ldots + x^{m-l})^r$

= Coefficient of x^{n-lr} in the expansion of $(1 + x + x^2 + \ldots + x^{m-l})^r$

= Coefficient of x^{n-lr} in the expansion of

$$\left(\frac{1 \cdot (1 - x^{m-l+1})}{(1 - x)}\right)^r$$

[sum of $m - l + 1$ terms of GP]

= Coefficient of x^{n-lr} in the expansion of

$$(1 - x^{m-l+1})^r (1 - x)^{-r}$$

Example 87. In how many ways can three persons, each throwing a single dice once, make a sum of 15 ?

Sol. Number on the faces of the dice are 1, 2, 3, 4, 5, 6 (least number 1, greatest number 6)

Here, $l = 1, m = 6, r = 3$ and $n = 15$

$\therefore$ Required number of ways = Coefficient of $x^{15 - 1 \times 3}$ in the expansion of $(1 - x^6)^3 (1 - x)^{-3}$

= Coefficient of x^{12} in the expansion of

$$(1 - 3x^6 + 3x^{12})(1 + {}^3C_1 x + {}^4C_2 x^2 + \ldots + {}^8C_6 x^6 + \ldots + {}^{14}C_{12} x^{12} + \ldots)$$

$$= {}^{14}C_{12} - 3 \times {}^8C_6 + 3 = {}^{14}C_2 - 3 \times {}^8C_2 + 3$$
$$= 91 - 84 + 3 = 10$$

Example 88. ***In how many ways in which an examiner can assign 30 marks to 8 questions, giving not less than 2 marks to any question.***

Sol. If examiner given marks any seven question 2 (each) marks, then marks on remaining questions given by examiner $= -7 \times 2 + 30 = 16$

If x_i are the marks assigned to ith question, then $x_1 + x_2 + x_3 + \ldots + x_8 = 30$ and $2 \leq x_i \leq 16$ for $i = 1, 2, 3, \ldots, 8$.

Here, $l = 2, m = 16, r = 8$ and $n = 30$

$\therefore$ Required number of ways

= Coefficient of $x^{30 - 2 \times 8}$ in the expansion of

$$(1 - x^{16-2+1})^8 (1 - x)^{-8}$$

= Coefficient of x^{14} in the expansion of

$$(1 - x^{15})^8 (1 + {}^8C_1 x + {}^9C_2 x^2 + \ldots + {}^{21}C_{14} x^{14} + \ldots)$$

= Coefficient of x^{14} in the expansion of

$$(1 + {}^8C_1 x + {}^9C_2 x^2 + \ldots + {}^{21}C_{14} x^{14} + \ldots)$$

$$= {}^{21}C_{14} = {}^{21}C_7$$

Note Coefficient of x^r in the expansion of $(1 - x)^{-n}$ is ${}^{n+r-1}C_r$.

(e) If a group has n things in which p are identical, then the number of ways of selecting r things from a group is

$$\sum_{r=0}^{r} {}^{n-p}C_r \text{ or } \sum_{r=r-p}^{r} {}^{n-p}C_r, \text{according as } r \le p \text{ or}$$

$r \ge p$.

Example 89. A bag has contains 23 balls in which 7 are identical. Then, find the number of ways of selecting 12 balls from bag.

Sol. Here, $n = 23,\ p = 7,\ r = 12\ (r > p)$

$$\therefore \text{ Required number of selections} = \sum_{r=5}^{12} {}^{16}C_r$$

$$= {}^{16}C_5 + {}^{16}C_6 + {}^{16}C_7 + {}^{16}C_8 + {}^{16}C_9 + {}^{16}C_{10} + {}^{16}C_{11} + {}^{16}C_{12}$$

$$= ({}^{16}C_5 + {}^{16}C_6) + ({}^{16}C_7 + {}^{16}C_8) + ({}^{16}C_9 + {}^{16}C_{10}) + ({}^{16}C_{11} + {}^{16}C_{12})$$

$$= {}^{17}C_6 + {}^{17}C_8 + {}^{17}C_{10} + {}^{17}C_{12} \qquad [\because {}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r]$$

$$= {}^{17}C_{11} + {}^{17}C_9 + {}^{17}C_{10} + {}^{17}C_{12} \qquad [\because {}^{n}C_r = {}^{n}C_{n-r}]$$

$$= ({}^{17}C_{11} + {}^{17}C_{12}) + ({}^{17}C_9 + {}^{17}C_{10})$$

$$= {}^{18}C_{12} + {}^{18}C_{10} = {}^{18}C_6 + {}^{18}C_8$$

Derangements Any change in the order of the things in a group is called a derangement.

Or

When 'n' things are to be placed at 'n' specific places but none of them is placed on its specified position, then we say that the 'n' things are deranged.

Or

Assume $a_1, a_2, a_3, \ldots, a_n$ be n distinct things such that their positions are fixed in a row. If we now rearrange $a_1, a_2, a_3, \ldots, a_n$ in such a way that no one occupy its original position, then such an arrangement is called a derangement.

Consider 'n' letters and 'n' corresponding envelops. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$$

Proof n letters are denoted by 1, 2, 3, ..., n. Let A_i denote the set of distribution of letters in envelopes (one letter in each envelop) so that the
i th letter is placed in the corresponding envelope, then

$$n(A_i) = 1 \times (n-1)!$$

[$\because$ the remaining $(n-1)$ letters can be placed in $(n-1)$ envelopes is $(n-1)!$]

and $n(A_i \cap A_j) = 1 \times 1 \times (n-2)!$ [$\because$ i and j can be placed in their corresponding envelopes and remaining $(n-2)$ letters can be placed in $(n-2)$ envelopes in $(n-2)!$ way]

Also, $n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$

Hence, the required number is

$$n(A_1' \cap A_2' \cap A_3' \cap \ldots \cap A_n')$$
$$= n(U) - n(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)$$
$$= n! - \Big\{\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) - \ldots + (-1)^n \sum n(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n)$$
$$= n! - \{{}^{n}C_1 \times (n-1)! - {}^{n}C_2 \times (n-2)! + {}^{n}C_3 \times (n-3)! - \ldots + (-1)^{n-1} \times {}^{n}C_n \times 1!\}$$
$$= n! - \left\{\frac{n \times (n-1)!}{1!} - \frac{n(n-1)}{2!} \times (n-2)! + \frac{n(n-1)(n-2)}{3!} \times (n-3)! - \ldots + (-1)^{n-1} \times 1\right\}$$
$$= n! - \left\{\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \ldots + (-1)^n \cdot 1\right\}$$
$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right]$$

Maha Short Cut Method

If $$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right]$$

Then, $\boldsymbol{D_{n+1} = (n+1)D_n + (-1)^{n+1}}$, $\forall\ \boldsymbol{x \in N}$

and $\boldsymbol{D_{n+1} = n\,(D_n + D_{n-1})}$, $\forall\ \boldsymbol{x \in N - \{1\}}$

where $D_1 = 0$

For $n = 1$, from result I

$$D_2 = 2D_1 + (-1)^2 = 0 + 1 = 1$$

For $n = 2$, from result I

$$D_3 = 3D_2 + (-1)^3 = 3 \times 1 - 1 = 2$$

For $n = 3$, from result I

$$D_4 = 4D_3 + (-1)^4 = 4 \times 2 + 1 = 9$$

For $n = 4$, from result I

$$D_5 = 5D_4 + (-1)^5 = 5 \times 9 - 1 = 44$$

For $n = 5$, from result I

$$D_6 = 6D_5 + (-1)^6 = 6 \times 44 + 1 = 265$$

Note $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, $D_4 = 9$, $D_5 = 44$, $D_6 = 265$ [Remember]

Remark

If r things goes to wrong place out of n things, then $(n - r)$ things goes to original place (here $r < n$).

