11 THREE-DIMENSIONAL GEOMETRY

Objectives

After studying the material of this chapter, you should be able to :

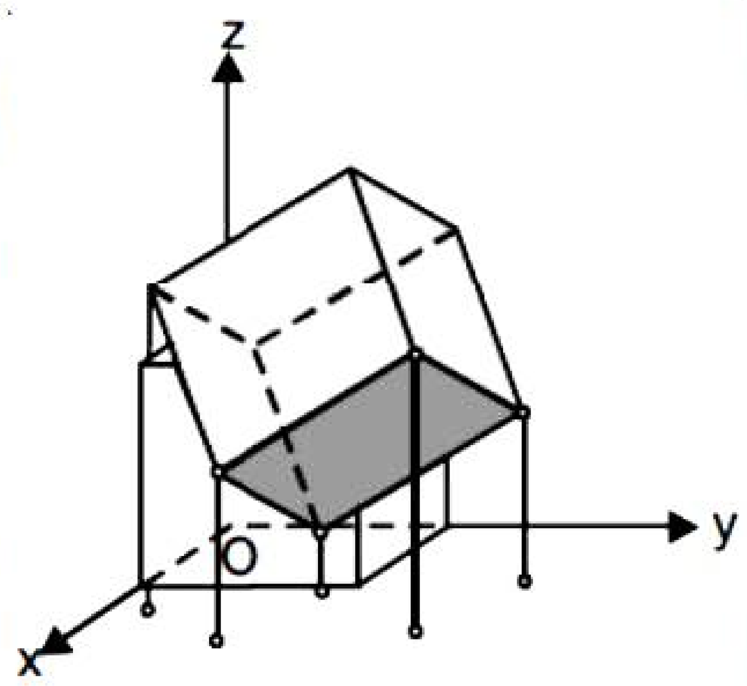
- Understand the direction -ratios and direction cosines of a line.
- Understand to find the equations of lines and planes in space under various conditions.
- Understand to find the angle between two lines, two planes and a line & a plane.
- Understand to find the shortest distance between two skew lines.
- Understand to find the distance of a point from a plane.

He who seeks for methods without having a definite problem in mind seeks for the most part in vain

—*D. Hilbert*

INTRODUCTION

In the previous class, we have already studied plane geometry, known as analytic geometry in 2-dimensions. Now we extend our scope of analytic geometry to 3-dimensions. In the case of plane analytic geometry we confined ourselves to co-ordinate methods. In the same class we had elementary idea regarding 3-dimensional geometry in Cartesian form. Now we shall see that the study of 3-dimensional geometry becomes very simple with the help of vectors. We shall obtain most of the results in vector form by using the techniques of vector algebra. Nevertheless, we shall also translate them in the Cartesian form, which presents a better geometric and analytic picture in many situations.



In this chapter, we will learn following concept :

- Direction cosines and direction ratios of a line joining two points.
- About the equations of lines and planes in space under different conditions
- Angle between two lines, two planes, a line and a plane.
- Shortest distance between two skew lines.
- Distance of a point from a plane.

SUB CHAPTER 11.0 Review

11.1. THREE DIMENSIONAL CARTESIAN FORM

Let us take three axes in such a way that they form a *right–handed* system. This means if a screw, placed at the origin, is turned in the sense from positive *x-axis* to positive *y*-axis, it moves in the direction of positive *z-axis. The three perpendicular co-ordinate axes define the co-ordinate planes.*

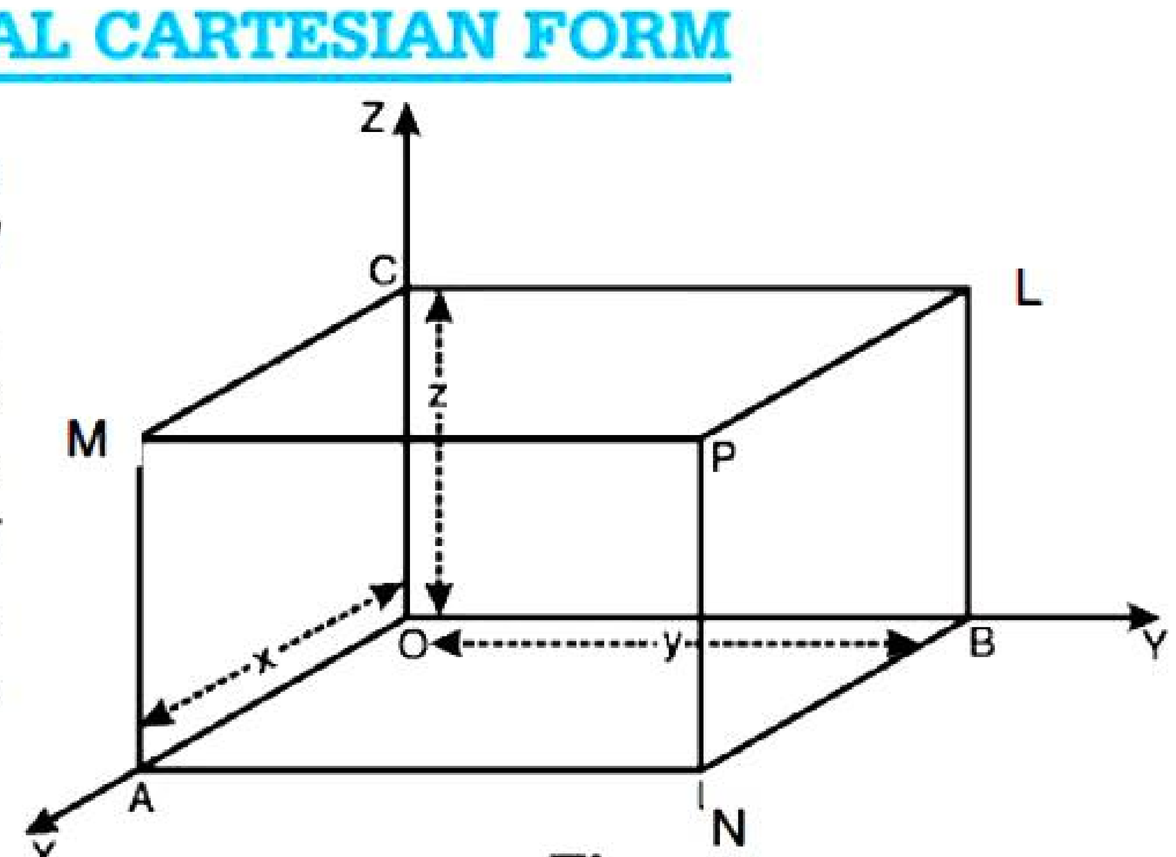


Fig.

Chapter at Glance

The plane, passing through OX and OY, is called **XY-plane** (or XOY plane or Z-plane).

Similarly, YZ- and ZX-planes.

It is obvious that these co-ordinate planes are mutually perpendicular.

Let us associate any point P with the co-ordinates (x, y, z) in the following way :

Thro' P, draw a perpendicular, meeting XY-plane in N. We take PN = z. *From N, draw perpendiculars on the X-axis and Y-axis meeting them in A and B respectively. Take NB = x and NA = y.*

Alternatively, thro' P, draw perps. on the X, Y, Z-axes meeting them in A, B, C respectively. Then OA = x, OB = y and OC = z. *The co-ordinates x, y, z* are taken as +ve or – ve according as the respective points are on the + ve or – ve side of the corresponding co-ordinate axes.

Now we have associated an ordered triplet (x, y, z) with any point P in space in a unique way. Thus there exists one–one correspondence between the points in space and the ordered triplets of real numbers.

11.1.1. DEFINITIONS

With the help of cartesian co–ordinates, we have a better picture with regard to fundamental concepts.

(*a*) Rectangular Axes.

Let X′OX, Y′OY and Z′OZ be three mutually perpendicular straight lines.

(I) The common point O is called the *origin.*

(II) X′OX is called the ***X-axis*** (or axis of X)

(III) Y′OY is called the ***Y-axis*** (or axis of Y)

(IV) Z′OZ is called the ***Z-axis*** (or axis of Z).

These three, taken together, are called *co-ordinate axes* (or simply *axes*).

Note. Since the axes are mutually perpendicular, therefore, they are called **rectangular axes.**

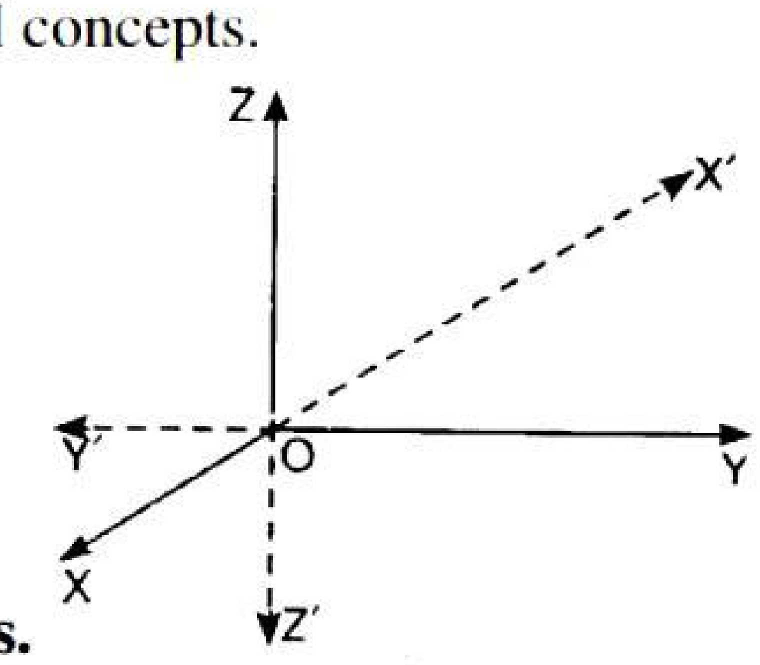


Fig.

KEY POINT

In the whole of this chapter, the treatment is with regard to rectangular axes. Thus by axes, we shall mean rectangular axes.

(*b*) Co-ordinate Planes.

(I) XOY, the plane containing X and Y-axes is called ***XY-plane.***

(II) YOZ, the plane containing Y and Z-axes is called ***YZ-plane.***

(III) ZOX, the plane containing Z and X-axes is called ***ZX-plane.***

These three, taken together, are called **co-ordinate planes.**

(*c*) Convention for Signs.

(I) Distances measured *upwards* of XY-plane are taken as +ve and *downwards* as – ve.

(II) Distances measured *in front* of YZ-plane are taken as +ve and *back* of it as – ve.

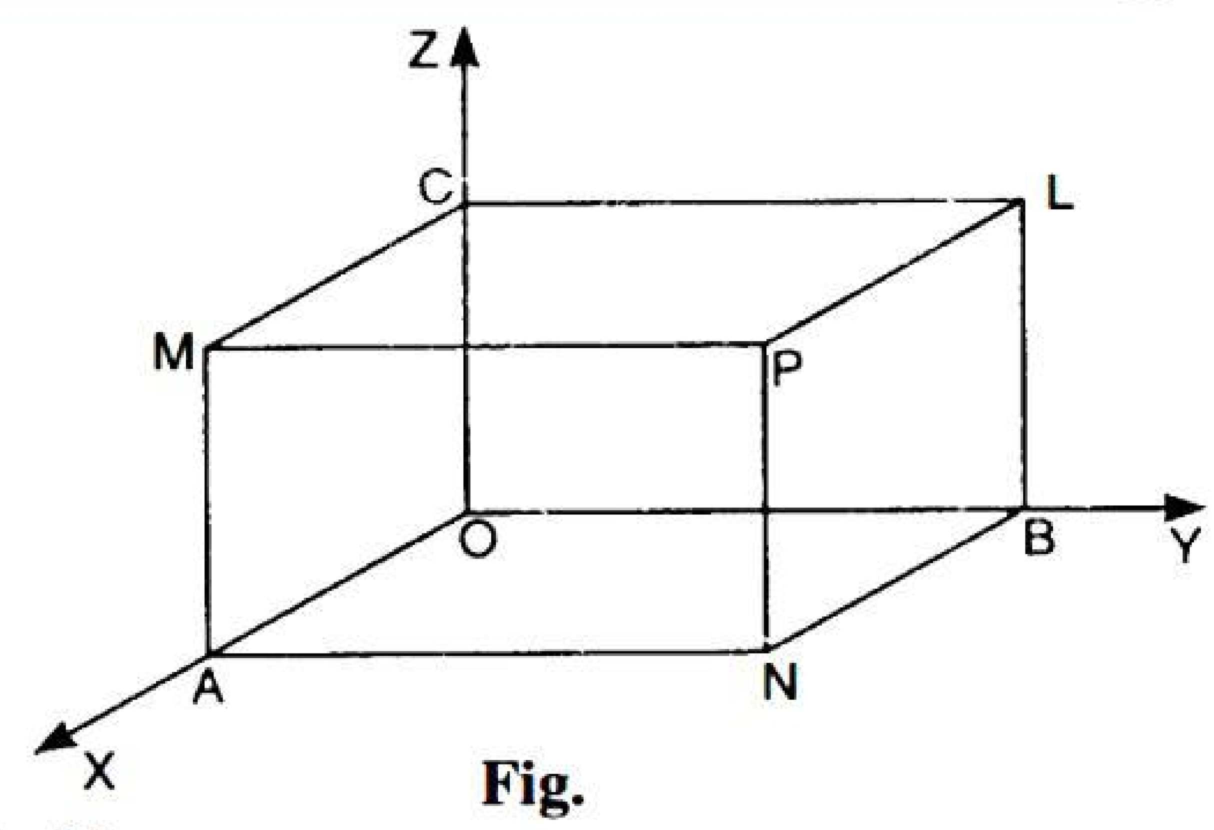


Fig.

(III) Distances measured to the *right* of ZX-plane are taken as +ve and *left* of it as – ve.

The three co-ordinate planes divide the whole space into eight compartments, known as **octants.**

(*d*) Co-ordinates of a point.

Let P be *any* point in space. Thro' P, draw three planes parallel to co–ordinate planes and meeting the axes in A, B and C respectively [(see Fig. of part (*b*)]. If x, y, z be the directed distances OA, OB, OC respectively, then the ordered triplet (x, y, z) are called cartesian rectangular co-ordinates of P and is denoted by P(x, y, z). From the definitions, we observe that : "*Given any point P in space, the ordered triplet (x, y, z) of real numbers is determined uniquely*."

Conversely, given the ordered triplet (x, y, z) of real numbers, we are to find the point of which these are the co-ordinates. Cut off from O on the co-ordinate axes OX, OY, OZ distances x, y, z respectively and find the respective points A, B, C. Thro' these points, draw planes parallel to YZ, ZX, XY-planes respectively. The point of intersection of these planes is the required point P whose co-ordinates are (x, y, z).

KEY POINT

There is one-one correspondence between the set of points in space and the set of ordered triplets of real numbers.

USEFUL FACTS

(I) The co-ordinates of the origin O are (0, 0, 0).

(II) If the point P (x, y, z) lies in the YZ-plane, then $x = 0$.

Conversely, if $x = 0$, then P lies in the YZ-plane.

Thus the **equation of YZ-plane is x = 0.**

Similarly, the **equation of ZX-plane is y = 0** and the **equation of XY-plane is z = 0.**

(III) Since X-axis is the common line of ZX and XY-planes,

∴ **equations of X-axis are y = 0 and z = 0.**

Similarly, the **equations of Y-axis are z = 0 and x = 0** and the **equations of Z-axis are x = 0 and y = 0.**

11.2. DISTANCE BETWEEN TWO POINTS

The distance between two distinct points whose co–ordinates are (x_1, y_1, z_1) and (x_2, y_2, z_2) is :

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$ **(In Cartesian Co–ordinates)**

NOTATIONS

(*i*) | AB | denotes the **distance between A and B**

(*ii*) [AB] denotes the **line segment AB.**

11.3. SECTION FORMULAE

The co–ordinates of the point, which divides the line segment joining two distinct points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $m_1 : m_2$ $(m_1 + m_2 \neq 0)$ are $\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_1z_2+m_2z_1}{m_1+m_2}\right)$.

GUIDE–LINES

Step (i) *Multiply m_1 by x_2 and m_2 by x_1 .*

Step (ii) *Add these two and divide the sum by $m_1 + m_2$.*

This gives *x*, the *x*-co-ordinate. Similarly, *y* and *z*-co-ordinates can be found.

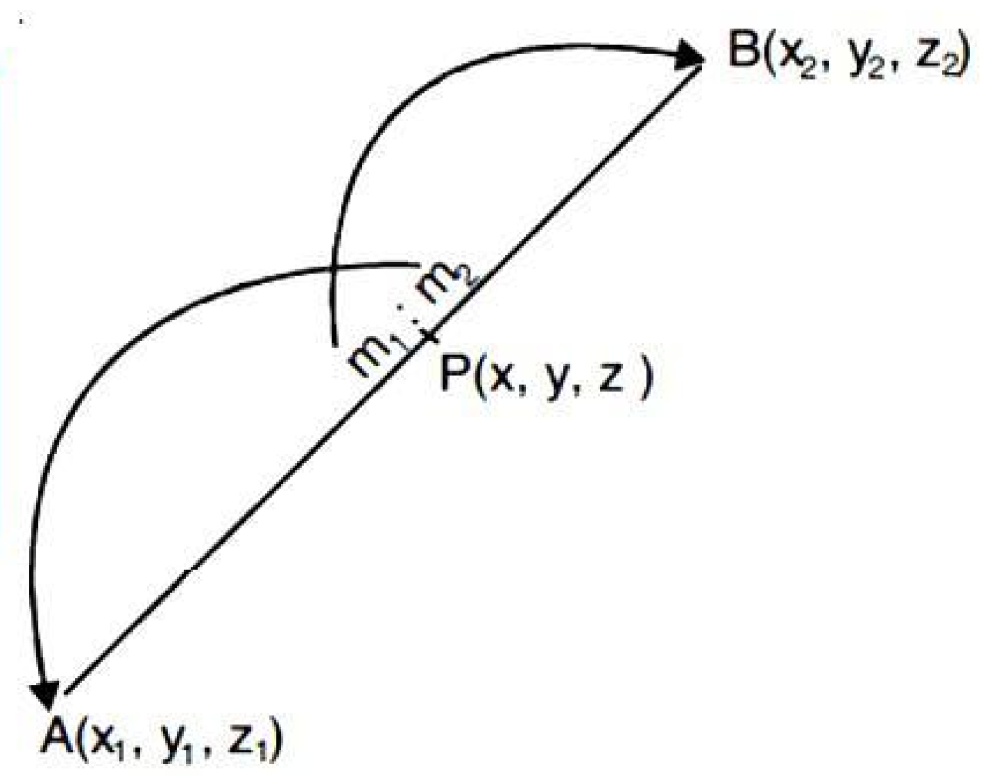


Fig.

Cor. 1. If Q divides the join of A and B in the ratio $m_1 : m_2$ externally, then the co-ordinates of Q are :

$$\left(\frac{m_1x_2-m_2x_1}{m_1-m_2}, \frac{m_1y_2-m_2y_1}{m_1-m_2}, \frac{m_1z_2-m_2z_1}{m_1-m_2}\right),$$ where $m_1 - m_2 \neq 0$ *i.e.* $m_1 \neq m_2$.

Cor. 2. Mid-point Formula.

If C (x, y, z) is the mid-point of [AB] with A (x_1, y_1, z_1) and B (x_2, y_2, z_2),

then the co-ordinates of C are $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

Cor. 3. Putting $\lambda = \frac{m_1}{m_2}, \lambda \neq -1, x = \frac{\lambda x_2 + x_1}{\lambda + 1}, y = \frac{\lambda y_2 + y_1}{\lambda + 1}, z = \frac{\lambda z_2 + z_1}{\lambda + 1}$. These are **parametric representations.**

11.4. CENTROID OF A TRIANGLE

Definition

Centroid of a triangle is the point, which divides all the medians in the ratio 2 : 1.

This is also called as **Centre of gravity** of the triangle.

The co-ordinates of the centroid (centre of gravity) of a triangle whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are :

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right).$$

KEY POINT

Medians of a triangle are concurrent.

SUB CHAPTER
11.1
Direction-Cosines

11.5. INTRODUCTION

In this section, we shall deal with direction-ratios and direction-cosines. Also we shall find the angle between the lines whose direction-ratios or direction-cosines are given.

11.6. DIRECTION-COSINES AND DIRECTION-RATIOS OF A LINE

Here we shall study the direction-cosines and direction-ratios of a line.

11.6.1. DIRECTION-COSINES OF A LINE

Let AB be a line in space. Through O, draw a line QP parallel to the line AB. Let the ray OP make angles α, β, γ with the rays OX, OY and OZ respectively.

Then the ray AB also makes same angles with the positive directions of the co-ordinate axes.

The cosines of these angles *i.e.* $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction-cosines of the ray AB.

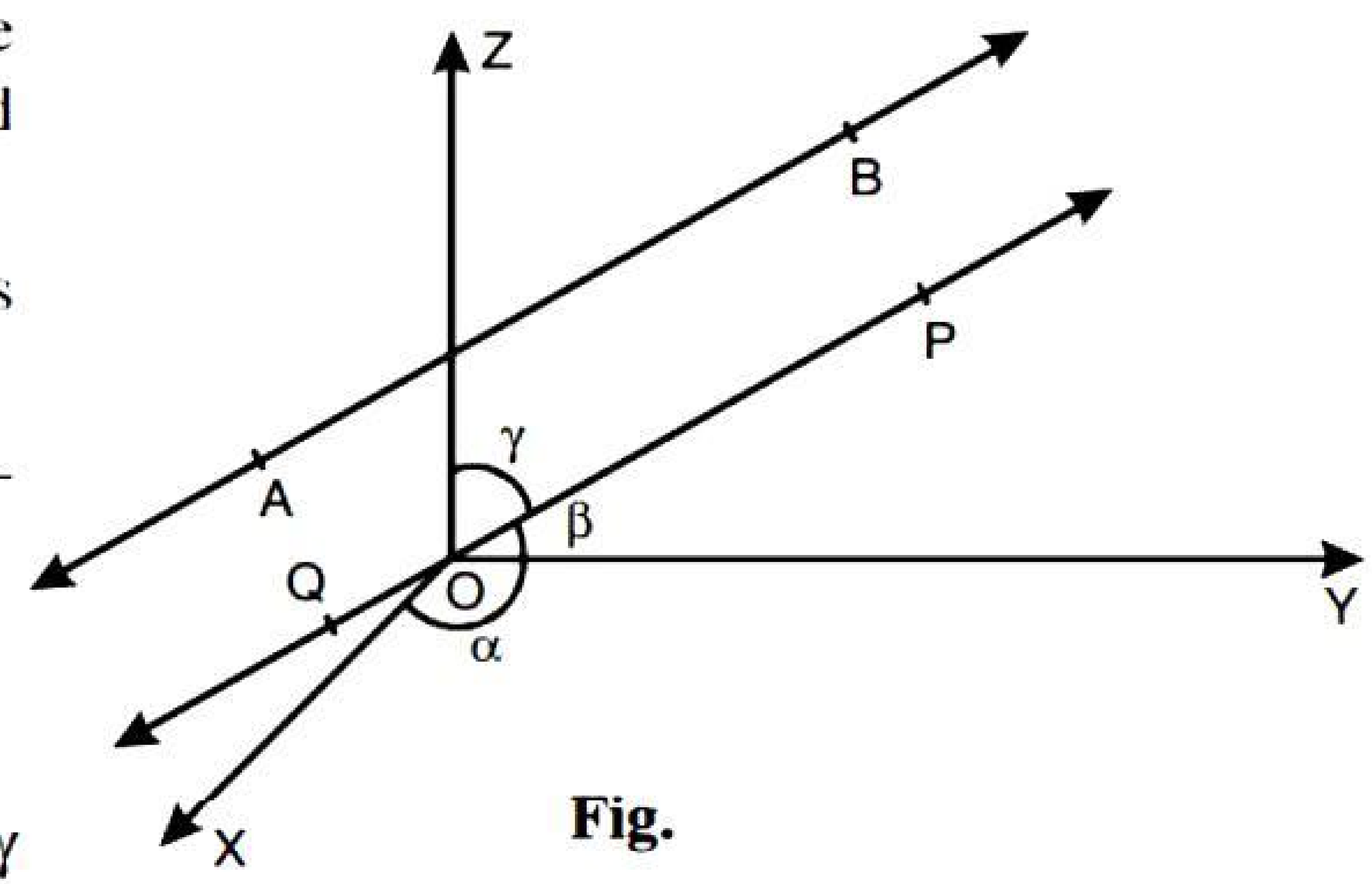


Fig.

Notation. The direction-cosines are usually denoted by $< l, m, n >$.

Thus $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$.

Observation. Clearly the ray OQ makes angles $\pi - \alpha$, $\pi - \beta$, $\pi - \gamma$ with the rays OX, OY and OZ respectively.

Thus the direction-cosines of the ray BA are :

$$< \cos(\pi - \alpha), \cos(\pi - \beta), \cos(\pi - \gamma) >$$

i.e., $< -\cos\alpha, -\cos\beta, -\cos\gamma >$ *i.e.*, $< -l, -m, -n >$.

Cor. The direction-cosines of the axes of co-ordinates. ***(N.C.E.R.T.)***

The x-axis makes angles 0°, 90°, 90° with the co-ordinate axes, its direction-cosines are $< \cos 0°, \cos 90°, \cos 90° >$

i.e., $\mathbf{< 1, 0, 0 >}$.

Similarly, the direction-cosines of y-axis are $< \mathbf{0, 1, 0} >$ and the direction-cosines of z-axis are $< \mathbf{0, 0, 1} >$.

11.6.2. RELATION BETWEEN DIRECTION-COSINES OF A LINE

Let AB be a line with direction-cosines $< l, m, n >$.

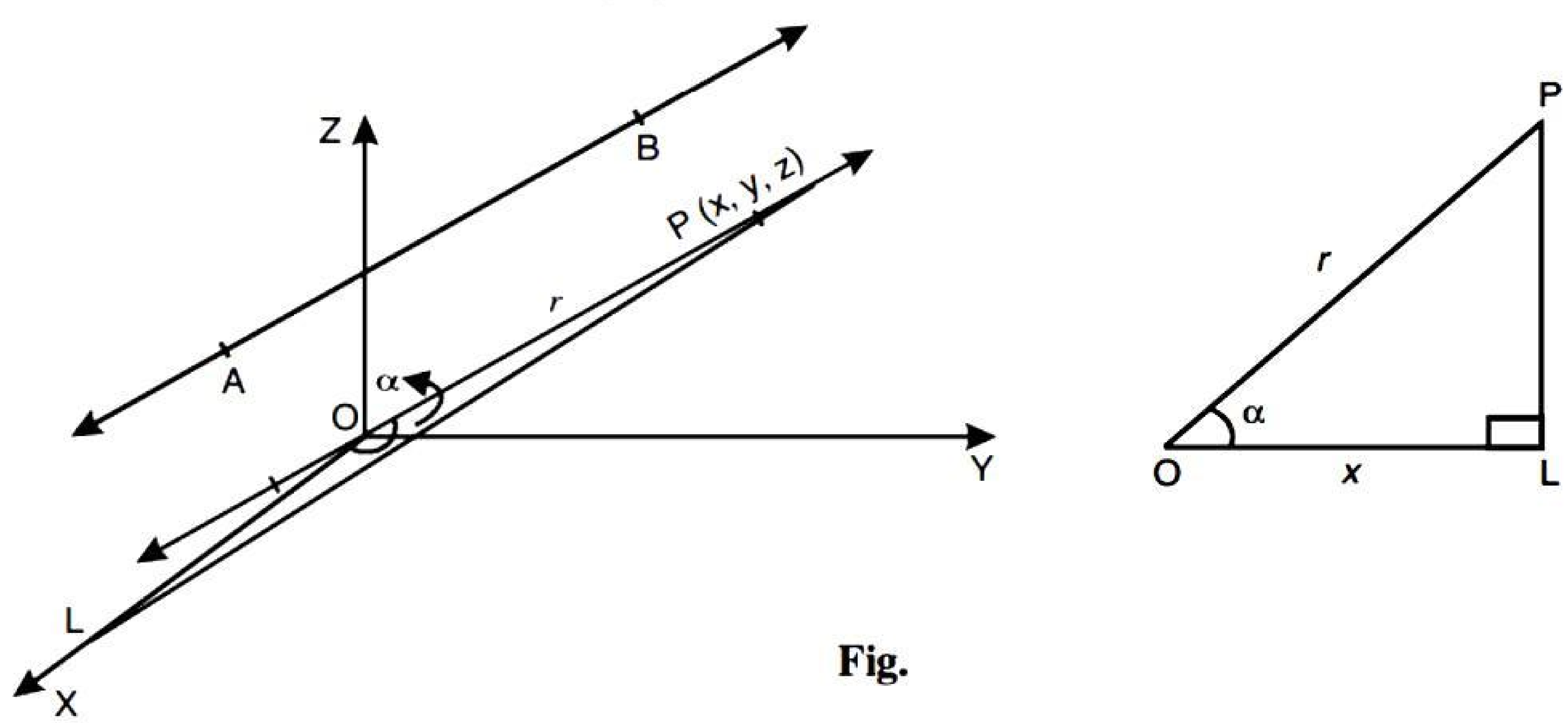


Fig.

Through O, draw a line parallel to AB.

Let P (x, y, z) be *any* point on this line.

From P, draw PL, perpendicular on the x-axis.

If $\quad OP = r$, then $\cos\alpha = \frac{OL}{OP} \quad \Rightarrow \quad l = \frac{x}{r}$

$$\Rightarrow \quad x = lr.$$

Similarly, $y = mr$ and $z = nr$.

Squaring and adding, $x^2 + y^2 + z^2 = r^2 (l^2 + m^2 + n^2)$

$\Rightarrow \qquad r^2 = r^2 (l^2 + m^2 + n^2)$.

Hence, $\quad \boxed{\mathbf{l^2 + m^2 + n^2 = 1.}}$

In Words : *Sum of the squares of the direction-cosines of any line is equal to 1.*

Another Form. If α, β, γ be the angles, which OP makes with the axes, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

11.6.3. DIRECTION-RATIOS OF A LINE

Definition

Any three numbers, which are proportional to the direction-cosines of a line, are called the direction-ratios of the line.

If $< a, b, c >$ are direction-ratios of a line, then $< ka, kb, kc >$ $(k \neq 0)$ are also direction-ratios of the line.

Thus there are infinitely many sets of direction-ratios of a line.

11.6.4. CONVERSION OF DIRECTION-RATIOS TO DIRECTION-COSINES

If the direction-ratios of a line are < a, b, c >, to find its direction-cosines.

Let $< l, m, n >$ be the direction-cosines (*d.c.'s*) of the line.

Then, $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = k$ (say), where $k \neq 0$

$\Rightarrow \qquad l = ak, m = bk, n = ck \qquad$ ...(1)

But $\qquad l^2 + m^2 + n^2 = 1$

$\Rightarrow \quad a^2k^2 + b^2k^2 + c^2k^2 = 1 \quad \Rightarrow \quad k = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$.

Hence, from (1), the direction-cosines of the line are :

$< \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}} >$, where signs should be taken all positive or all negative.

Divide each of a, b, c by $\pm\sqrt{a^2+b^2+c^2}$.

KEY POINT

$\mathbf{l^2 + m^2 + n^2 = 1}$ **but** $\mathbf{a^2 + b^2 + c^2 \neq 1.}$

11.6.5. DIRECTION-RATIOS OF LINE JOINING TWO POINTS

To find the direction-ratios of the line joining two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Let P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) be any two points on the line whose direction-cosines are $< l, m, n >$.

Then PQ $\cos\alpha = x_2 - x_1$, PQ $\cos\beta = y_2 - y_1$, PQ $\cos\gamma = z_2 - z_1$, where α, β, γ are the angles, which the line makes with the axes

$$\Rightarrow \quad PQ = \frac{x_2 - x_1}{\cos\alpha},\ PQ = \frac{y_2 - y_1}{\cos\beta},\ PQ = \frac{z_2 - z_1}{\cos\gamma}$$

$$\Rightarrow \quad \frac{x_2 - x_1}{\cos\alpha} = \frac{y_2 - y_1}{\cos\beta} = \frac{z_2 - z_1}{\cos\gamma} \ (= PQ)$$

$$\Rightarrow \quad \frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} \ (= PQ)$$

$$\therefore \quad l = \frac{x_2 - x_1}{PQ} = \frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}. \qquad \left[\because PQ = \sqrt{\Sigma(x_2 - x_1)^2}\right]$$

Similarly, $$m = \frac{y_2 - y_1}{PQ} = \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}} \text{ and } n = \frac{z_2 - z_1}{PQ} = \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}.$$

Hence, the direction-ratios of the line PQ are :

$$< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$$

and the direction-cosines are : $$\left\langle \frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}} \right\rangle.$$

11.7. ANGLE BETWEEN TWO LINES

(*a*) To determine the angle between two lines L_1 and L_2 whose direction-cosines are $< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$ respectively.

Let PQ ($\equiv L_1$) and ST ($\equiv L_2$) be two lines whose direction-cosines are given by : $< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$ respectively.

Let 'θ' be the required angle between these lines.

Through O, draw OA ($= r$) $\parallel$ to PQ and OB ($= r$) $\parallel$ to ST.

Thus $\angle AOB = \theta$.

[$\because$ *Angle between two lines = Angle between their parallels*]

$\therefore$ The co-ordinates of A and B are $(l_1 r, m_1 r, n_1 r)$ and $(l_2 r, m_2 r, n_2 r)$ respectively.

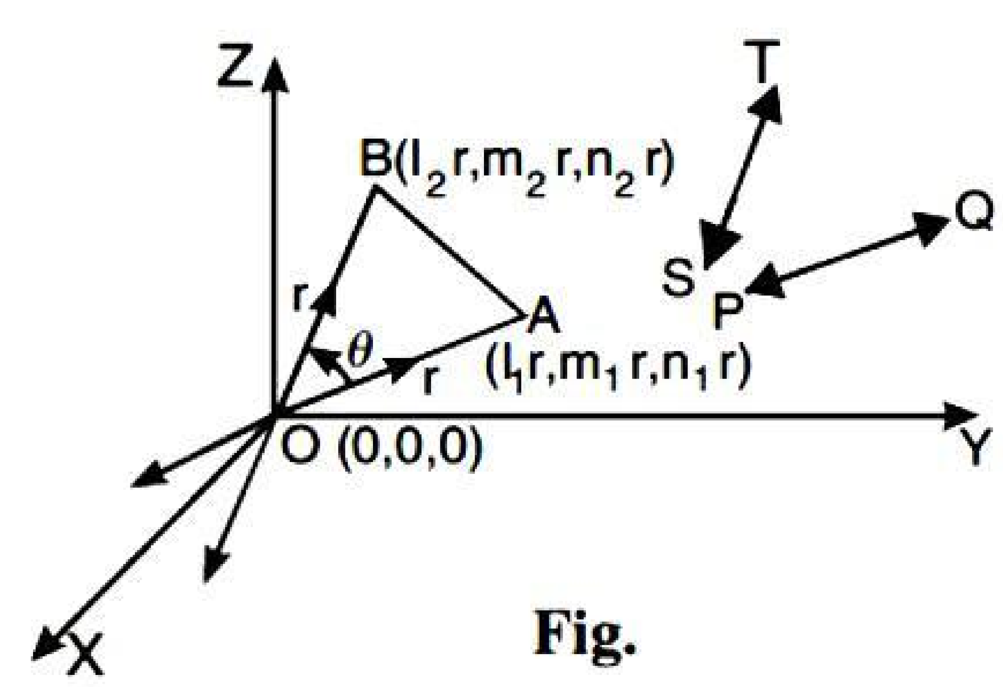


Fig.

$$\therefore \quad |AB| = \sqrt{(l_2 r - l_1 r)^2 + (m_2 r - m_1 r)^2 + (n_2 r - n_1 r)^2}$$

$$= r\sqrt{(l_2 - l_1)^2 + (m_2 - m_1)^2 + (n_2 - n_1)^2}$$

$$= r\sqrt{l_2^2 + l_1^2 - 2l_2 l_1 + m_2^2 + m_1^2 - 2m_2 m_1 + n_2^2 + n_1^2 - 2n_2 n_1}$$

$$= r\sqrt{(l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) - 2(l_1 l_2 + m_1 m_2 + n_1 n_2)}$$

$$= r\sqrt{1 + 1 - 2(l_1 l_2 + m_1 m_2 + n_1 n_2)} = r\sqrt{2 - 2(l_1 l_2 + m_1 m_2 + n_1 n_2)}.$$

Now, by *Cosine-Formula* in ΔOAB, we have :

$$\cos\theta = \frac{OA^2 + OB^2 - AB^2}{2|OA|.|OB|} = \frac{r^2 + r^2 - r^2[2 - 2(l_1l_2 + m_1m_2 + n_1n_2)]}{2r.r}$$

$$= \frac{1 + 1 - [2 - 2(l_1l_2 + m_1m_2 + n_1n_2)]}{2} = \frac{2(l_1l_2 + m_1m_2 + n_1n_2)}{2}.$$

Hence, $$\mathbf{\cos\theta = |\, l_1l_2 + m_1m_2 + n_1n_2\,|.}$$

i.e. $$\theta = \cos^{-1}(l_1l_2 + m_1m_2 + n_1n_2).$$

Sine Form :

Since $\sin^2\theta = 1 - \cos^2\theta$,

$\therefore$ $$\sin^2\theta = 1 - (l_1l_2 + m_1m_2 + n_1n_2)^2 \qquad [\because \cos\theta = |\, l_1l_2 + m_1m_2 + n_1n_2\,|]$$

$$= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2 \qquad \textbf{\textit{(Note this step)}}$$

$$= l_1^2l_2^2 + l_1^2m_2^2 + l_1^2n_2^2 + m_1^2l_2^2 + m_1^2m_2^2 + m_1^2n_2^2 + n_1^2l_2^2 + n_1^2m_2^2 + n_1^2n_2^2$$
$$- (l_1^2l_2^2 + m_1^2m_2^2 + n_1^2n_2^2 + 2l_1l_2m_1m_2 + 2\,m_1m_2n_1n_2 + 2\,n_1n_2l_1l_2)$$

$$= (m_1^2n_2^2 + m_2^2n_1^2 - 2\,m_1m_2n_1n_2) + (n_1^2l_2^2 + n_2^2l_1^2 - 2\,n_1n_2l_1l_2)$$
$$+ (l_1^2m_2^2 + l_2^2m_1^2 - 2\,l_1l_2m_1m_2)$$

$$= (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2.$$

But $0 \le \theta < \pi$, so that $\sin\theta > 0$.