If D_n = Number of ways, if all n things goes to wrong places.

and D_r = Number of ways, if r things goes to wrong places.

If r goes to wrong places out of n, then $(n - r)$ goes to correct places.

Then, $D_n = {}^nC_{n-r}\, D_r$

where, $D_r = r!\left(1 - \frac{1}{1!} + \frac{1}{2!}! - \frac{1}{3!} + \dots + (-1)^r \frac{1}{r!}\right)$

If atleast p things goes to wrong places, then $D_n = \sum_{r=p}^{n} {}^nC_{n-r} \cdot D_r$

Example 90. A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that (i) atleast two of them are in the wrong envelopes. (ii) all the letters are in the wrong envelopes.

Sol. (i) The number of ways in which atleast two of them in the wrong envelopes

$$= \sum_{r=2}^{6} {}^6C_{6-r} \cdot D_r$$

$$= {}^6C_4 \times D_2 + {}^6C_3 \times D_3 + {}^6C_2 \times D_4 + {}^6C_1 \times D_5 + {}^6C_0 \times D_6$$

$$= 15D_2 + 20D_3 + 15D_4 + 6D_5 + D_6 \quad \text{[from note]}$$

$$= 15 \times 1 + 20 \times 2 + 15 \times 9 + 6 \times 44 + 265$$

$$= 719$$

(ii) The number of ways in which all letters be placed in wrong envelopes $= D_6 = 265$ [from note]

Aliter

(i) The number of all the possible ways of putting 6 letters into 6 envelopes is $6!$.

Number of ways to place all letters correctly into corresponding envelopes $= 1$

and number of ways to place one letter in the wrong envelope and other 5 letters in the write envelope $= 0$

[$\because$ It is not possible that only one letter goes in the wrong envelope, when if 5 letters goes in the right envelope, then remaining one letter also goes in the write envelope]

Hence, number of ways to place atleast two letters goes in the wrong envelopes

$$= 6! - 0 - 1 = 6! - 1 = 720 - 1 = 719$$

(ii) The number of ways 1 letter in 1 address envelope, so that one letter is in wrong envelope $= 0$ ...(i)

[because it is not possible that only one letter goes in the wrong envelope]

The number of ways to put 2 letters in 2 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction − The number of ways in which all are in correct envelopes − The number of ways in which 1 letter is in the correct envelope

$$= 2! - 1 - 0 = 2 - 1$$

$$= 1 \quad \text{...(ii) [from Eq. (i)]}$$

The number of ways to put 3 letters in 3 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction − The number of ways in which all are in correct envelopes − The number of ways in which 1 letter is in the correct envelope − The number of ways in which 2 letter are in correct envelope

$$= 3! - 1 - {}^3C_1 \times 1 - 0 \quad \text{[from Eqs. (i) and (ii)]}$$

$$= 2$$

[${}^3C_1$ means that select one envelope to put the letter correctly]

The number of ways to put 4 letters in 4 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction − The number of ways in which all are in correct envelopes − The number of ways in which 1 letter is in the correct envelope − The number of ways in which 2 letters are in correct envelopes − The number of ways in which 3 letters are in correct envelopes

$$= 4! - 1 - {}^4C_1 \times 2 - {}^4C_2 \times 1 - {}^4C_3 \times 0$$

[from Eqs. (i), (ii) and (iii)]

$$= 24 - 1 - 8 - 6 - 0 = 9 \quad \text{...(iv)}$$

The number of ways to put 5 letters in 5 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction − The number of ways in which all are in correct envelopes − The number of ways in which 1 letter is in the correct envelopes − The number of ways in which 2 letters are in correct envelopes − The number of ways in which 3 letters are in correct envelopes − The number of ways in which 4 letters are in correct envelopes

$$= 5! - 1 - {}^5C_1 \times 9 - {}^5C_2 \times 2 - {}^5C_3 \times 1 - {}^5C_4 \times 0$$

[from Eqs. (i), (ii), (iii) and (iv)]

$$= 120 - 1 - 45 - 20 - 10 - 0 = 44$$

The number of ways to put 6 letters in 6 addressed envelopes so that all are in wrong envelopes

= The number of ways without restriction – The number of ways in which all are in correct envelopes – The number of ways in which 1 letter is in the correct envelope – The number of ways in which 2 letters are in correct envelopes – The number of ways in which 3 letters are in correct envelopes – The number of ways in which 4 letters are in correct envelopes – The number of ways in which 5 letters are in correct envelopes.

$= 6! - 1 - {}^6C_1 \times 44 - {}^6C_2 \times 9 - {}^6C_3 \times 2 - {}^6C_4 \times 1 - {}^6C_5 \times 0$

[from Eqs. (i), (ii), (iii), (iv) and (v)]

$= 720 - 1 - 264 - 135 - 40 - 15 = 720 - 455 = 265$

Multinomial Theorem

(i) If there are l objects of one kind, m objects of second kind, n objects of third kind and so on, then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + \dots$) is the coefficient of x^r in the expansion of

$$(1 + x + x^2 + x^3 + \dots + x^l)(1 + x + x^2 + \dots + x^m)(1 + x + x^2 + \dots + x^n)$$

Further, if one object of each kind is to be included, then the number of ways of choosing r objects out of these objects (i.e., $l + m + n + \dots$) is the coefficient of x^r in the expansion of

$$(x + x^2 + x^3 + \dots + x^l)(x + x^2 + x^3 + \dots + x^m)(x + x^2 + x^3 + \dots + x^n)\dots$$

(ii) If there are l objects of one kind, m objects of second kind, n objects of third kind and so on, then the number of possible arrangements/permutations of r objects out of these objects (i.e., $l + m + n + \dots$) is the coefficient of x^r in the expansion of

$$r!\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^l}{l!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^m}{m!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}\right)\dots$$

Different Cases of Multinomial Theorem

***Case* I If upper limit of a variable is more than or equal to the sum required, then the upper limit of that variable can be taken as infinite.**

Example 91. In how many ways the sum of upper faces of four distinct die can be five?