Hence, $$\mathbf{\sin\theta = \sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}.}$$

Tangent Form :

$$\mathbf{\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2}}{l_1l_2 + m_1m_2 + n_1n_2}}.$$

Cor. 1. Condition of Perpendicularity.

The two lines are perpendicular

iff $\theta = \frac{\pi}{2}$

iff $\mathbf{l_1l_2 + m_1m_2 + n_1n_2 = 0.}$

Cor. 2. Conditions of Parallelism.

The two lines are parallel

iff $\theta = 0°$ iff $\sin\theta = 0$

$$\Leftrightarrow \sqrt{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2} = 0$$

$$\Leftrightarrow (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 0$$

$$\Leftrightarrow m_1n_2 - m_2n_1 = 0,\ n_1l_2 - n_2l_1 = 0,\ l_1m_2 - l_2m_1 = 0$$

$$\Leftrightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} \Leftrightarrow \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{\sqrt{l_1^2 + m_1^2 + n_1^2}}{\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$\Leftrightarrow \quad \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{1}{1} \quad \Leftrightarrow \quad \boxed{\mathbf{l_1 = l_2, m_1 = m_2 \text{ and } n_1 = n_2.}}$$

(*b*) If < a_1, b_1, c_1 > and < a_2, b_2, c_2 > are direction-ratios of lines L_1 and L_2 respectively, then the angles between them are given by :

$$\cos\theta = \pm\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Since the direction-ratios of L_1 and L_2 are < a_1, b_1, c_1 > and < a_2, b_2, c_2 > respectively,

∴ the direction-cosines of L_1 and L_2 are :

$$<\pm\frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \pm\frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \pm\frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}>$$

and
$$<\pm\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}}, \pm\frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}}, \pm\frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}}>.$$

[The signs, to be taken, are all + ve or all – ve]

∴ The angles between the lines are given by :

$$\cos\theta = \pm\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

KEY POINT

If the lines L_1 and L_2 are non-perpendicular, then the acute angle between them is given by :

$$\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

and obtuse angle between them is given by :

$$\cos\theta = -\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Sine Form :

Since $\sin^2\theta = 1 - \cos^2\theta$,

$$\therefore \quad \sin^2\theta = 1 - \frac{(a_1a_2+b_1b_2+c_1c_2)^2}{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}$$

$$= \frac{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)-(a_1a_2+b_1b_2+c_1c_2)^2}{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}$$

$$= \frac{(a_1b_2-a_2b_1)^2+(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2}{(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)}.$$

But $0 \le \theta < \pi$, so that $\sin\theta > 0$.

Hence, $\sin\theta = \sqrt{\dfrac{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}{(a_1^2 + b_1^2 + c_1^2)\,(a_2^2 + b_2^2 + c_2^2)}}$.

Tangent Form :

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\sqrt{\frac{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}{a_1a_2 + b_1b_2 + c_1c_2}}.$$

Cor. 1. Condition of Perpendicularity.

The two lines L_1 and L_2 are perpendicular

iff $\theta = \dfrac{\pi}{2}$ iff $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Cor. 2. Condition of Parallelism.

The two lines L_1 and L_2 are parallel

iff $\theta = 0°$ iff $\sin\theta = 0$

$$\Leftrightarrow \sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2} = 0$$

$$\Leftrightarrow (a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 = 0$$

$$\Leftrightarrow a_1b_2 - a_2b_1 = 0,\ b_1c_2 - b_2c_1 = 0,\ c_1a_2 - c_2a_1 = 0$$

$$\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

11.8. PROJECTION

(a) Projection of a point on a line.

Definition

> *The projection of a point P on a line L is defined as P′, the foot of the perpendicular from P on L.*

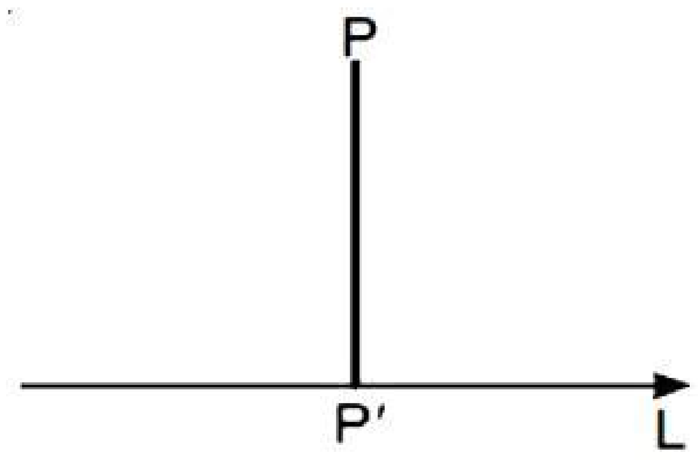


Fig.

(b) Projection of a point on a plane.

Definition

> *Let π be any plane and P be a given point, not on the plane π, then P′, the foot of perpendicular on the plane, is called orthogonal projection of P on the plane π.*

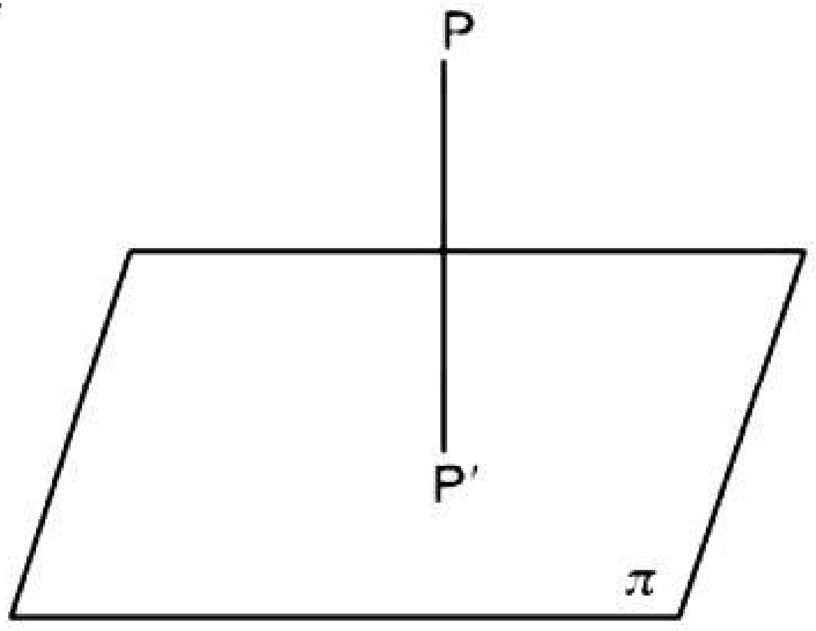


Fig.

(c) Projection of a line segment on a line.

Definition

> *The projection of the line segment [PQ] on a line L is the segment [P′Q′], where P′, Q′ are the feet of perpendiculars from P, Q respectively on the line L.*

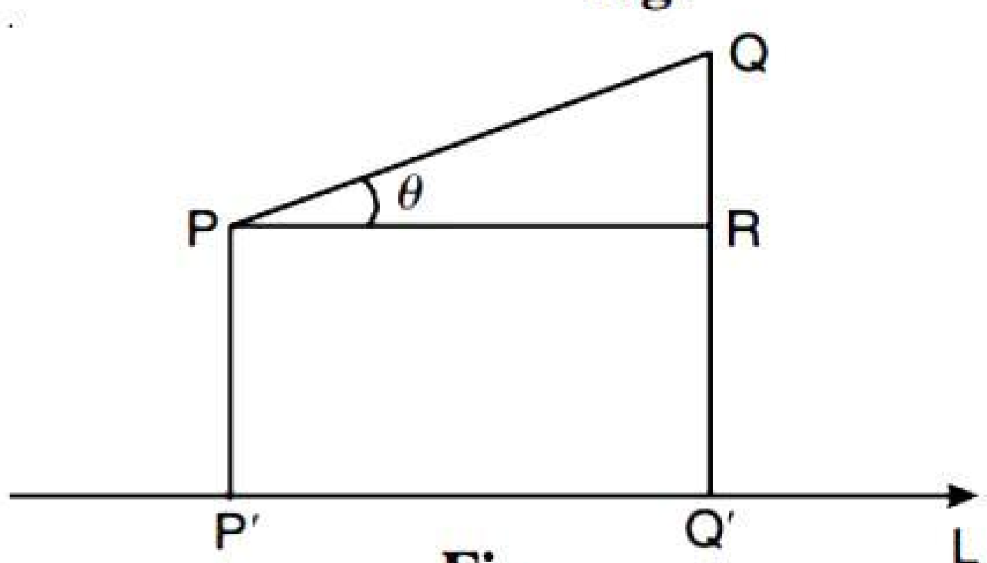


Fig.

If 'θ' is the angle between the line segment [PQ] and the line L, then $\angle QPR = \theta$, where PR || L, meeting QQ′ in R.

$\therefore$ $P'Q' = PR = PQ \cos\theta$.

Hence, the projection of line segment [PQ] on the line L is **PQ cos θ**, where 'θ' is the angle between the line segment [PQ] and the line L.

11.9. PROJECTION OF A SEGMENT

To find the projection of a line segment [AB] on line with direction-cosines <l, m, n>, where (x_1, y_1, z_1) and (x_2, y_2, z_2) are the co-ordinates of A and B respectively.

Let [AB] be the given segment. Then [A′B′] is its projection on the line L having direction-cosines $< l, m, n >$.

The direction-cosines of AB are :

$$<\frac{x_2-x_1}{|AB|}, \frac{y_2-y_1}{|AB|}, \frac{z_2-z_1}{|AB|}>.$$

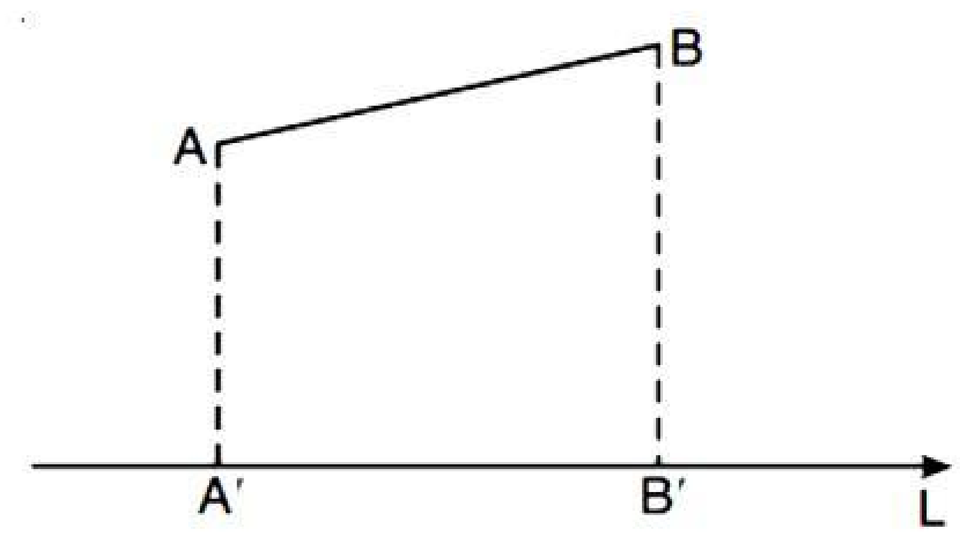


Fig.

Now $|A'B'| = |AB|\cos\theta$ ...(1),

where 'θ' is the angle between L and AB.

$\therefore$ $\cos\theta = |l_1l_2 + m_1m_2 + n_1n_2|$

$$= \left|\left(\frac{x_2-x_1}{|AB|}\right)l + \left(\frac{y_2-y_1}{|AB|}\right)m + \left(\frac{z_2-z_1}{|AB|}\right)n\right|$$

$\Rightarrow |AB|\cos\theta = |(x_2-x_1)\,l + (y_2-y_1)\,m + (z_2-z_1)n|$.

Hence, $\mathbf{|A'B'| = |(x_2-x_1)\,l + (y_2-y_1)\,m + (z_2-z_1)\,n|}$. [*Using (1)*]

Frequently Asked Questions

FAQs

Example 1. If a line makes angles of 90°, 60° and 30° with the positive x, y and z-axis respectively, find its direction-cosines. *(N.C.E.R.T.)*

Solution. Direction-cosines are :

$$< \cos 90°, \cos 60°, \cos 30° > \text{ i.e. } <0, \frac{1}{2}, \frac{\sqrt{3}}{2}>.$$

Example 2. Find the acute angle which the line with direction-cosines $<\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, n>$ makes wtith positive direction of z–axis. *(C.B.S.E. Sample Paper 2019)*

Solution. Since, $l^2 + m^2 + n^2 = 1$,

$$\therefore \quad \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + n^2 = 1$$

$$\Rightarrow \frac{1}{3}+\frac{1}{6}+n^2 = 1 \quad \Rightarrow \quad \frac{1}{3}+\frac{1}{6}+n^2 = 1$$

$$\Rightarrow \quad n^2 = 1-\frac{1}{2} \quad \Rightarrow \quad n^2 = \frac{1}{2}$$

$$\Rightarrow \quad n = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \cos\gamma = \frac{1}{\sqrt{2}},$$

where 'γ' is the angle, which the line makes with z–axis.

Hence, $\gamma = 45°$ or $\frac{\pi}{4}$.

Example 3. If a line has direction-cosines : $<-\frac{9}{11}, \frac{6}{11}, -\frac{2}{11}>$, then what are its direction-ratios ? *(N.C.E.R.T.)*

Solution. Given : Direction-cosines are :

$$<\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}>.$$

Clearly, $< -9, 6, -2 >$ is one set of direction-ratios.

All sets of direction-ratios are given by :

$< -9k, 6k, -2k >$, where $k \neq 0$.

Example 4. Find the direction-cosines of the line joining the points (– 2, 4, – 5) and (1, 2, 3). *(N.C.E.R.T.)*

Solution. We know that the direction-cosines of the line joining P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are :

$$<\frac{x_2-x_1}{|PQ|}, \frac{y_2-y_1}{|PQ|}, \frac{z_2-z_1}{|PQ|}>,$$

where $|PQ| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

Here P is (– 2, 4, – 5) and Q is (1, 2, 3).

$$\therefore \quad |PQ| = \sqrt{(1-(-2))^2+(2-4)^2+(3-(-5))^2}$$

$$= \sqrt{9+4+64} = \sqrt{77}.$$

Hence, the direction-cosines of the line are :

$$<\frac{1-(-2)}{\sqrt{77}}, \frac{2-4}{\sqrt{77}}, \frac{3-(-5)}{\sqrt{77}}>$$

i.e. $<\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}>$.

Example 5. If α, β, γ are direction-angles of a line, prove that cos 2α + cos 2β + cos 2γ + 1 = 0. *(N.C.E.R.T.)*

Solution. Since α, β, γ are direction-angles of a line,

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\Rightarrow \quad \frac{1+\cos 2\alpha}{2} + \frac{1+\cos 2\beta}{2} + \frac{1+\cos 2\gamma}{2} = 1$$

$$\Rightarrow \quad 1 + \cos 2\alpha + 1 + \cos 2\beta + 1 + \cos 2\gamma = 2$$

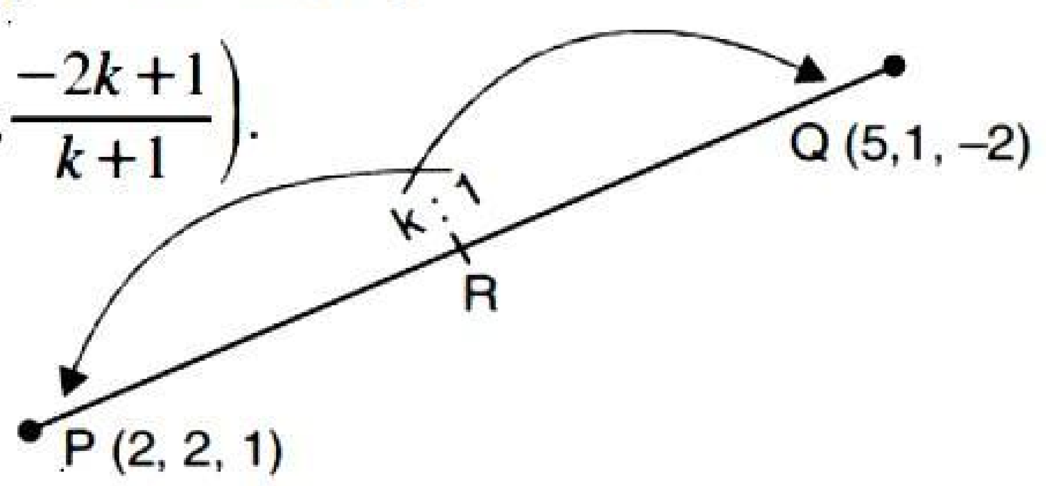
$$\Rightarrow \quad \cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1 = 0,$$

which is true.

Example 6. The x-cordinate of a point on the line joining the points P(2, 2, 1) and Q(5, 1, – 2) is 4. Find its z-co-ordinate. *(A.I.C.B.S.E. 2017)*

Solution. Any point R on [PQ] is :

$$\left(\frac{5k+2}{k+1}, \frac{k+2}{k+1}, \frac{-2k+1}{k+1}\right).$$



By the question, $\dfrac{5k+2}{k+1} = 4$

$\Rightarrow 5k + 2 = 4k + 4 \Rightarrow k = 2.$

Hence, the z-co-ordinate is : $\dfrac{-2(2)+1}{2+1}$ *i.e.* $\dfrac{-4+1}{3}$ *i.e.* – 1.

Example 7. Show that the points :

A(1, – 2, – 8); B(5, 0, – 2) and C(11, 3, 7) are collinear, and find the ratio in which B divides AC. *(Jammu B. 2012)*

Solution. (*i*) The direction-ratios of AB are :

$< 5 - 1, 0 + 2, - 2 + 8 >$ *i.e.* $< 4, 2, 6 >$ *i.e.* $< 2, 1, 3 >$

and the direction-ratios of BC are :

$< 11-5, 3 - 0, 7 + 2 >$ *i.e.* $< 6, 3, 9 >$ *i.e.* $< 2, 1, 3 >$.

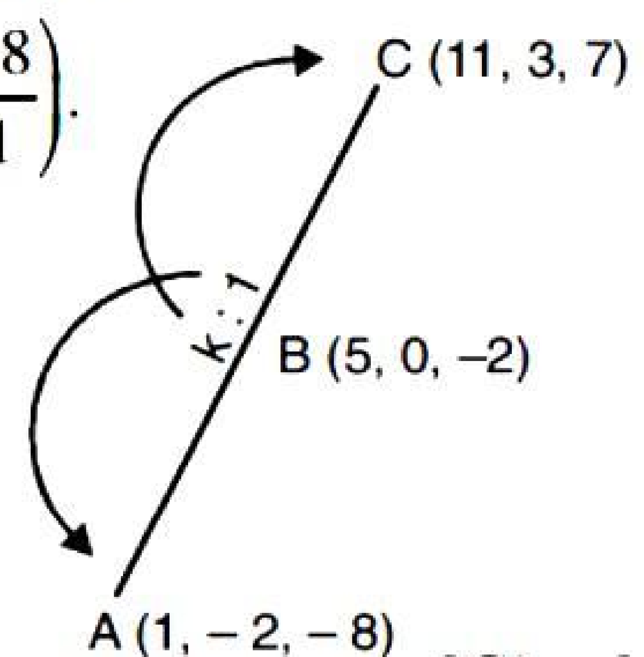
Thus AB, BC are either parallel or coincident lines.

But B is the common point.

Hence, the lines are coincident lines and consequently the three points A, B and C are collinear.

(*ii*) Let B divide [AC] in the ratio $k : 1$.

$$\therefore \text{ B is } \left(\frac{11k+1}{k+1}, \frac{3k-2}{k+1}, \frac{7k-8}{k+1}\right).$$



But B is (5, 0, –2). [*Given*]

$$\therefore \quad \frac{11k+1}{k+1} = 5, \frac{3k-2}{k+1} = 0 \text{ and } \frac{7k-8}{k+1} = -2$$

$\Rightarrow 11k + 1 = 5k + 5,\ 3k - 2 = 0$ and $7k - 8 = -2k - 2$

$\Rightarrow 6k = 4,\quad 3k = 2$ and $9k = 6$

$\Rightarrow k = \dfrac{2}{3},\quad k = \dfrac{2}{3}$ and $k = \dfrac{2}{3}$.

Hence, the reqd. ratio is $\dfrac{2}{3} : 1$ *i.e.* 2 : 3.

Example 8. Find the acute angle between the lines whose direction-ratios are :

< 1, 1, 2 > and < – 3, – 4, 1 >.

Solution. If 'θ' be the reqd. angle between the lines, then :

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$= \frac{|(1)(-3) + (1)(-4) + (2)(1)|}{\sqrt{1+1+4}\sqrt{9+16+1}}$$

$$= \frac{|-3-4+2|}{\sqrt{6}\sqrt{26}} = \frac{5}{\sqrt{156}}.$$

Hence, $\theta = \cos^{-1}\left(\dfrac{5}{\sqrt{156}}\right)$.

Example 9. Find the angle between the lines whose direction-cosines are given by the equations :

3l + m + 5n = 0, 6mn – 2nl + 5lm = 0.

Solution. We have : $3l + m + 5n = 0$ ...(1)

and $6mn - 2nl + 5lm = 0$ ...(2)

From (1), $m = -(3l + 5n)$ ...(3)

Putting in (2), we get :

$$-6(3l + 5n)n - 2nl - 5l(3l + 5n) = 0$$

$$\Rightarrow \quad -18ln - 30n^2 - 2nl - 15l^2 - 25nl = 0$$

$$\Rightarrow \quad -30n^2 - 45nl - 15l^2 = 0$$

$$\Rightarrow \quad 2n^2 + 3nl + l^2 = 0$$

$$\Rightarrow \quad (2n + l)(n + l) = 0.$$

$\therefore$ Either $2n + l = 0$ or $n + l = 0$.

(I) When $2n + l = 0$ *i.e.* $l = -2n$.

From (3), $m = -(-6n + 5n) = n$.

(II) When $n + l = 0$ *i.e.* $l = -n$.

From (3), $m = -(-3n + 5n) = -2n$.

$\therefore$ Direction–ratios of two lines are :

$< -2n, n, n >$ and $< -n, -2n, n >$

i.e. $< -2, 1, 1 >$ and $< 1, 2, -1 >$.

If 'θ' be the angle between the two lines,

$$\text{then } \cos\theta = \frac{|(-2)(1) + (1)(2) + (1)(-1)|}{\sqrt{4+1+1}\sqrt{1+4+1}} = \frac{1}{6}.$$

Hence, $\theta = \cos^{-1}\dfrac{1}{6}$.

Example 10. Find the length of the projection of the line segment joining the points P (3, – 1, 2) and Q (2, 4, –1) on the line with direction-ratios < – 1, 2, –2 >.

Solution. The direction-ratios of the line are < –1, 2,– 2 >.

Its direction-cosines are :

$$< \frac{-1}{\sqrt{1+4+4}}, \frac{2}{\sqrt{1+4+4}}, \frac{-2}{\sqrt{1+4+4}} >$$

$$i.e. < \frac{-1}{3}, \frac{2}{3}, \frac{-2}{3} >.$$

$\therefore$ The length of projection of [PQ] on the given line

$= |(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n|$

$$= \left|(2-3)\left(-\frac{1}{3}\right) + (4+1)\left(\frac{2}{3}\right) + (-1-2)\left(\frac{-2}{3}\right)\right|$$

$$= \left|\frac{1}{3} + \frac{10}{3} + \frac{6}{3}\right| = \left|\frac{17}{3}\right| = \frac{17}{3} \text{ units.}$$

Example 11. Find the area of the triangle ABC whose vertices are :

A (1, 2, 4); B (–2, 1, 2) and C (2, 4, –3).

Solution. Area of ΔABC

$$= \frac{1}{2} |AB|\,|AC| \sin A \quad ...(1)$$

Now $|AB| = \sqrt{(-2-1)^2+(1-2)^2+(2-4)^2}$

$= \sqrt{9+1+4} = \sqrt{14}$

and $|AC| = \sqrt{(2-1)^2+(4-2)^2+(-3-4)^2}$

$= \sqrt{1+4+49} = \sqrt{54}$.

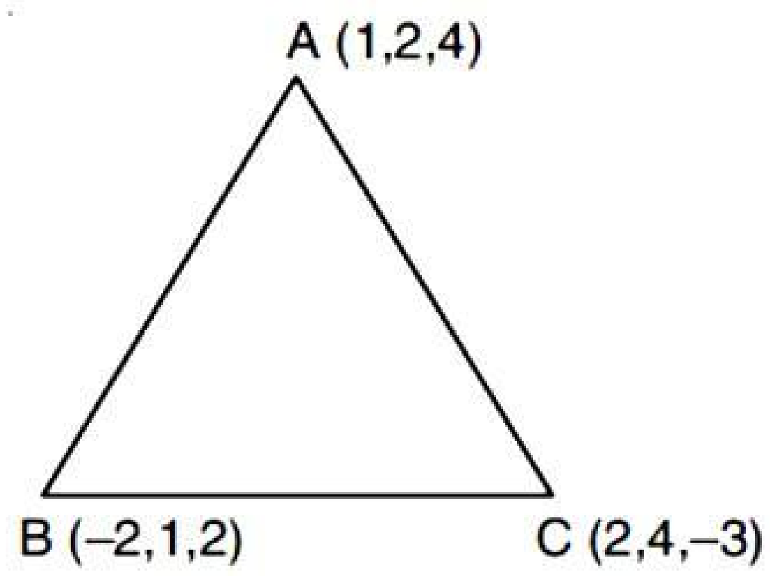


Fig.

Now direction-ratios of AB are :

$< -2-1, 1-2, 2-4 >$ *i.e.* $< 3, 1, 2 >$

and direction-ratios of AC are :

$< 2-1, 4-2, -3-4 >$ *i.e.* $< 1, 2, -7 >$.

$$\therefore \sin A = \frac{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$= \frac{\sqrt{(-7-4)^2+(2+21)^2+(6-1)^2}}{\sqrt{9+1+4}\sqrt{1+4+49}}$$

$$= \frac{\sqrt{121+529+25}}{\sqrt{14}\sqrt{54}} = \frac{\sqrt{675}}{\sqrt{14}\sqrt{54}}.$$

From (1), area of $\Delta ABC = \frac{1}{2} \times \sqrt{14} \times \sqrt{54} \times \frac{\sqrt{675}}{\sqrt{14}\sqrt{54}}$

$$= \frac{15\sqrt{3}}{2} \text{ sq. units.}$$

Definition

(a) (i) Parallelopiped. *It is a figure bounded by three parallel planes. Thus the parallelopiped is as shown in the adjoining figure :*

It has *six faces, viz.* || gms.

OCLB, AMPN, OBNA, CLPM, OAMC, BNPL.

It has *four diagonals viz.* OP, AL, BM, CN.

(ii) Rectangular Parallelopiped.

When the faces are rectangles, then the parallelopiped is a rectangular one.

(b) Cube. *It is a parallelopiped with all its faces as squares.*

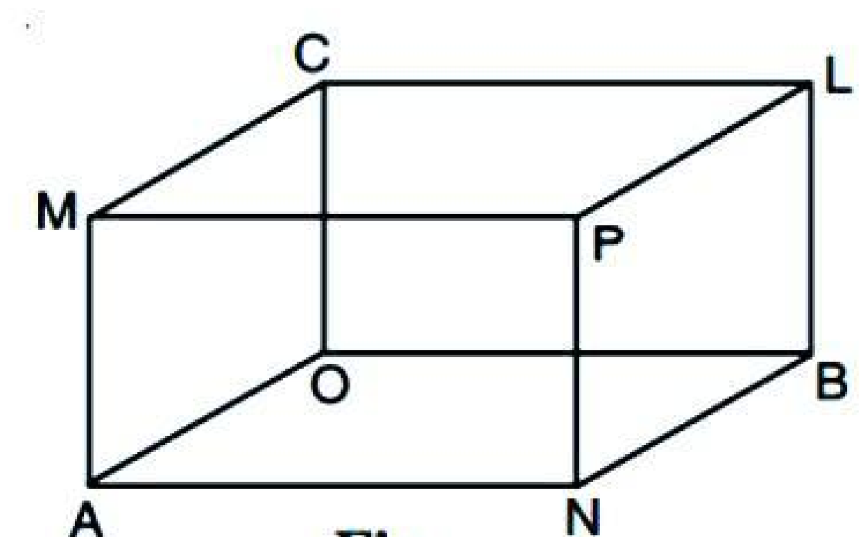


Fig.

Example 12. A line makes angles α, β, γ, δ with the four diagonals of a cube, prove that :

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

(N.C.E.R.T.)

Solution. Let O be the origin and OA, OB, OC (each = a) be the axes.

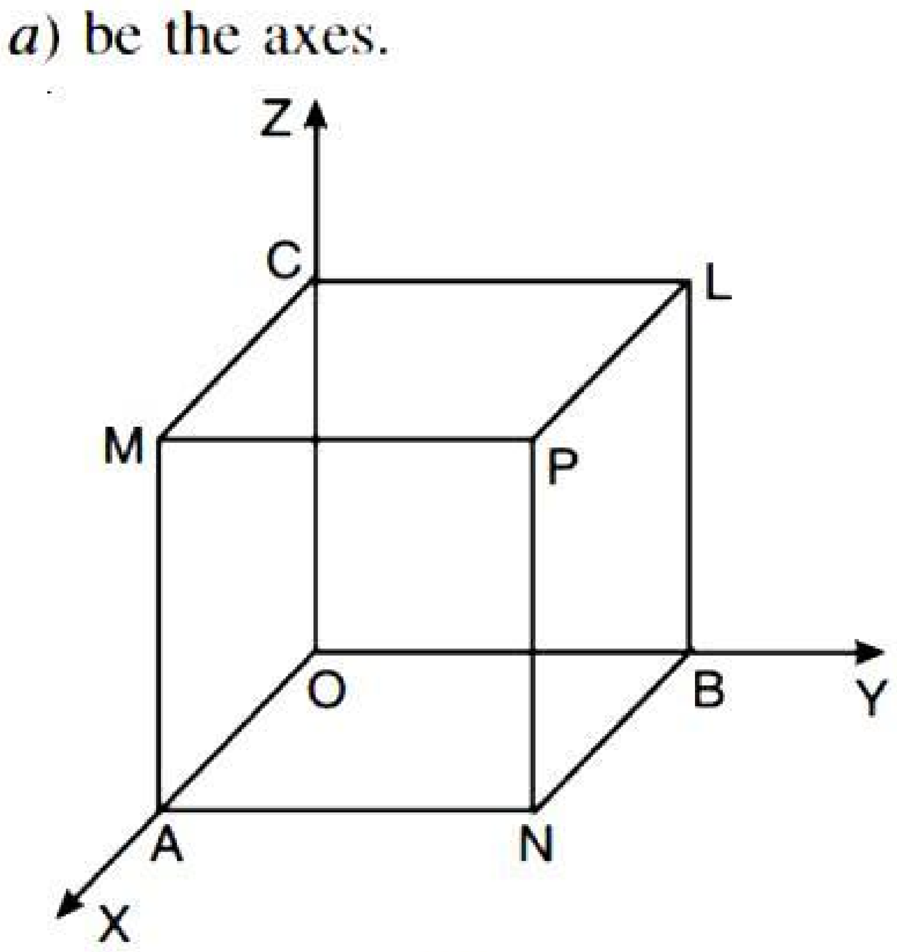


Fig.

Thus the co-ordinates of the points are :

O (0, 0, 0), A (a, 0, 0), B (0, a, 0), C (0, 0, a),

P (a, a, a), L (0, a, a), M (a, 0, a), N (a, a, 0).

Here OP, AL, BM and CN are four diagonals.

Let $< l, m, n >$ be the direction-cosines of the given line.

Now direction-ratios of OP are $< a-0, a-0, a-0 >$

i.e. $< a, a, a >$ *i.e.* $< 1, 1, 1 >$,

direction-ratios of AL are :

$< 0-a, a-0, a-0 >$ *i.e.* $< -a, a, a >$ *i.e.* $< -1, 1, 1 >$,

direction-ratios of BM are :

$< a-0, 0-a, a-0 >$ *i.e.* $< a, -a, a >$ *i.e.* $< 1, -1, 1 >$

and direction-ratios of CN are :

$< a-0, a-0, 0-a >$ *i.e.* $< a, a, -a >$ *i.e.* $< 1, 1, -1 >$.

Thus the direction-cosines of OP are :

$$< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >;$$

the direction-cosines of AL are $< \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$;

the direction-cosines of BM are $< \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$

and the direction-cosines of CN are :

$$< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} >.$$

If the given line makes an angle 'α' with OP, then :

$$\cos\alpha = \left| l\left(\frac{1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) \right|.$$

$$\therefore \quad \cos\alpha = \frac{|l+m+n|}{\sqrt{3}} \quad ...(1)$$

If the given line makes an angle 'β' with AL, then :

$$\cos\beta = \left| l\left(-\frac{1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) \right|$$

$$\therefore \quad \cos\beta = \frac{|-l+m+n|}{\sqrt{3}} \quad ...(2)$$

Similarly, $\cos\gamma = \frac{|l-m+n|}{\sqrt{3}}$ ...(3)

and $\cos\delta = \frac{|l+m-n|}{\sqrt{3}}$ ...(4)

Squaring and adding (1), (2), (3) and (4), we get :

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2]$$

$$= \frac{1}{3}[4(l^2+m^2+n^2)] = \frac{1}{3}[4(1)].$$

Hence, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.

EXERCISE 11 (a)

Fast Track Answer Type Questions

FTATQ

1. (*a*) Direction-cosines of (*i*) x-axis (*ii*) y-axis (*iii*) z-axis are .... **(Fill in the blank).** *(Kashmir B. 2017, 16)*

(*b*) Find the distance of the point (2, 3, 4) from the x-axis. *(C.B.S.E. 2010 C)*

2. (*i*) If a line makes angles 90°, 60° and θ with x, y and z-axis respectively, where θ is acute, then find 'θ'. *(C.B.S.E. 2015)*

(*ii*) If a line makes angles 90° and 60° respectively, with the positive directions of x and y axes, find the angle which it makes with the positive direction of z-axis. *(C.B.S.E. 2017)*

3. If a line has direction-cosines $< \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} >$, then find the direction-ratios.

4. If a line has direction-ratios < 2, – 1, – 2 >, determine its direction-cosines. *(N.C.E.R.T.; Jharkhand B. 2016; Uttarakhand B. 2013, 15; C.B.S.E. 2012)*

5. (*a*) Find the direction-cosines of a line passing through the points (1, 0, 0) and (0, 1, 1). *(A.I.C.B.S.E. 2011)*

(*b*) Find the direction-cosines of the lines joining the points :
(– 1, – 1, – 1) and (2, 3, 4). *(Rajasthan B. 2012)*

(*c*) Find the direction ratios and direction cosines of the vector joining the points (4, 7, 2) and (5, 11, – 4). *(Meghalaya. B. 2018; Nagaland. B. 2016)*

(*d*) Find the direction cosines of a line segment joining the points A(2, 5, 7) and B(3, 2, 9). *(Nagaland. B. 2018)*

6. Write the direction-cosines of the vector :

(*i*) $-2\hat{i} + \hat{j} - 5\hat{k}$ *(C.B.S.E. 2011)*

(*ii*) $\hat{i} + 2\hat{j} + 3\hat{k}$. *(Assam B. 2015; Karnataka B. 2014)*

7. Find the length of the projection of the line segment joining (3, 4, 5) and (4, 6, 3) on the straight line :

$\frac{x-4}{2} = \frac{y-5}{3} = \frac{z-6}{6}$. *(Tripura B. 2016)*

Very Short Answer Type Questions

VSATQ

8. Show that the following points are collinear :
(1, 2, 7) ; (2, 6, 3) ; (3, 10, –1). *(Jammu B. 2015,13)*

9. Find the acute angle between two lines whose direction-ratios are :
< 2, 3, 6 > and < 1, 2, – 2 >.

10. Find the obtuse angle between two lines whose direction-ratios are :
< 3, – 6, 2 > and <1, – 2, – 2 >.

11. Find the angle between the lines whose direction-ratios are :
< a, b, c > and < $b - c$, $c - a$, $a - b$ >. *(N.C.E.R.T.)*

Short Answer Type Questions

SATQ

12. Find the direction-cosines of the sides of the triangle whose vertices are (3, 5, – 4), (– 1, 1, 2) and (– 5, – 5, – 2). *(N.C.E.R.T.)*

13. Show that the lines with direction-cosines :

$$< \frac{12}{13}, -\frac{3}{13}, -\frac{4}{13} >; < \frac{4}{13}, \frac{12}{13}, \frac{3}{13} >; < \frac{3}{13}, \frac{-4}{13}, \frac{12}{13} >$$

are mutually perpendicular.

14. Find the angle between the lines whose direction-cosines are given by :

(*i*) $l + m + n = 0$, $l^2 + m^2 - n^2 = 0$

(*ii*) $2l - m + 2n = 0$, $mn + nl + lm = 0$.

15. Find the area of the triangle whose vertices are : A (1, 2, 3) ; B (2, –1, 4) and C (4, 5, –1). *(C.B.S.E. 2017, 13)*

16. Show that the join of the points (4, 7, 8) and (2, 3, 4) is parallel to the join of the points (–1, –2, 1) and (1, 2, 5).

17. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line through the points (3, 5, – 1) and (4, 3, – 1).

18. Determine the value of 'k' so that the line joining the points A (k, 1, – 1), B (2, 0, $2k$) is perpendicular to the line joining the points C (4, $2k$, 1) and D (2, 3, 2).

19. Prove that the angle between any two diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

Long Answer Type Questions

LATQ

20. Find the projection of the line segment joining the points :

(*i*) (2, –3, 0), (0, 4, 5) on the line with direction-cosines $<\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}>$

(*ii*) (1, 2, 3), (4, 3, 1) on the line with direction-ratios $< 3, – 6, 2 >$.

21. If the edges of a rectangular parallelopiped are a, b and c, show that the angles between the four diagonals are given by :

$$\cos^{-1}\frac{a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}.$$

Answers

1. (*a*) (*i*) <1,0,0> (*ii*) <0,1,0> (*iii*) <0,0,1> (*b*) 5.

2. (*i*) $\theta = 30°$ (*ii*) 30° . 3. $< 2k, – k, – 2k >$; $k \neq 0$.

4. $<\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}>$.

5. (*a*) $<-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}>$ (*b*) $<\frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{5}{5\sqrt{2}}>$

(*c*) $<1, 4, -6>; <\frac{1}{\sqrt{53}}, \frac{4}{\sqrt{53}}, \frac{-6}{\sqrt{53}}>$

(*d*) $<\frac{1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{2}{\sqrt{14}}>$.

6. (*i*) $<-\frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{5}{\sqrt{30}}>$ (*ii*) $<\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}>$.

7. $\frac{4}{7}$.

9. $\cos^{-1}\frac{4}{21}$.

10. $\cos^{-1}\left(-\frac{11}{21}\right)$.

11. 90°.

12. $<\frac{2}{\sqrt{17}}, \frac{2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}>, <\frac{2}{\sqrt{17}}, \frac{3}{\sqrt{17}}, \frac{2}{\sqrt{17}}>$, $<\frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}>$.

14. (*i*) 60° (*ii*) 90°.

15. $\sqrt{\frac{137}{2}}$ sq. units.

18. $k = 1$.

20. (*i*) $\frac{13}{7}$ (*ii*) $\frac{1}{7}$.

Hints to Selected Questions

14. (*i*) $l + m + n = 0 \Rightarrow n = -(l + m)$.

$l^2 + m^2 - n^2 = 0 \Rightarrow l^2 + m^2 - (l + m)^2 = 0 \Rightarrow l = 0, m = 0$.

When $l = 0$, then $m + n = 0 \Rightarrow m = -n$.

When $m = 0$, then $l + n = 0 \Rightarrow l = -n$.

∴ Direction cosines of two lines are :

$< 0, -n, n>$ and $<-n, 0, n>$

i.e. $< 0, -1, 1 >$ and $<-1, 0, 1>$.

$$\therefore \cos\theta = \frac{|(0)(-1)+(-1)(0)+(1)(1)|}{\sqrt{0+1+1}\sqrt{1+0+1}} = \frac{1}{2} \Rightarrow \theta = 60°.$$

15. As in **Ex. 11.**

19. Refer **Ex. 12.**

If 'θ' be the angle between OP and AL,

then $\cos\theta = \left(\frac{1}{\sqrt{3}}\right)\left(-\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{3}$; etc.

21. As in **Ex. 12.**

SUB CHAPTER

11.2 Straight Line in Space

11.10. INTRODUCTION

We know that a straight line is uniquely determined in space if it :

(*a*) passes through a given point and has a given direction or (*b*) passes through two given points.

We shall determine the vector equation as well as cartesian equation of a straight line under different conditions.

11.11. SYMMETRICAL (CANONICAL) FORM

(*a*) To find the equation of a straight line passing through a fixed point A and parallel to a given vector $\vec{m}$.

Let $\vec{a}$ be the position vector of the fixed point A and $\vec{r}$, the position vector of *any* point P, where O is the origin.

Since $\overrightarrow{AP}$ is parallel to $\vec{m}$,

$\therefore \quad \overrightarrow{AP} = \lambda \vec{m}$ for some scalar λ

$\Rightarrow \quad \overrightarrow{OP} - \overrightarrow{OA} = \lambda \vec{m}$

$\Rightarrow \quad \vec{r} - \vec{a} = \lambda \vec{m}$

$\Rightarrow \quad \boxed{\vec{\mathbf{r}} = \vec{\mathbf{a}} + \lambda \vec{\mathbf{m}}}$ ...(1),

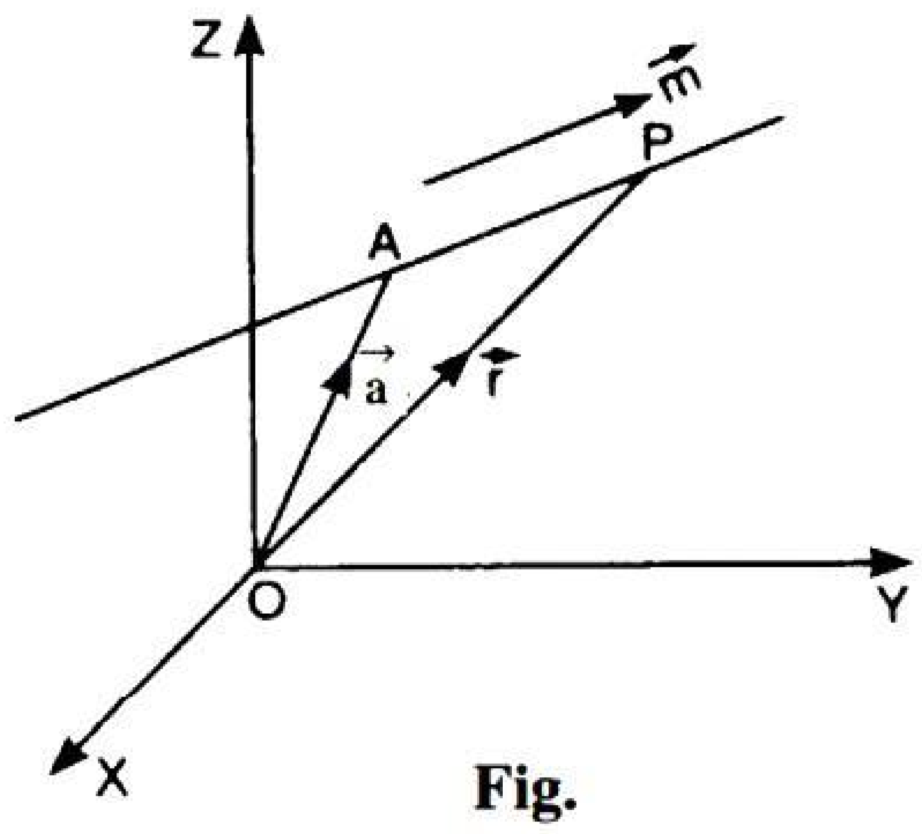


Fig.

which is the reqd. equation.

Cor. *The vector equation of a straight line through the origin and parallel to the vector* $\vec{m}$ *is* $\vec{r} = \lambda \vec{m}$.

Cartesian Form.

Let A (x_1, y_1, z_1) be the fixed point and let $< a, b, c >$ be the direction–ratios of the line.

Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$.

Putting in (1), $(x\hat{i} + y\hat{j} + z\hat{k}) = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k})$.

Comparing coeffs. of $\hat{i}, \hat{j}, \hat{k}$, $x = x_1 + \lambda a, y = y_1 + \lambda b, z = z_1 + \lambda c$ (*Parametric Form*)

$$\Rightarrow \quad \frac{\mathbf{x} - \mathbf{x_1}}{\mathbf{a}} = \frac{\mathbf{y} - \mathbf{y_1}}{\mathbf{b}} = \frac{\mathbf{z} - \mathbf{z_1}}{\mathbf{c}} (= \lambda) \qquad \text{(Symmetrical Form)}$$

Cor. *The equations of a straight line whose direction cosines are < l, m, n > and passing through the point* (x_1, y_1, z_1) *are* :

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

This result follows the fact that the direction-cosines of a line may also be direction-ratios of the line.

(b) To find the equation of a straight line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Let $\vec{a}, \vec{b}$ be the position vectors of the fixed points A and B respectively.

Let $\vec{r}$ be the position vector of *any* point P.

Since A, B, P are collinear,

$$\therefore \quad \overrightarrow{AP} = \lambda\, \overrightarrow{AB} \text{ for some scalar } \lambda$$

$$\Rightarrow \quad \overrightarrow{OP} - \overrightarrow{OA} = \lambda\,(\overrightarrow{OB} - \overrightarrow{OA})$$

$$\Rightarrow \quad \vec{r} - \vec{a} = \lambda\,(\vec{b} - \vec{a})$$

$$\Rightarrow \quad \boxed{\vec{\mathbf{r}} = \vec{\mathbf{a}} + \lambda\,(\vec{\mathbf{b}} - \vec{\mathbf{a}})} \qquad ...(1),$$

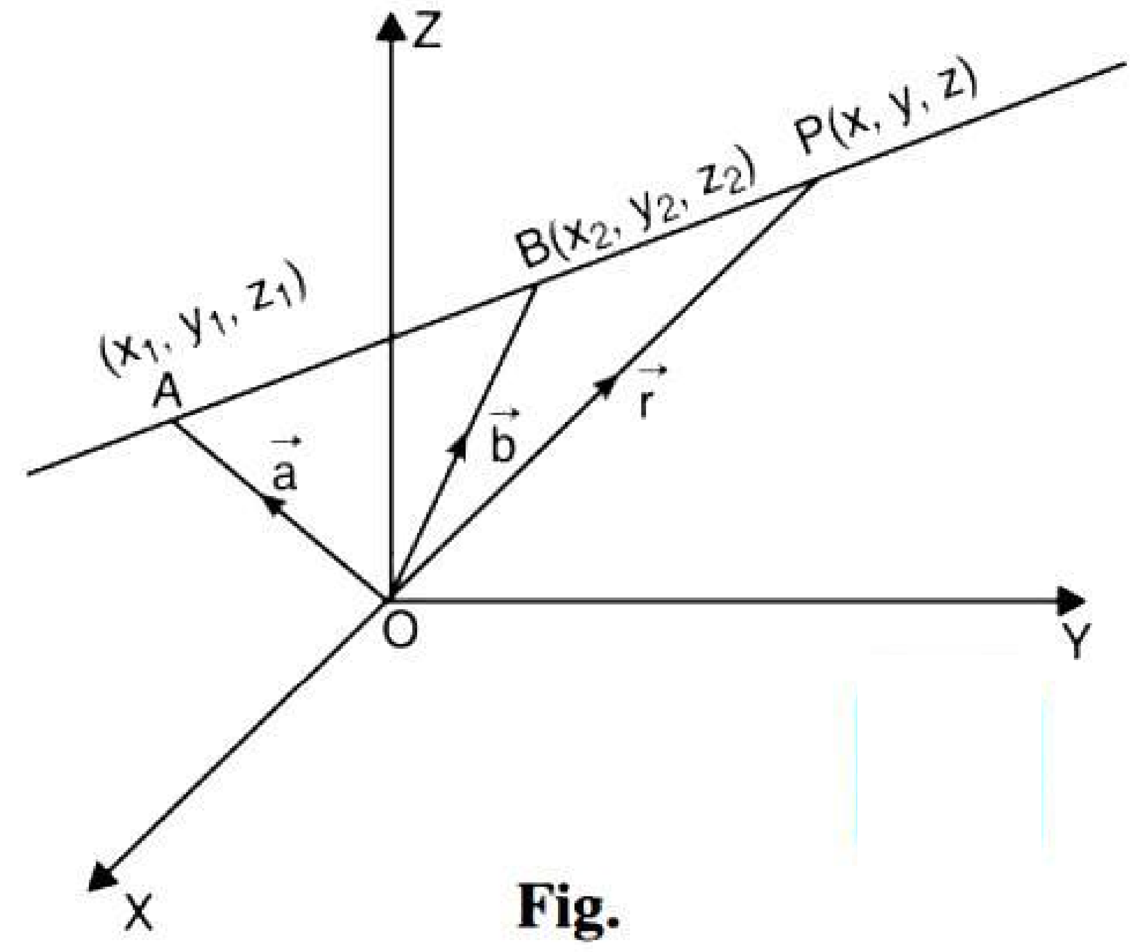


Fig.

which is the reqd. equation.

Cartesian Form :

Here $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}, \vec{a} = x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}, \vec{b} = x_2\,\hat{i} + y_2\,\hat{j} + z_2\,\hat{k}.$

Putting in (1), $x\,\hat{i} + y\,\hat{j} + z\,\hat{k} = (x_1\,\hat{i} + y_1\,\hat{j} + z_1\,\hat{k}) + \lambda\,((x_2 - x_1)\,\hat{i} + (y_2 - y_1)\,\hat{j} + (z_2 - z_1)\,\hat{k}).$

Comparing coeffs. of $\hat{i}, \hat{j}, \hat{k}$, $x = x_1 + \lambda\,(x_2 - x_1)$, $y = y_1 + \lambda\,(y_2 - y_1)$, $z = z_1 + \lambda\,(z_2 - z_1)$

$$\Rightarrow \quad \frac{\mathbf{x - x_1}}{\mathbf{x_2 - x_1}} = \frac{\mathbf{y - y_1}}{\mathbf{y_2 - y_1}} = \frac{\mathbf{z - z_1}}{\mathbf{z_2 - z_1}} \quad (= \lambda).$$

Cor. COLLINEARITY OF THREE POINTS

Find the equation of the st. line through any two given points. If the remaining third point satisfies the equation, then the three points are collinear.

11.12. ANGLE BETWEEN TWO LINES

(a) Vectorially :

Let $\quad \vec{r} = \vec{a} + \lambda\,\vec{b} \qquad ...(1)$

and $\quad \vec{r} = \vec{a}' + \mu\,\vec{b}' \qquad ...(2)$

be two straight lines in space.

Clearly, (1) and (2) are st. lines in the directions of $\vec{b}$ and $\vec{b}'$ respectively.

If 'θ' be the angle between the lines (1) and (2), then 'θ' is the angle between the directions of $\vec{b}$ and $\vec{b}'$.

[∵ *We know that the angle between two st. lines depends on their directions and not on their positions.*]

Now $\quad \vec{b}.\vec{b}' = |\vec{b}||\vec{b}'|\cos\theta$, which gives $\cos\theta = \dfrac{\vec{b}.\vec{b}'}{|\vec{b}||\vec{b}'|} \qquad ...(A)$

(b) Cartesian Form :

Let $\quad \dfrac{x - x_1}{b_1} = \dfrac{y - y_1}{b_2} = \dfrac{z - z_1}{b_3} \qquad ...(1)$

and $\quad \dfrac{x - x_1'}{b_1'} = \dfrac{y - y_1'}{b_2'} = \dfrac{z - z_1'}{b_3'} \qquad ...(2)$

be two straight lines in space.

Then $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{b'} = b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k}$, so that :

$$\vec{b}\,.\,\vec{b'} = (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})\,.\,(b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k}) = b_1b_1' + b_2b_2' + b_3b_3'$$

and $|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$ and $|\vec{b'}| = \sqrt{b_1'^2 + b_2'^2 + b_3'^2}$.

The result (A) of part (*a*) becomes : $\cos\theta = \dfrac{b_1b_1' + b_2b_2' + b_3b_3'}{\sqrt{b_1^2 + b_2^2 + b_3^2}\sqrt{b_1'^2 + b_2'^2 + b_3'^2}}$ ...(B)

Cor. 1. If < *l, m, n* > and < *l', m', n'* > be the direction–cosines of two lines, then :

$$l = \frac{b_1}{\sqrt{b_1^2 + b_2^2 + b_3^2}},\quad m = \frac{b_2}{\sqrt{b_1^2 + b_2^2 + b_3^2}},\quad n = \frac{b_3}{\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

and $l' = \dfrac{b_1'}{\sqrt{b_1'^2 + b_2'^2 + b_3'^2}}, m' = \dfrac{b_2'}{\sqrt{b_1'^2 + b_2'^2 + b_3'^2}}, n' = \dfrac{b_3'}{\sqrt{b_1'^2 + b_2'^2 + b_3'^2}}$.

The result (B) of part (*b*) becomes $\cos\theta = ll' + mm' + nn'$. ...(C)

Cor. 2. Condition of Perpendicularity (Orthogonality).

When the angle between two lines is $\theta = \dfrac{\pi}{2}$, then from (B), $b_1b_1' + b_2b_2' + b_3b_3' = 0$

and from (C), $ll' + mm' + nn' = 0$. $\left[\because \cos\dfrac{\pi}{2} = 0\right]$

ILLUSTRATIVE EXAMPLES

Example 1. Find the direction-cosines of the line :

$$\mathbf{\frac{x-1}{2} = -y = \frac{z+1}{2}}.$$ *(C.B.S.E. Sample Paper 2019)*

Solution. The given line is $\dfrac{x-1}{2} = \dfrac{y}{-1} = \dfrac{z+1}{2}$.

Its direction-ratios are <2, –1, 2>.

Hence, the direction-cosines of the line are :

$$<\frac{2}{\sqrt{4+1+4}}, \frac{-1}{\sqrt{4+1+4}}, \frac{2}{\sqrt{4+1+4}}>$$

i.e., $<\dfrac{2}{\sqrt{9}}, \dfrac{-1}{\sqrt{9}}, \dfrac{2}{\sqrt{9}}>$

i.e., $<\dfrac{2}{3}, -\dfrac{1}{3}, \dfrac{2}{3}>$ or $<-\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{-2}{3}>$.

Example 2. If the cartesian equations of a line are :

$$\mathbf{\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}},$$

write the vector equation for the line. *(A.I.C.B.S.E. 2014)*

Solution. The given line is $\dfrac{3-x}{5} = \dfrac{y+4}{7} = \dfrac{2z-6}{4}$

$$\Rightarrow \quad \frac{x-3}{-5} = \frac{y-(-4)}{7} = \frac{z-3}{2}.$$

Its vector equation is :

$$\vec{r} = (3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k}).$$

Example 3. Find the equation of the line, which passes through the point (1, 2, 3) and is parallel to the vector :

$$\mathbf{3\hat{i} + 2\hat{j} - 2\hat{k}}.$$ *(Kashmir B. 2017)*

Solution. The equation of the line is $\vec{r} = \vec{a} + \lambda\vec{b}$

i.e., $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$.

Example 4. Find the vector of the line joining (1, 2, 3) and (–3, 4, 3) and show that it is perpendicular to the *z*–axis. *(C.B.S.E. Sample Paper 2019)*

Solution. The given points are $\vec{a}$ = (1, 2, 3) and $\vec{b}$ = (– 3, 4, 3)

∴ Vector equation of the line joining (1, 2, 3) and (– 3, 4, 3) is :

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

$$\Rightarrow \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda((-3\hat{i} + 4\hat{j} + 3\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}))$$

$$\Rightarrow \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-4\hat{i} + 2\hat{j}) \quad ...(1)$$

Equation of z–axis is $\vec{r} = \mu\hat{k}$ ...(2)

Since $(-4\hat{i} + 2\hat{j}).\hat{k} = 0$

$\therefore$ Hence, (1) is perpendicular to z–axis.

Example 5. Find the vector equation of the line through (4, 3, – 1) and parallel to the line :

$$\mathbf{\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} - \hat{j} + 4\hat{k})}.$$ *(J. & K. B. 2011)*

Solution. The equation of the line is $\vec{r} = \vec{a} + \lambda\vec{b}$

i.e. $\vec{r} = (4\hat{i} + 3\hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j} + 4\hat{k})$.

Example 6. Find the angle between the following pair of lines :

$$\mathbf{\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})},$$
$$\mathbf{\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \mu(3\hat{i} + \hat{j} - 2\hat{k})}.$$

Solution. The given pair of lines is :

$$\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$$

and $\vec{r} = 2\hat{i} - \hat{j} + \hat{k} + \mu(3\hat{i} + \hat{j} - 2\hat{k})$.

Here $\vec{b} = \hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{b}' = 3\hat{i} + \hat{j} - 2\hat{k}$,

where $|\vec{b}| = \sqrt{1+9+4} = \sqrt{14}$

and $|\vec{b}'| = \sqrt{9+1+4} = \sqrt{14}$

so that $\vec{b}\cdot\vec{b}' = (\hat{i} - 3\hat{j} + 2\hat{k})\cdot(3\hat{i} + \hat{j} - 2\hat{k})$

$= (1)(3) + (-3)(1) + (2)(-2)$

$= 3 - 3 - 4 = -4$.

If 'θ' be the required angle, then :

$$\cos\theta = \frac{\vec{b}\cdot\vec{b}'}{|\vec{b}||\vec{b}'|} = \frac{-4}{\sqrt{14}\sqrt{14}} = \frac{-4}{14} = \frac{-2}{7}.$$

Hence, $\theta = \cos^{-1}\left(\frac{-2}{7}\right) = \pi - \cos^{-1}\left(\frac{2}{7}\right)$.

Example 7. Find the angle between the following pair of lines :

$$\mathbf{\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}} \textbf{ and } \mathbf{\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}}$$

and check whether the lines are parallel or perpendicular. *(C.B.S.E. 2011)*

Solution. The given lines can be rewritten as :

$$\frac{x-2}{2} = \frac{y-1}{7} = \frac{z+3}{-3} \quad ...(1)$$

and $$\frac{x+2}{-1} = \frac{y-4}{2} = \frac{z-5}{4} \quad ...(2)$$

Here < 2, 7, – 3 > and < – 1, 2, 4 > are direction-ratios of lines (1) and (2) respectively.

$$\therefore \quad \cos\theta = \frac{(2)(-1) + (7)(2) + (-3)(4)}{\sqrt{4+49+9}\sqrt{1+4+16}}$$

$$= \frac{-2+14-12}{\sqrt{62}\sqrt{21}} = 0$$

$$\Rightarrow \quad \theta = \frac{\pi}{2}.$$

Hence, the given lines are perpendicular.

Example 8. Find the point on the line :

$$\mathbf{\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}}$$

at a distance 5 from the point (1, 3, 3). *(A.I.C.B.S.E. 2010)*

Solution. Any point on the line :

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} \quad ....(1)$$

is P $(3k - 2, 2k - 1, 2k + 3)$ ...(2)

The given point is A (1, 3, 3).

By the question, | AP | = 5

$$\Rightarrow \sqrt{(3k-2-1)^2 + (2k-1-3)^2 + (2k+3-3)^2} = 5$$

$$\Rightarrow (3k-3)^2 + (2k-4)^2 + 4k^2 = 25$$

$$\Rightarrow 9k^2 - 18k + 9 + 4k^2 - 16k + 16 + 4k^2 = 25$$

$$\Rightarrow \quad 17k^2 - 34k = 0 \Rightarrow k = 0, 2.$$

Putting in (2), the reqd. point is :

(– 2, – 1, 3) or (6 – 2, 4 – 1, 4 + 3) *i.e.* (4, 3, 7).

Example 9. Find the equations of the line passing through the point (–1, 3, –2) and perpendicular to the lines :

$$\mathbf{\frac{x}{1} = \frac{y}{2} = \frac{z}{3}} \textbf{ and } \mathbf{\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}}.$$ *(C.B.S.E. 2012)*

Solution. The given lines are $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ ...(1)

and $$\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5} \quad ...(2)$$

Any line through (–1, 3, –2) is $\frac{x+1}{a} = \frac{y-3}{b} = \frac{z+2}{c}$ ...(3)

Since (3) is perpendicular to (1) and (2),

$\therefore \quad a(1) + b(2) + c(3) = 0 \Rightarrow a + 2b + 3c = 0$ ...(4)

and $a(-3) + b(2) + c(5) = 0 \Rightarrow 3a - 2b - 5c = 0$ ...(5)

Solving (4) and (5), $\frac{a}{-10+6} = \frac{b}{9+5} = \frac{c}{-2-6}$

$$\Rightarrow \quad \frac{a}{-4} = \frac{b}{14} = \frac{c}{-8} \Rightarrow \frac{a}{2} = \frac{b}{-7} = \frac{c}{4} \quad ...(6)$$

From (3) and (6), $\frac{x+1}{2}=\frac{y-3}{-7}=\frac{z+2}{4}$,

which are the reqd. equations.

Example 10. Find the Vector and Cartesian equations of the line through the point (1, 2, – 4) and perpendicular to the lines:

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$**.**

Solution. Let the line be :

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(a\hat{i} + b\hat{j} + c\hat{k}) \quad ...(1)$$

Since (1) is perpendicular to :

$$\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$$

and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$,

$\therefore \quad a(3) + b(-16) + c(7) = 0 \quad ...(2)$

and $a(3) + b(8) + c(-5) = 0 \quad ...(3)$

Solving, $\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$

$\Rightarrow \quad \frac{a}{24} = \frac{b}{36} = \frac{c}{72}$

$\Rightarrow \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{6} \quad ...(4)$

From (1) and (4), the vector equation of the line is:

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}).$$

Cartesian Form:

$$x\hat{i} + y\hat{j} + z\hat{k} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

$\Rightarrow x = 1 + 2\lambda, \quad y = 2 + 3\lambda \quad$ and $z = -4 + 6\lambda$

$\Rightarrow \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6} (= \lambda)$,

which is the required Cartesian equation.

Example 11. Show that, if the axes are rectangular, the equations of the line through (x_1, y_1, z_1) at right angles to the lines :

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}, \frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$$

are $\frac{x - x_1}{m_1n_2 - m_2n_1} = \frac{y - y_1}{n_1l_2 - n_2l_1} = \frac{z - z_1}{l_1m_2 - l_2m_1}$**.**

Solution. The two given lines are :

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1} \quad ...(1)$$

and $\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2} \quad ...(2)$

Any line through (x_1, y_1, z_1) is :

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad ...(3)$$

Since (3) is perp. to (1), $\therefore ll_1 + mm_1 + nn_1 = 0 \quad ...(4)$

Since (3) is perp. to (2), $\therefore ll_2 + mm_2 + nn_2 = 0 \quad ..(5)$

Solving (4) and (5),

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1} \quad ...(6)$$

From (3) and (6), $\frac{x - x_1}{m_1n_2 - m_2n_1} = \frac{y - y_1}{n_1l_2 - n_2l_1} = \frac{z - z_1}{l_1m_2 - l_2m_1}$,

which are the reqd. equations.

EXERCISE 11 (b)

Very Short Answer Type Questions

VSATQ

1. Write the vector equation of the line :

$\frac{x-5}{3} = \frac{y+4}{7} = \frac{6-z}{2}$.

(Tripura B. 2016; C.B.S.E. 2010)

2. Show that the three lines with direction-cosines :

$$\left\langle \frac{12}{13}, \frac{-3}{13}, \frac{-4}{13} \right\rangle; \left\langle \frac{4}{13}, \frac{12}{13}, \frac{3}{13} \right\rangle; \left\langle \frac{3}{13}, \frac{-4}{13}, \frac{12}{13} \right\rangle$$

are mutually perpendicular. *(N.C.E.R.T.)*

3. Express the following equations of the lines into vector form :

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$

and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$. *(Kerala B. 2013)*

4. (*a*) Find the cartesian as well as the vector equation of the line passing through :

(*i*) (–2, 4, –5) and parallel to the line :

$\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$ *(C.B.S.E. 2013)*

(*ii*) (0, – 1, 4) and parallel to the straight line :

$\frac{-x-2}{1} = \frac{y+3}{7} = \frac{2z-6}{3}$. *(Bihar B. 2014)*

(*iii*) (–1, 2, 3) and parallel to the line :

$\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-1}{6}$. *(H.B. 2013)*

(*b*) The cartesian equations of a line are :

(*i*) $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$

(N.C.E.R.T.; Jammu B. 2015; A.I.C.B.S.E. 2011)

(*ii*) $\frac{x+3}{2}=\frac{y-5}{4}=\frac{z+6}{2}$. (N.C.E.R.T.)

Find the vector equation of the lines.

(*c*) Find the vector equation of the line passing through the point A(1, 2, – 1) and parallel to the line :

$5x - 25 = 14 - 7y = 35z$. (C.B.S.E. 2017)

5. (*a*) Find the equation of a line parallel to *x*–axis and passing through the origin. (N.C.E.R.T.)

(*b*) Find the direction-cosines of a line parallel to the line :

$$\frac{2x-5}{4}=\frac{y+4}{3}=\frac{6-z}{6}.$$

(*c*) Write the direction-cosines of a line parallel to the line :

$$\frac{3-x}{3}=\frac{y+2}{-2}=\frac{z+2}{6}.$$ (C.B.S.E. 2009 C)

6. (*a*) Find the vector and cartesian equations of the line through the point (5, 2, – 4) and which is parallel to the vector $3\hat{i}+2\hat{j}-8\hat{k}$. (N.C.E.R.T.)

(*b*) Find the equation of a line passing through the point P(2, –1, 3) and perpendicular to the lines :

$$\vec{r}=(\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$$

and $$\vec{r}=(2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k}).$$

(C.B.S.E. 2012)

7. Find the equation of the line in vector and in cartesian form that passes through the point with position vector $2\hat{i}-\hat{j}+4\hat{k}$ and is in the direction of $\hat{i}+2\hat{j}-\hat{k}$.

(N.C.E.R.T.)

8. Find the vector equation for the line through the points :

(– 1, 0, 2) and (3, 4, 6).

(N.C.E.R.T.; Assam B. 2018; Kashmir B. 2016, Jammu B. 2016)

9. Find the vector and cartesian equations of the line that passes through :

(*i*) the origin and (5, – 2, 3) (N.C.E.R.T.)

(*ii*) the points (1, 2, 3) and (2, – 1, 4). (J. & K. B. 2011)

10. (*a*) Find the equation of a st. line through (– 1, 2, 3) and equally inclined to the axes.

(*b*) Find the equation of a line parallel to *x*-axis and passing through the origin.

11. Find the angle between the pairs of lines with direction–ratios :

(*i*) $< 5, - 12, 13 >$; $< - 3, 4, 5 >$

(*ii*) $< a, b, c >$; $< b - c, c - a, a - b >$. (N.C.E.R.T.)

12. Find the angle between a line with direction–ratios $< 2, 2, 1 >$ and a line joining (3, 1, 4) to (7, 2, 12).

13. Find the angle between the following pairs of lines :

(*i*) $\vec{r}=3\hat{i}+2\hat{j}-4\hat{k}+\lambda(\hat{i}+2\hat{j}+2\hat{k})$,

$\vec{r}=5\hat{j}-2\hat{k}+\mu(3\hat{i}+2\hat{j}+6\hat{k})$

(N.C.E.R.T. ; Karnataka B. 2014; Kerala B. 2014; Kashmir B. 2011)

(*ii*) $\vec{r}=3\hat{i}+\hat{j}-2\hat{k}+\lambda(\hat{i}-\hat{j}-2\hat{k})$

and $\vec{r}=(2\hat{i}-\hat{j}-56\hat{k})+\mu(3\hat{i}-5\hat{j}-4\hat{k})$

(H.P.B. 2016)

(*iii*) $\frac{x-2}{2}=\frac{y-1}{5}=\frac{z+3}{-3}$

and $\frac{x+2}{-1}=\frac{y-4}{8}=\frac{z-5}{4}$.

(Kerala B. 2018; H.P.B. 2017, 16, 13)

(*iv*) $\frac{x-4}{3}=\frac{y+1}{4}=\frac{z-6}{5}$ and $\frac{x-5}{1}=\frac{2y+5}{-2}=\frac{z-3}{1}$

(Meghalaya B. 2017)

(*v*) $\frac{5-x}{3}=\frac{y+3}{-4}$, $z=7$ and $x=\frac{1-y}{2}=\frac{z-6}{2}$

(H.P.B. 2016; Meghalaya B. 2015)

(*vi*) $\frac{x+3}{3}=\frac{y-1}{5}=\frac{z+3}{4}$ and $\frac{x+1}{1}=\frac{y-4}{1}=\frac{z-5}{2}$.

(N.C.E.R.T.; H.P.B. 2013 S, 13 ; Kashmir B. 2011)

14. Show that the lines :

(*i*) $\frac{x-5}{7}=\frac{y+2}{-5}=\frac{z}{1}$ and $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

(N.C.E.R.T.; H.B. 2017, 15; Kashmir B. 2011)

(*ii*) $\frac{x-3}{2}=\frac{y+1}{-3}=\frac{z-2}{4}$ and $\frac{x+2}{2}=\frac{y-4}{4}=\frac{z+5}{2}$

(Meghalaya B. 2013)

are perpendicular to each other.

15. (*i*) Find the value of '*p*' so that the lines :

$l_1:\frac{1-x}{3}=\frac{7y-14}{2p}=\frac{z-3}{2}$ and $l_2:\frac{7-7x}{3p}=\frac{y-5}{1}=\frac{6-z}{5}$

are at right angles. (Mizoram B. 2018)

Also find the equations of the line passing through (3, 2, – 4) and parallel to line l_1.

(N.C.E.R.T.; A.I.C.B.S.E. 2014)

(*ii*) Find '*k*' so that the lines :

$\frac{x-3}{2}=\frac{y+1}{3}=\frac{z-2}{2k}$ and $\frac{x+2}{1}=\frac{4-y}{k}=\frac{z+5}{1}$

are perpendicular to each other. (Nagaland B. 2015)

16. Show that the line through the points :

(*a*) (1, – 1, 2) , (3, 4, – 2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6)

(*b*) (4, 7, 8) , (2, 3, 4) is parallel to the line through the points (– 1, – 2, 1) and (1, 2, 5). (N.C.E.R.T.)

Short Answer Type Questions

SATQ

17. The cartesian equations of a line are :

$$3x + 1 = 6y - 2 = 1 - z.$$

Find the fixed point through which it passes, its direction-ratios and also its vector equation.