Sol. Here, the number of required ways will be equal to the number of solutions of $x_1 + x_2 + x_3 + x_4 = 5$ i.e., $1 \le x_i \le 6$ for $i = 1, 2, 3, 4$.

Since, upper limit is 6, which is greater than required sum, so upper limit taken as infiite. So, number of Sols is equal to coefficient of α^5 in the expansion of $(1 + \alpha + \alpha^2 + \dots + \infty)^4$

$= $ Coefficient of α^5 in the expansion of $(1 - \alpha)^{-4}$

$= $ Coefficient of α^5 in the expansion of $(1 + {}^4C_1\alpha + {}^5C_2\alpha^2 + \dots)$

$$= {}^8C_5 = {}^8C_3 = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

***Case* II If the upper limit of a variable is less than the sum required and the lower limit of all variables is non-negative, then the upper limit of that variable is that given in the problem.**

Example 92. ***In an examination, the maximum marks each of three papers is* 50 *and the maximum mark for the fourth paper is* 100*. Find the number of ways in which the candidate can score* 60% *marks in aggregate.***

Sol. Aggregate of marks $= 50 \times 3 + 100 = 250$

$$\therefore \; 60\% \text{ of the aggregate} = \frac{60}{100} \times 250 = 150$$

Let the marks scored by the candidate in four papers be x_1, x_2, x_3 and x_4. Here, the number of required ways will be equal to the number of Sols of $x_1 + x_2 + x_3 + x_4 = 150$ i.e., $0 \le x_1, x_2, x_3 \le 50$ and $0 \le x_4 \le 100$.

Since, the upper limit is 100 < required sum (150).

The number of solutions of the equation is equal to coefficient of α^{150} in the expansion of

$$(\alpha^0 + \alpha^1 + \alpha^2 + \dots + \alpha^{50})^3(\alpha^0 + \alpha^1 + \alpha^2 + \dots + \alpha^{100})$$

$= $ Coefficient of α^{150} in the expansion of $(1 - \alpha^{51})^3(1 - \alpha^{10})(1 - \alpha)^{-4}$

$= $ Coefficient of α^{150} in the expansion of $(1 - 3\alpha^{51} + 3\alpha^{102})(1 - \alpha^{101})(1 + {}^4C_1\alpha + {}^5C_2\alpha^2 + \dots + \infty)$

$= $ Coefficient of α^{150} in the expansion of $(1 - 3\alpha^{51} - \alpha^{101} + 3\alpha^{102})(1 + {}^4C_1\alpha + {}^5C_2\alpha^2 + \dots + \infty)$

$= {}^{153}C_{150} - 3 \times {}^{102}C_{99} - {}^{52}C_{49} + 3 \times {}^{51}C_{48}$

$= {}^{153}C_3 - 3 \times {}^{102}C_3 - {}^{52}C_3 + 3 \times {}^{51}C_3$

$= 110556$

Very Important Trick

On multiplying $p_0 + p_1\alpha + p_2\alpha^2 + p_3\alpha^3 + \ldots + p_n\alpha^n$ by $(1+\alpha)$, we get

$$p_0 + (p_0 + p_1)\alpha + (p_1 + p_2)\alpha^2 + (p_2 + p_3)\alpha^3 + \ldots + (p_{n-2} + p_{n-1})\alpha^{n-1} + (p_{n-1} + p_n)\alpha^n + p_n\alpha^{n+1}$$

i.e., we just add coefficient of α^r with coefficient of α^{r-1} (i.e., previous term) to get coefficient α^r in product.

Now, coefficient of $\alpha^r = p_{r-1} + p_r$

On multiplying $p_0 + p_1\alpha + p_2\alpha^2 + p_3\alpha^3 + \ldots + p_n\alpha^n$ by $(1+\alpha+\alpha^2)$

we get, $p_0 + (p_0 + p_1)\alpha + (p_0 + p_1 + p_2)\alpha^2 + (p_1 + p_2 + p_3)\alpha^3 + (p_2 + p_3 + p_4)\alpha^4 + \ldots$

i.e., to find coefficient of α^r in product and add this with 2 preceding coefficients.

Now, coefficient of $\alpha^r = p_{r-2} + p_{r-1} + p_r$

Similarly, in product of $p_0 + p_1\alpha + p_2\alpha^2 + \ldots$ with $(1+\alpha+\alpha^2+\alpha^3)$, the coefficient of α^r in product will be

$$\underbrace{p_{r-3} + p_{r-2} + p_{r-1}}_{3 \text{ preceding coefficients}} + p_r$$

and in product of $p_0 + p_1\alpha + p_2\alpha^2 + \ldots$ with $(1+\alpha+\alpha^2+\alpha^3+\alpha^4)$, the coefficient of α^r in product will be $\underbrace{p_{r-4} + p_{r-3} + p_{r-2} + p_{r-1}}_{4 \text{ preceding coefficients}} + p_r$

Finally, in product of $p_0 + p_1\alpha + p_2\alpha^2 + \ldots$ with $(1+\alpha+\alpha^2+\alpha^3+\ldots+\text{upto}\,\infty)$, the coefficient of α^r in product will be $\underbrace{p_0 + p_1 + p_2 + \ldots + p_{r-1}}_{\text{all preceding coefficients}} + p_r$

Example 93. ***Find the coefficient of*** α^6 ***in the product*** $(1+\alpha+\alpha^2)(1+\alpha+\alpha^2)(1+\alpha+\alpha^2+\alpha^3)(1+\alpha)(1+\alpha)(1+\alpha)$.

Sol. The given product can be written as

$$(1+\alpha+\alpha^2)(1+\alpha+\alpha^2)(1+\alpha+\alpha^2+\alpha^3)(1+\alpha)^3$$

$$\text{or } (1+\alpha+\alpha^2)(1+\alpha+\alpha^2)(1+\alpha+\alpha^2+\alpha^3)(1+3\alpha+3\alpha^2+\alpha^3)$$

Multiplying Synthetically

1	α	α^2	α^3	α^4	α^5	α^6	...
1	3	3	1	0	0	0	

... on multiplying by $1+\alpha+\alpha^2+\alpha^3 \rightarrow$ To each coefficient add 3 preceding coefficients

1	4	7	8	7	4	1	...

...on multiplying by $1+\alpha+\alpha^2 \rightarrow$ To each coefficient add 2 preceding coefficients.

1	5	12	19	22	19	12	...

...on multiplying by $1+\alpha+\alpha^2 \rightarrow$ To each coefficient add 2 preceding coefficients.

...	...	...	...	...	...	53	...

Hence, required coefficient is 53.