18. The points A (4, 5, 10), B (2, 3, 4) and C (1, 2, – 1) are three vertices of a parallelogram ABCD. Find the vector and cartesian equations for the sides AB and BC and find the co-ordinates of D. *(C.B.S.E. 2010)*

19. Write the equation of a line, parallel to the line $\frac{x-2}{-3} = \frac{y+3}{2} = \frac{z+3}{6}$ and passing through the point (1, 2, 3). *(A.I.C.B.S.E. 2009 C)*

20. Find the equation of the line perpendicular to the lines :

$$\vec{r} = (3\hat{i}+2\hat{j}-4\hat{k})+\lambda(\hat{i}+2\hat{j}-2\hat{k})$$

and $\vec{r} = (5\hat{j}-2\hat{k})+\mu(3\hat{i}+2\hat{j}+6\hat{k})$

and passing through the point (1, 1, 1). *(Kerala B. 2014)*

21. (*i*) Find the equations of the straight line passing through the point (2, 3, – 1) and is perpendicular to the lines : $\frac{x-2}{2} = \frac{y+1}{1} = \frac{z-3}{-3}$ and $\frac{x-3}{1} = \frac{y+2}{1} = \frac{z-1}{1}$. *(P.B. 2012, 10 S)*

(*ii*) Find the equation of the line which intersects the lines :

$$\frac{x+2}{1} = \frac{y-3}{2} = \frac{z+1}{4} \text{ and } \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

perpendicularly and passes through the point (1, 1, 1). *(C.B.S.E. Sample Paper 2018; W. Bengal 2018)*

22. Find the equation in vector and cartesian form of the line passing through the point :

(*i*) (2, – 1, 3) and perpendicular to the lines :

$\vec{r} = (\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$ and

$\vec{r} = (2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$ *(A.I.C.B.S.E. 2014; C.B.S.E. 2012)*

(*ii*) (2, – 1, 3) and perpendicular to the lines :

$\vec{r} = (\hat{i}-\hat{j}+\hat{k})+\lambda(2\hat{i}+\hat{j}+3\hat{k})$ and

$\vec{r} = (\hat{i}+\hat{j}-\hat{k})+\mu(\hat{i}+\hat{j}+\hat{k})$. *(P.B. 2011)*

23. Prove that the points (1, 2, 3), (4, 0, 4), (– 2, 4, 2) and (7, – 2, 5) are collinear.

24. Show that the following points whose position vectors are given are collinear :

(*i*) $5\hat{i}+5\hat{k}, 2\hat{i}+\hat{j}+3\hat{k}$ and $-4\hat{i}+3\hat{j}-\hat{k}$

(*ii*) $-2\hat{i}+3\hat{j}+5\hat{k}, \hat{i}+2\hat{j}+3\hat{k}$ and $7\hat{i}-\hat{k}$. *(P.B. 2014 S)*

25. Find the points on the line through the points A(1, 2, 3) and B (5, 8, 15) at a distance of 14 units from the mid-point of AB. *(Meghalaya B. 2016)*

26. Find the equations of the perpendicular from the point (3, –1, 11) to the line :

$$\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}.$$

27. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points : (3, 5, – 1), (4, 3, – 1). *(N.C.E.R.T.)*

Long Answer Type Questions

LATQ

28. (*i*) Find the vector and cartesian equation of the line passing through the point (1, 2, – 4) and perpendicular to the two lines :

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

(N.C.E.R.T.; C.B.S.E. 2017, 12; H.P.B. 2015; Jammu B. 2015, 13, 12; Meghalaya B. 2015; P.B. 2012)

(*ii*) Find the vector and cartesian equations of the line passing through the point (2, 1, 3) and perpendicular to the lines :

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3} \text{ and } \frac{x}{-3} = \frac{y}{2} = \frac{z}{5}.$$

(Type : Assam B. 2017, A.I.C.B.S.E. 2014)

29. (*i*) Find the vector equation of a line passing through a point with position vector $2\hat{i}-\hat{j}+\hat{k}$ and parallel to the line joining the points with position vectors $-\hat{i}+4\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+2\hat{k}$. Also, find the cartesian equivalent of the equation.

(*ii*) Find the vector equation of a line passing through the point with position vector $\hat{i}-2\hat{j}-3\hat{k}$ and parallel to the line joining the points with position vectors $\hat{i}-\hat{j}+4\hat{k}$ and $2\hat{i}+\hat{j}+2\hat{k}$. Also find the cartesian form of the equation.

HOTS

Answers

1. $\vec{r} = (5\hat{i}-4\hat{j}+6\hat{k})+\lambda(3\hat{i}+7\hat{j}-2\hat{k})$.

3. $\vec{r} = (3\hat{i}+8\hat{j}+3\hat{k})+\lambda(3\hat{i}-\hat{j}+\hat{k})$ and $\vec{r} = (-3\hat{i}-7\hat{j}+6\hat{k})+\mu(-3\hat{i}+2\hat{j}+4\hat{k})$.

4. (*a*) (*i*) $\frac{x+2}{3} = \frac{y-4}{-5} = \frac{z+5}{6}$;

$\vec{r} = (-2\hat{i} + 4\hat{j} - 5\hat{k}) + \lambda(3\hat{i} - 5\hat{j} + 6\hat{k})$

(*ii*) $\frac{x}{-1} = \frac{y+1}{7} = \frac{2(z-4)}{3}$;

$\vec{r} = (-\hat{j} + 4\hat{k}) + \lambda\left(-\hat{i} + 7\hat{j} + \frac{3}{2}\hat{k}\right)$

(*iii*) $\frac{x+1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$;

$\vec{r} = (-\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

(*b*) (*i*) $\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$

(*ii*) $\vec{r} = -3\hat{i} + 5\hat{j} - 6\hat{k} + \lambda(2\hat{i} + 4\hat{j} + 2\hat{k})$

(*c*) $\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k})$.

5. (*a*) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$ (*b*) $< \frac{2}{7}, \frac{3}{7}, \frac{-6}{7} >$

(*c*) $< -\frac{3}{7}, \frac{-2}{7}, \frac{6}{7} >$.

6. (*a*) $\vec{r} = (5+3\lambda)\hat{i} + (2+2\lambda)\hat{j} + (-4-8\lambda)\hat{k}$;

$\frac{x-5}{3} = \frac{y-2}{2} = \frac{z+4}{-8}$

(*b*) $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(2\hat{i} + \hat{j} - 2\hat{k})$.

7. $\vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + 2\hat{j} - \hat{k})$;

$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$.

8. $\vec{r} = -\hat{i} + 2\hat{k} + \lambda(4\hat{i} + 4\hat{j} + 4\hat{k})$.

9. (*i*) $\vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$; $\frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$

(*ii*) $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + \hat{k})$;

$\frac{x-1}{1} = \frac{y-2}{-3} = \frac{z-3}{1}$.

10. (*a*) $x + 1 = y - 2 = z - 3$ (*b*) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$.

11. (*i*) $\cos^{-1}\left(\frac{1}{65}\right)$ (*ii*) $\frac{\pi}{2}$.

12. $\cos^{-1}\frac{2}{3}$.

13. (*i*) $\cos^{-1}\left(\frac{19}{21}\right)$ (*ii*) $\cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$

(*iii*) $\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$ (*iv*) $\cos^{-1}\left(\frac{4}{5\sqrt{6}}\right)$

(*v*) $\cos^{-1}\left(\frac{5}{9}\right)$ (*vi*) $\cos^{-1}\left(\frac{8}{5\sqrt{2}}\right)$.

15. (*i*) $p = \frac{70}{11}$; $\frac{3-x}{3} = \frac{11y-22}{140} = \frac{z+4}{2}$

(*ii*) $k = 2$.

17. $\left(-\frac{1}{3}, \frac{1}{3}, 1\right)$; $< 2, 1, -6 >$;

$\vec{r} = (-\frac{1}{3}\hat{i} + \frac{1}{3}\hat{j} + \hat{k}) + \lambda(2\hat{i} + \hat{j} - 6\hat{k})$.

18. $\vec{r} = 4\hat{i} + 5\hat{j} + 10\hat{k} + \lambda(\hat{i} + \hat{j} + 3\hat{k})$;

$4 - x = 5 - y = \frac{10-z}{3}$;

$\vec{r} = 2\hat{i} + 3\hat{j} + 4\hat{k} + (\hat{i} + \hat{j} + 5\hat{k})$;

$2 - x = 3 - y = \frac{4-z}{5}$; (3, 4, 5).

19. $\frac{x-1}{-3} = \frac{y-2}{2} = \frac{z-3}{6}$.

20. $\vec{r} = (\hat{i} + \hat{j} + \hat{k}) + \lambda(16\hat{i} - 12\hat{j} - 4\hat{k})$.

21. (*i*) $\frac{x-2}{4} = \frac{y-3}{-5} = \frac{z+1}{1}$

(*ii*) $\frac{x-1}{4} = \frac{y-1}{-4} = \frac{z+1}{1}$.

22. (*i*) $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(2\hat{i} + \hat{j} - 2\hat{k})$;

$\frac{x-2}{2} = \frac{y+1}{1} = \frac{z-3}{-2}$

(*ii*) $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(4\hat{i} - 5\hat{j} + \hat{k})$;

$\frac{x-2}{4} = \frac{y+1}{-5} = \frac{z-3}{1}$.

25. (7, 11, 21) and (–1, –1, –3).

26. $\frac{x-3}{2} = \frac{y+1}{5} = \frac{z-11}{7}$.

28. (*i*) $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$;

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$

(*ii*) $\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k})$;

$\frac{x-2}{2} = \frac{y-1}{-7} = \frac{z-3}{4}$.

29. (*i*) $\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$;

$\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$

(*ii*) $\vec{r} = \hat{i} - 2\hat{j} - 3\hat{k} + \lambda(\hat{i} + 2\hat{j} - 2\hat{k})$;

$\frac{x-1}{1} = \frac{y+2}{2} = \frac{z+3}{-2}$.

Hints to Selected Questions

4. (*b*) (*i*) The line passes through (5, – 4, 6) and has direction-cosines < 3, 7, 2 >.

10. (*a*) Equations of the line are $\frac{x+1}{1}=\frac{y-2}{1}=\frac{z-3}{1}$

$\Rightarrow\ x+1=y-2=z-3.$

15. (*i*) Lines are $l_1:\frac{x-1}{-3}=\frac{y-2}{2p/7}=\frac{z-3}{2}$

and $l_2:\frac{x-1}{-3p/7}=\frac{y-5}{1}=\frac{z-6}{-5}.$

Lines are perpendicular

if $(-3)\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right)(1)+(2)(-5)=0$

if $p=\frac{70}{11}.$

18. Mid–points of [AC] and [BD] are same.

23. Equations of AB are $\frac{x-1}{3}=\frac{y-2}{-2}=\frac{z-3}{1}.$

C lies on it.

29. (*i*) Here $\vec{b}=(\hat{i}+\hat{i})+(2\hat{j}-4\hat{j})+(2\hat{k}-\hat{k})$

$=2\hat{i}-2\hat{j}+\hat{k}$ and $\vec{a}=2\hat{i}-\hat{j}+\hat{k}.$

∴ The equation of the line is :

$\vec{r}=2\hat{i}-\hat{j}+\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$; etc.

11.13. PERPENDICULAR DISTANCE OF A POINT FROM A LINE

To find the perpendicular distance of the point (α , β , γ) from the line :

$$\frac{\mathbf{x}-\mathbf{x_1}}{\mathbf{l}}=\frac{\mathbf{y}-\mathbf{y_1}}{\mathbf{m}}=\frac{\mathbf{z}-\mathbf{z_1}}{\mathbf{n}}.$$

The given line L is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ...(1)

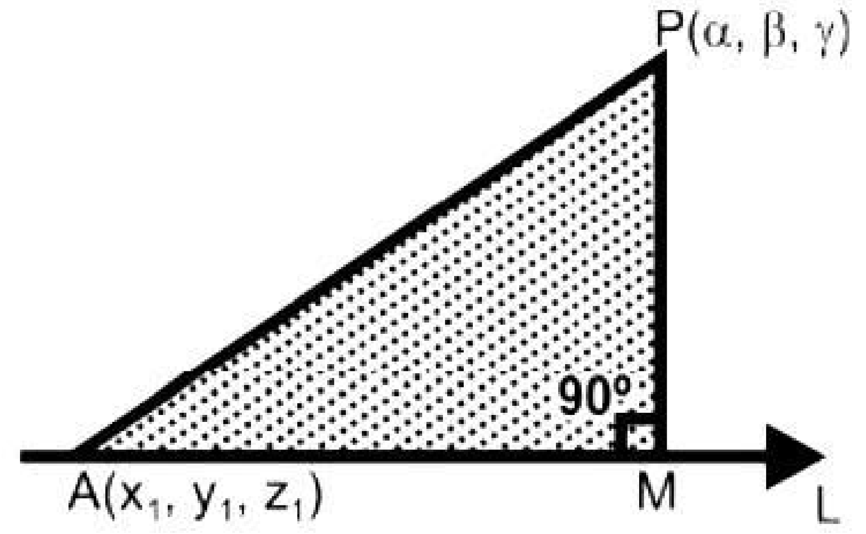


Fig.

Let P (α, β, γ) be the given point.

Let A be the point (x_1, y_1, z_1).

Join AP. Draw PM perp. on the line L.

In ΔAPM, *by Pythagoras' Theorem,* we have :

$PM^2 + AM^2 = AP^2$ *i.e.* $PM^2 = AP^2 - AM^2$ ...(2)

But $AP^2 = (x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2$

and | AM | = Projection [AP] on the line L = $(x_1-\alpha)\,l+(y_1-\beta)\,m+(z_1-\gamma)\,n$,

where < *l, m, n* > are direction-cosines of L

$\Rightarrow AM^2=[(x_1-\alpha)\,l+(y_1-\beta)\,m+(z_1-\gamma)\,n]^2.$

Putting in (2), we get : $PM^2=[(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2]-[(x_1-\alpha)\,l+(y_1-\beta)\,m+(z_1-\gamma)\,n]^2$,

which gives the required perpendicular distance.

GUIDE LINES

Step (i) *Find | AP |, by distance formula, and hence find AP^2.*

Step (ii) *Find | AM |, by projection formula, and hence find AM^2.*

Step (iii) *Find | PM |, from the relation $PM^2 = AP^2 - AM^2$.*

ILLUSTRATIVE EXAMPLES

Example 1. Find the length of the perpendicular from the point (3, 4, 5) on the line $\frac{x-2}{2}=\frac{y-3}{5}=\frac{z-1}{3}$.

Solution. Let P (3, 4, 5) be the given point and L the given line : $\frac{x-2}{2}=\frac{y-3}{5}=\frac{z-1}{3}$.

Let A be a point (2, 3, 1) on L.

The direction-ratios of L are < 2, 5, 3 >.

Its direction-cosines are :

$$<\frac{2}{\sqrt{4+25+9}}, \frac{5}{\sqrt{4+25+9}}, \frac{3}{\sqrt{4+25+9}}>$$

i.e. $<\frac{2}{\sqrt{38}}, \frac{5}{\sqrt{38}}, \frac{3}{\sqrt{38}}>$.

Now $PM^2 = AP^2 - AM^2$ ...(1)

[*Pythagoras' Theorem*]

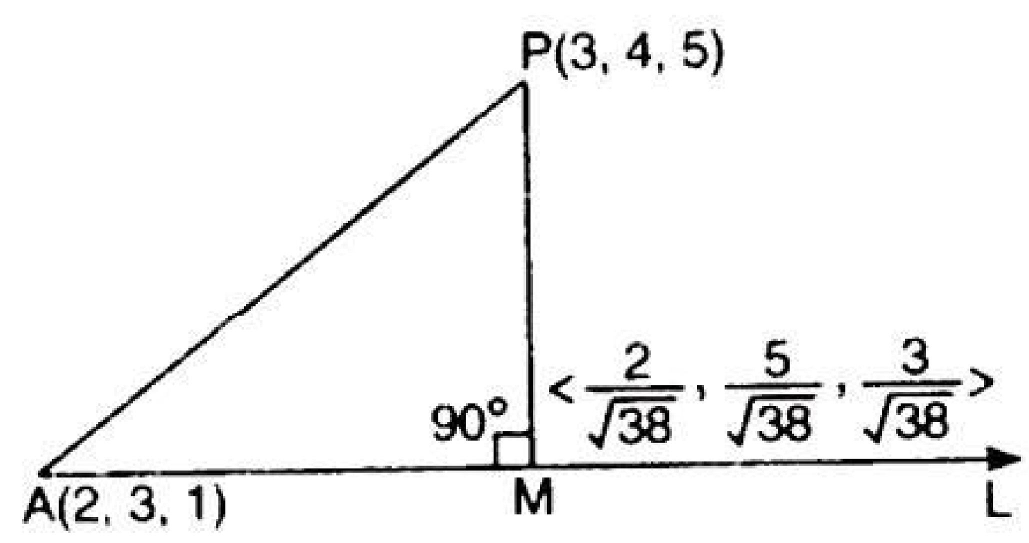


Fig.

But $|AP| = \sqrt{(3-2)^2+(4-3)^2+(5-1)^2}$

[*Distance Formula*]

$$= \sqrt{1+1+16} = \sqrt{18}$$

and | AM | = Projection of [AP] on AL

$$= (3-2)\frac{2}{\sqrt{38}} + (4-3)\frac{5}{\sqrt{38}} + (5-1)\frac{3}{\sqrt{38}}$$

[*Using* $(x_2-x_1)\,l + (y_2-y_1)\,m + (z_2-z_1)\,n$]

$$= \frac{2}{\sqrt{38}} + \frac{5}{\sqrt{38}} + \frac{12}{\sqrt{38}} = \frac{19}{\sqrt{38}}.$$

Putting these values in (1), we get :

$$PM^2 = (\sqrt{18})^2 - \left(\frac{19}{\sqrt{38}}\right)^2 = 18 - \frac{361}{38} = 18 - \frac{19}{2} = \frac{17}{2}.$$

Hence, $|PM| = \sqrt{\frac{17}{2}}$ units,

which is the required perpendicular distance.

Example 2. Find the co-ordinates of the foot of the perpendicular drawn from the point A (1, 8, 4) to the line joining B (0, – 1, 3) and C (2, – 3, – 1). *(A.I.C.B.S.E. 2016)*

Solution.

Any point on BC, which divides [BC] in the ratio $k : 1$, is :

$$\left(\frac{2k}{k+1}, \frac{-3k-1}{k+1}, \frac{-k+3}{k+1}\right) \quad ...(1)$$

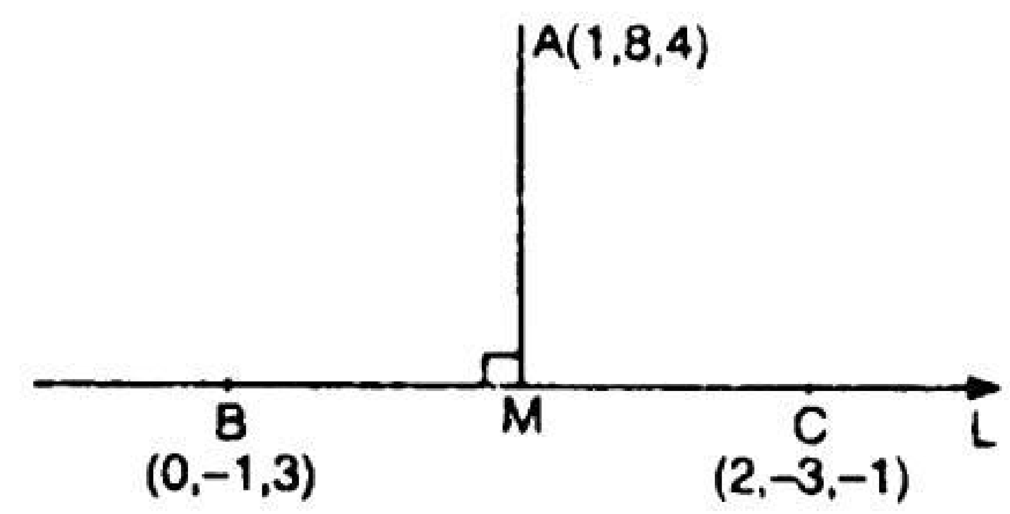


Fig.

This becomes M, the foot of perp. from A on BC

if AM ⊥ BC ...(2)

But direction-ratios of BC are :

< 2 – 0, – 3 + 1, – 1 – 3 > *i.e.* < 2, – 2, – 4 >

i.e. < 1, – 1, – 2 >

and direction-ratios of AM are :

$$<\frac{2k}{k+1}-1, \frac{-3k-1}{k+1}-8, \frac{-k+3}{k+1}-4>$$

i.e. $< k-1, -11k-9, -5k-1 >$

∴ Due to (2), $(1)(k-1) + (-1)(-11k-9) + (-2)(-5k-1) = 0$

$\Rightarrow \quad k - 1 + 11k + 9 + 10k + 2 = 0$

$\Rightarrow \quad 22k + 10 = 0 \quad \Rightarrow \quad k = -\frac{5}{11}.$

∴ From (1), the co-ordinates of M, the foot of perp. are :

$$\left(\frac{\frac{-10}{11}}{\frac{-5}{11}+1}, \frac{\frac{15}{11}-1}{\frac{-5}{11}+1}, \frac{\frac{5}{11}+3}{\frac{-5}{11}+1}\right)$$

i.e. $\left(\frac{-10}{6}, \frac{4}{6}, \frac{38}{6}\right)$ *i.e.* $\left(\frac{-5}{3}, \frac{2}{3}, \frac{19}{3}\right)$.

Example 3. Find the vector equation of the line parallel to the line :

$$\frac{x-1}{5}=\frac{3-y}{2}=\frac{z+1}{4}$$

and passing through (3, 0, – 4).

Also, find the distance between these two lines.

Solution. (*i*) The given line is $\frac{x-1}{5}=\frac{y-3}{-2}=\frac{z-(-1)}{4}$ ...(1)

Its direction–ratios are < 5, – 2, 4 >.

The vector equation of the line through (3, 0, – 4) parallel to (1) is :

$$\vec{r}=\left(3\hat{i}-4\hat{k}\right)+\lambda\left(5\hat{i}-2\hat{j}+4\hat{k}\right) \quad ...(2)$$

(*ii*) The cartesian equations of (2) are :

$$\frac{x-3}{5}=\frac{y-0}{-2}=\frac{z+4}{4} \quad ...(3)$$

To find the distance between (1) and (3) :

A (1, 3, – 1) is a point on (1).

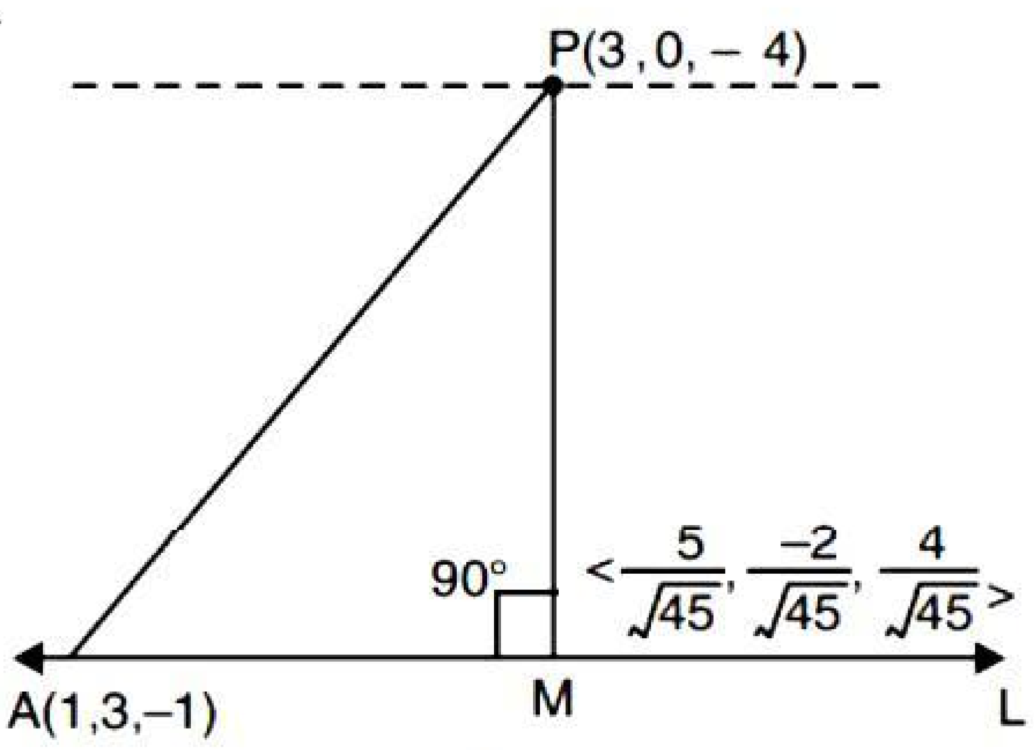


Fig.

The direction-cosines of (1) are :

$$\left\langle\frac{5}{\sqrt{25+4+16}},\frac{-2}{\sqrt{25+4+16}},\frac{4}{\sqrt{25+4+16}}\right\rangle$$

i.e. $\left\langle\frac{5}{\sqrt{45}},\frac{-2}{\sqrt{45}},\frac{4}{\sqrt{45}}\right\rangle$.

Now $\quad PM^2 = AP^2 - AM^2$ ...(4)

[*Pythagoras' Theorem*]

But $\quad |AP| = \sqrt{(3-1)^2+(0-3)^2+(-4+1)^2}$

$= \sqrt{4+9+9}=\sqrt{22}$

and $\quad |AM| = (3-1)\left(\frac{5}{\sqrt{45}}\right)+(0-3)\left(-\frac{2}{\sqrt{45}}\right)+(-4+1)\left(\frac{4}{\sqrt{45}}\right)$

$= \frac{10+6-12}{\sqrt{45}}=\frac{4}{\sqrt{45}}$.

Putting in (4), $PM^2 = 22-\frac{16}{45}=\frac{990-16}{45}=\frac{974}{45}$.

Hence, $\quad |PM| = \sqrt{\frac{974}{45}}$ units.

Example 4. Find the equations of the perpendicular drawn from the point P (2, 4, – 1) to the line :

$$\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}.$$

Also, write down the co-ordinates of the foot of the perpendicular from P to the line.

Solution. The given line AB is $\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$ ...(1)

Any point on line (1) is :
$(k-5, 4k-3, -9k+6)$.

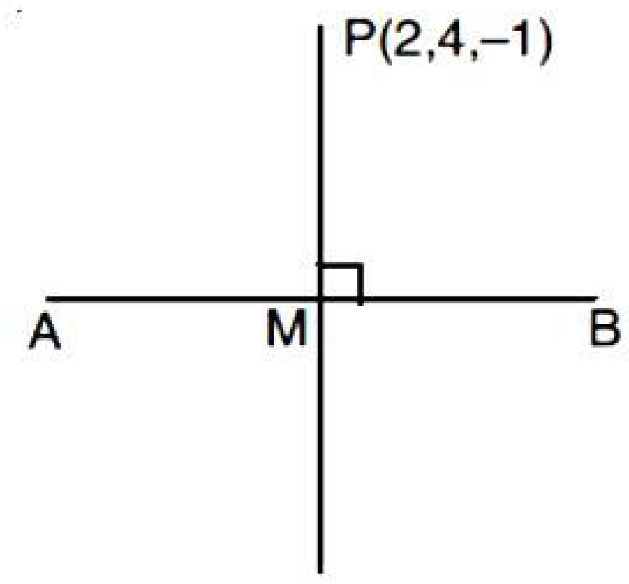


Fig.

For some value of k, this point is M such that PM is perp. to line (1).

Now direction-ratios of the line are < 1, 4, – 9 >

and direction-ratios of PM are :

$< k-5-2, 4k-3-4, -9k+6+1 >$

i.e. $< k-7, 4k-7, -9k+7 >$.

$\therefore$ (1) $(k-7)$ + (4) $(4k-7)$ + (– 9) $(-9k+7) = 0$

$\Rightarrow k-7+16k-28+81k-63=0$

$\Rightarrow \quad 98k = 98 \Rightarrow k = 1$.

$\therefore$ M is (1 – 5, 4 – 3, – 9 + 6) *i.e.* (– 4, 1, – 3).

$\therefore$ The equations of PM are :

$$\frac{x-2}{-4-2}=\frac{y-4}{1-4}=\frac{z+1}{-3+1} \Rightarrow \frac{x-2}{-6}=\frac{y-4}{-3}=\frac{z+1}{-2}$$

i.e. $\frac{x-2}{6}=\frac{y-4}{3}=\frac{z+1}{2}$.

Hence, the equations of PM are :

$$\frac{x-2}{6}=\frac{y-4}{3}=\frac{z+1}{2}$$

and foot of perpendicular is (– 4, 1, –3).

REFLECTION OR IMAGE OF A POINT ON A STRAIGHT LINE

If the perpendicular PM from P on the st. line AB be produced to P' such that PM = MP', then P' is called the image or reflection of P in the given line.

[For fig., see Ex. 5]

Example 5. (a) Find the image of the point (1, 6, 3) in the line :

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}.$$

(C.B.S.E. 2010 C)

(b) Also, write the equation of the line joining the given point and its image and find the length of the segment joining the given point and its image. *(C.B.S.E. 2010 C)*

Solution. (*a*) Let P be the given point (1, 6, 3) and M, the foot of perpendicular from P on the given line AB :

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3} \quad (= k \text{ (say)})$$

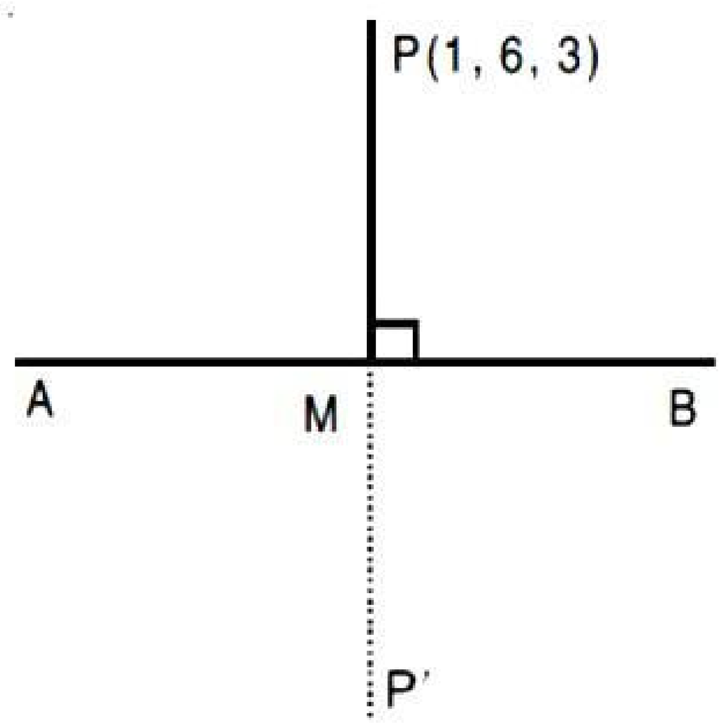


Fig.

Any point on the given line is :

$$(k\ ,\ 1 + 2k\ ,\ 2 + 3k).$$

For some value of k, let the point be M.

$\therefore$ Direction-ratios of PM are :

$$< k - 1\ ,\ 1 + 2k - 6,\ 2 + 3k - 3 >$$

i.e. $< k - 1,\ 2k - 5,\ 3k - 1 >$.

Since PM $\perp$ AB ,

$\therefore$ (1) $(k - 1)$ + (2) $(2k - 5)$ + (3) $(3k - 1) = 0$

$\Rightarrow$ $k - 1 + 4k - 10 + 9k - 3 = 0$

$\Rightarrow$ $14k = 14$ $\Rightarrow$ $k = 1$.

$\therefore$ Foot of perpendicular M is (1, 1+2, 2+3) *i.e.* (1,3, 5).

Let P′(α, β, γ) be the image of P in the given line.

Then M is the mid-point of [PP′].

$$\therefore \quad \frac{\alpha+1}{2}=1, \frac{\beta+6}{2}=3, \frac{\gamma+3}{2}=5$$

$\Rightarrow$ $\alpha + 1 = 2,\ \beta + 6 = 6,\ \gamma + 3 = 10$

$\Rightarrow$ $\alpha = 1,\ \beta = 0,\ \gamma = 7$.

Hence, the reqd. image is (1, 0, 7).

(*b*) (*i*) The equations of the line PP′ are :

$$\frac{x-1}{1-1}=\frac{y-6}{0-6}=\frac{z-3}{7-3}$$

$$\text{i.e.} \quad \frac{x-1}{0}=\frac{y-6}{-6}=\frac{z-3}{4}$$

$$\Rightarrow \quad \frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}.$$

(*ii*) Length of segment [PP′]

$$= \sqrt{(1-1)^2+(0-6)^2+(7-3)^2}$$

$$= \sqrt{0+36+16} = \sqrt{52} = 2\sqrt{13} \text{ units.}$$

Another Form

Find the image of the point (1, 6, 3) in the line :

$$\vec{r} = \hat{j} + 2\hat{k} + \lambda(\hat{i} + 2\hat{j} + 3\hat{k}).$$

Solution. The cartesian equations of given vector equation are :

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}.$$

Now it is same as above.

Example 6. Find the co-ordinates of the foot of perpendicular and the length of the perpendicular drawn from the point P (5, 4, 2) to the line :

$$\vec{r} = -\hat{i} + 3\hat{j} + \hat{k} + \lambda(2\hat{i} + 3\hat{j} - \hat{k}).$$

Also, find the image of P in this line.

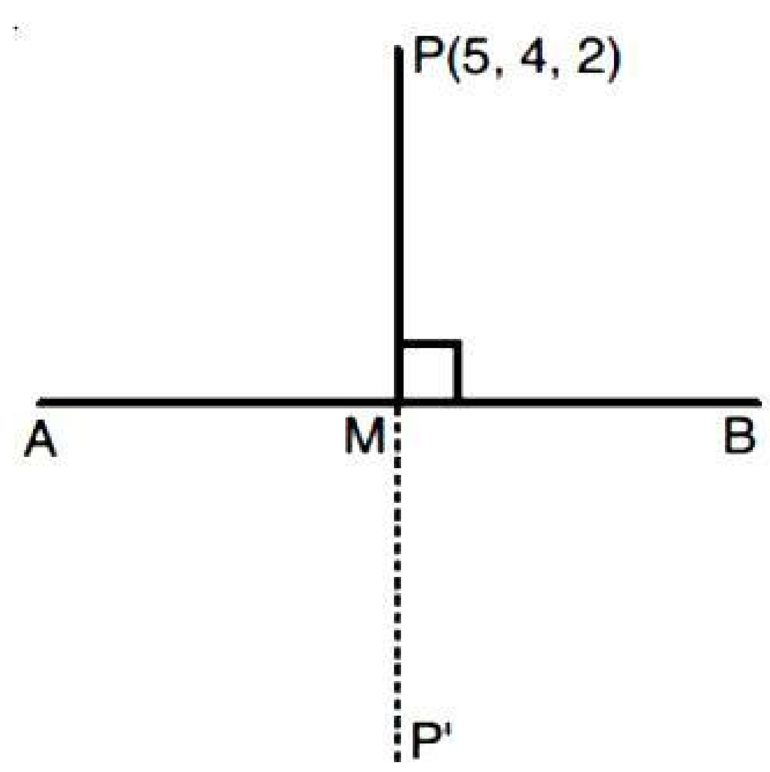
(Mizoram B. 2016; A.I.C.B.S.E. 2012)

Solution. (*i*) The given line is :

$$\vec{r} = -\hat{i}+3\hat{j}+\hat{k}+\lambda(2\hat{i}+3\hat{j}-\hat{k})$$

$$\text{i.e.} \quad \frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1} \qquad ...(1)$$

The given point is P (5, 4, 2).



Let M be the foot of perpendicular from P on the given line AB.

Any point on (1) is $(-1 + 2k,\ 3 + 3k,\ 1 - k)$.

For some value of k, let the point be M.

$\therefore$ Direction-ratios of PM are :

$< -1 + 2k - 5,\ 3 + 3k - 4,\ 1 - k - 2>$

i.e. $< 2k - 6,\ 3k - 1,\ - k - 1>$.

Since PM $\perp$ AB,

$\therefore$ $2(2k - 6) + 3(3k - 1) + (-1)(-k - 1) = 0$

$\Rightarrow$ $4k - 12 + 9k - 3 + k + 1 = 0$

$\Rightarrow$ $14k - 14 = 0 \Rightarrow k = 1$.

$\therefore$ Foot of perpendicular M is (– 1 + 2, 3 + 3, 1 – 1) *i.e.* (1, 6, 0).

(*ii*) Length of Perpendicular

$$= \sqrt{(5-1)^2+(4-6)^2+(2-0)^2}$$

$$= \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6} \text{ units.}$$

(*iii*) Let P′ (α, β, γ) be the image of P in the given line. Then M is the mid-points of [PP′].

$$\therefore \quad \frac{\alpha+5}{2} = 1, \frac{\beta+4}{2} = 6, \frac{\gamma+2}{2} = 0$$

$\Rightarrow \quad \alpha + 5 = 2, \beta + 4 = 12, \gamma + 2 = 0$

$\Rightarrow \quad \alpha = -3, \beta = 8, \gamma = -2.$

Hence, the reqd. image is (–3, 8, –2).

EXERCISE 11 (c)

Short Answer Type Questions

SATQ

1. Find the distance of the point (1, 2, 3) from the line joining the points (– 1, 2, 5) and (2, 3, 4).

2. Find the distance of the point (1, 2, 3) from the co-ordinate axes.

3. Find the distance of (– 1, 2, 5) from the line passing through the point (3, 4, 5) and whose direction-ratios are < 2, – 3, 6 >.

4. Find the perpendicular distance of the point (1, 0, 0) from the line :

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}.$$

Also find the co-ordinates of the foot of the perpendicular.

5. (*a*) Find the length of the perpendicular from the point (1, 2, 3) to the line :

$$\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}.$$

(*b*) Find the perpendicular distance from the point (1, 2, 3) to the line :

$$\vec{r} = 6\hat{i} + 7\hat{j} + 7\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k}).$$

6. (*a*) Find the foot of the perpendicular from the point (*i*) (2, – 1, 5) on the line :

$$\frac{x-11}{10} = \frac{y+2}{-5} = \frac{z+8}{11}$$ ***(C.B.S.E. 2009 C)***

(*ii*) (0, 2, 3) on the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$.

(J. & K.B. 2011; C.B.S.E. 2009)

(*b*) Also find the length of perpendicular in part (*ii*). ***(C.B.S.E. 2009)***

7. Find the co-ordinates of the foot of the perpendicular from the point A (1, 0, 3) to the line joining B (4, 7, 1) and C (3, 5, 3).

8. A (1, 0, 4), B (0, – 11, 3), C (2, – 3, 1) are three points and D is the foot of the perpendicular from A on BC. Find the co-ordinates of D.

9. Find the perpendicular distance of an angular point of a cube from a diagonal, which does not pass through that angular point.

Long Answer Type Questions

LATQ

10. Find the image of the point :

(*a*) (*i*) (2, 0, 1) in the line $\frac{x-3}{1} = \frac{y+2}{-2} = \frac{z-3}{5}$ ***(P.B. 2014)***

(*ii*) (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$. ***(H.B. 2018)***

(*b*) Find the image of the point A(–1, 8, 4) in the line joining the points B (0, –1, 3) and C (2, –3, –1). ***(A.I.C.B.S.E. 2016)***

11. Let the point P (5, 9, 3) lie on the top of Qutub Minar, Delhi. Find the image of the point on the line :

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}.$$

12. Find the foot of the perpendicular from the point (1, 2, 3) to the line joining the points (6, 7, 7) and (9, 9, 5). ***(Nagaland B. 2018)***

13. Find the length and the foot of the perpendicular drawn from the point (2, – 1, 5) to the line :

$$\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}.$$ ***(H.B. 2018)***

14. Find the equations of the perpendicular drawn from the point (2, 4, – 1) to the line :

$$\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}.$$

15. Find the perpendicular distance of the point (2, 3, 4) from the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$.

Also find the co-ordinates of the foot of the perpendicular. ***(C.B.S.E. 2009 C)***

16. Find the equations of the perpendicular from the point (3, – 1, 11) to the line :

$$\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}.$$

Also, find the foot of the perpendicular and the length of the perpendicular. ***(H.B. 2018)***

17. A line passing through the point A with position vector $\vec{a} = 4\hat{i} + 2\hat{j} + 2\hat{k}$ is parallel to the vector $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. Find the length of the perpendicular drawn on this line from a point P with position vector $\vec{r}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$. ***(A.I.C.B.S.E. 2015)***

Answers

1. $\sqrt{\frac{24}{11}}$.
2. $\sqrt{13}, \sqrt{10}, \sqrt{5}$.
3. $\frac{\sqrt{976}}{7}$.
4. $2\sqrt{6}, (3, -4, -2)$.
5. (*a*) – (*b*) 7.
6. (*a*) (*i*) $\left(\frac{531}{41}, \frac{-122}{41}, \frac{-240}{41}\right)$

(*ii*) (2, 3, – 1) (*b*) $\sqrt{21}$.

7. $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$.
8. $\left(\frac{22}{9}, -\frac{1}{9}, \frac{5}{9}\right)$.
9. $a\sqrt{\frac{2}{3}}$, where a is the edge of the cube.
10. (*a*) (*i*) (3, –2, 0) (*ii*) (1, 0, 7) (*b*) (– 3, – 6, 10).
11. (3, 5, 7).

(*b*) (– 3, –6, 10).

12. (3, 5, 9)
13. $\sqrt{14}$ units; (1, 2, 3).
14. $\frac{x-2}{6} = \frac{y-4}{3} = \frac{z+1}{2}$.
15. $\left(\frac{170}{49}, \frac{78}{49}, \frac{10}{49}\right); \frac{3}{7}\sqrt{101}$.
16. $\frac{x-3}{1} = \frac{y+1}{-6} = \frac{z-11}{4}$; (2, 5, 7) ; $\sqrt{53}$.
17. $\frac{1}{7}\sqrt{530}$ units.

11.14. COPLANAR LINES

Let the two given lines be $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} = r_1$ (say) ...(1)

and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} = r_2$ (say) ...(2)

When the lines are coplanar i.e. they lie in the same plane, then either they are parallel or they intersect.

If $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$, then the two lines are parallel and consequently they lie in the same plane.

In other words, two parallel lines are always coplanar. In case the two lines are not parallel, they are coplanar if they intersect.

If two non-parallel lines do not intersect, they are not coplanar. Such lines are called **skew lines.**

Definition

Two lines, which are neither parallel nor intersecting, are called skew lines.

11.15. SHORTEST DISTANCE BETWEEN TWO LINES

Definition

(*i*) **Line of Shortest Distance**

If L_1 and L_2 are two skew lines, then there is one and only one line which is perpendicular to both and is known as the line of shortest distance.

(*ii*) **Shortest Distance.**

The shortest distance between two lines L_1 and L_2 is the distance | PQ |, where P, Q are points at which the line of shortest distance meets L_1 and L_2 respectively.

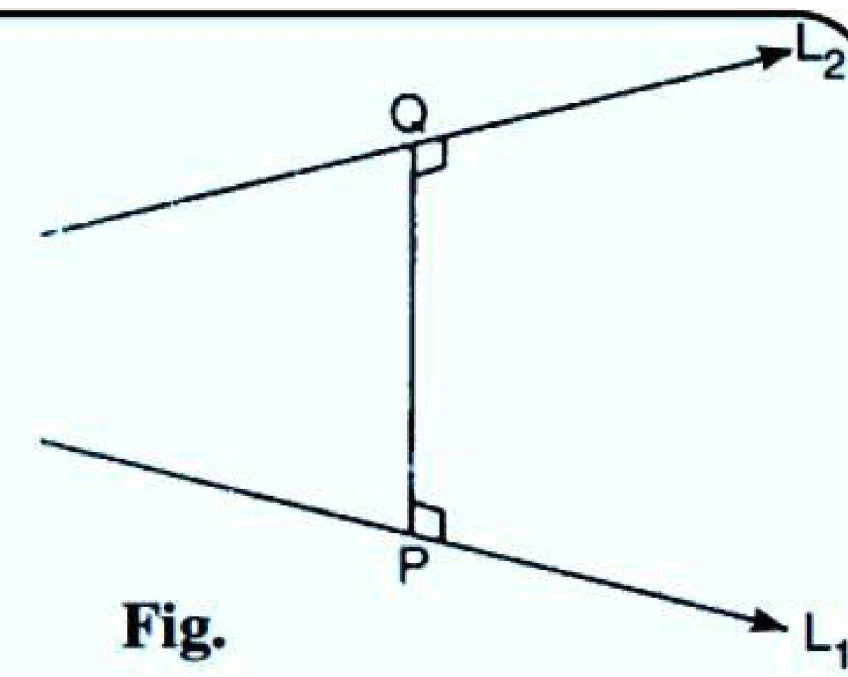


Fig.

KEY POINT

If two lines in space intersect at a point, then the shortest distance between them is zero.

(a) To find the shortest distance between two skew lines $\vec{\mathbf{r}} = \vec{\mathbf{a}}_1 + \lambda \vec{\mathbf{b}}_1$ and $\vec{\mathbf{r}} = \vec{\mathbf{a}}_2 + \lambda \vec{\mathbf{b}}_2$.

Let two skew lines L_1 and L_2 be $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ ...(1)

and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ ...(2)

Take any point S $(\vec{a}_1)$ on L_1 and T$(\vec{a}_2)$ on L_2.

Let $\overrightarrow{PQ}$ be the shortest distance vector between them.

By *def.*, $\overrightarrow{PQ}$ is perp. to (1) and (2)

$\Rightarrow$ $\overrightarrow{PQ}$ is perp. to both $\vec{b}_1$ and $\vec{b}_2$

$\Rightarrow$ $\overrightarrow{PQ}$ is parallel to $\vec{b}_1 \times \vec{b}_2$.

The unit vector $\hat{n}$ along PQ is given by $\hat{n} = \dfrac{\vec{b}_1 \times \vec{b}_2}{|\vec{b}_1 \times \vec{b}_2|}$.

Let $\overrightarrow{PQ} = d\,\hat{n}$, where d is the magnitude of the shortest distance vector.

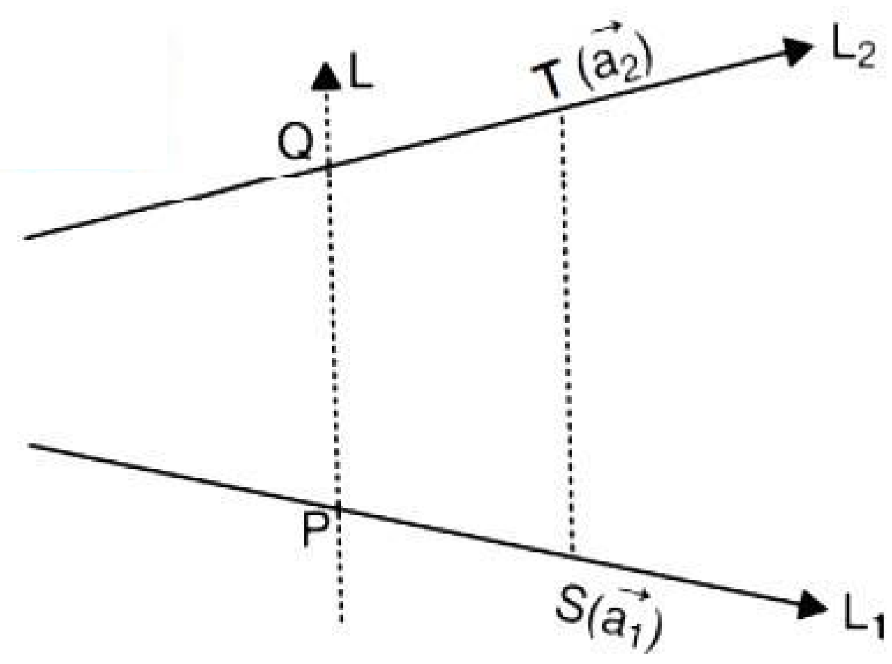


Fig.

Clearly, PQ is projection of $\overrightarrow{ST}$ on $\overrightarrow{PQ}$.

Now if 'θ' be the angle between $\overrightarrow{PQ}$ and $\overrightarrow{ST}$, then PQ = ST cos θ.

But $\cos\theta = \dfrac{\overrightarrow{PQ}\,.\,\overrightarrow{ST}}{|\overrightarrow{PQ}||\overrightarrow{ST}|} = \dfrac{d\,\hat{n}\,.(\vec{a}_2 - \vec{a}_1)}{d\,|\vec{b}_1 \times \vec{b}_2|}$.

Hence, $d = \text{PQ} = \text{ST}\cos\theta = \dfrac{(\vec{b}_1 \times \vec{b}_2)\,.\,(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}$.

Since d is always to be taken as positive, $\therefore$ $\mathbf{d} = \left|\dfrac{(\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2)\,.\,(\vec{\mathbf{a}}_2 - \vec{\mathbf{a}}_1)}{|\vec{\mathbf{b}}_1 \times \vec{\mathbf{b}}_2|}\right|$.

Cor. If two lines intersect, then $d = 0$

i.e. $(\vec{b}_1 \times \vec{b}_2)\,.\,(\vec{a}_2 - \vec{a}_1) = 0$.

(b) To find the shortest distance between two parallel lines : $\vec{\mathbf{r}} = \vec{\mathbf{a}}_1 + \lambda\,\vec{\mathbf{b}}$ and $\vec{\mathbf{r}} = \vec{\mathbf{a}}_2 + \mu\,\vec{\mathbf{b}}$.

Let two parallel lines L_1 and L_2 be : $\vec{r} = \vec{a}_1 + \lambda\,\vec{b}$ ...(1)

and $\vec{r} = \vec{a}_2 + \lambda\,\vec{b}$ ...(2)

These are clearly coplanar.

Clearly, either of L_1 and L_2 is parallel to $\vec{b}$ and they pass through the points S $(\vec{a}_1)$ and T $(\vec{a}_2)$.

Let $\overrightarrow{PQ}$ be the shortest distance vector between them.

$\therefore \quad d = PQ = ST \cos(90° - \theta) = ST \sin \theta$

$$= (ST)\frac{|\vec{b} \times \vec{ST}|}{|\vec{b}|(ST)} = \frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|}.$$

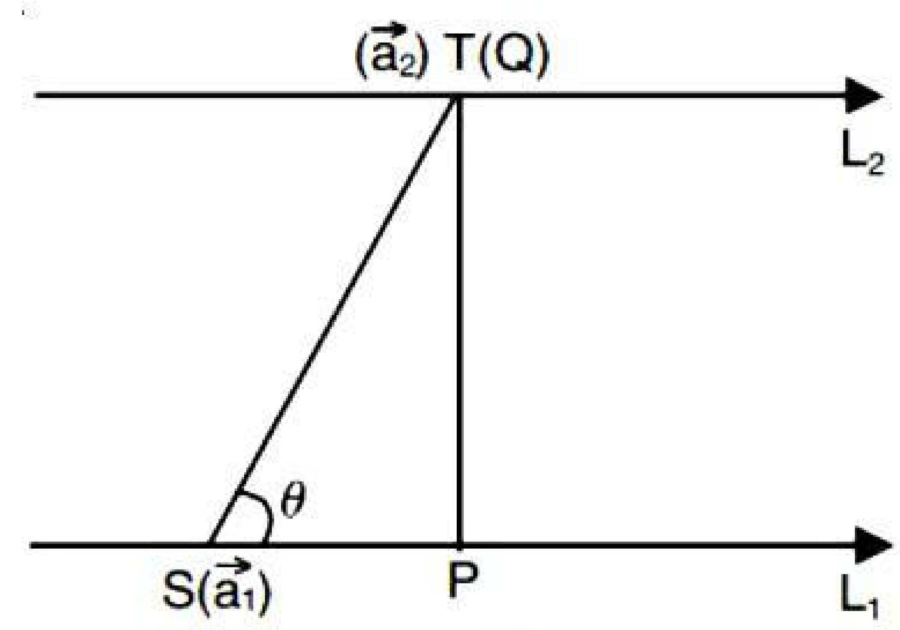


Fig.

Since d is always to be taken as positive, $\therefore$ $\mathbf{d} = \left|\frac{\vec{\mathbf{b}} \times (\vec{\mathbf{a_2}} - \vec{\mathbf{a_1}})}{|\vec{\mathbf{b}}|}\right|$.

(*c*) To find the shortest distance between two straight lines whose equations are :

$$\frac{\mathbf{x - x_1}}{\mathbf{l_1}} = \frac{\mathbf{y - y_1}}{\mathbf{m_1}} = \frac{\mathbf{z - z_1}}{\mathbf{n_1}}$$

and $$\frac{\mathbf{x - x_2}}{\mathbf{l_2}} = \frac{\mathbf{y - y_2}}{\mathbf{m_2}} = \frac{\mathbf{z - z_2}}{\mathbf{n_2}}.$$

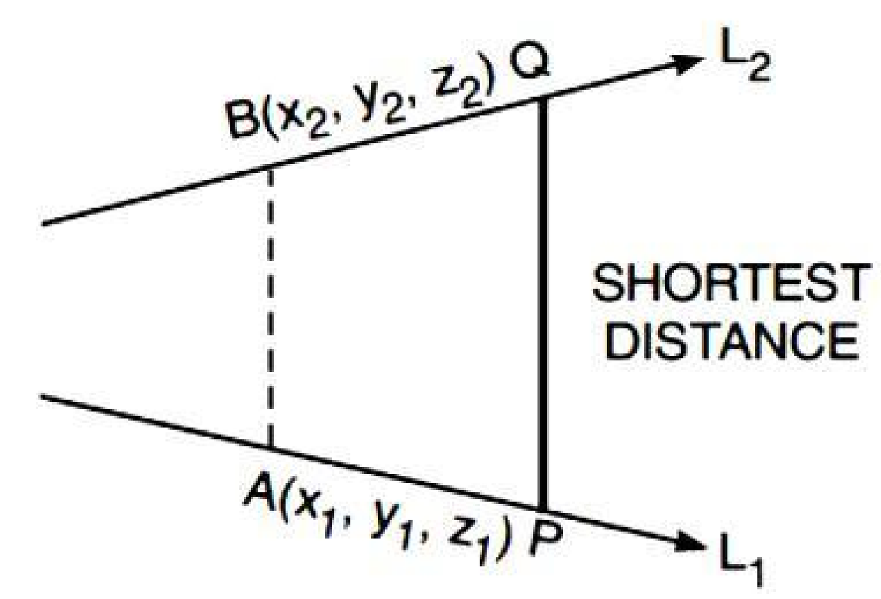


Fig.

Let PQ be the S.D.

Let $< l, m, n >$ be its direction-cosines.

Then $\quad ll_1 + mm_1 + nn_1 = 0$ ...(3)

and $\quad ll_2 + mm_2 + nn_2 = 0$ ...(4)

Solving, $\dfrac{l}{m_1n_2 - m_2n_1} = \dfrac{m}{n_1l_2 - n_2l_1} = \dfrac{n}{l_1m_2 - l_2m_1}$.

$\therefore$ Direction-ratios of PQ are :

$$< m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1 > .$$

$\therefore$ Direction-cosines of PQ are :

$$< \frac{m_1n_2 - m_2n_1}{\sqrt{\Sigma (m_1n_2 - m_2n_1)^2}}, \frac{n_1l_2 - n_2l_1}{\sqrt{\Sigma (m_1n_2 - m_2n_1)^2}}, \frac{l_1m_2 - l_2m_1}{\sqrt{\Sigma (m_1n_2 - m_2n_1)^2}} >.$$

$\therefore$ **Length of the S.D.** = | PQ |= Projection of [AB] on PQ

$$= \frac{(x_2 - x_1)(m_1n_2 - m_2n_1) + (y_2 - y_1)(n_1l_2 - n_2l_1) + (z_2 - z_1)(l_1m_2 - l_2m_1)}{\sqrt{\Sigma (m_1n_2 - m_2n_1)^2}}.$$

Cor. If two lines intersect, then :

$(x_2 - x_1)(m_1n_2 - m_2n_1) + (y_2 - y_1)(n_1l_2 - n_2l_1) + (z_2 - z_1)(l_1m_2 - l_2m_1) = 0$

$$\Rightarrow \quad \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

11.16. CO-PLANARITY OF TWO LINES

Consider the two lines :

$$\vec{r} = \vec{a_1} + \lambda \vec{b_1} \quad ...(1)$$

and $$\vec{r} = \vec{a_2} + \mu \vec{b_2} \quad ...(2)$$

(1) passes thro' A having position vector $\vec{a_1}$ and is parallel to $\vec{b_1}$ and

(2) passes thro' B having position vector $\vec{a_2}$ and is parallel to $\vec{b_2}$.

Thus $\quad \overrightarrow{AB} = \text{p.v. of B} - \text{p.v. of A} = \vec{a_2} - \vec{a_1}$.

The given lines are coplanar if and only if $\overrightarrow{AB}$ is perp. to $\vec{b_1} \times \vec{b_2}$

i.e. $\overrightarrow{AB} \cdot (\vec{b_1} \times \vec{b_2}) = 0 \qquad \Rightarrow (\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2}) = 0$

i.e. **d, the shortest distance between (1) and (2) = 0.**

CARTESIAN FORM :

Let A and B have co-ordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively.

Let $\vec{b_1}$ and $\vec{b_2}$ have direction-ratios :

$< a_1, b_1, c_1 >$ and $< a_2, b_2, c_2 >$ respectively.

$\therefore \quad \overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$,

where $\quad \vec{b_1} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$

and $\quad \vec{b_2} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.

The given lines are coplanar if and only if $\overrightarrow{AB} \cdot (\vec{b_1} \times \vec{b_2}) = 0$ [*As above*]

This can be expressed, in cartesian form, as :

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Frequently Asked Questions

FAQs

Example 1. The vector equations of two lines are :

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) \text{ and}$$
$$\vec{r} = 2\hat{i} + 4\hat{j} + 5\hat{k} + \mu(3\hat{i} + 4\hat{j} + 5\hat{k}).$$

Find the shortest distance between these lines.

(P.B. 2016)

Solution. Comparing given equations with :

$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, we have :

$\vec{b_1} = 2\hat{i} + 3\hat{j} + 4\hat{k}$,

$\vec{b_2} = 3\hat{i} + 4\hat{j} + 5\hat{k}$

and $\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$,

$\vec{a_2} = 2\hat{i} + 4\hat{j} + 5\hat{k}$.

Now $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$

$= \hat{i}(15 - 16) - \hat{j}(10 - 12) + \hat{k}(8 - 9)$

$= -\hat{i} + 2\hat{j} - \hat{k}$.

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(-1)^2 + (2)^2 + (-1)^2}$

$= \sqrt{1 + 4 + 1} = \sqrt{6}$.

Also, $\vec{a_2} - \vec{a_1} = (2 - 1)\hat{i} + (4 - 2)\hat{j} + (5 - 3)\hat{k}$

$= \hat{i} + 2\hat{j} + 2\hat{k}$.

$\therefore$ *d*, the shortest distance between the given lines is given by :

$$d = \left| \frac{(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$

$$= \left| \frac{(-\hat{i} + 2\hat{j} - \hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})}{\sqrt{6}} \right|$$

$$= \left| \frac{(-1)(1) + (2)(2) + (-1)(2)}{\sqrt{6}} \right|$$

$$= \left| \frac{-1 + 4 - 2}{\sqrt{6}} \right| = \left| \frac{1}{\sqrt{6}} \right| = \frac{1}{\sqrt{6}} = \frac{\sqrt{6}}{6} \text{ units.}$$

Example 2. Find the distance between the lines L_1 and L_2 given by :

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}) \text{ and}$$

$$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k}).$$ *(N.C.E.R.T.)*

Solution. Clearly, L_1 and L_2 are parallel.

Comparing given equations with :

$\vec{r} = \vec{a_1} + \lambda\vec{b}$ and $\vec{r} = \vec{a_2} + \mu\vec{b}$, we have :

$\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, so that

$$|\vec{b}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$$

and $\vec{a_1} = \hat{i} + 2\hat{j} - 4\hat{k}$; $\vec{a_2} = 3\hat{i} + 3\hat{j} - 5\hat{k}$.

Now $\vec{a_2} - \vec{a_1} = (3-1)\hat{i} + (3-2)\hat{j} + (-5+4)\hat{k}$

$= 2\hat{i} + \hat{j} - \hat{k}$.

$$\therefore \vec{b} \times (\vec{a_2} - \vec{a_1}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}$$

$= \hat{i}(-3-6) - \hat{j}(-2-12) + \hat{k}(2-6)$

$= -9\hat{i} + 14\hat{j} - 4\hat{k}$.

$\therefore$ *d*, the distance between the given lines is given by :

$$d = \left|\frac{\vec{b} \times (\vec{a_2} - \vec{a_1})}{|\vec{b}|}\right| = \left|\frac{-9\hat{i} + 14\hat{j} - 4\hat{k}}{7}\right|$$

$$= \frac{1}{7}|-9\hat{i} + 14\hat{j} - 4\hat{k}|$$

$$= \frac{1}{7}\sqrt{81 + 196 + 16}$$

$$= \frac{1}{7}\sqrt{293} \text{ units.}$$

Example 3. Show that the two lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect.

Find also the point of intersection of these lines.

(Mizoram B. 2017; Meghalaya B. 2014; P.B. 2012)

Solution. The given lines are :

$$L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \qquad ...(1)$$

and $$L_2 : \frac{x-4}{5} = \frac{y-1}{2} = \frac{z}{1} \qquad ...(2)$$

Any point on L_1 is $(2\lambda + 1, 3\lambda + 2, 4\lambda + 3)$ ...(3)

Any point on L_2 is $(5\mu + 4, 2\mu + 1, \mu)$ ...(4)

The lines L_1 and L_2 will intersect iff points (3) and (4) coincide

iff $2\lambda + 1 = 5\mu + 4$, $3\lambda + 2 = 2\mu + 1$, $4\lambda + 3 = \mu$

Taking first two, $2\lambda - 5\mu = 3$ ...(5)

Taking middle two, $3\lambda - 2\mu = -1$ ...(6)

Taking last two, $4\lambda - \mu = -3$ ...(7)

Solving (5) and (6), $\lambda = -1$ and $\mu = -1$.

Putting in (7), $4(-1) + 1 = -3$

$\Rightarrow -3 = -3$, which is true.

Hence, the given lines L_1 and L_2 intersect.

Putting $\lambda = -1$ in (3), [or $\mu = -1$ in (4)],

we get the reqd. point of intersection as $(-1, -1, -1)$.

Example 4. Show that the lines :

$$\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5} \text{ and } \frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$$

are coplanar. Also, find the equation of the plane.

(N.C.E.R.T.)

Solution. Comparing the given equations with

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and $$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2},$$

we get :

$x_1 = -3, y_1 = 1, z_1 = 5$; $x_2 = -1, y_2 = 2, z_2 = 5$

and $a_1 = -3, b_1 = 1, c_1 = 5$; $a_2 = -1, b_2 = 2, c_2 = 5$.

$$\text{Now} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} -1+3 & 2-1 & 5-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 0 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix}$$

$= 2(5 - 10) - 1 \,.\, (-15 + 5) + 0 = -10 + 10 + 0 = 0$.

Hence, the given lines are coplanar.

(*ii*) The equation of the plane containing given lines is :

$$\begin{vmatrix} x+3 & y-1 & z-5 \\ -3 & 1 & 5 \\ -1 & 2 & 5 \end{vmatrix} = 0 \qquad [\textit{Ref. Art. 11.17 (e)}]$$

$\Rightarrow (x + 3)(5 - 10) - (y - 1)(-15 + 5) + (z - 5)(-6 + 1) = 0$

$\Rightarrow - 5x - 15 + 10y - 10 - 5z + 25 = 0$

$\Rightarrow - 5x + 10y - 5z = 0 \Rightarrow x - 2y + z = 0.$

Example 5. Find whether the lines :

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \lambda (2\hat{i} + \hat{j})$$

and $\vec{r} = (2\hat{i} - \hat{j}) + \mu (\hat{i} + \hat{j} - \hat{k})$

intersect or not. If intersecting, find their point of intersection.

Solution. The given lines are :

$$\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \lambda (2\hat{i} + \hat{j})$$

and $$\vec{r} = (2\hat{i} - \hat{j}) + \mu (\hat{i} + \hat{j} - \hat{k})$$

i.e. $$\vec{r} = (1 + 2\lambda)\hat{i} + (-1 + \lambda)\hat{j} - \hat{k} \quad \text{....(1)}$$

and $$\vec{r} = (2 + \mu)\hat{i} + (-1 + \mu)\hat{j} - \mu\hat{k} \quad \text{...(2)}$$

If the lines (1) and (2) intersect, then for some values of λ and μ, we have :

$$1 + 2\lambda = 2 + \mu \quad \text{...(3)}$$

$$-1 + \lambda = -1 + \mu \quad \text{...(4)}$$

and $$-1 = -\mu \Rightarrow \mu = 1 \quad \text{...(5)}$$

Putting in (4), $-1 + \lambda = -1 + 1 \Rightarrow \lambda = 1$.

Thus $\lambda = 1$ and $\mu = 1$.

These also satisfy (3). [$\because$ *1 + 2 (1) = 2 + 1 i.e. 3 = 3*]

Hence, the lines intersect.

Putting $\lambda = 1$ in (1),

$$\vec{r} = (1 + 2)\hat{i} + (-1 + 1)\hat{j} - \hat{k} = 3\hat{i} - \hat{k}.$$

Putting $\mu = 1$ in (2),

$$\vec{r} = (2 + 1)\hat{i} + (-1 + 1)\hat{j} - \hat{k} = 3\hat{i} - \hat{k}.$$

Hence, the point of intersection is (3, 0, – 1).

Example 6. Find the shortest distance and the vector equation of the line of shortest distance between the lines given by :

$$\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + \lambda (3\hat{i} - \hat{j} + \hat{k}) \text{ and}$$

$$\vec{r} = (-3\hat{i} - 7\hat{j} + 6\hat{k}) + \mu (-3\hat{i} + 2\hat{j} + 4\hat{k}).$$

Solution. The given equations in the cartesian form are :

$$L_1 : \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} (= \lambda) \quad \text{...(1)}$$

and $$L_2 : \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} (= \mu) \quad \text{...(2)}$$

Any point on L_1 is $(3\lambda + 3, -\lambda + 8, \lambda + 3)$.

Any point on L_2 is $(-3\mu - 3, 2\mu - 7, 4\mu + 6)$.

If the line of shortest distance intersects (1) in P and (2) in Q, then the direction-ratios of $\overrightarrow{PQ}$ are :

$< -3\mu - 3 - 3\lambda - 3, 2\mu - 7 + \lambda - 8, 4\mu + 6 - \lambda - 3 >$

i.e. $< 3\lambda + 3\mu + 6, -\lambda - 2\mu + 15, \lambda - 4\mu - 3 >$.

Since PQ is perp. to line (1),

$\therefore (3)(3\lambda + 3\mu + 6) + (-1)(-\lambda - 2\mu + 15) + (1)(\lambda - 4\mu - 3) = 0$

$$\Rightarrow \quad 11\lambda + 7\mu = 0 \quad \text{...(3)}$$

Since PQ is perp. to line (2),

$\therefore (-3)(3\lambda + 3\mu + 6) + 2(-\lambda - 2\mu + 15) + 4(\lambda - 4\mu - 3) = 0$

$$\Rightarrow \quad -7\lambda - 29\mu = 0 \quad \text{...(4)}$$

Solving (3) and (4), $\lambda = 0, \mu = 0$.

$\therefore$ Points P and Q are (3, 8, 3) and (– 3, – 7, 6) respectively.

$\therefore$ S.D. = | PQ |

$$= \sqrt{(-3-3)^2 + (-7-8)^2 + (6-3)^2}$$

$$= \sqrt{36 + 225 + 9} = \sqrt{270} = 3\sqrt{30} \text{ units}$$

and the vector equation of line of S.D. is :

$$\vec{r} = (3\hat{i} + 8\hat{j} + 3\hat{k}) + \mu (-6\hat{i} - 15\hat{j} + 3\hat{k}).$$

[*Using* $\vec{r} = \vec{a} + \lambda (\vec{b} - \vec{a})$]

EXERCISE 11 (d)

Short Answer Type Questions

SATQ

Find the shortest distance between the following (1 – 4) lines whose vector equations are :

1. $\vec{r} = \hat{i} + \hat{j} + \lambda (2\hat{i} - \hat{j} + \hat{k})$

and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu (3\hat{i} - 5\hat{j} + 2\hat{k})$.

(N.C.E.R.T. ; Kerala B. 2016; Kashmir B. 2016, 12, 11; P.B. 2015, 10; H.P.B. 2013 S, 11; H.B. 2010)

2. (*i*) $\vec{r} = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k})$

and $\vec{r} = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k})$

(P.B. 2012)

(*ii*) $(\hat{i} - \hat{j} + 2\hat{k}) + \lambda(-2\hat{i} + \hat{j} + 3\hat{k})$

and $(2\hat{i} + 3\hat{j} - \hat{k}) + \mu(3\hat{i} - 2\hat{j} + 2\hat{k})$.

(Assam B. 2016)

(*iii*) $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$

and $\vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(2\hat{i} + 3\hat{j} + \hat{k})$.

3. (*i*) $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$

and $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$

(N.C.E.R.T.; H.P.B. 2017, 15; P.B. 2013 ; Jammu B. 2012 ; C.B.S.E. (F) 2011)

(*ii*) $\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$

and $\vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})$

(C.B.S.E. 2018; H.B. 2010)

(*iii*) $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

and $\vec{r} = (3\hat{i} + 3\hat{j} + 5\hat{k}) + \mu(-2\hat{i} + 3\hat{j} + 6\hat{k})$

(H.P.B. 2017; H.B. 2017, 15)

(*iv*) $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$.

(Jharkhand B. 2016)

4. (*i*) $\vec{r} = (\lambda - 1)\hat{i} + (\lambda - 1)\hat{j} - (1 + \lambda)\hat{k}$

and $\vec{r} = (1 - \mu)\hat{i} + (2\mu - 1)\hat{j} + (\mu + 2)\hat{k}$

(*ii*) $\vec{r} = (1 + \lambda)\hat{i} + (2 - \lambda)\hat{j} + (1 + \lambda)\hat{k}$

and $\vec{r} = 2(1 + \mu)\hat{i} - (1 - \mu)\hat{j} + (-1 + 2\mu)\hat{k}$.

5. Consider the equations of the straight lines given by :

$L_1 : \vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$

$L_2 : \vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

If $\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k},\ \vec{b}_1 = \hat{i} - \hat{j} + \hat{k},$

$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k},\ \vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$, then find :

(*i*) $\vec{a}_2 - \vec{a}_1$ (*ii*) $\vec{b}_2 - \vec{b}_1$

(*iii*) $\vec{b}_1 \times \vec{b}_2$ (*iv*) $\vec{a}_1 \times \vec{a}_2$

(*v*) $(\vec{b}_1 \times \vec{b}_2).(\vec{a}_2 - \vec{a}_1)$

(*vi*) the shortest distance between L_1 and L_2.

(Jammu B. 2018, 16, 14, 12, 11; Assam B. 2018; Karnataka B. 2017; H.B. 2017; H.P.B. 2016; Bihar 2014)

Find the shortest distance between the following (6 – 7) lines whose vector equations are :

6. (*i*) $\vec{r} = (1 - t)\hat{i} + (t - 2)\hat{j} + (3 - 2t)\hat{k}$

and $\vec{r} = (s + 1)\hat{i} + (2s - 1)\hat{j} - (2s + 1)\hat{k}$

(N.C.E.R.T.; Kashmir B. 2018, 11; Jammu B. 2016, 15W, 14, 13; H.P.B. 2013, 12, 10 S, 10; A.I.C.B.S.E. 2011)

(*ii*) $\vec{r} = (3 - t)\hat{i} + (4 + 2t)\hat{j} + (t - 2)\hat{k}$

and $\vec{r} = (1 + s)\hat{i} + (3s - 7)\hat{j} + (2s - 2)\hat{k}$,

where t and s are scalars.

(H.P.B. Model Paper 2018; H.P.B. 2018, 17, 16, 14, 13; Kerala B. 2018)

7. (*i*) $\vec{r} = (8 + 3\lambda)\hat{i} - (9 + 16\lambda)\hat{j} + (10 + 7\lambda)\hat{k}$

and $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$

(*ii*) $\vec{r} = 3\hat{i} - 15\hat{j} + 9\hat{k} + \lambda(2\hat{i} - 7\hat{j} + 5\hat{k})$

and $\vec{r} = (2\mu - 1)\hat{i} + (1 + \mu)\hat{j} + (9 - 3\mu)\hat{k}$.

8. Find the S.D. between the lines :

(*i*) $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+4}{2}$

(H.B. 2016)

(*ii*) $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{2}$ and $\frac{x+1}{3} = \frac{y-1}{2} = \frac{z-1}{5}$

(Jammu B. 2013)

(*iii*) $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

(H.P.B. 2018, P.B. 2017)

(*iv*) $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.

(Kerala B. 2013)

Determine whether or not the following (9 – 11) pairs of lines intersect :

9. $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + \hat{j} - 2\hat{k})$

and $\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + \mu(\hat{i} + 2\hat{j} - 2\hat{k})$.

10. $\vec{r} = (2\lambda + 1)\hat{i} - (\lambda + 1)\hat{j} + (\lambda + 1)\hat{k}$.

and $\vec{r} = (3\mu + 2)\hat{i} - (5\mu + 5)\hat{j} + (2\mu - 1)\hat{k}$.

11. $\frac{x-1}{2} = \frac{y+1}{3} = z$; $\frac{x+1}{5} = \frac{y-2}{1}$; $\frac{z-2}{0}$.

12. Prove that the lines : $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar.

Long Answer Type Questions

LATQ

13. Find the shortest distance and the equation of the shortest distance between the following two lines :

$$\vec{r} = (-\hat{i} + \hat{j} + 9\hat{k}) + \lambda(2\hat{i} + \hat{j} - 3\hat{k})$$

and $\vec{r} = (3\hat{i} - 15\hat{j} + 9\hat{k}) + \mu(2\hat{i} - 7\hat{j} + 5\hat{k})$.

14. Find the shortest distance and the vector equation of the line of shortest distance between the lines given by :

(*i*) $\vec{r} = (-4\hat{i} + 4\hat{j} + \hat{k}) + \lambda(\hat{i} + \hat{j} - \hat{k})$

and $\vec{r} = (-3\hat{i} - 8\hat{j} - 3\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 3\hat{k})$

(*ii*) $\vec{r} = (-\hat{i} + 5\hat{j}) + \lambda(-\hat{i} + \hat{j} + \hat{k})$

and $\vec{r} = (-\hat{i} - 3\hat{j} + 2\hat{k}) + \mu(3\hat{i} + 2\hat{j} + \hat{k})$.

15. Write the vector equations of the following lines and hence determine the distance between them :

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6} ; \frac{x-3}{4} = \frac{y-3}{6} = \frac{z+5}{12}.$$

(C.B.S.E. 2010)

16. Show that the lines : $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$

and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$

intersect each other. Also, find their point of intersection. ***(C.B.S.E. 2014; P.B. 2012)***

17. Show that the lines :

(*i*) $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$

and $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$ ***(A.I.C.B.S.E. 2013)***

(*ii*) $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$

and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ ***(C.B.S.E. 2014)***

are intersecting. Hence, find their point of intersection.

18. Show that the lines :

(*a*) $\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$ and $\frac{x-2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$

(*b*) $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{k})$

and $\vec{r} = (2\hat{i} - \hat{j}) + \mu(\hat{i} + \hat{k} - \hat{k})$

do not intersect. ***(Meghalaya B. 2015)***

19. Find the S.D. between the lines :

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5},$$

find also its equations. ***(P.B. 2014 S)***

20. Show that the following lines are coplanar :

(*i*) $\frac{5-x}{-4} = \frac{y-7}{-4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$

(C.B.S.E. 2014)

(*ii*) $\frac{x+3}{-3} = \frac{y-1}{1} = \frac{z-5}{5}$ and $\frac{x+1}{-1} = \frac{y-2}{2} = \frac{z-5}{5}$.