Example 94. ***Find the number of different selections of*** 5 ***letters which can be made from*** 5*A****'s,*** 4*B* ***'s,*** 3*C****'s,*** 2*D****'s and*** 1*E*

Sol. All selections of 5 letters are given by 5th degree terms in

$$(1 + A + A^2 + A^3 + A^4 + A^5)(1 + B + B^2 + B^3 + B^4)$$

$$(1 + C + C^2 + C^3)(1 + D + D^2)(1 + E)$$

$\therefore$ Number of 5 letter selections

$= \text{Coefficient of } \alpha^5 \text{ in } (1+\alpha+\alpha^2+\alpha^3+\alpha^4+\alpha^5)(1+\alpha+\alpha^2+\alpha^3+\alpha^4)(1+\alpha+\alpha^2+\alpha^3)(1+\alpha+\alpha^2)(1+\alpha)$

Multiplying synthetically

1	α	α^2	α^3	α^4	α^5 ...
1	1	1	1	1	1
---	---	---	---	---	$\times\, 1 + \alpha + \alpha + \alpha^2 + \alpha^3 + \alpha^4$
1	2	3	4	5	5 ...
---	---	---	---	---	$\times\, 1 + \alpha + \alpha^2 + \alpha^3$
1	3	6	10	14	17 ...
---	---	---	---	---	$\times\, 1 + \alpha + \alpha^2$
1	4	10	19	30	41 ...
---	---	---	---	---	$\times\, 1 + \alpha$
1	5	14	29	49	**71**

Hence, required coefficient is 71.

Example 95. Find the number of combinations and permutations of 4 letters taken from the word EXAMINATION.

Sol. There are 11 letters

A, A, N, N, X, M, T, O.

Then, number of combinations

$= \text{coefficient of } x^4 \text{ in } (1 + x + x^2)^3(1 + x)^5$

[$\because$ 2A's, 2I 's, 2N's, 1E, 1X, 1M, 1T and 1O]

$= \text{Coefficient of } x^4 \text{ in } \{(1 + x)^3 + x^6 + 3(1 + x)^2 x^2$

$$+3(1+x)x^4\}(1+x)^5$$

= Coefficient of x^4 in

$$\{(1+x)^8+x^6(1+x)^5+3x^2(1+x)^7+3x^4(1+x)^6\}$$

$$= {}^8C_4+0+3\cdot{}^7C_2+3=\frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}+3\cdot\frac{7\cdot6}{1\cdot2}+3=70+63+3$$

= 136

and number of permutations

$$= \text{Coefficient of } x^4 \text{ in } 4!\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)^3\left(1+\frac{x}{1!}\right)^5$$

$$= \text{Coefficient of } x^4 \text{ in } 4!\left(1+x+\frac{x^2}{2}\right)^3(1+x)^5$$

= Coefficient of x^4 in

$$4!\left\{(1+x)^3+\frac{x^6}{8}+\frac{3}{2}(1+x)^2x^2+\frac{3}{4}x^4(1+x)\right\}(1+x)^5$$

= Coefficient of x^4 in

$$4!\left\{(1+x)^8+\frac{x^6}{8}(1+x)^5+\frac{3}{2}x^2(1+x)^7+\frac{3}{4}x^4(1+x)^6\right\}$$

$$=4!\left\{{}^8C_4+0+\frac{3}{2}\cdot{}^7C_2+\frac{3}{4}\right\}=24\left\{\frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}+\frac{3}{2}\cdot\frac{7\cdot6}{1\cdot2}+\frac{3}{4}\right\}$$

$$=8\cdot7\cdot6\cdot5+6(3\cdot7\cdot6)+6\cdot3=1680+756+18=2454$$

Aliter There are 11 letters:

A, A, I, I, N, N, E, X, M, T, O

The following cases arise:

***Case* I All letters different** The required number of choosing 4 different letters from 8 different (A, I, N, E, X, M, T, O) types of the letters

$$= {}^8C_4=\frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}=70$$

and number of permutations = ${}^8P_4=8\cdot7\cdot6\cdot5=1680$

***Case* II Two alike of one type and two alike of another type** This must be 2A's, 2I's or 2I's, 2N's, or 2N's, 2A's.

∴ Number of selections = ${}^3C_2=3$

For example, [for arrangements]

A	A	I	I

and number of permutations $=3\cdot\frac{4!}{2!2!}=18$

***Case* III Two alike and two different** This must be 2A's or 2I's or 2N's

and for each case 7 different letters.

For example, for 2A's, 7 differents's are I, N, E, X, M, T, O

For example, [for arrangements]

A	X	A	O

∴ Number of selections = ${}^3C_1\times{}^7C_2=3\times\frac{7\times6}{1\times2}=63$

and number of permutations $=63\cdot\frac{4!}{2!}=756$

From Case I, II and III

The required number of combinations $=70+3+63=136$

and number of permutations $=1680+18+756=2454$

Note Number of combinations and permutations of 4 letters taken from the word MATHEMATICS are 136 and 2454 respectively, as like of EXAMINATION.

Number of Solutions with the Help of Multinomial Theorem

***Case* I** If the equation

$$\alpha+2\beta+3\gamma+\ldots+q\theta=n \qquad \ldots\text{(i)}$$

(a) If zero included, the number of solution of Eq. (i)

$$= \text{Coefficient of } x^n \text{ in } (1+x+x^2+\ldots)(1+x^2+x^4+\ldots)(1+x^3+x^6+\ldots)\ldots(1+x^q+x^{2q}+\ldots)$$

$$= \text{Coefficient of } x^n \text{ in } (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\ldots(1-x^q)^{-1}$$

(b) If zero excluded, then the number of solutions of Eq. (i)

$$= \text{Coefficient of } x^n \text{ in } (x+x^2+x^3+\ldots)(x^2+x^4+x^6+\ldots)(x^3+x^6+x^9+\ldots)\ldots(x^q+x^{2q}+\ldots)$$

$$= \text{Coefficient of } x^n \text{ in } x^{1+2+3+\ldots+q}(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\ldots(1-x^q)^{-1}$$

$$= \text{Coefficient of } x^{n-\frac{q(q+1)}{2}} \text{ in } (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\ldots(1-x^q)^{-1}$$

Example 96. Find the number of non-negative integral solutions of $x_1+x_2+x_3+4x_4=20$.

Sol. Number of non-negative integral solutions of the given equation

$$= \text{Coefficient of } x^{20} \text{ in } (1-x)^{-1}(1-x)^{-1}(1-x)^{-1}(1-x^4)^{-1}$$

$$= \text{Coefficient of } x^{20} \text{ in } (1-x)^{-3}(1-x^4)^{-1}$$

$$= \text{Coefficient of } x^{20} \text{ in } (1+{}^3C_1+{}^4C_2x^2+{}^5C_3x^3+{}^6C_4x^4+\ldots+{}^{10}C_8x^8+\ldots+{}^{14}C_{12}x^{12}+\ldots+{}^{18}C_{16}x^{16}+\ldots+{}^{22}C_{20}x^{20}+\ldots)(1+x^4+x^8+x^{12}+x^{16}+x^{20}+\ldots)$$

$$=1+{}^6C_4+{}^{10}C_8+{}^{14}C_{12}+{}^{18}C_{16}+{}^{22}C_{20}$$

$$=1+{}^6C_2+{}^{10}C_2+{}^{14}C_2+{}^{18}C_2+{}^{22}C_2$$

$$= 1 + \left(\frac{6\cdot 5}{1\cdot 2}\right) + \left(\frac{10\cdot 9}{1\cdot 2}\right) + \left(\frac{14\cdot 13}{1\cdot 2}\right) + \left(\frac{18\cdot 17}{1\cdot 2}\right) + \left(\frac{22\cdot 21}{1\cdot 2}\right)$$

$$= 1 + 15 + 45 + 91 + 153 + 231 = 536$$

Example 97. Find the number of positive unequal integral solutions of the equation $x + y + z + w = 20$.

Sol. We have, $x + y + z + w = 20$ ...(i)

Assume $x < y < z < w$. Here, $x, y, z, w \geq 1$

Now, let $x = x_1, y - x = x_2, z - y = x_3$ and $w - z = x_4$

$\therefore$ $x = x_1, y = x_1 + x_2, z = x_1 + x_2 + x_3$ and $w = x_1 + x_2 + x_3 + x_4$

From Eq. (i), $4x_1 + 3x_2 + 2x_3 + x_4 = 20$

Then, $x_1, x_2, x_3, x_4 \geq 1$

$\because$ $4x_1 + 3x_2 + 2x_3 + x_4 = 20$ ...(ii)

$\therefore$ Number of positive integral solutions of Eq. (ii)

= Coefficient of x^{20-10} in

$$(1 - x^4)^{-1}(1 - x^3)^{-1}(1 - x^2)^{-1}(1 - x)^{-1}$$

= Coefficient of x^{10} in

$$(1 - x^4)^{-1}(1 - x^3)^{-1}(1 - x^2)^{-1}(1 - x)^{-1}$$

= Coefficient of x^{10} in $(1 + x^4 + x^8 + x^{12} + \dots)$

$$\times (1 + x^3 + x^6 + x^9 + x^{12} + \dots) \times (1 + x^2 + x^4 + x^6 + x^8 + x^{10} + \dots) \times (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + \dots)$$

= Coefficient of x^{10} in

$$(1 + x^3 + x^6 + x^9 + x^4 + x^7 + x^{10} + x^8) \times (1 + x^2 + x^4 + x^6 + x^8 + x^{10})(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})$$

[neglecting higher powers]

= Coefficient of x^{10} in

$$(1 + x^2 + x^4 + x^6 + x^8 + x^{10} + x^3 + x^5 + x^7 + x^9 + x^6 + x^8 + x^{10} + x^9 + x^4 + x^6 + x^8 + x^{10} + x^7 + x^9 + x^{10} + x^8 + x^{10})(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})$$

[neglecting higher powers]

$$= 1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1 = 23$$

But x, y, z and w can be arranged in ${}^4P_4 = 4! = 24$

Hence, required number of Sols = (23)(24) = 552

Example 98. In how many ways can 15 identical blankets be distribted among six beggars such that everyone gets atleast one blanket and two particular beggars get equal blankets and another three particular beggars get equal blankets.

Sol. The number of ways of distributing blankets is equal to the number of solutions of the equation $3x + 2y + z = 15$, where $x \geq 1, y \geq 1, z \geq 1$ which is equal to coefficient of α^{15} in

$$(\alpha^3 + \alpha^6 + \alpha^9 + \alpha^{12} + \alpha^{15} + \dots) \times (\alpha^2 + \alpha^4 + \alpha^6 + \alpha^8 + \alpha^{10} + \alpha^{12} + \alpha^{14} + \dots) \times (\alpha + \alpha^2 + \alpha^3 + \dots + \alpha^{15} + \dots)$$

= Coefficient of α^9 in $(1 + \alpha^3 + \alpha^6 + \alpha^9)$

$$\times (1 + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^8) \times (1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^9)$$

[neglecting higher powers]

= Coefficient of α^9 in $(1 + \alpha^2 + \alpha^4 + \alpha^6 + \alpha^8 + \alpha^3 + \alpha^5 + \alpha^7 + \alpha^9 + \alpha^6 + \alpha^8 + \alpha^9) \times (1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^9)$

[neglecting higher powers]

$$= 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 = 12$$

***Case* II** If the inequation

$$x_1 + x_2 + x_3 + \dots + x_m \leq n \quad \text{...(i)}$$

[when the required sum is not fixed]

In this case, we introduce a dummy variable x_{m+1}.

So that,

$$x_1 + x_2 + x_3 + \dots + x_m + x_{m+1} = n,$$

$$x_{m+1} \geq 0 \quad \text{...(ii)}$$

Here, the number of Sols of Eqs. (i) and (ii) will be same.

Example 99. Find the number of positive integral solutions of the inequation $3x + y + z \leq 30$.

Sol. Let dummy variable w, then

$$3x + y + z + w = 30, w \geq 0 \quad \text{...(i)}$$

Now, let $a = x - 1, b = y - 1, c = z - 1, d = w$, then

$$3a + b + c + d = 25, \text{ where } a, b, c, d \geq 0 \quad \text{...(ii)}$$

$\therefore$ Number of positive integral solutions of Eq. (i)

= Number of non-negative integral solutions of Eq. (ii)

= Coefficient of α^{25} in $(1 + \alpha^3 + \alpha^6 + \dots)(1 + \alpha + \alpha^2 + \dots)^3$

= Coefficient of α^{25} in $(1 + \alpha^3 + \alpha^6 + \dots)(1 - \alpha)^{-3}$

= Coefficient of α^{25} in

$$(1 + \alpha^3 + \alpha^6 + \dots)(1 + {}^3C_1\alpha + {}^4C_2\alpha^2 + \dots)$$

$$= {}^{27}C_{25} + {}^{24}C_{22} + {}^{21}C_{19} + {}^{18}C_{16} + {}^{15}C_{13} + {}^{12}C_{10} + {}^9C_7 + {}^6C_4 + {}^3C_1$$

$$= {}^{27}C_2 + {}^{24}C_2 + {}^{21}C_2 + {}^{18}C_2 + {}^{15}C_2 + {}^{12}C_2 + {}^9C_2 + {}^6C_2 + {}^3C_1$$

$$= 351 + 276 + 210 + 153 + 105 + 66 + 36 + 15 + 3 = 1215$$

Aliter

From Eq. (ii), $3a + b + c + d = 25$, where $a, b, c, d \geq 0$

Clearly, $0 \leq a \leq 8$, if $a = k$, then

$$b + c + d = 25 - 3k \quad \text{...(iii)}$$

Hence, number of non-negative integral solutions of Eq. (iii) is

$$^{25-3k+3-1}C_{3-1} = {}^{27-3k}C_2 = \frac{(27-3k)(26-3k)}{2}$$

$$= \frac{3}{2}(3k^2 - 53k + 234)$$

Therefore, required number is

$$\frac{3}{2}\sum_{k=0}^{8}(3k^2 - 53k + 234)$$

$$= \frac{3}{2}\left[3\cdot\left(\frac{8\times 9\times 17}{6}\right) - 53\cdot\left(\frac{8\times 9}{2}\right) + 234\times 9\right] = 1215$$

Example 100. In how many ways can we get a sum of atmost 15 by throwing six distinct dice ?