(Jammu B. 2018; Uttrakhand B. 2013 ; Assam B. 2013)

21. Show that the lines :

$\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$ and

$\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar.

(N.C.E.R.T.)

22. Find the equations of the lines joining the following pair of vertices and then find its shortest distance between the lines :

(*i*) (0, 0, 0), (1, 0, 2) (*ii*) (1, 3, 0), (0, 3, 0).

Answers

1. $\frac{10}{\sqrt{59}}$.

2. (i) $3\sqrt{30}$ (ii) $\frac{57}{\sqrt{234}}$ (iii) $\frac{3}{19}\sqrt{19}$.

3. (i) 9 (ii) $\frac{6\sqrt{5}}{5}$ (iii) $\frac{3\sqrt{5}}{5}$ (iv) 0.

4. (i) $\frac{5\sqrt{2}}{2}$ (ii) $\frac{3\sqrt{2}}{2}$.

5. (i) $\hat{i}-3\hat{j}-2\hat{k}$ (ii) $\hat{i}+2\hat{j}+\hat{k}$ (iii) $-3\hat{i}+3\hat{k}$

(iv) $-\hat{i}+3\hat{j}-5\hat{k}$ (v) -9 (vi) $\frac{3}{2}\sqrt{2}$ units.

6. (i) $\frac{8\sqrt{29}}{29}$ (ii) $\sqrt{35}$.

7. (i) 14 (ii) $4\sqrt{3}$.

8. (i) $\frac{1}{3}\sqrt{3}$ (ii) $\frac{\sqrt{2336}}{81}$ (iii) – (iv) $3\sqrt{30}$.

9. No. **10.** No. **11.** No.

13. $4\sqrt{3}$; $\vec{r} = (3\hat{i}+3\hat{j}-3\hat{k}) - \mu(4\hat{i}+4\hat{j}+4\hat{k})$.

14. (i) $\sqrt{62}$; $\vec{r} = (-5\hat{i}+3\hat{j}+2\hat{k}) + \mu(\hat{i}-2\hat{j}+3\hat{k})$

(ii) $\sqrt{42}$, $\vec{r} = (\hat{i}+3\hat{j}-2\hat{k}) + \mu(\hat{i}-4\hat{j}+5\hat{k})$.

15. $\vec{r} = \left(\hat{i}+2\hat{j}-4\hat{k}\right) + \lambda\left(2\hat{i}+3\hat{j}+6\hat{k}\right)$

and $\vec{r} = \left(3\hat{i}+3\hat{j}-5\hat{k}\right) + \mu\left(4\hat{i}+6\hat{j}+12\hat{k}\right)$;

$\frac{1}{7}\sqrt{293}$ units.

16. $\left(\frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}\right)$.

17. (i) (–1, –6, –12) (ii) (4, 0, – 1).

19. $\frac{1}{\sqrt{6}}$; $6x-9=10-3y=6z-25$.

22. (i) $\vec{r} = \lambda(\hat{i}+2\hat{k})$; $\frac{x}{1}=\frac{y}{0}=\frac{z}{2}$

(ii) $\vec{r} = (\hat{i}+3\hat{j}) - \mu\hat{i}$; $\frac{x-1}{-1}=\frac{y-3}{0}=\frac{z}{0}$.

S.D. = 3 units.

Hints to Selected Questions

9. Show that S.D. $\neq$ 0.

16. Show that S.D. = 0.

SUB CHAPTER 11.3 The Plane

INTRODUCTION

A plane is a surface such that if we take any two distinct points on it, then the line joining these points lies wholly on it.

We can specify a particular plane in several ways as :

(i) *One and only one plane can be drawn through three non–collinear points.*

Thus three given non–collinear points specify a particular plane.

(ii) *One and only one plane can be drawn to contain two concurrent lines.*

Thus two given concurrent lines specify a particular plane.

(iii) *One and only one plane can be drawn perpendicular to given direction at a given distance from the origin.*

Thus the normal to the plane and the distance of the plane from the origin specify a particular plane.

(iv) *One and only one plane can be drawn through a given point and perpendicular to a given direction.*

Thus a point on the plane and a normal to the plane specify a particular plane.

Of the above, (iii) and (iv) are most useful.

11.17. EQUATIONS OF PLANES

(a) STANDARD FORM (NORMAL FORM) *(Assam B. 2015; Karnataka B. 2014)*

To find the equation of a plane, which is at a distance 'p' from the origin and perpendicular to the unit vector $\hat{n}$ (being directed away from O).

Let O be the origin and 'p', the length of perpendicular OL from O to the given plane. Let $\hat{n}$ be the unit vector normal to the plane in the direction away from O *i.e.* from O to L.

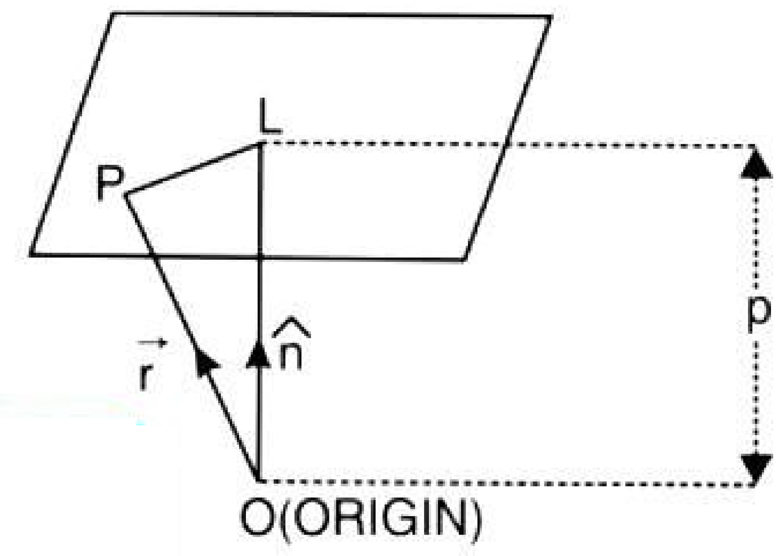


Fig.

Then $\overrightarrow{OL} = p\hat{n}$.

Let P be *any* point on the plane whose position vector is $\vec{r}$.

Clearly, $\overrightarrow{LP}$ is $\perp \overrightarrow{OL}$ *.

$\therefore$ P is a point on the plane $\Leftrightarrow \overrightarrow{LP} \cdot \overrightarrow{OL} = 0$ ...(A) **(Scalar-Product Form)**

Now $\overrightarrow{LP} = \overrightarrow{OP} - \overrightarrow{OL} = \vec{r} - p\hat{n}$.

$\therefore$ From (A), $\left(\vec{r} - p\hat{n}\right) \cdot \hat{n} = 0$

$\Rightarrow \vec{r} \cdot \hat{n} - p\hat{n} \cdot \hat{n} = 0$

$\Rightarrow \vec{r} \cdot \hat{n} - p = 0$ $[\because \hat{n} \cdot \hat{n} = 1]$

$\Rightarrow$ $\mathbf{\vec{r} \cdot \hat{n} = p}$ ...(B) **(Standard Form)**

Cartesian Form :

Let $\vec{r} \cdot \hat{n} = p$ ...(1) be the vector equation of the plane, where $\hat{n}$ is the unit vector normal to the plane as shown in the figure.

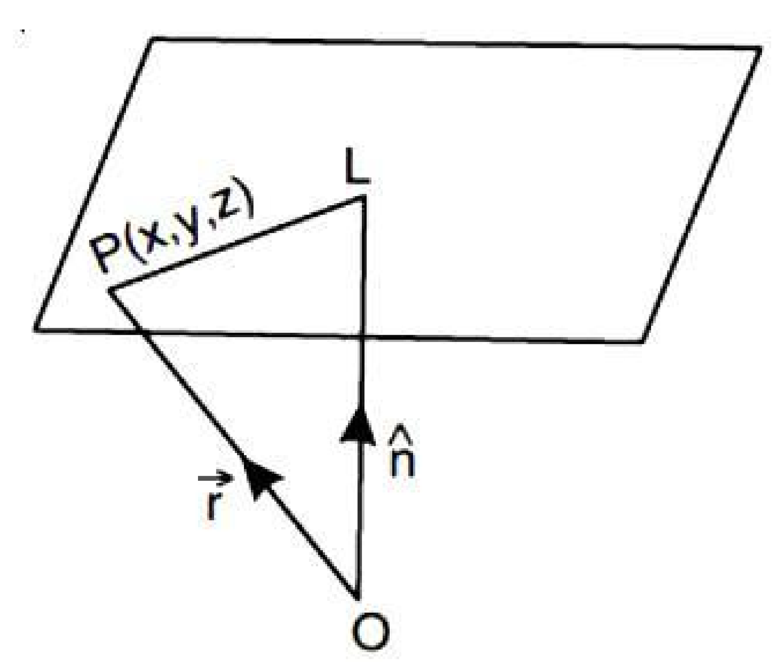


Fig.

Let P (x, y, z) be *any* point on the plane, then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

If $< l, m, n >$ are direction-cosines of $\hat{n}$, then $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$.

Putting in (1), we get : $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (l\hat{i} + m\hat{j} + n\hat{k}) = p$

$\Rightarrow$ **lx + my + nz = p.**

Here $< l, m, n >$ are direction-cosines of the normal to the plane and 'p' is the length of the perpendicular from the origin to the plane.

(b) GENERAL FORM

To prove that the general equation of the first degree in x, y, z represents a plane.

Let the general equation of the first degree in x, y, z be $ax + by + cz + d = 0$ ...(1),

where a, b, c are not all zero *i.e.* $a^2 + b^2 + c^2 \neq 0$.

Then $ax + by + cz = -d$

*If P is not in the plane, $\overrightarrow{LP}$ is not perp. to $\overrightarrow{OL}$.

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}).(a\hat{i} + b\hat{j} + c\hat{k}) = -d$$

$$\Rightarrow \vec{r}.\left\{\frac{a}{\sqrt{a^2+b^2+c^2}}\hat{i} + \frac{b}{\sqrt{a^2+b^2+c^2}}\hat{j} + \frac{c}{\sqrt{a^2+b^2+c^2}}\hat{k}\right\} = \frac{-d}{\sqrt{a^2+b^2+c^2}}$$

$$\Rightarrow \vec{r}.(l\hat{i} + m\hat{j} + n\hat{k}) = p, \text{ where } \frac{-d}{\sqrt{a^2+b^2+c^2}} = p$$

$\Rightarrow \vec{r}.\hat{n} = p$, where $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$, which represents a plane.

Hence, $ax + by + cz + d = 0$ represents a plane, the length of whose perpendicular from the origin is $p = \dfrac{-d}{\sqrt{a^2+b^2+c^2}}$ and the direction–ratios of the normal are $< a, b, c >$.

Cor. Reduction to Normal Form.

$$ax + by + cz + d = 0 \qquad \Rightarrow \qquad -ax - by - cz = d$$

$$\Rightarrow \frac{-a}{\sqrt{a^2+b^2+c^2}}x + \frac{-b}{\sqrt{a^2+b^2+c^2}}y + \frac{-c}{\sqrt{a^2+b^2+c^2}}z = \frac{d}{\sqrt{a^2+b^2+c^2}}$$

[*Dividing by* $\sqrt{(-a)^2+(-b)^2+(-c)^2}$]

$\Rightarrow lx + my + nz = p$,

where $l\left(=\dfrac{-a}{\sqrt{a^2+b^2+c^2}}\right), m\left(=\dfrac{-b}{\sqrt{a^2+b^2+c^2}}\right), n\left(=\dfrac{-c}{\sqrt{a^2+b^2+c^2}}\right)$

are the direction-cosines of the normal to the plane and $p\left(=\dfrac{d}{\sqrt{a^2+b^2+c^2}}\right)$ is the length of the perpendicular from the origin to the plane.

KEY POINT

In the normal form of the equation of the plane $lx + my + nz = p$, the following characteristics may be noted with care :

(1) *p is always positive.*

(2) (*coeff. of x*)^2 + (*coeff. of y*)^2 + (*coeff. of z*)^2 = *1*.

Another Form. The equation of the plane in the normal form is also of the form :

$$x\cos\alpha + y\cos\beta + z\cos\gamma = p,$$

where $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.

(c) ONE–POINT FORM

Equation of a plane passing through a given point and perpendicular to a given vector.

(Assam B. 2015; Kashmir B. 2011)

To prove that the vector equation of a plane passing through a given point $\vec{r_1}$ and perpendicular to given vector $\hat{n}$ is $(\vec{r} - \vec{r_1}).\hat{n} = 0$.

Let O be the origin. Let $\vec{r_1}$ be the position vector of a point A, lying on the plane. Let P be *any* point on the plane whose position vector is $\vec{r}$.

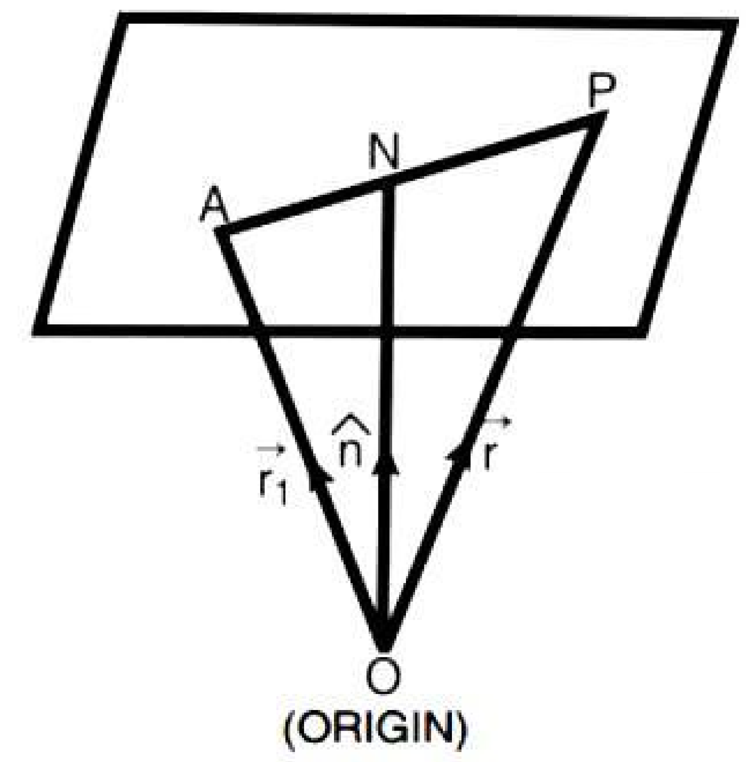


Fig.

$\therefore \qquad \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \vec{r} - \vec{r_1}$.

Since $\overrightarrow{AP}$ lies in the plane and $\hat{n}$ is normal to the plane,

$\therefore$ $\mathbf{(\vec{r} - \vec{r_1}) \cdot \hat{n} = 0}$, which is the reqd. equation.

Cartesian Form. Equation of a plane through (x_1, y_1, z_1) and normal to the plane having direction-ratios < a, b, c >.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$; $\vec{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\hat{n} = a\hat{i} + b\hat{j} + c\hat{k}$.

Then $(\vec{r} - \vec{r_1}) \cdot \hat{n} = 0$ becomes : $(\vec{r} - \vec{r_1}) \cdot \dfrac{\vec{n}}{|\vec{n}|} = 0$

$\Rightarrow [(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}] \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = 0 \Rightarrow \mathbf{a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.}$

(*d*) Equation of a plane through a point and parallel to two given lines.

To find the equation of a plane through a point having position vector $\vec{a}$ and parallel to the lines : $\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{a'} + \mu\vec{b'}$.

Let A be the point whose position vector is $\vec{a}$.

Let P be *any* point in the plane whose position vector is $\vec{r}$.

Now P lies in the plane iff $\overrightarrow{AP}$, $\vec{b}$ and $\vec{b'}$ are coplanar

iff $\quad \overrightarrow{AP} \cdot (\vec{b} \times \vec{b'}) = 0$

iff $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{b'}) = 0$ ...(1), which is the reqd. vector equation of the plane.

Cartesian Form. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

and $\qquad \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \vec{b'} = b_1'\hat{i} + b_2'\hat{j} + b_3'\hat{k}$.

Putting in (1), we get :

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ b_1 & b_2 & b_3 \\ b_1' & b_2' & b_3' \end{vmatrix} = 0,$$ which is the reqd. cartesian equation of the plane.

Example : Find the vector and cartesian equations of the plane passing through the point (1, 2, – 4) and parallel to the lines :

$$\vec{\mathbf{r}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}} + \lambda\,(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$$

and $$\vec{\mathbf{r}} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 5\hat{\mathbf{k}} + \mu\,(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}).$$

Solution. We have : $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ and $\vec{b'} = \hat{i} + \hat{j} - \hat{k}$.

(*i*) The vector equation of the plane is :

$$(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{b'}) = 0$$

$$\Rightarrow \quad [\vec{r} - (\hat{i} + 2\hat{j} - 4\hat{k})] \cdot [(2\hat{i} + 3\hat{j} + 6\hat{k}) \times (\hat{i} + \hat{j} - \hat{k})] = 0.$$

(*ii*) The cartesian equation of the plane is :

$$\begin{vmatrix} x-1 & y-2 & z+4 \\ 2 & 3 & 6 \\ 1 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (x-1)(-3-6) - (y-2)(-2-6) + (z+4)(2-3) = 0$$

$$\Rightarrow \quad -9x + 9 + 8y - 16 - z - 4 = 0$$

$$\Rightarrow \quad 9x - 8y + z + 11 = 0.$$

(*e*) Equation of a plane through two lines.

To find the equation of a plane through two lines :

$$\vec{\mathbf{r}} = \vec{\mathbf{a}} + \lambda\vec{\mathbf{b}} \text{ and } \vec{\mathbf{r}} = \vec{\mathbf{a}'} + \mu\vec{\mathbf{b}'}.$$

The given lines are $\vec{r} = \vec{a} + \lambda\vec{b}$ ...(1)

and $\vec{r} = \vec{a'} + \mu\vec{b'}$ ...(2)

(1) is a line through $A(\vec{a})$ and parallel to vector $\vec{b}$.

(2) is a line through $A'(\vec{a'})$ and parallel to vector $\vec{b'}$.

Let $P(\vec{r})$ be *any* point in the plane.

Now P lies in the plane iff $\overrightarrow{AP}$, $\vec{b}$ and $\vec{b'}$ are coplanar

iff $(\vec{r} - \vec{a}) \cdot (\vec{b} \times \vec{b'}) = 0$ ...(3), which is the reqd. vector equation of the plane.

Cartesian Form. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

and $\vec{b} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$, $\vec{b'} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$.

Putting in (3), we get :

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

Example : Find the vector and cartesian forms of the equation of the plane containing two lines :

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda\,(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and $$\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu\,(-2\hat{i} + 3\hat{j} + 8\hat{k}).$$

Solution. We have : $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ and $\vec{b'} = -2\hat{i} + 3\hat{j} + 8\hat{k}$.

(*i*) The vector equation of the plane is :

$$[\vec{r} - (\hat{i} + 2\hat{j} - 4\hat{k})].[(2\hat{i} + 3\hat{j} + 6\hat{k}) \times (-2\hat{i} + 3\hat{j} + 8\hat{k})] = 0.$$

(*ii*) The cartesian equation of the plane is $\begin{vmatrix} x-1 & y-2 & z+4 \\ 2 & 3 & 6 \\ -2 & 3 & 8 \end{vmatrix} = 0$

$\Rightarrow \quad (x-1)(24-18) - (y-2)(16+12) + (z+4)(6+6) = 0$

$\Rightarrow \quad 6x - 6 - 28y + 56 + 12z + 48 = 0$

$\Rightarrow \quad 6x - 28y + 12z + 98 = 0$

$\Rightarrow \quad 3x - 14y + 6z + 49 = 0.$

(*f*) THREE–POINT FORM

Equation of a plane passing through three points.

To find the equation of a plane through three points having position vectors $\vec{r_1}, \vec{r_2}$ and $\vec{r_3}$.

Let A, B, C be three given points having position vectors $\vec{r_1}, \vec{r_2}, \vec{r_3}$ respectively. Let $\vec{r}$ be the position vector of *any* point P on the plane.

Then $\overrightarrow{AP}, \overrightarrow{BP}$ and $\overrightarrow{CP}$ are coplanar vectors $\Rightarrow \quad \overrightarrow{AP}.(\overrightarrow{BP} \times \overrightarrow{CP}) = 0$

$$\Rightarrow (\vec{r} - \vec{r_1}).[(\vec{r} - \vec{r_2}) \times (\vec{r} - \vec{r_3})] = 0.$$

Cartesian Form. Equation of the plane through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3).

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{r_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, $\vec{r_3} = x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$.

Then $(\vec{r} - \vec{r_1}) = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$, $(\vec{r} - \vec{r_2}) = (x - x_2)\hat{i} + (y - y_2)\hat{j} + (z - z_2)\hat{k}$

and $(\vec{r} - \vec{r_3}) = (x - x_3)\hat{i} + (y - y_3)\hat{j} + (z - z_3)\hat{k}$

$$\Rightarrow \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \\ x - x_3 & y - y_3 & z - z_3 \end{vmatrix} = 0.$$

Note. In numerical examples, in order to find the equation of a plane passing through three given points, it is a shorter to proceed as in **Ex. 12.**

(*g*) Equation of a plane through two given points and parallel to a given vector.

To prove that the equation of the plane through two points A and B having position vectors $\vec{r_1}$ and $\vec{r_2}$ respectively and parallel to a given vector $\vec{m}$ is $(\vec{r} - \vec{r_1}).[(\vec{r_2} - \vec{r_1}) \times \vec{m}] = 0$.

Since the points A and B lie in the plane, [*Given*]

$\therefore\ \overrightarrow{AB}$ is parallel to the plane $\Rightarrow \vec{r_2} - \vec{r_1}$ is parallel to the plane.

But $\vec{m}$ is parallel to the plane. [*Given*]

$\therefore\ (\vec{r_2} - \vec{r_1}) \times \vec{m}$ is normal to the plane.

$\therefore$ The equation of the plane through $\vec{r_1}$ with $(\vec{r_2} - \vec{r_1}) \times \vec{m}$ as the normal is $(\vec{r} - \vec{r_1}).[(\vec{r_2} - \vec{r_1}) \times \vec{m}] = 0$,

which is the reqd. equation of the plane.

Cartesian Form. Equation of a plane passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) and parallel to the line having direction–ratios < a, b, c >.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $\vec{r_1} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{r_2} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $\vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$.

Then $r - \vec{r_1} = (x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$

and $\vec{r_2} - \vec{r_1} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{j}$ and $\vec{m} = a\hat{i} + b\hat{j} + c\hat{k}$.

Then the equation of the plane is $(\vec{r} - \vec{r_1}).[(\vec{r_2} - \vec{r_1}) \times \vec{m}] = 0$

$\Rightarrow [\vec{r} - \vec{r_1}\ \ \vec{r_2} - \vec{r_1}\ \ \vec{m}] = 0$

$$\Rightarrow \begin{vmatrix} \mathbf{x - x_1} & \mathbf{y - y_1} & \mathbf{z - z_1} \\ \mathbf{x_2 - x_1} & \mathbf{y_2 - y_1} & \mathbf{z_2 - z_1} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \end{vmatrix} = \mathbf{0.}$$ **| Refer 10.33 Def.**

(*h*) INTERCEPT FORM*

To find the equation of the plane, which cuts off intercepts *a, b, c* on the co–ordinate axes.

Let the required plane (not passing through origin O) meet the co–ordinate axes in points A, B, C.

Then OA = a, OB = b and OC = c.

$\therefore$ The co–ordinates of A, B and C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

Let the equation of the required plane be $A'x + B'y + C'z + D' = 0$ ...(1)

[We use dashes to avoid confusion]

Since (1) passes through the points A $(a, 0, 0)$, B $(0, b, 0)$ and C $(0, 0, c)$,

$\therefore\quad A'(a) + D' = 0 \quad \Rightarrow \quad A' = -\frac{D'}{a}$,

$B'(b) + D' = 0 \quad \Rightarrow \quad B' = -\frac{D'}{b}$

and $\quad C'(c) + D' = 0 \quad \Rightarrow \quad C' = -\frac{D'}{c}$.

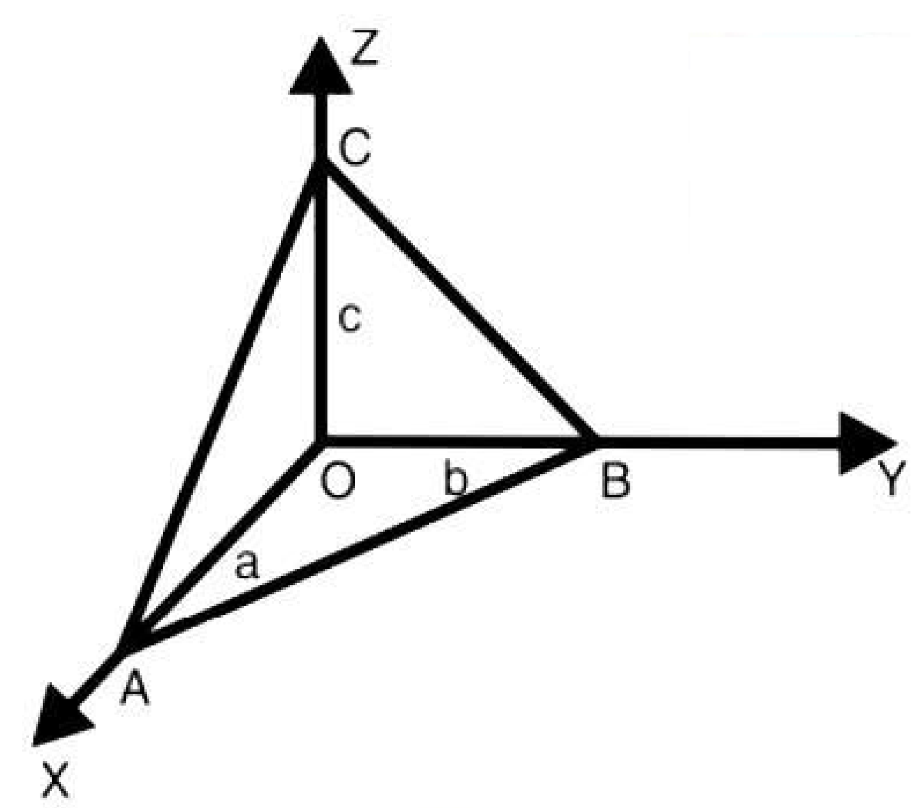


Fig.

Putting these values in (1), we get $-\frac{D'}{a}x - \frac{D'}{b}y - \frac{D'}{c}z + D' = 0$

$$\Rightarrow \boxed{\frac{\mathbf{x}}{\mathbf{a}} + \frac{\mathbf{y}}{\mathbf{b}} + \frac{\mathbf{z}}{\mathbf{c}} = \mathbf{1},}$$ [$\because$ *D'* $\neq$ *0*]

which is the required equation.

---

*If a plane (not passing through the origin) meets the co–ordinate axes in the points A, B and C respectively, then the directed distances OA, OB and OC are called *intercepts* made by the plane on X–axis, Y–axis and Z–axis respectively.

These are usually denoted by *a, b* and *c* respectively.

11.18. EQUATION OF A PLANE THROUGH A GIVEN POINT AND PARALLEL TO TWO GIVEN LINES

To find the vector equation of a plane through a given point $(\vec{r_1})$ and parallel to $\vec{a}$ and $\vec{b}$.

Let $\vec{r_1}$ be the position vector of the given point C on the plane and P be *any* point on it. Now the vector $\overrightarrow{CP}$ is coplanar with $\vec{a}$ and $\vec{b}$.

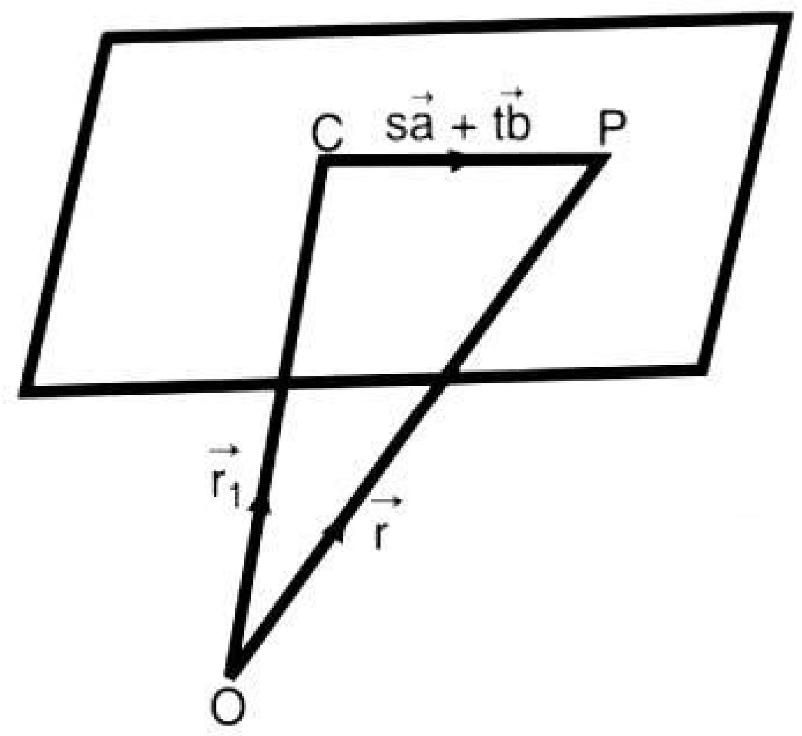


Fig.

$$\therefore \quad \overrightarrow{CP} = s\vec{a} + t\vec{b}.$$

If $\vec{r}$ be the position vector of P, then $\vec{r} = \overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}$

$$= \vec{r_1} + s\vec{a} + t\vec{b}.$$

Hence, the reqd. equation of the plane is $\vec{r} = \vec{r_1} + s\vec{a} + t\vec{b}$, where s and t are some scalars.

Cor. Equation of a plane containing two lines.

Let the two lines be $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$.

The equation of the plane containing these lines is $\vec{r} = \vec{a_1} + s\vec{b_1} + t\vec{b_2}$.

11.19. INTERSECTION OF TWO PLANES

When two planes intersect, they always intersect in a line.
Let π_1 and π_2 be two intersecting planes.

Let $\vec{a}$ be the position vector of a point, which is common to both the planes.

$\therefore$ The equations of the planes are $\pi_1 : (\vec{r} - \vec{a}) . \vec{n_1} = 0$ and $\pi_2 : (\vec{r} - \vec{a}) . \vec{n_2} = 0$.

Clearly, $(\vec{r} - \vec{a})$ is perpendicular to both $\vec{n_1}$ and $\vec{n_2}$

$\Rightarrow$ $(\vec{r} - \vec{a})$ is parallel to $\vec{n_1} \times \vec{n_2}$ $\Rightarrow$ $(\vec{r} - \vec{a}) = \lambda(\vec{n_1} \times \vec{n_2})$ for some scalar λ

$\Rightarrow$ $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\vec{b} = \vec{n_1} \times \vec{n_2}$, which is a st. line.

Hence, the result.

11.20. PLANE THROUGH THE INTERSECTION OF TWO PLANES

To prove that the equation of the plane through the intersection of two planes $\vec{r} . \vec{n_1} = p_1$ and $\vec{r} . \vec{n_2} = p_2$ is :

$\vec{r} = (\vec{n_1} + \lambda\vec{n_2}) = p_1 + \lambda p_2$.

Let π_1 and π_2 be two intersecting planes $\vec{r} . \vec{n_1} = p_1$ and $\vec{r} . \vec{n_2} = p_2$.

Let $\vec{r_1}$ be the position vector of a point, which is common to both of them.

Then $\vec{r_1} . \vec{n_1} = p_1$ and $\vec{r_1} . \vec{n_2} = p_2$

$\Rightarrow$ $\vec{r_1} . \vec{n_1} + \lambda(\vec{r_1} . \vec{n_2}) = p_1 + \lambda p_2$ $\quad\Rightarrow\quad$ $\vec{r_1} . (\vec{n_1} + \lambda\vec{n_2}) = p_1 + \lambda p_2$.

Hence, the required equation of the plane is $\vec{r} . (\vec{n_1} + \lambda\vec{n_2}) = p_1 + \lambda p_2$.

Cartesian Form. To prove that the equation of any plane through the intersection of two planes :

$$\mathbf{a_1x + b_1y + c_1z + d_1 = 0 \text{ and } a_2x + b_2y + c_2z + d_2 = 0}$$

is a plane of the form :

$$\mathbf{(a_1x + b_1y + c_1z + d_1) + \lambda (a_2x + b_2y + c_2z + d_2) = 0.}$$

The given planes are $a_1x + b_1y + c_1z = -d_1$ and $a_2x + b_2y + c_2z = -d_2$

$\Rightarrow$ $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (a_1\hat{i} + b_1\hat{j} + c_1\hat{k}) = -d_1$ and $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (a_2\hat{i} + b_2\hat{j} + c_2\hat{k}) = -d_2$

i. e. $\vec{r} \cdot \vec{n_1} = p_1$ and $\vec{r} \cdot \vec{n_2} = p_2$,

where $\vec{n_1} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}, \vec{n_2} = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}, p_1 = -d_1, p_2 = -d_2$.

$\therefore$ The reqd. equation of the plane is $\vec{r} \cdot (\vec{n_1} + \lambda \vec{n_2}) = p_1 + \lambda p_2$

$\Rightarrow$ $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot [(a_1 + \lambda a_2)\hat{i} + (b_1 + \lambda b_2)\hat{j} + (c_1 + \lambda c_2)\hat{k}] = (-d_1) + \lambda(-d_2)$

$\Rightarrow$ $(a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)z + (d_1 + \lambda d_2) = 0$

$\Rightarrow$ $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0.$

11.21. ANGLE BETWEEN TWO PLANES

Let α_1 and α_2 be the two given planes. The angle between these two planes is zero when α_1 and α_2 are parallel. When α_1 and α_2 are not parallel, then :

"*the angle between the two planes is defined to be the angle between their normals.*"

Thus if 'θ_1' and 'θ_2' be the two angles between two non–parallel planes, then we have the relation :

$$\theta_1 + \theta_2 = \pi, 0 < \theta_1, \theta_2 < \pi.$$

(*a*) To find the angle between the two planes $\vec{r} \cdot \hat{n}_1 = p_1$ and $\vec{r} \cdot \hat{n}_2 = p_2$.

Let π_1 and π_2 be two given planes.

Then the angle between π_1 and π_2 = angle between normals to planes π_1 and π_2 *i.e.* the angle between $\hat{n}_1$ and $\hat{n}_2$.

If 'θ' be the angle between planes π_1 and π_2, then $\hat{n}_1 \cdot \hat{n}_2 = \cos\theta$.

$[\because \hat{n}_1 \cdot \hat{n}_2 = (1)(1)\cos\theta = \cos\theta]$

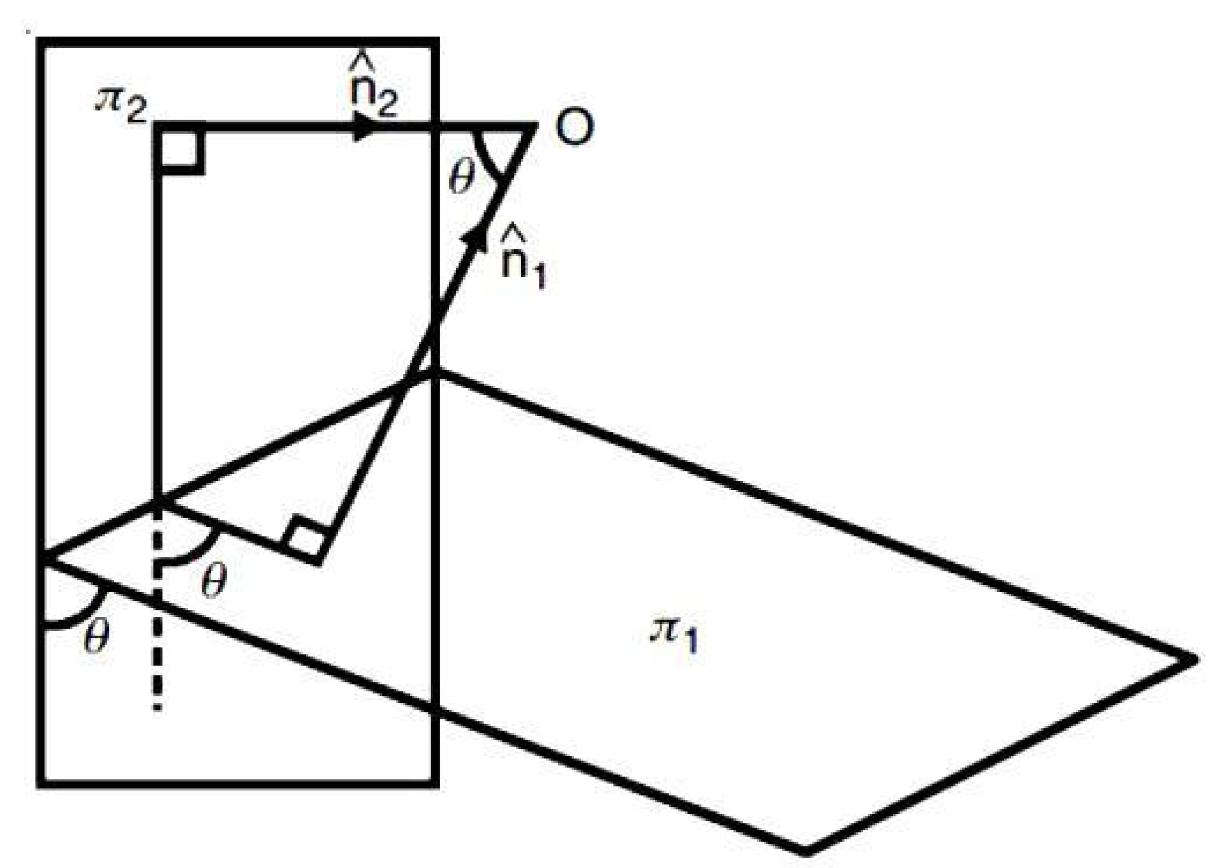


Fig.

Cor. 1. When planes are perpendicular.

Here $\theta = \pi/2$, then $\hat{n}_1 \cdot \hat{n}_2 = 0$.

Cor. 2. When planes are parallel.

Here $\theta = 0°$, then $\hat{n}_1 = \hat{n}_2$.

(*b*) To find the angle between the two planes :

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

The given planes are $a_1x + b_1y + c_1z + d_1 = 0$ ....(1)

and $a_2x + b_2y + c_2z + d_2 = 0$ .... (2)

Direction–ratios of the normals to these planes are $< a_1, b_1, c_1 >$ and $< a_2, b_2, c_2 >$ respectively.

But the angle between the two planes = the angle between their normals.

$\therefore$ If 'θ' be the angle between the two planes, then $\cos\theta = \pm \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

KEY POINT

If the two planes are non–perpendicular, then the acute angle between them is given by :

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$ and the obtuse angle between them is given by :

$$\cos\theta = -\frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}.$$

Cor. 1. Condition of Parallelism.

The two planes are parallel iff normals to these planes are parallel iff $\mathbf{\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}}$.

Cor. 2. Condition of Perpendicularity.

The two planes are perpendicular iff $\theta_1 = \theta_2 = \frac{\pi}{2}$ iff $\cos\theta_1 = \cos\theta_2 = 0$

iff $\mathbf{a_1a_2 + b_1b_2 + c_1c_2 = 0}$.

KEY POINT

Two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are identical iff $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2}$.

Cor. 3. Equation of any plane parallel to a given plane.
The equation of any plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is any constant.

For, the planes $ax + by + cz + d = 0$, $ax + by + cz + k = 0$ are parallel because $\frac{a}{a} = \frac{b}{b} = \frac{c}{c} = 1$, which is true.

Rule to write :
In the equation of the given plane, change only the constant term to a new constant.

11.22. DISTANCE OF A POINT FROM A PLANE

(a) To find the distance of a point having position vector $\vec{a}$ from the plane whose equation $\vec{r}\cdot\hat{n} = p$.

Let π_1 be the plane $\vec{r}\cdot\hat{n} = p$ ...(1)

Let P be the point whose position vector is $\vec{a}$.
Consider the plane π_2 through P parallel to plane π_1.

$\therefore$ The unit normal to π_2 plane is also $\hat{n}$.

$\therefore$ Its equation is $(\vec{r} - \vec{a})\cdot\hat{n} = 0$

$\Rightarrow \vec{r}\cdot\hat{n} - \vec{a}\cdot\hat{n} = 0 \Rightarrow \vec{r}\cdot\hat{n} = \vec{a}\cdot\hat{n}$

$\Rightarrow$ distance OL of this plane from the origin O = $\vec{a}\cdot\hat{n}$

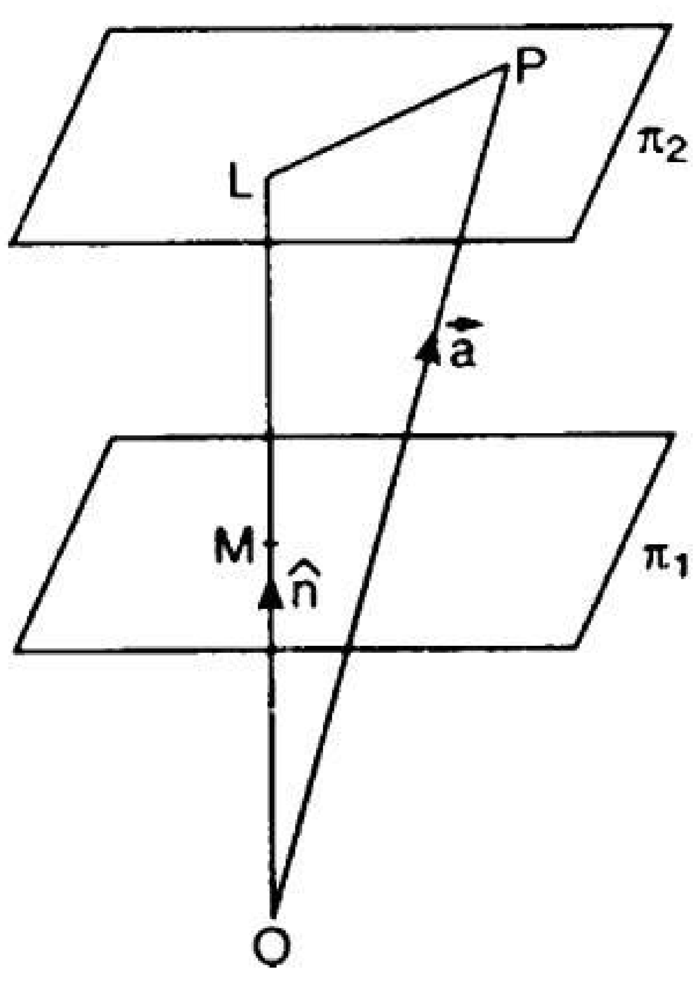


Fig.

Case I. *When P and O are on opposite sides of π_1,*

then distance LM of P from plane π_1 is $\vec{r}\cdot\hat{n} = d$.

Case II. *When P and O are on same side of π_1, the above formula gives a negative result.*

(*b*) To find the perpendicular distance of the point (x_1, y_1, z_1) from the plane lx + my + nz = p (p > 0).

Let P (x_1, y_1, z_1) be the given point.

Let $lx + my + nz = p$ ($p > 0$) be the given plane ABC.

Thro' P, draw PL ⊥ on the plane ABC. Let = $|\overrightarrow{PL}| = d$ (say).

Let OM be perp. from O on the plane ABC so that $|\overrightarrow{OM}| = p$.

Thus the direction-cosines of $\overrightarrow{OM}$ are $< l, m, n >$.

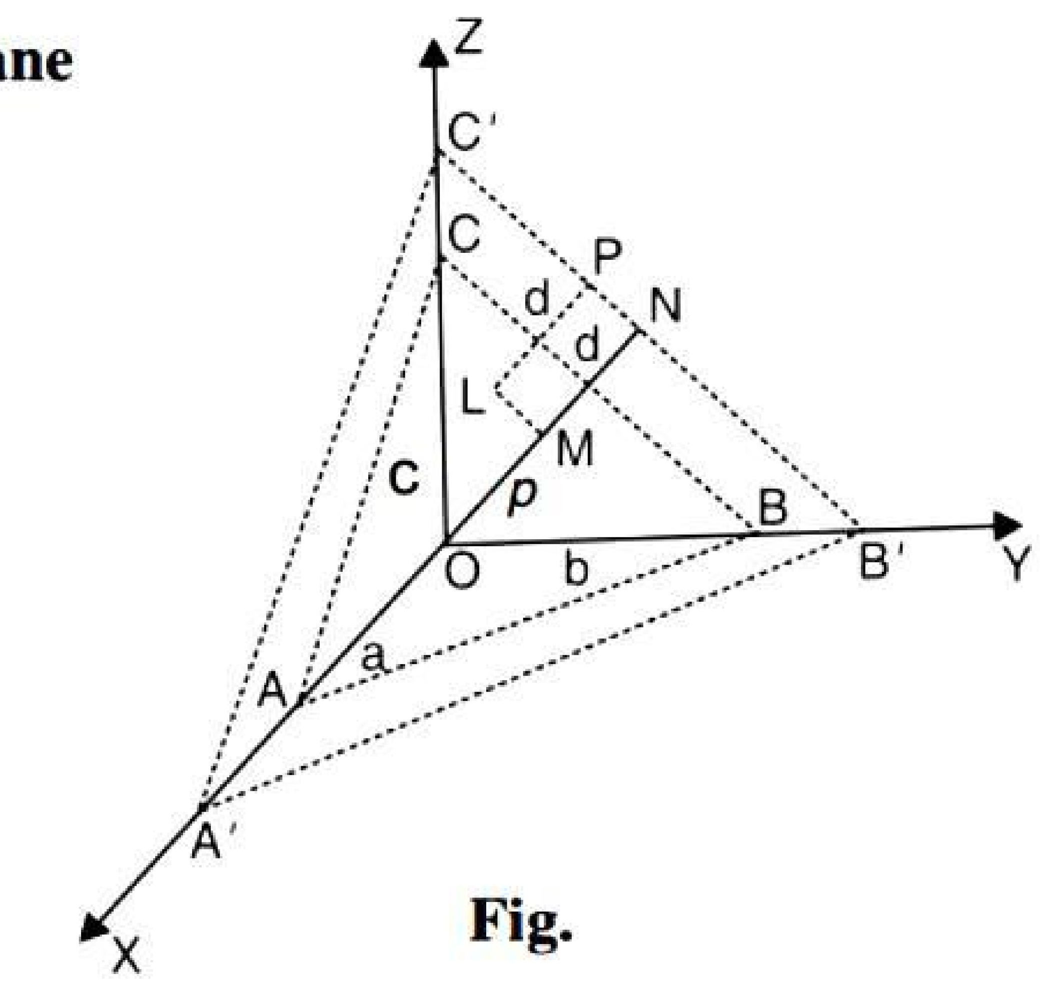


Fig.

Thro' P, draw a plane A′B′C′ parallel to the given plane ABC so as to meet OM (produced) in N.

Now $|\overrightarrow{ON}| = |\overrightarrow{OM}| + |\overrightarrow{MN}| = |\overrightarrow{OM}| + |\overrightarrow{LP}| = p + d$.

The equation of the plane A′B′C′ is $lx + my + nz = p + d$.

Since this plane passes thro' P (x_1 , y_1, z_1), ∴ $lx_1 + my_1 + nz_1 = p + d$.

Hence, $\mathbf{d = lx_1 + my_1 + nz_1 - p}$, the numerical value of which gives the required perpendicular distance.

RULE to write down :

*Put the co-ordinates of the point in the **LHS** of the equation of the plane (after making **RHS** = 0). The result is the required perpendicular distance.*

(*c*) To find the perpendicular distance of the point (x_1, y_1, z_1) from the plane ax + by + cz + d = 0.

The given equation of the plane is $ax + by + cz + d = 0$ ...(1)

To reduce (1) to normal form :

Dividing (1) by $\sqrt{a^2 + b^2 + c^2}$, we get :

$$\frac{a}{\sqrt{a^2+b^2+c^2}}x + \frac{b}{\sqrt{a^2+b^2+c^2}}y + \frac{c}{\sqrt{a^2+b^2+c^2}}z + \frac{d}{\sqrt{a^2+b^2+c^2}} = 0 \quad ...(2)$$

(2) is of the normal form if we transpose the constant term to **RHS.**

If D is the reqd. perpendicular distance, then :

$$D = \frac{a}{\sqrt{a^2+b^2+c^2}}x_1 + \frac{b}{\sqrt{a^2+b^2+c^2}}y_1 + \frac{c}{\sqrt{a^2+b^2+c^2}}z_1 + \frac{d}{\sqrt{a^2+b^2+c^2}}$$

$$= \pm \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2+b^2+c^2}}.$$ [*Considering both signs of d*]

Hence, $$\mathbf{D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}}.$$

GUIDE-LINES

Step (i) : *Make **RHS** of the equation of the plane as zero.*

Step (ii) : *Put the co-ordinates of the point in the **LHS** of the equation of the plane.*

Step (iii) : *Divide the result by* $\sqrt{(\text{coeff. of } x)^2 + (\text{coeff. of } y)^2 + (\text{coeff. of } z)^2}$.

The numerical value of the result is the required perpendicular distance.

11.23. PLANES BISECTING THE ANGLES BETWEEN TWO PLANES

To find the equations of the planes, which bisect the angles between the planes :

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$.

The given planes are $a_1x + b_1y + c_1z + d_1 = 0$ ...(1)

and $a_2x + b_2y + c_2z + d_2 = 0$ ...(2)

Let P (x, y, z) be *any* point on either of the planes bisecting the angles between the given planes.

Then the perpendicular distance of P from plane (1)

= Perpendicular distance of P from plane (2)

$$\Rightarrow \frac{|a_1x + b_1y + c_1z + d_1|}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{|a_2x + b_2y + c_2z + d_2|}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\Rightarrow \frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}},$$

which are the required equations of the planes.

Note. One of these two bisecting planes bisects the acute angle and the other bisects the obtuse angle between the two given planes.

11.24. GENERAL (UNSYMMETRICAL) FORM

Two general equations of the first degree in x, y, z, taken together, represent a straight line.

Consider the equations of two planes :

$$a_1x + b_1y + c_1z + d_1 = 0 \quad ...(1)$$

and $a_2x + b_2y + c_2z + d_2 = 0$ ...(2)

If these planes are not parallel *i.e.* $a_1 : b_1 : c_1 \neq a_2 : b_2 : c_2$, then at least two of the numbers

$a_1b_2 - a_2b_1$, $b_1c_2 - b_2c_1$, $c_1a_2 - c_2a_1$ are not zero.

Without any loss of generality, let $a_1b_2 - a_2b_1 \neq 0$, $b_1c_2 - b_2c_1 \neq 0$.

Multiplying (1) by c_2 and (2) by c_1 and subtracting, we get :

$(a_1c_2 - a_2c_1)\, x + (\, b_1c_2 - b_2c_1)\, y + (d_1c_2 - d_2c_1) = 0$

$$\Rightarrow \frac{a_1c_2 - a_2c_1}{b_1c_2 - b_2c_1}\, x + y + \frac{d_1c_2 - d_2c_1}{b_1c_2 - b_2c_1} = 0 \qquad [\because\ b_1c_2 - b_2c_1 \neq 0]$$

$$\Rightarrow \quad y = Ax + B \quad ...(3),$$

where $A = -\dfrac{a_1c_2 - a_2c_1}{b_1c_2 - b_2c_1}$ and $B = -\dfrac{d_1c_2 - d_2c_1}{b_1c_2 - b_2c_1}$.

Again multiplying (1) by a_2 and (2) by a_1 and subtracting, we get :

$$(a_2b_1 - a_1b_2)y + (c_1a_2 - c_2a_1)\, z + (d_1a_2 - d_2a_1) = 0$$

$$\Rightarrow \quad y - \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\, z - \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1} = 0 \qquad [\because\ a_1b_2 - a_2b_1 \neq 0]$$

$$\Rightarrow \quad y = Cz + D \quad ...(4),$$

where $C = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$ and $D = \dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}$.

From (3) and (4), $Ax + B = y = Cz + D$

$$\Rightarrow \qquad \frac{x + B/A}{C} = \frac{y}{AC} = \frac{z + D/C}{A} \qquad ...(5)$$

Thus (1) and (2), taken together, are equivalent to (5).

But (5) represents a st. line.

Thus planes (1) and (2), taken together, represent a st. line.

Hence, planes (1) and (2) intersect in a st. line.

11.25. TRANSFORMATION OF UNSYMMETRICAL FORM TO SYMMETRICAL FORM

To transform the equations :

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ to the symmetrical form.

The equations of a line in unsymmetrical form are given as :

$$a_1x + b_1y + c_1z + d_1 = 0 \qquad ...(1)$$

and $$a_2x + b_2y + c_2z + d_2 = 0 \qquad ...(2)$$

To transform (1) and (2) to symmetrical form, we need :

(*i*) direction-ratios of the line (*ii*) co–ordinates of *any* point on the line.

Let $< a, b, c >$ be the direction–ratios of the line represented by (1) and (2), taken together.

Since the line lies in both planes (1) and (2),

$\therefore$ the normals to the two planes must be perpendicular to this line.

$\therefore$ $$aa_1 + bb_1 + cc_1 = 0 \qquad ...(3)$$

and $$aa_2 + bb_2 + cc_2 = 0 \qquad ...(4)$$

Solving, $$\frac{a}{b_1c_2 - b_2c_1} = \frac{b}{c_1a_2 - c_2a_1} = \frac{c}{a_1b_2 - a_2b_1} = k \text{ (say)},$$

where k is any non-zero constant.

$\therefore$ $$a = k\,(b_1c_2 - b_2c_1),\ b = k\,(c_1a_2 - c_2a_1),\ c = k\,(a_1b_2 - a_2b_1).$$

For convenience, we find the point of intersection of this line with the XY–plane (*i.e.* $z = 0$ plane).

Putting $z = 0$ in (1) and (2), we get :

$$a_1x + b_1y + d_1 = 0 \qquad ...(5)$$

and $$a_2x + b_2y + d_2 = 0 \qquad ...(6)$$

Solving, $$\frac{x}{b_1d_2 - b_2d_1} = \frac{y}{d_1a_2 - d_2a_1} = \frac{1}{a_1b_2 - a_2b_1}.$$

$\therefore$ $$x = \frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}.$$

Hence, the co–ordinates of a point on the line are $\left(\frac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \frac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}, 0\right)$, where $a_1b_2 - a_2b_1 \neq 0$.

Hence, the equations of the line in the symmetrical form are :

$$\frac{x - \dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}}{b_1c_2 - b_2c_1} = \frac{y - \dfrac{d_1a_2 - d_2a_1}{a_1b_2 - a_2b_1}}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}.$$ [*Cancelling k because $k \neq 0$*]

Frequently Asked Questions

FAQs

Example 1. Write the sum of intercepts cut off by the plane $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})-5=0$ on the three axes. *(A.I.C.B.S.E. 2016)*

Solution. The given plane is $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})-5=0$.

In Cartesian System. $2x+y-z=5$

$$\Rightarrow \frac{x}{5/2}+\frac{y}{5}+\frac{z}{-5}=1.$$

Its intercepts are $\frac{5}{2}$, 5 and –5.

Hence, the sum of intercepts $=\frac{5}{2}+5+(-5)=\frac{5}{2}$.

Example 2. Find the distance of the plane : $2x-3y+4z-6=0$ from the origin. *(N.C.E.R.T.)*

Solution. The given equation of the plane is :

$$2x-3y+4z-6=0.$$

∴ Distance of the plane from the origin

$$=\frac{|2(0)-3(0)+4(0)-6|}{\sqrt{4+9+16}}=\frac{|-6|}{\sqrt{29}}=\frac{6}{\sqrt{29}}=\frac{6\sqrt{29}}{29}\text{ units.}$$

Example 3. Find the distance between the planes : $2x-y+2z=5$ and $5x-2.5y+5z=20$. *(A.I.C.B.S.E. 2017)*

Solution. Any point on $2x-y+2z=5$ is $(0,-5,0)$.

$$\text{Required distance}=\frac{|5(0)-2.5(-5)+0-20|}{\sqrt{(5)^2+\left(-\frac{5}{2}\right)^2+5^2}}$$

$$=\frac{\left|\frac{5}{2}\times5-20\right|}{(5)\sqrt{1+\left(-\frac{1}{2}\right)^2+1}}=\frac{\frac{15}{2}}{5\sqrt{2+\frac{1}{4}}}$$

$$=\frac{3}{2}\times\frac{2}{\sqrt{9}}=1\text{ unit.}$$

Example 4. Find the equation of the plane with intercepts 2, 3 and 4 on the x, y and z–axis respectively. *(N.C.E.R.T.)*

Solution. Here 'a' = 2, 'b' = 3 and 'c' = 4.

∴ The equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

i.e. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$ *i.e.* $6x+4y+3z=12$.

Example 5. Find the co-ordinates of the point where the line through the points A (3, 4, 1) and B (5, 1, 6) crosses the xy-plane. *(N.C.E.R.T.; H.P.B. 2013 S ; A.I.C.B.S.E. 2012)*

Solution. The equations of the line through A (3, 4, 1) and B (5, 1, 6) are :

$$\frac{x-3}{5-3}=\frac{y-4}{1-4}=\frac{z-1}{6-1}$$

$$\Rightarrow \frac{x-3}{2}=\frac{y-4}{-3}=\frac{z-1}{5} \quad ...(1)$$

Any point on (1) is $(3+2k, 4-3k, 1+5k)$ ...(2)

This lies on XY–plane ($z=0$).

$\therefore\ 1+5k=0 \Rightarrow k=-\frac{1}{5}$.

Putting in (2), $\left(3-\frac{2}{5},4+\frac{3}{5},1-1\right)$ *i.e.* $\left(\frac{13}{5},\frac{23}{5},0\right)$,

which are the reqd. co-ordinates of the point.

Example 6. Find the vector equation of a plane passing through the point having position vector $2\hat{i}+\hat{j}+\hat{k}$ and perpendicular to the vector : $4\hat{i}-2\hat{j}+3\hat{k}$. *(H.B. 2016)*

Solution. The required vector equation of the plane is :

$$(\vec{r}-\vec{r_1})\cdot\hat{n}=0$$

$$\Rightarrow (\vec{r}-(2\hat{i}+\hat{j}+\hat{k}))\cdot\frac{4\hat{i}-2\hat{j}+3\hat{k}}{\sqrt{16+4+9}}=0$$

$$\Rightarrow \vec{r}\cdot(4\hat{i}-2\hat{j}+3\hat{k})-(2\hat{i}+\hat{j}+\hat{k})\cdot(4\hat{i}-2\hat{j}+3\hat{k})=0$$

$$\Rightarrow \vec{r}\cdot(4\hat{i}-2\hat{j}+3\hat{k})-[(2)(4)+(1)(-2)+(1)(3)]=0$$

$$\Rightarrow \vec{r}\cdot(4\hat{i}-2\hat{j}+3\hat{k})=9.$$

Example 7. Find the vector equation of the plane, which is at a distance of $\frac{6}{\sqrt{29}}$ units from the origin and its normal vector from the origin is $2\hat{i}-3\hat{j}+4\hat{k}$. Also, find its cartesian form. *(N.C.E.R.T.; W. Bengal 2018)*

Solution. Let $\vec{n}=2\hat{i}-3\hat{j}+4\hat{k}$.

Then $|\vec{n}|=\sqrt{4+9+16}=\sqrt{29}$.

Now $\hat{n}=\frac{\vec{n}}{|\vec{n}|}=\frac{2\hat{i}-3\hat{j}+4\hat{k}}{\sqrt{29}}$.

Hence, the reqd. equation of the plane is :

$$\vec{r}\cdot\left(\frac{2}{\sqrt{29}}\hat{i}-\frac{3}{\sqrt{29}}\hat{j}+\frac{4}{\sqrt{29}}\hat{k}\right)=\frac{6}{\sqrt{29}}.$$

In Cartesian Form :

$$(x\hat{i} + y\hat{j} + z\hat{k}).\left(\frac{2}{\sqrt{29}}\hat{i} - \frac{3}{\sqrt{29}}\hat{j} + \frac{4}{\sqrt{29}}\hat{k}\right) = \frac{6}{\sqrt{29}}$$

$$\Rightarrow \quad (x)\left(\frac{2}{\sqrt{29}}\right) + y\left(\frac{-3}{\sqrt{29}}\right) + z\left(\frac{4}{\sqrt{29}}\right) = \frac{6}{\sqrt{29}}$$

$$\Rightarrow \quad 2x - 3y + 4z = 6.$$

Example 8. Find the direction-cosines of the unit vector perpendicular to the plane $\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$ through the origin. *(N.C.E.R.T.)*

Solution. The given plane is $\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$

$$\Rightarrow \vec{r}.(-6\hat{i} + 3\hat{j} + 2\hat{k}) = 1 \quad ...(1)$$

Now $|-6\hat{i} + 3\hat{j} + 2\hat{k}| = \sqrt{36+9+4} = \sqrt{49} = 7$.

Dividing (1) by 7, $\vec{r}.\left(-\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}\right) = \frac{1}{7}$,

which is the equation of the plane in the form $\vec{r}.\hat{n} = p$.

Thus $\hat{n} = -\frac{6}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{2}{7}\hat{k}$,

which is the unit vector perpendicular to the plane through the origin.

Hence, the direction-cosines of $\hat{n}$ are $< -\frac{6}{7}, \frac{3}{7}, \frac{2}{7} >$.

Example 9. Find the angle between two planes :
2x + y – 2z = 5 and 3x – 6y – 2z = 7,
using vector method. *(N.C.E.R.T.)*

Solution. We know that the angle between two planes = the angle between their normals.

From the given equations,

$\vec{n_1} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{n_2} = 3\hat{i} - 6\hat{j} - 2\hat{k}$.

If 'θ' be the reqd. angle, then :

$$\cos\theta = \left|\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right|$$

$$= \left|\frac{(2\hat{i} + \hat{j} - 2\hat{k})\cdot(3\hat{i} - 6\hat{j} - 2\hat{k})}{\sqrt{4+1+4}\sqrt{9+36+4}}\right|$$

$$= \left|\frac{(2)(3) + (1)(-6) + (-2)(-2)}{(3)(7)}\right|$$

$$= \left|\frac{6-6+4}{21}\right| = \left|\frac{4}{21}\right| = \frac{4}{21}.$$

Hence, $\theta = \cos^{-1}\left(\frac{4}{21}\right)$.

Example 10. Find the equation of the plane through the points (1, –1, 2) and (2, – 2, 2) and perpendicular to the plane 6x – 2y + 2z = 9. *(Meghalaya B. 2013)*

Solution. Any plane through (1, – 1, 2) is :

$$a(x-1) + b(y+1) + c(z-2) = 0 \quad ...(1)$$

[*One–point Form*]

Since it passes through (2, – 2, 2),

$\therefore a(2-1) + b(-2+1) + c(2-2) = 0$

$$\Rightarrow \quad a - b + 0.c = 0 \quad ...(2)$$

Also the plane (1) and the given plane $6x - 2y + 2z = 9$ are perpendicular,

∴ their normals are also perpendicular

$\Rightarrow a(6) + b(-2) + c(2) = 0$

$$\Rightarrow \quad 3a - b + c = 0 \quad ...(3)$$

Solving (2) and (3), $\frac{a}{-1+0} = \frac{b}{0-1} = \frac{c}{-1+3}$

$\Rightarrow \quad \frac{a}{1} = \frac{b}{1} = \frac{c}{-2} = k$ (say), where $k \neq 0$

$\therefore \quad a = k, b = k, c = -2k.$

Putting these values of a, b, c in (1), we get :

$k(x-1) + k(y+1) - 2k(z-2) = 0$

$\Rightarrow \quad (x-1) + (y+1) - 2(z-2) = 0 \quad [\because k \neq 0]$

$\Rightarrow \quad x - 1 + y + 1 - 2z + 4 = 0$

$\Rightarrow \quad x + y - 2z + 4 = 0,$

which is the reqd. equation.

Example 11. Find the equation of a plane through the points (2, 1, 0), (3, –2, –2) and (3, 1, 7).

Solution. Any plane through (2, 1, 0) is :

$a(x-2) + b(y-1) + c(z-0) = 0$ [*One-point Form*]

$$\Rightarrow \quad a(x-2) + b(y-1) + cz = 0 \quad ...(1)$$

Since the plane passes through the points (3, – 2, – 2) and (3, 1, 7),

$\therefore a(3-2) + b(-2-1) + c(-2) = 0$

and $\quad a(3-2) + b(1-1) + c(7) = 0$

$$\Rightarrow \quad a - 3b - 2c = 0 \quad ...(2)$$

$$\text{and} \quad a + 0b + 7c = 0 \quad ...(3)$$

Solving (2) and (3), $\frac{a}{-21+0} = \frac{b}{-2-7} = \frac{c}{0+3}$

$\Rightarrow \quad \frac{a}{-21} = \frac{b}{-9} = \frac{c}{3} \quad \Rightarrow \quad \frac{a}{7} = \frac{b}{3} = \frac{c}{-1} = k$ (say),

where $k \neq 0$.

$\therefore \quad a = 7k, b = 3k, c = -k.$

Putting these values of a, b, c in (1), we get :

$7k(x-2) + 3k(y-1) - k(z) = 0$

$\Rightarrow \quad 7(x-2) + 3(y-1) - z = 0 \quad [\because k \neq 0]$

$\Rightarrow \quad 7x - 14 + 3y - 3 - z = 0$

$\Rightarrow \quad 7x + 3y - z - 17 = 0,$

which is the reqd. equation.

Example 12. Find the equation of the plane determined by the points A (3, –1, 2), B (5, 2, 4) and C (–1, –1, 6) and hence find the distance between the plane and the point P (6, 5, 9). *(C.B.S.E. 2013, 12)*

Solution. (*i*) Any plane through A (3, –1, 2) is :

$$a(x-3)+b(y+1)+c(z-2)=0 \quad ...(1)$$

Since the plane passes through the points B (5, 2, 4) and C (–1, –1, 6),

$\therefore \quad a(5-3)+b(2+1)+c(4-2)=0$

and $a(-1-3)+b(-1+1)+c(6-2)=0$

$\Rightarrow \quad 2a+3b+2c=0 \quad ...(2)$

and $-4a+0b+4c=0 \quad ...(3)$

Solving (2) and (3), $\dfrac{a}{12-0}=\dfrac{b}{-8-8}=\dfrac{c}{0+12}$

$\Rightarrow \dfrac{a}{12}=\dfrac{b}{-16}=\dfrac{c}{12} \Rightarrow \dfrac{a}{3}=\dfrac{b}{-4}=\dfrac{c}{3}=k$ (say), where $k \neq 0$.

$\therefore \quad a=3k,\ b=-4k,\ c=3k.$

Putting these values of *a*, *b*, *c* in (1), we get :

$3k(x-3)-4k(y+1)+3k(z-2)=0$

$\Rightarrow \quad 3(x-3)-4(y+1)+3(z-2)=0 \quad [\because k \neq 0]$

$\Rightarrow \quad 3x-9-4y-4+3z-6=0$

$\Rightarrow \quad 3x-4y+3z-19=0,$

which is the reqd. equation of the plane.

(*ii*) Distance of P (6, 5, 9) from the above plane

$$=\frac{|3(6)-4(5)+3(9)-19|}{\sqrt{9+16+9}}=\frac{|18-20+27-19|}{\sqrt{34}}$$

$$=\frac{|45-39|}{\sqrt{34}}=\frac{6}{\sqrt{34}}=\frac{6\sqrt{34}}{34}=\frac{3}{17}\sqrt{34} \text{ units.}$$

Example 13. Find the equation of the plane passing through the line of intersection of the planes :

$$\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1 \text{ and } \vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})+4=0 \text{ and}$$

parallel to x-axis. *(A.I.C.B.S.E. 2011)*

Solution. The equation of the plane passing through the line of intersection of the given planes :

$\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=1$ and $\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})+4=0$ is :

$$[\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})-1]+\lambda[\vec{r}\cdot(2\hat{i}+3\hat{j}-\hat{k})+4]=0$$

$$\Rightarrow \vec{r}\cdot[(1+2\lambda)\hat{i}+(1+3\lambda)\hat{j}+(1-\lambda)\hat{k}]+(-1+4\lambda)=0 \quad ...(1)$$

Since the plane (1) is parallel to *x*-axis,

$\therefore (1+2\lambda)(1)+(1+3\lambda)(0)+(1-\lambda)(0)=0$

$\Rightarrow \quad 1+2\lambda=0 \Rightarrow \lambda=-\dfrac{1}{2}.$

Putting the value of λ in (1), we get :

$$\vec{r}\cdot\left[\left(1+2\left(-\frac{1}{2}\right)\right)\hat{i}+\left(1+3\left(-\frac{1}{2}\right)\right)\hat{j}+\left(1+\frac{1}{2}\right)\hat{k}\right]+\left(-1+4\left(-\frac{1}{2}\right)\right)=0$$

$$\Rightarrow \vec{r}\cdot\left[0\hat{i}-\frac{1}{2}\hat{j}+\frac{3}{2}\hat{k}\right]+(-1-2)=0$$

$$\Rightarrow \vec{r}\cdot\left(-\frac{\hat{j}}{2}+\frac{3}{2}\hat{k}\right)-3=0 \Rightarrow \vec{r}\cdot(\hat{j}-3\hat{k})+6=0,$$

which is the reqd. equation.

Example 14. Find the equation of the plane through the line of intersection of :

$$\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1 \text{ and } \vec{r}\cdot(\hat{i}-\hat{j})+4=0$$

and perpendicular to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+8=0$.

Hence, find whether the plane thus obtained contains the line :

x – 1 = 2y – 4 = 3z – 12. *(C.B.S.E. 2017)*

Solution. The given planes are :

$$\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1 \quad ...(1)$$

and $$\vec{r}\cdot(\hat{i}-\hat{j})+4=0 \quad ....(2)$$

Any plane through the intersection of planes (1) and (2) is :

$$[\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})-1]+\lambda[\vec{r}\cdot(\hat{i}-\hat{j})+4]=0$$

$$\Rightarrow \vec{r}\cdot[(2+\lambda)\hat{i}+(-3-\lambda)\hat{j}+4\hat{k}]=1-4\lambda \quad ...(3)$$

Since the plane (3) is perpendicular to the plane :

$$\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+8=0 \quad ...(4)$$

$\therefore (2+\lambda)(2)+(-3-\lambda)(-1)+4(1)=0$

$\Rightarrow 4+2\lambda+3+\lambda+4=0 \Rightarrow 3\lambda+11=0$

$\Rightarrow \lambda=\dfrac{-11}{3}.$

Putting this value of λ in (3), we get :

$$\vec{r}\cdot\left[\left(2-\frac{11}{3}\right)\hat{i}+\left(-3+\frac{11}{3}\right)\hat{j}+4\hat{k}\right]=1-4\times\frac{-11}{3}$$

$$\Rightarrow \quad \vec{r}\cdot\left(-\frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+4\hat{k}\right)=\frac{47}{3}$$

$$\Rightarrow \quad \vec{r}\cdot(-5\hat{i}+2\hat{j}+12\hat{k})=47,$$

which is the required equation of the plane.

Now, the given line is $x - 1 = 2y - 4 = 3z - 12$

i.e. $$\frac{x-1}{1} = \frac{y-2}{1/2} = \frac{z-4}{1/3}$$

$$\Rightarrow \quad \frac{x-1}{6} = \frac{y-2}{3} = \frac{z-4}{2}.$$

The line passes through the point (1, 2, 4) and satisfies the equation of the plane.

Hence, the plane contains the line.

Example 15. Find the equation of the plane which contains the line of intersection of the planes:

$$\vec{r}.(\hat{i} - 2\hat{j} + 3\hat{k}) - 4 = 0 \text{ and}$$

$$\vec{r}.(-2\hat{i} + \hat{j} + \hat{k}) + 5 = 0$$ **and whose intercept on x–axis is equal to that of on y-axis.** *(A.I.C.B.S.E. 2016)*

Solution. The equation of the plane which contains the line of intersection of the planes :

$$\vec{r}.(\hat{i} - 2\hat{j} + 3\hat{k}) - 4 = 0 \text{ and}$$

$$\vec{r}.(-2\hat{i} + \hat{j} + \hat{k}) + 5 = 0 \text{ is :}$$

$$\left[\vec{r}.(\hat{i} - 2\hat{j} + 3\hat{k}) - 4\right] + \lambda\left[\vec{r}.(-2\hat{i} + \hat{j} + \hat{k}) + 5\right] = 0$$

$$\Rightarrow \vec{r}\cdot\left[(1-2\lambda)\hat{i} + (-2+\lambda)\hat{j} + (3+\lambda)\hat{k}\right] + (-4 + 5\lambda) = 0 \quad ...(1)$$

i.e., $(1 - 2\lambda)x + (-2 + \lambda)y + (3 + \lambda)z = 4 - 5\lambda$

$$\Rightarrow \quad \frac{x}{\frac{4-5\lambda}{1-2\lambda}} + \frac{y}{\frac{4-5\lambda}{-2+\lambda}} + \frac{z}{\frac{4-5\lambda}{3+\lambda}} = 1.$$

By the question, $\dfrac{4-5\lambda}{1-2\lambda} = \dfrac{4-5\lambda}{-2+\lambda}$

$\Rightarrow \quad -2 + \lambda = 1 - 2\lambda \Rightarrow 3\lambda = 3 \Rightarrow \lambda = 1.$

Putting in (1), $\vec{r}.\left[(1-2)\hat{i} + (-2+1)\hat{j} + (3+1)\hat{k}\right] + (-4 + 5) = 0$

$$\Rightarrow \quad \vec{r}.(-\hat{i} - \hat{j} + 4\hat{k}) + 1 = 0.$$

Hence, the vector equation of the required plane is :

$$\vec{r}.(\hat{i} + \hat{j} - 4\hat{k}) - 1 = 0.$$

Example 16. If the lines :

$$\frac{\mathbf{x-1}}{\mathbf{-3}} = \frac{\mathbf{y-2}}{\mathbf{-2k}} = \frac{\mathbf{z-3}}{\mathbf{2}} \textbf{ and } \frac{\mathbf{x-1}}{\mathbf{k}} = \frac{\mathbf{y-2}}{\mathbf{1}} = \frac{\mathbf{z-3}}{\mathbf{5}}$$

are perpendicular, find the value of 'k' and hence find the equation of plane containing these lines.