Sol. Let x_1, x_2, x_3, x_4, x_5 and x_6 be the number that appears on the six dice.

The number of ways = Number of solutions of the inequation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 15$$

Introducing a dummy variable $x_7 (x_7 \geq 0)$, the inequation becomes an equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 15$$

Here, $1 \leq x_i \leq 6$ for $i = 1, 2, 3, 4, 5, 6$ and $x_7 \geq 0$.

Therefore, number of solutions

$= \text{Coefficient of } x^{15} \text{ in } (x + x^2 + x^3 + x^4 + x^5 + x^6)^6 \times (1 + x + x^2 + \dots)$

$= \text{Coefficient of } x^9 \text{ in } (1 - x^6)^6 (1 - x)^{-7}$

$= \text{Coefficient of } x^9 \text{ in } (1 - 6x^6)(1 + {}^7C_1 x + {}^8C_2 x^2 + \dots)$ [neglecting higher powers]

$= {}^{15}C_9 - 6 \times {}^9C_3 = {}^{15}C_6 - 6 \times {}^9C_3$

$= 5005 - 504 = 4501$

***Case* III** If the inequation

$$x_1 + x_2 + x_3 + \dots + x_n \geq n$$

[when the values of $x_1, x_2, \dots, x_n$ are restricted]

In this case first find the number of solutions of $x_1 + x_2 + x_3 + \dots + x_n \leq n - 1$ and then subtract it from the total number of solutions.

Example 101. In how many ways can we get a sum greater than 15 by throwing six distinct dice?

Sol. Let x_1, x_2, x_3, x_4, x_5 and x_6 be the number that appears on the six dice.

The number of ways = Number of solutions of the inequation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 > 15$$

Here, $1 \leq x_i \leq 6$, $i = 1, 2, 3, 4, 5, 6$

Total number of cases $= 6^6 = 2^6 \times 3^6 = 64 \times 729 = 46656$

and number of ways to get the sum less than or equal to 15, which is 4501 [from Example 100]

Hence, the number of ways to get a sum greater than 15 is $46656 - 4501 = 42155$

***Case* IV** If the equation

$$x_1 x_2 x_3 \dots x_n = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \dots$$

where $\alpha_1, \alpha_2, \alpha_3, \dots$ are natural numbers.

In this case number of positive integral solutions $(x_1, x_2, x_3, \dots, x_n)$ are

$$({}^{\alpha_1 + n - 1}C_{n-1})({}^{\alpha_2 + n - 1}C_{n-1})({}^{\alpha_3 + n - 1}C_{n-1}) \dots$$

Example 102. Find the total number of positive integral solutions for (x, y, z) such that $xyz = 24$.

Sol. $\because \; xyz = 24 = 2^3 \times 3^1$

Hence, total number of positive integral solutions

$$= ({}^{3+3-1}C_{3-1})({}^{1+3-1}C_{3-1})$$

$$= {}^5C_2 \times {}^3C_2 = 30$$

Aliter

$\because \quad xyz = 24 = 2^3 \times 3^1$

Now, consider three boxes x, y, z.

3 can be put in any of the three boxes.

Also, 2, 2, 2 can be distributed in the three boxes in ${}^{3+3-1}C_{3-1} = {}^5C_2 = 10$ ways. Hence, the total number of positive integral solutions = the number of distributions which is given by $3 \times 10 = 30$.

Geometrical Problems

(a) If there are n points in a plane out of these points no three are in the same line except m points which are collinear, then

(i) Total number of different lines obtained by joining these n points is ${}^nC_2 - {}^mC_2 + 1$

(ii) Total number of different triangles formed by joining these n points is ${}^nC_3 - {}^mC_3$

(iii) Total number of different quadrilateral formed by joining these n points is

$$^nC_4 - ({}^mC_3 \cdot {}^nC_1 + {}^mC_4 \cdot {}^nC_0)$$

Example 103. There are 10 points in a plane out of these points no three are in the same straight line except 4 points which are collinear. How many

(i) straight lines (ii) trian-gles

(iii) quadrilateral, by joining them?

Sol. (i) Required number of straight lines

$= {}^{10}C_2 - {}^4C_2 + 1 = \dfrac{10 \cdot 9}{1 \cdot 2} - \dfrac{4 \cdot 3}{1 \cdot 2} + 1 = 45 - 6 + 1 = 40$

(ii) Required number of triangles

$= {}^{10}C_3 - {}^4C_3 = \dfrac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} - {}^4C_1 = 120 - 4 = 116$

(iii) Required number of quadrilaterals

$= {}^{10}C_4 - ({}^4C_3 \cdot {}^6C_1 + {}^4C_4 \cdot {}^6C_0)$

$= \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} - ({}^4C_1 \cdot {}^6C_1 + 1.1)$

$= 210 - (4 \times 6 + 1) = 210 - 25 = 185$

(b) If there are n points in a plane out of these points no any three are collinear, then

(i) Total points of intersection of the lines joining these n points $= {}^pC_2$, where $p = {}^nC_2$

(ii) If n points are the vertices of a polygon, then total number of diagonals $= {}^nC_2 - n = \dfrac{n(n-3)}{2}$

Example 104. How many number of points of intersection of n straight lines, if n satisfies ${}^{n+5}P_{n+1} = \dfrac{11(n-1)}{2} \times {}^{n+3}P_n$?

Sol. We have, ${}^{n+5}P_{n+1} = \dfrac{11(n-1)}{2} \times {}^{n+3}P_n$

$\Rightarrow \quad \dfrac{(n+5)!}{4!} = \dfrac{11(n-1)}{2} \times \dfrac{(n+3)!}{3!}$

$\Rightarrow \quad \dfrac{(n+5)(n+4)}{4} = \dfrac{11(n-1)}{2}$

$\Rightarrow \quad n^2 - 13n + 42 = 0 \Rightarrow (n-6)(n-7) = 0$

$\Rightarrow \quad n = 6$ or $n = 7$

The number of points of intersection of lines is ${}^6C_2$ or ${}^7C$

$= 15$ or 21

Example 105. The interior angles of a regular polygon measure 150° each. Then, find the number of diagonals of the polygon.

Sol. Each exterior angle = 30°

$\therefore$ Number of sides $= \dfrac{360^\circ}{30^\circ} = \dfrac{360 \times \frac{\pi}{180}}{30 \times \frac{\pi}{180}} = 12$

$\therefore$ Number of diagonals $= {}^{12}C_2 - 12 = 66 - 12 = 54$

Example 106. In a polygon the number of diagonals is 77. Find the number of sides of the polygon.

Sol. Let number of sides of the polygon $= n$, then ${}^nC_2 - n = 77$

$\Rightarrow \quad \dfrac{n(n-1)}{2} - n = 77 \Rightarrow \dfrac{n(n-3)}{2} = \dfrac{14 \times 11}{2}$

we get, $n = 14$

(c) n straight lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. Then, number of parts into which these lines divides the plane is equal to

$$1 + \sum_{k=1}^{n} k, \text{ .e. } \frac{(n^2 + n + 2)}{2}$$

Example 107. If n lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent, such that these lines divide the plane in 67 parts, then find number of different points at which these lines will cut.

Sol. Given number of straight lines $= n$, then

$$1 + \sum_{k=1}^{n} k = 67 \Rightarrow \frac{n^2 + n + 2}{2} = 67$$

$\Rightarrow \quad n^2 + n - 132 = 0 \Rightarrow (n+12)(n-11) = 0$

$\therefore \quad n = 11, n \neq -12$

Hence, required number of points $= {}^nC_2 = {}^{11}C_2 = \dfrac{11 \cdot 10}{2}$

$= 55$

(d) If m parallel lines in a plane are intersected by a family of other n parallel lines. Then, total number of parallelograms so formed

$$= {}^mC_2 \cdot {}^nC_2 \text{ i.e., } \frac{mn(m-1)(n-1)}{4}$$

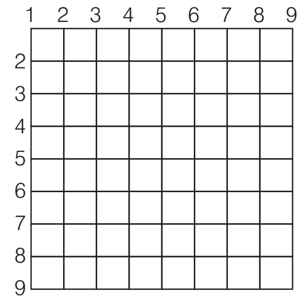
Example 108. Find number of rectangles in a chess board, which are not a square.

Sol. Number of rectangles $= {}^9C_2 \times {}^9C_2 = (36)^2 = 1296$

Number of squares $= 8 \times 8 + 7 \times 7 + 6 \times 6 + \ldots + 1 \times 1$

$= 204$

$\therefore$ Required number $= 1296 - 204 = 1092$



Square can be formed as follows :

To form the smallest square, select any two consecutive lines from the given (here 9) vertical and horizontal lines. This can be done in 8×8 ways (1-2, 2-3, 3-4, ..., 8-9)

Again, to form the square consists of four small squares, select the lines as follows (1-3, 2-4, 3-5,..., 7-9) from both vertical and horizontal lines, thus 7×7 squares are obtained. Proceed in the same way)

Note If n parallel lines are intersected by another n parallel lines, then number of rhombus $= \sum (n-1)^2 = \dfrac{(n-1)\,n\,(2n-1)}{6}$

(e) Number of Rectangles and Squares

(i) Number of rectangles of any size in a square of $n \times n$ is $\sum_{r=1}^{n} r^3$ and number of squares of any size is $\sum_{r=1}^{n} r^2$.

(ii) In a rectangle of $n \times p\ (n < p)$ number of rectangles of any size is $\dfrac{np}{4}(n+1)(p+1)$ and number of squares of any size is

$$\sum_{r=1}^{n} (n+1-r)(p+1-r).$$

Example 109. Find the number of rectangles excluding squares from a rectangle of size 9×6.

Sol. Here, $n = 6$ and $p = 9$

$\therefore$ Number of rectangles excluding square

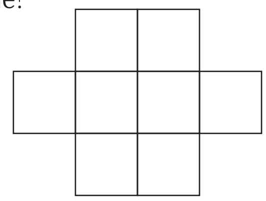
$$= \frac{6 \cdot 9}{4}(6+1)(9+1) - \sum_{r=1}^{6}(7-r)(10-r)$$

$$= 945 - \sum_{r=1}^{6}(70 - 17r + r^2) = 945 - 154 = 791$$

(f) If there are n rows, first row has α_1 squares, 2nd row has α_2 squares, 3rd row has α_3 squares, ... and nth row has α_n squares. If we have to filled up the squares with $\beta\ X_s$ such that each row has atleast one X. The number of ways $=$ Coefficient of x^{β} in

$$({}^{\alpha_1}C_1 x + {}^{\alpha_2}C_2 x^2 + \ldots + {}^{\alpha_1}C_{\alpha_1} x^{\alpha_1})$$
$$\times ({}^{\alpha_2}C_1 x + {}^{\alpha_2}C_2 x^2 + \ldots + {}^{\alpha_2}C_{\alpha_2} x^{\alpha_2})$$
$$\times ({}^{\alpha_3}C_1 x + {}^{\alpha_3}C_2 x^2 + \ldots + {}^{\alpha_3}C_{\alpha_3} x^{\alpha_3}) \times$$
$$\ldots \times ({}^{\alpha_n}C_1 x + {}^{\alpha_n}C_2 x^2 + \ldots + {}^{\alpha_n}C_{\alpha_n} x^{\alpha_n})$$

Example 110. Six X's have to be placed in the squares of the figure below, such that each row contains atleast one X. In how many different ways can this be done?



Sol. The required number of ways

$$= \text{Coefficient of } x^6 \text{ in } ({}^2C_1 x + {}^2C_2 x^2)({}^4C_1 x + {}^4C_2 x^2 + {}^4C_3 x^3 + {}^4C_4 x^4)({}^2C_1 x + {}^2C_2 x^2)$$

$= \text{Coefficient of } x^3 \text{ in } (2+x)^2 (4 + 6x + 4x^2 + x^3)$

$= \text{Coefficient of } x^3 \text{ in } (4 + 4x + x^2)(4 + 6x + 4x^2 + x^3)$

$= 4 + 16 + 6$

$= 26$

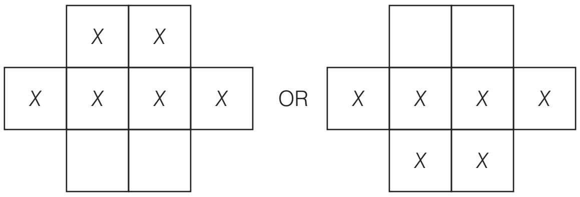
Aliter

In the given figure there are 8 squares and we have to place $6X$'s this can be done in

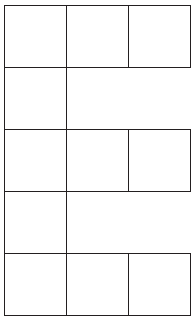
$${}^8C_6 = {}^8C_2 = \frac{8 \cdot 7}{1 \cdot 2} = 28 \text{ ways}$$

But these include the possibility that either headed row or lowest row may not have any X. These two possibilities are to be excluded.

$\therefore$ Required number of ways $= 28 - 2 = 26$



Example 111. In how many ways the letters of the word DIPESH can be placed in the squares of the adjoining figure so that no row remains empty?



Sol. If all letters are same, then number of ways

$= \text{Coefficient of } x^6 \text{ in } ({}^3C_1 x + {}^3C_2 x^2 + {}^3C_3 x^3)^3 ({}^1C_1 x)^2$

$= \text{Coefficient of } x \text{ in } (3 + 3x + x^2)^3$

$= \text{Coefficient of } x \text{ in } (3 + 3x)^3$

[neglecting higher degree term]

$= 27 \times {}^3C_1 = 81$

But in DIPESH all letters are different.

$\therefore$ Required number of ways $= 81 \times 6!$

Exercise for Session 6

1. If number of ways in which 7 different balls can be distributed into 4 different boxes, so that no box remains empty is 100 λ, the value of λ is

(a) 18 (b) 108 (c) 1008 (d) 10008

2. If number of ways in which 7 different balls can be distributed into 4 boxes, so that no box remains empty is 48 λ, the value of λ is

(a) 231 (b) 331 (c) 431 (d) 531

3. If number of ways in which 7 identical balls can be distributed into 4 boxes, so that no box remains empty is 4λ, the value of λ is

(a) 5 (b) 7 (c) 9 (d) 11

4. Number of non-negative integral solutions of the equation $a + b + c = 6$ is

(a) 28 (b) 32 (c) 36 (d) 56

5. Number of integral solutions of $a + b + c = 0$, $a \geq -5$, $b \geq -5$ and $c \geq -5$, is

(a) 272 (b) 136 (c) 240 (d) 120

6. If a, b and c are integers and $a \geq 1$, $b \geq 2$ and $c \geq 3$. If $a + b + c = 15$, the number of possible solutions of the equation is

(a) 55 (b) 66 (c) 45 (d) None of these

7. Number of integral solutions of $2x + y + z = 10$ $(x \geq 0,\ y \geq 0,\ Z \geq 0)$ is

(a) 18 (b) 27 (c) 36 (d) 51

8. A person writes letters to six friends and addresses the corresponding envelopes. Let x be the number of ways so that atleast two of the letters are in wrong envelopes and y be the number of ways so that all the letters are in wrong envelopes. Then, $x - y$ is equal to

(a) 719 (b) 265 (c) 454 (d) None of these

9. A person goes for an examination in which there are four papers with a maximum of m marks from each paper. The number of ways in which one can get $2m$ marks, is

(a) ${}^{2m+3}C_3$ (b) $\left(\frac{1}{3}\right)(m+1)(2m^2+4m+1)$

(c) $\left(\frac{1}{3}\right)(m+1)(2m^2+4m+3)$ (d) None of these

10. The number of selections of four letters from the letters of the word ASSASSINATION, is

(a) 72 (b) 71 (c) 66 (d) 52

11. The number of positive integral solutions of $2x_1 + 3x_2 + 4x_3 + 5x_4 = 25$, is

(a) 20 (b) 22 (c) 23 (d) None of these

12. If a, b, and c are positive integers such that $a + b + c \leq 8$, the number of possible values of the ordered triplet (a, b, c) is

(a) 84 (b) 56 (c) 83 (d) None of these

13. The total number of positive integral solutions of $15 < x_1 + x_2 + x_3 \leq 20$ is equal to

(a) 685 (b) 785 (c) 1125 (d) None of these

14. The total number of integral solutions for (x, y, z) such that $xyz = 24$, is

(a) 36 (b) 90 (c) 120 (d) None of these

15. There are 12 points in a plane in which 6 are collinear. Number of different straight lines that can be drawn by joining them, is

(a) 51 (b) 52 (c) 132 (d) 18

16. 4 points out of 11 points in a plane are collinear. Number of different triangles that can be drawn by joining them, is

(a) 165 (b) 161 (c) 152 (d) 159

17. The number of triangles that can be formed with 10 points as vertices, n of them being collinear, is 110. Then, n is

(a) 3 (b) 4 (c) 5 (d) 6

18. $ABCD$ is a convex quadrilateral. 3, 4, 5 and 6 points are marked on the sides AB, BC, CD and DA, respectively. The number of triangles with vertices on different sides, is

(a) 270 (b) 220 (c) 282 (d) None of these

19. There are 10 points in a plane of which no three points are collinear and 4 points are concyclic. The number of different circles that can be drawn through atleast 3 points of these points, is

(a) 116 (b) 120 (c) 117 (d) None of these

20. 4 points out of 8 points in a plane are collinear. Number of different quadrilateral that can be formed by joining them, is

(a) 56 (b) 60 (c) 76 (d) 53

21. There are $2n$ points in a plane in which m are collinear $(n > m > 4)$. Number of quadrilateral formed by joining these lines

(a) is equal to ${}^{2n}C_4 - {}^{m}C_4$ (b) is greater than ${}^{2n}C_4 - {}^{m}C_4$

(c) is less than ${}^{2n}C_4 - {}^{m}C_4$ (d) None of these

22. In a polygon the number of diagonals is 54. The number of sides of the polygon, is

(a) 10 (b) 12 (c) 9 (d) None of these

23. In a polygon no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon be 70, then the number of diagonals of the polygon, is

(a) 20 (b) 28 (c) 8 (d) None of these

24. n lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. The number of different points at which these lines will cut, is

(a) $\sum_{k=1}^{n-1} k$ (b) $n(n-1)$ (c) n^2 (d) None of these

25. Six straight lines are drawn in a plane such that no two lines are parallel and no three lines are concurrent. Then, the number of parts into which these lines divide the plane, is

(a) 15 (b) 22 (c) 29 (d) 36

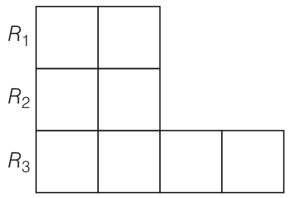
26. A parallelogram is cut by two sets of m lines parallel to its sides. The number of parallelogram thus formed, is

(a) $({}^{m}C_2)^2$ (b) $({}^{m+1}C_2)^2$ (c) $({}^{m+2}C_2)^2$ (d) None of these

27. The number of rectangles excluding squares from a rectangle of size 11×8 is 48λ, then the value of λ is

(a) 13 (b) 23 (c) 43 (d) 53

28. The number of ways the letters of the word PERSON can be placed in the squares of the figure shown so that no row remains empty, is



(a) $24 \times 6\,!$ (b) $26 \times 6\,!$ (c) $26 \times 7\,!$ (d) $27 \times 7\,!$

Answers

Exercise for Session 6

1. (c)	*2.* (c)	**3.** (a)	**4.** (a)	**5.** (b)	*6.* (a)
7. (c)	**8.** (c)	**9.** (c)	**10.** (a)	**11.** (d)	**12.** (b)
13. (a)	**14.** (c)	**15.** (b)	**16.** (b)	**17.** (c)	**18.** (d)
19. (c)	**20.** (d)	**21.** (c)	**22.** (b)	**23.** (a)	**24.** (a)
25. (b)	**26.** (c)	**27.** (c)	**28.** (b)		