(A.I.C.B.S.E. 2012)

Solution. The given lines are $\dfrac{x-1}{-3} = \dfrac{y-2}{-2k} = \dfrac{z-3}{2}$ ...(1)

and $\dfrac{x-1}{k} = \dfrac{y-2}{1} = \dfrac{z-3}{5}$ ...(2)

are perpendicular $\Rightarrow (-3)(k) + (-2k)(1) + (2)(5) = 0$

$\Rightarrow \quad -3k - 2k + 10 = 0 \quad \Rightarrow 5k = 10 \Rightarrow k = 2.$

$\therefore$ The given lines become : $\dfrac{x-1}{-3} = \dfrac{y-2}{-4} = \dfrac{z-3}{2}$ ...(3)

and $\dfrac{x-1}{2} = \dfrac{y-2}{1} = \dfrac{z-3}{5}$ ...(4)

The equation of the reqd. plane is :

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ -3 & -4 & 2 \\ 2 & 1 & 5 \end{vmatrix} = 0$$

$\Rightarrow (x-1)(-20-2) - (y-2)(-15-4) + (z-3)(-3+8) = 0$

$\Rightarrow \quad -22x + 22 + 19y - 38 + 5z - 15 = 0$

$\Rightarrow \quad 22x - 19y - 5z + 31 = 0.$

Example 17. Find the vector equation of the line passing through the point (1, 2, 3) and parallel to the planes :

$$\vec{r}.(\hat{i} - \hat{j} + 2\hat{k}) = 5 \textbf{ and } \vec{r}.(3\hat{i} + \hat{j} + \hat{k}) = 6.$$

(A.I.C.B.S.E. 2013; Kashmir B. 2013)

Solution. The given planes are :

$$\vec{r}.(\hat{i} - \hat{j} + 2\hat{k}) = 5 \text{ and } \vec{r}.(3\hat{i} + \hat{j} + \hat{k}) = 6$$

i.e. $\quad x - y + 2z = 5$ and $3x + y + z = 6$

$\Rightarrow \quad x - y + 2z - 5 = 0$ ...(1)

and $\quad 3x + y + z - 6 = 0$ ...(2)

Let the line through (1, 2, 3) be :

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z-3}{c} \quad ...(3)$$

Since (3) is parallel to (1),

$\therefore \quad (a)(1) + (b)(-1) + (c)(2) = 0$

i.e. $\quad a - b + 2c = 0$ ...(4)

Since (3) is parallel to (2),

$\therefore \quad (a)(3) + (b)(1) + (c)(1) = 0$

i.e. $\quad 3a + b + c = 0$ ...(5)

Solving (4) and (5), $\dfrac{a}{-1-2} = \dfrac{b}{6-1} = \dfrac{c}{1+3}$

$\Rightarrow \quad \dfrac{a}{-3} = \dfrac{b}{5} = \dfrac{c}{4}$ ...(6)

From (3) and (6), equation to the line is :

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}.$$

Its vector equation is :

$$\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(-3\hat{i} + 5\hat{j} + 4\hat{k}\right).$$

Example 18. From the point P (1, 2, 4), a perpendicular is drawn on the plane 2x + y – 2z + 3 = 0. Find the equations, the length and co-ordinates of the foot of the perpendicular.

Solution. The given plane is $2x + y - 2z + 3 = 0$ ....(1)

Let PM be the perpendicular from P (1, 2, 4) on the plane.

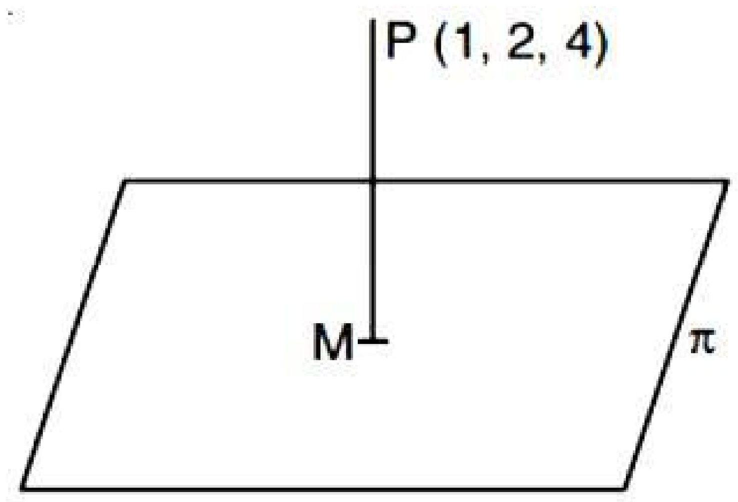


Fig.

The equations of PM are :

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-4}{-2} \quad ....(2),$$

which are the reqd. equations of the perpendicular.

Any point on (2) is $(2k + 1, k + 2, - 2k + 4)$ ...(3)

This point is point M

if $2(2k + 1) + k + 2 - 2(-2k + 4) + 3 = 0$

if $4k + 2 + k + 2 + 4k - 8 + 3 = 0$

if $9k \quad 1 = 0$ if $k = \frac{1}{9}$.

Putting in (3), the point M is $\left(\frac{2}{9}+1, \frac{1}{9}+2, -\frac{2}{9}+4\right)$

i.e. $\left(\frac{11}{9}, \frac{19}{9}, \frac{34}{9}\right)$.

And length of perpendicular

$$= |PM| = \sqrt{\left(\frac{11}{9}-1\right)^2 + \left(\frac{19}{9}-2\right)^2 + \left(\frac{34}{9}-4\right)^2}$$

$$= \sqrt{\frac{4}{81}+\frac{1}{81}+\frac{4}{81}} = \sqrt{\frac{1}{9}} = \frac{1}{3} \text{ unit.}$$

Example 19. Find the co-ordinates of the point P, where the line through A(3, – 4, – 5) and B(2, – 3, 1) crosses the plane passing through three points L(2, 2, 1), M(3, 0, 1) and N(4, – 1, 0). Also, find the ratio in which P divides the line segment AB. ***(C.B.S.E. 2016)***

Solution. Any plane through L(2, 2, 1) is:

$$a(x - 2) + b(y - 2) + c(z - 1) = 0 \quad ...(1)$$

Since (1) passes through M(3, 0, 1),

$\therefore \quad a(3 - 2) + b(0 - 2) + c(1 - 1) = 0$

$\Rightarrow \quad a - 2b + 0 \,.\, c = 0$ ...(2)

Since (1) passes through N (4, – 1, 0),

$\therefore \quad a(4 - 2) + b(-1 - 2) + c(0 - 1) = 0$

$\Rightarrow \quad 2a - 3b - c = 0$ ...(3)

Solving (2) and (3), $\frac{a}{2+0} = \frac{b}{0+1} = \frac{c}{-3+4} = k$ (say)

$\Rightarrow a = 2k, b = k, c = k.$

Putting in (1), $2k(x - 2) + k(y - 2) + k(z - 1) = 0$

$\Rightarrow 2(x - 2) + (y - 2) + (z - 1) = 0$ $[\because k \neq 0]$

$\Rightarrow 2x + y + z - 7 = 0$ ...(4)

Now equations of AB are :

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5}$$

$$\Rightarrow \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = r \text{ (say)}$$

$\Rightarrow x = 3 - r, y = -4 + r$ and $z = -5 + 6r$.

Any point on AB is $P(-r + 3, r - 4, 6r - 5)$ ...(5)

This lies on plane (4).

$\therefore \quad 2(-r + 3) + (r - 4) + (6r - 5) - 7 = 0$

$\Rightarrow 5r = 10 \Rightarrow r = 2.$

Putting in (5), the co-ordinates of P are (1, – 2, 7).

Let P divide the segment [AB] in the ratio of $k : 1$.

$$\therefore \left(\frac{k\times 2+1\times 3}{k+1}, \frac{k\times(-3)+1\times(-4)}{k+1}, \frac{k\times 1+1\times(-5)}{k+1}\right) = (1, -2, 7).$$

Comparing, $\frac{3+2k}{k+1} = 1 \Rightarrow 3 + 2k = k + 1$

$\Rightarrow k = -2.$

Hence, P divides the line segment [AB] in the ratio 2 : 1 externally.

Example 20. Find the distance of the point (1, –2, 3) from the plane x – y + z = 5, measured parallel to the line :

$$\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}. \quad \textit{(Mizoram. B. 2018)}$$

Solution. Let A (1,– 2, 3) be the given point.

The given plane is $x - y + z = 5$ ...(1)

and the given line is $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ ...(2)

The equations of the line through A and parallel to (2) are :

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6}.$$

Any point on it is $(2k + 1, 3k - 2, -6k + 3)$.

This is P if it lies on (1) if $(2k+1)-(3k-2)+(-6k+3)=5$

if $-7k+6=5$ if $-7k=-1$ if $k=\frac{1}{7}$.

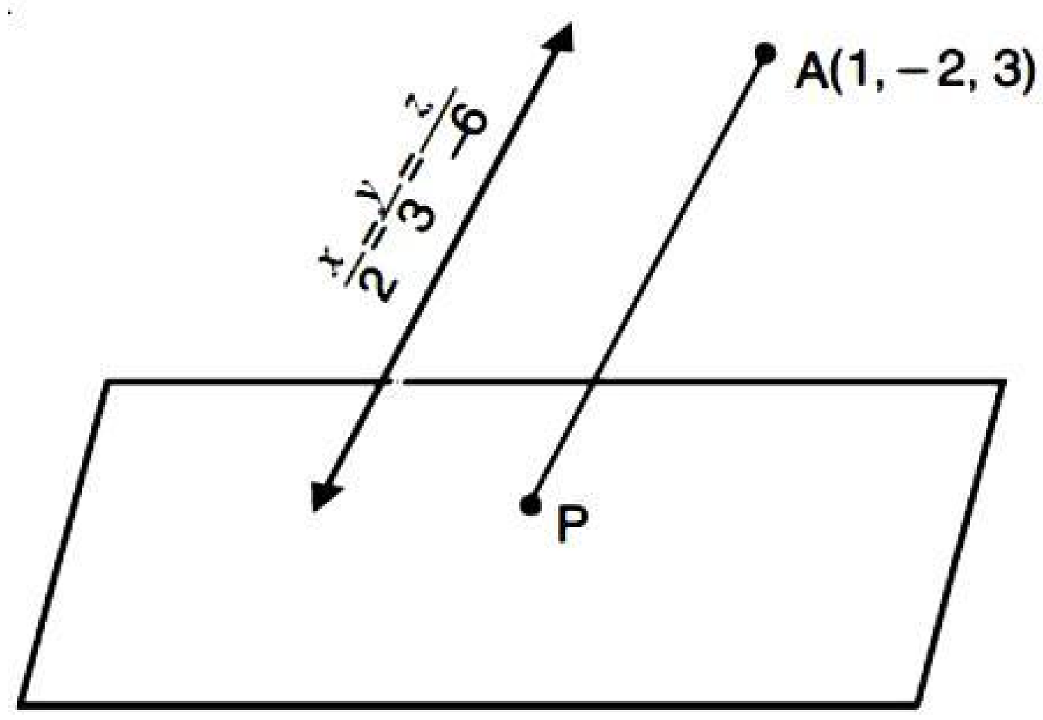


Fig.

$\therefore$ The point P is $\left(\frac{2}{7}+1, \frac{3}{7}-2, -\frac{6}{7}+3\right)$

i.e. $\left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$.

$\therefore$ Reqd. distance = |AP|

$$= \sqrt{\left(\frac{9}{7}-1\right)^2+\left(\frac{-11}{7}+2\right)^2+\left(\frac{15}{7}-3\right)^2}$$

$$= \sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}} = \sqrt{\frac{49}{49}} = 1 \text{ unit.}$$

<u>IMAGE OF A POINT IN A PLANE</u>

Let P and P' be two points and π be the plane such that :

(i) line PP' is perpendicular to π-plane

(ii) mid–point of [PP'] lies on π-plane.

Then each point is the image of the other in π-plane.

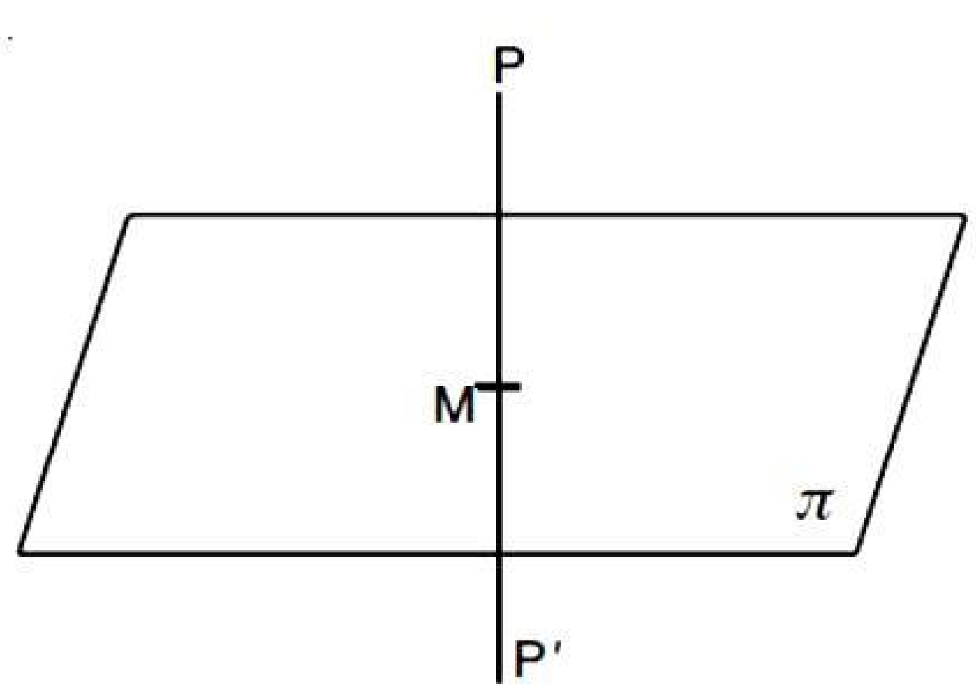


Fig.

Example 21. Find the co-ordinates of the foot of the perpendicular and the perpendicular distance of the point (1, 3, 4) from the plane 2x – y + z + 3 = 0. Find also, the image of the point in the plane.

(Mizoram B. 2017, 15; A.I.C.B.S.E. 2009 C)

Solution. The given plane is $2x-y+z+3=0$ ...(1)

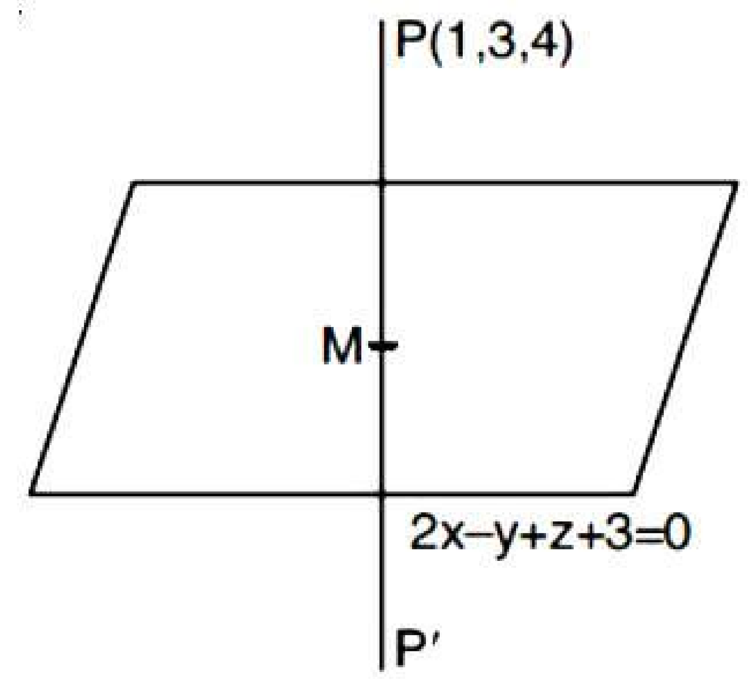


Fig.

Let P (1, 3, 4) be the given point.

Let M be the foot of perpendicular from P on plane (1),

Let P′ be the image of P in the plane (1),

$\therefore$ The equations of PM are $\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}$ ...(2)

Any point on (2) is $(1+2k, 3-k, 4+k)$ ...(3)

This point is M if it lies on (1)

if $2(1+2k)-(3-k)+(4+k)+3=0$

if $6k=-6$ if $k=-1$.

Putting in (3), the point M is :

(1 + 2 (1), 3 (1), 4 1) *i.e.* (1, 4, 3).

Hence, the co-ordinates of M, the foot of perpendicular are (– 1, 4, 3).

And perpendicular distance = | PM |

$$= \sqrt{(1+1)^2+(3-4)^2+(4-3)^2}$$

$$= \sqrt{4+1+1} = \sqrt{6} \text{ units.}$$

Now M is the mid-point of [PP′].

If P′ is (α, β, γ), then :

$$\frac{1+\alpha}{2}=-1, \frac{3+\beta}{2}=4, \frac{4+\gamma}{2}=3$$

$\Rightarrow$ $1+\alpha=-2, 3+\beta=8, 4+\gamma=6$

$\Rightarrow$ $\alpha=-3, \beta=5, \gamma=2$.

Hence, the reqd. image is P′ (– 3, 5, 2).

Example 22. Find the equation of the plane, which meets the axes in A, B, C, given that the centroid of the triangle ABC is the point (α, β, γ).

Solution. Let the equation of the plane be :

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad ...(1),$$

where A (a, 0, 0), B (0, b, 0) and C (0, 0, c) are the points on the axes.

Since (α , β , γ) is the centroid of ΔABC, [*Given*]

$$\therefore \quad \alpha = \frac{a+0+0}{3}, \beta = \frac{0+b+0}{3} \text{ and } \gamma = \frac{0+0+c}{3}$$

$\Rightarrow$ $a = 3\alpha$, $b = 3\beta$ and $c = 3\gamma$.

Putting in (1), $\frac{x}{3\alpha} + \frac{y}{3\beta} + \frac{z}{3\gamma} = 1$

$\Rightarrow \frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$, which is the reqd. equation.

Example 23. Find the equations of the planes bisecting the angles between the planes :

x + 2y + 2z – 3 = 0, 3x + 4y + 12z + 1 = 0

and specify the plane, which bisects the acute angle.

Solution. (*i*) The given planes are :

$$x + 2y + 2z - 3 = 0 \qquad ...(1)$$

and $$3x + 4y + 12z + 1 = 0 \qquad ...(2)$$

The equations of two bisecting planes are :

$$\frac{x+2y+2z-3}{\sqrt{1+4+4}} = \pm\frac{3x+4y+12z+1}{\sqrt{9+16+144}}$$

$$\Rightarrow \quad \frac{x+2y+2z-3}{3} = \pm\frac{3x+4y+12z+1}{13}$$

$\Rightarrow$ $13x + 26y + 26z - 39 = \pm\ (9x + 12y + 36z + 3)$.

Taking +ve sign,

$13x + 26y + 26z - 39 = 9x + 12y + 36z + 3$

$\Rightarrow$ $4x + 14y - 10z - 42 = 0$

$\Rightarrow$ $2x + 7y - 5z = 21$ ...(3)

Taking – ve sign,

$13x + 26y + 26z - 39 = -9x - 12y - 36z - 3$

$\Rightarrow$ $22x + 38y + 62z - 36 = 0$

$\Rightarrow$ $11x + 19y + 31z = 18$ ...(4)

Hence, (3) and (4) are the required equations of the bisecting planes.

(*ii*) Let 'θ' be the angle between planes (1) and (4).

$$\therefore \quad \cos\theta = \frac{(1)(11)+(2)(19)+(2)(31)}{\sqrt{1+4+4}\sqrt{121+361+961}}$$

$$\left[\textit{Using } \cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right]$$

$$= \frac{11+38+62}{\sqrt{9}\sqrt{1443}} = \frac{111}{3\sqrt{1443}} < \frac{1}{\sqrt{2}}$$

$\Rightarrow$ $\theta > 45°$.

Thus (4) bisects the obtuse angle.

Hence, (3) bisects the acute angle.

Example 24. Show that the line of intersection of the planes :

x + 2y + 3z = 8 and 2x + 4y + 4z = 11

is coplanar with the line $\frac{x+1}{1} + \frac{y+1}{2} = \frac{z+1}{3}$.

Also, find the equation of the plane containing them.

(C.B.S.E. Sample Paper 2019)

Solution. The given line is $\frac{x+1}{1} + \frac{y+1}{2} = \frac{z+1}{3}$ ...(1)

The given planes are :

$$x + 2y + 3z - 8 = 0 \qquad ...(2)$$

and $$2x + 3y + 4z - 11 = 0 \qquad ...(3)$$

In order to prove that the line (1) is coplanar with the line determined by (2) and (3), we shall show that there exists a plane passing through the intersection of planes (2) and (3) containing line (1).

Equation of the plane through the intersection of (2) and (3) is :

$(x + 2y + 3z - 8) + k\,(2x + 3y + 4z - 11) = 0$ ...(4)

Let us find the value for which the plane passes through the point (–1, –1, –1), lying on line (1).

Putting in (4), $(-1-2-3-8) + k\,(-2-3-4-11) = 0$

$\Rightarrow -14 - 20k = 0 \Rightarrow k = -\frac{7}{10}$.

Putting the value of k in (4), we get :

$(x + 2y + 3z - 8) - \frac{7}{10}(2x + 3y + 4z - 11) = 0$

$\Rightarrow (10x + 20y + 30z - 80) - (14x + 21y + 28z - 77) = 0$

$\Rightarrow 4x + y - z + 3 = 0$ ...(5)

If $< a_1, b_1, c_1 >$ are d-ratios of line (1)

and $< a_2, b_2, c_2 >$ are d-ratios of normal to plane (5),

then $a_1a_2 + b_1b_2 + c_1c_2 = (1)(4) + (2)(1) + (3)(-2)$

$= 4 + 2 - 6 = 0$.

Thus, line (1) lies in the plane (5).

Hence, the two lines are coplanar and the equation of the plane containing them is $4x + y - 2z + 3 = 0$

EXERCISE 11 (e)

Fast Track Answer Type Questions

FTATQ

1. (*i*) Find the equation of a plane which makes x, y, z intercepts respectively as 1, 2, 3. *(Kerala B. 2018)*

(*ii*) Find the equation of a plane passing through the point (1, 2, 3), which is parallel to above plane. *(Kerala B. 2018)*

(*iii*) Find the equation of the plane with intercept 3 on the y–axis and parallel to ZOX plane. *(N.C.E.R.T.)*

(*iv*) Find the equation of the plane with intercept 4 on the z-axis and parallel to XOY plane. *(Karnataka B. 2014)*

(*v*) Find the intercepts cut off by the plane :

$7x + y - z = 5$. *(Kashmir B. 2017)*

2. (*i*) Find the vector equation of a plane, which is at a distance of 7 units from the origin and which is normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$. *(N.C.E.R.T.; H.P.B. 2010)*

(*ii*) Find the vector equation of a plane, which is at a distance of 5 units from the origin and its normal vector is

$2\hat{i} - 3\hat{j} + 6\hat{k}$. *(C.B.S.E. 2016)*

3. Find the vector equation of the plane whose cartesian form of equation is :

(*i*) $5x - 7y + 2z = 3$ (*ii*) $x - 2y + 3z + 1 = 0$.

4. Find the cartesian equations of the following planes :

(*i*) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$

(*ii*) $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

(*iii*) $\vec{r} \cdot [(s - 2t)\hat{i} + (3 - t)\hat{j} + (2s + t)\hat{k}] = 15$. *(N.C.E.R.T.)*

5. (*a*) What are the direction-cosines of the normal to the plane $3x + 2y - 3z = 8$? *(Assam B. 2016)*

(*b*) Find the direction–cosines of the perpendicular from the origin to the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - 2\hat{k}) = 18$.

6. Find the vector equation of the line through the origin, which is perpendicular to the plane $\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 3$.

7. (*i*) Find the distance of the point (2, 3, 4) from the plane :

$\vec{r} \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) = -11$.

(*ii*) Find the distance of a point (2, 5, –3) from the plane :

$\vec{r} \cdot (6\hat{i} - 3\hat{j} + 2\hat{k}) = 4$.

(A.I.C.B.S.E. 2015; Kashmir B. 2015)

8. (*i*) Find the distance from (1, 2, 3) to the plane $2x + 3y - z + 2 = 0$. *(Kerala B. 2015)*

(*ii*) Find the length of the perpendicular drawn from the origin to the plane $2x - 3y + 6z + 21 = 0$. *(A.I.C.B.S.E. 2013)*

(*iii*) Find the distance of the plane $2x - 4y + 12z = 3$ from the origin. *(A.I.C.B.S.E. 2012)*

9. Find a unit vector normal to the plane :

$\vec{r} \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) + 14 = 0$. *(Mizoram B. 2015)*

Very Short Answer Type Questions

VSATQ

Find the angle between the planes (10 – 11) :

10. (*i*) $3x - 6y - 2z = 7$ and $2x + y - 2z = 5$ *(N.C.E.R.T.; H.P.B. 2016)*

(*ii*) $4x + 8y + z = 8$ and $y + z = 4$ *(Meghalaya B. 2016)*

(*iii*) $2x - y + z = 6$ and $x + y + 2z = 7$. *(H.B. 2013)*

11. (*i*) $\vec{r} \cdot (\hat{i} + \hat{j}) = 3$ and $\vec{r} \cdot (\hat{i} + \hat{k}) = 3$

(*ii*) $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$. *(N.C.E.R.T.; Kashmir B. 2017)*

12. Find the value of 'k' for which the planes :

$3x - 6y - 2z = 7$ and $2x + y - kz = 5$

are perpendicular to each other.

13. (*i*) The position vectors of two points A and B are $3\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}-2\hat{j}-4\hat{k}$ respectively. Find the vector equation of the plane passing thro' B and perpendicular to $\overrightarrow{AB}$. *(Bihar B. 2014)*

(*ii*) Find the vector equation of the plane through the point (2, 0, – 1) and perpendicular to the line joining the two points (1, 2, 3) and (3, – 1, 6).

14. Find the equation of the plane passing through the point (1, 2, 1) and perpendicular to the line joining the points (1, 4, 2) and (2, 3, 5). Also, find the perpendicular distance of the plane from the origin.

15. Find the vector and cartesian equations of the plane passing through the point (5, 2, – 4) and perpendicular to the line with direction ratios < 2, 3, – 1 >. *(N.C.E.R.T.)*

16. Find the vector and cartesian equations of the plane :

(*i*) that passes through the point (5, 2, – 4) and perpendicular to the line with direction–ratios <2, 3, – 1>. *(H.P.B. 2018)*

(*ii*) that passes through the point (1, 0, – 2) and the normal to the plane is $\hat{i}+\hat{j}-\hat{k}$ *(H.P.B. 2018, 10)*

(*iii*) that passes through the point (1, 4, 6) and the normal vector to the plane is $\hat{i}-2\hat{j}+\hat{k}$. *(N.C.E.R.T.; H.P.B. 2018; H.B. 2014)*

17. Find the length of the perpendicular from the point (2, 3, 7) to the plane $3x - y - z = 7$. Also find the co-ordinates of the foot of the perpendicular.

18. In the following, find the distance of each of the given points from the corresponding given planes :

	Point	Plane	
(*i*)	(0, 0, 0)	$2x - y + 2z + 1 = 0$	*(A.I.C.B.S.E. 2010)*
(*ii*)	(3, –2, 1)	$2x - y + 2z + 3 = 0$	
(*iii*)	(– 6, 0, 0)	$2x - 3y + 6z - 2 = 0$	*(N.C.E.R.T.)*
(*iv*)	(2, 3, –5)	$x + 2y - 2z = 9$.	*(N.C.E.R.T.)*

19. In the following, determine the direction- cosines of the normal to the plane and the distance from the origin :

(*i*) $z = 2$ (*ii*) $5y + 8 = 0$. *(N.C.E.R.T.)*

20. If the points (1, 1, *p*) and (– 3, 0, 1) be equidistant from the plane $\vec{r}\cdot(3\hat{i}+4\hat{j}-12\hat{k})+13=0$, then find the value of '*p*'. *(N.C.E.R.T. ; Jammu B. 2012)*

21. In the following cases, find the co-ordinates of the foot of the perpendicular drawn from the origin :

(*i*) $2x + 3y + 4z - 12 = 0$ *(H.P.B. 2012)*

(*ii*) $3y + 4z - 6 = 0$ *(N.C.E.R.T.)*

(*iii*) $x + y + z = 1$ *(N.C.E.R.T.)*

(*iv*) $5y + 8 = 0$. *(N.C.E.R.T.)*

22. Find the length and the foot of the perpendicular from the point (7, 14, 5) to the plane $2x + 4y - z = 2$.

23. (*i*) Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes :

$$\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=5 \text{ and } \vec{r}\cdot(3\hat{i}+\hat{j}+\hat{k})=6.$$

(H.P.B. 2015)

(*ii*) Find the vector equation of the straight line passing through (1, 2, 3) and perpendicular to the plane :

$\vec{r}\cdot(\hat{i}+2\hat{j}-5\hat{k})+9=0$. *(N.C.E.R.T.; H.P.B. 2012)*

24. (*i*) Find the equation of the plane passing through (*a*, *b*, *c*) and parallel to the vector $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$. *(N.C.E.R.T.; H.P.B. 2015, 12)*

(*ii*) Find the vector equation of the plane through the point $\hat{i}+\hat{j}+\hat{k}$ and parallel to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=5$. *(W. Bengal B. 2017)*

25. Find the vector and cartesian equations of the plane containing the lines :

$$\vec{r}=2\hat{i}+\hat{j}-3\hat{k}+\lambda(\hat{i}+2\hat{j}+5\hat{k})$$

and $\vec{r}=3\hat{i}+3\hat{j}-7\hat{k}+\mu(3\hat{i}-2\hat{j}+5\hat{k})$.

26. Find the angle between the lines :

$x - 2y + z = 0 = x + 2y - 2z$

and $x + 2y + z = 0 = 3x + 9y + 5z$.

27. Show that the lines $3x - 2y + 5 = 0$, $y + 3z - 15 = 0$ and $\frac{x-1}{5}=\frac{y+5}{-3}=\frac{z}{1}$ are perpendicular to each other.

28. Find the equations of the line passing through the point (1, – 2, 3) and parallel to the planes :

$x - y + 2z = 5$ and $3x + 2y - z = 6$.

29. Find the equation of the plane, which bisects the line joining the points (–1, 2, 3) and (3, –5, 6) at right angles.

Short Answer Type Questions

SATQ

30. (*a*) Find the equation of the plane through the intersection of the planes :

$3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$

and the point (2, 2, 1). *(H.P.B. 2016, 11, 10)*

(*b*) Find the vector equation of the plane through the intersection of the planes :

$\vec{r}.(\hat{i}+\hat{j}+\hat{k})=6$ and $\vec{r}.(2\hat{i}+3\hat{j}+4\hat{k})=-5$

at the point (1, 1, 1). *(Uttarakhand B. 2013)*

31. Find the vector equation of the following plane in scalar product form :

$$\vec{r}=\hat{i}-\hat{j}+\lambda(\hat{i}+\hat{j}+\hat{k})+\mu(\hat{i}-2\hat{j}+3\hat{k}).$$

32. (*a*) Find the equation of the plane through three non-collinear points :

(*i*) (0, – 1, – 1) ; (4, 5, 1) ; (3, 9, 4) *(N.C.E.R.T. ; P.B. 2013, 12, 10)*

(*ii*) (1, 1, 0), (1, 2, 1) and (–2, 2, –1). *(H.P.B. 2018; Jammu B. 2015W)*

(*iii*) (2, 5, –3), (–2, –3, 5) and (5, 3, –3) *(H.P.B. 2018; Jammu B. 2018, 16; Kerala B. 2016)*

(*iv*) (0, – 1, 0) ; (1, 1, 1) ; (3, 3, 0) *(P.B. 2013, 12, 10)*

(*v*) (3, – 1, 2); (5, 2, 4); (–1, –1, 6) *(Kerala B. 2013)*

(*vi*) (2, 2, – 1); (3, 4, 2); (7, 0, 6) *(Kashmir B. 2012; C.B.S.E. (F) 2011)*

(*vii*) (2, 1, –1); (6, 5, 0); (2, – 1, 5) *(P.B. 2011)*

(*viii*) (2, 1, 0); (3, – 2, – 2); (3, 1, 7). *(H.B. 2018)*

(*b*) Find the equation of the plane through the points (3, – 2, 4), (– 13, 17, – 1) and (– 6, 3, 2).

Show that it passes through (5, 7, 3). *(J. & K. B. 2011)*

(*c*) Find the equation of the plane through three points :

(– 3, 5, 1), (4, – 1, 2) and (2, 3, 4).

Also find the angles which the normal to this plane makes with axes. *(J. & K. B. 2010)*

(*d*) Find the vector equation of the plane passing through the points :

R (2, 5, – 3), S (– 2, – 3, 5) and T (5, 3, – 3). *(N.C.E.R.T.)*

33. Find the equations of the faces of the tetrahedron whose vertices are the points :

(0, 0, 0); (0, 3, 0) ; (2, 1, 0) ; (1, 1, 2).

34. (*i*) Find the distance of the point P (6, 5, 9) from the plane determined by the points A (3, – 1, 2), B (5, 2, 4) and C (– 1, –1, 6).

(N.C.E.R.T.; C.B.S.E. (F) 2012 ; A.I.C.B.S.E. 2010)

(*ii*) Find the distance between the point (7, 2, 4) and the plane determined by the points :

A(2, 5, – 3), B (–2, – 3, 5) and C (5, 3, – 3). *(C.B.S.E. 2014)*

35. (*i*) Find the equation of the plane through the points (2, – 3, 1) and (5, 2, – 1) and perpendicular to the plane $x - 2y + 4z = 10$. *(A.I.C.B.S.E. 2013)*

(*ii*) Find the vector equation of the plane through the points (2, 1, – 1) and (– 1, 3, 4) and perpendicular to the plane $x - 2y + 4z = 10$.

(A.I.C.B.S.E. 2013 ; C.B.S.E.(F) 2012)

36. Find the equation of the plane through the points (0, 0, 0) and (3,–1, 2) and parallel of the line $\dfrac{x-4}{1}=\dfrac{y+3}{-4}=\dfrac{z+1}{7}$. *(C.B.S.E. 2010)*

37. (*a*) Show that the following four points are coplanar :

(*i*) (4, 5, 1), (0, – 1, – 1), (3, 9, 4) and (– 4, 4, 4) *(A.I.C.B.S.E. 2016)*

(*ii*) (0, – 1, 0), (2, 1, – 1), (1, 1, 1) and (3, 3, 0).

(*b*) Show that the four points : (0, –1, –1), (4, 5, 1), (3, 9, 4) and (–4, 4, 4) are coplanar. *(J. & K. B. 2010)*

Also, find the equation of the plane containing them.

38. The foot of the perpendicular drawn from the origin to a plane is (2, – 3, – 4). Find the equation of the plane. *(J. & K. B. 2013)*

39. (*i*) Find the foot and length of the perpendicular from the point (3, 4, 5) to the plane :

$2x - 5y + 3z = 39$. *(Nagaland B. 2015)*

(*ii*) Find the length and the foot of the perpendicular from the point (7, 14, 5) to the plane $2x + 4y - z = 2$. *(H.B. 2016, 11)*

40. Find the co-ordinates of the points, where the line through the points (3, – 4, –5) and (2, –3, 1) crosses the plane $2x + y + z = 7$. *(N.C.E.R.T.; H.B. 2014; A.I.C.B.S.E. 2012)*

41. Find the distance between two parallel planes :

$2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$. *(Meghalaya B. 2014)*

42. (*i*) Find the equation of the plane passing through the intersection of the planes :

$2x - 7y + 4z = 3$ and $3x - 5y + 4z + 11 = 0$ and the point (–2, 1, 3). *(W. Bengal 2018; H.B. 2017, 10)*

(*ii*) Find the equation of plane through the intersection of planes :

$3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$

and the point (2, 2, 1). *(Kashmir B. 2016, 15)*

43. (*i*) Find the vector equation of the plane through the intersection of the planes :

$$\vec{r}.(\hat{i}+\hat{j}+\hat{k})=6,\quad \vec{r}.(2\hat{i}+3\hat{j}+4\hat{k})=-5$$

and the point (1, 1,1).

(N.C.E.R.T.; H.B. 2017; Meghalaya B. 2013 ; H.P.B. 2011; Kashmir B. 2011)

(*ii*) Find the equation of the plane which contains the line of intersection of the planes :

$$\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4=0, \vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5=0$$

and which is perpendicular to the plane :

$$\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8=0.$$

(Assam B. 2018; H.P.B. 2016)

(*iii*) Find the equation of the plane passing through the intersection of the planes $x+y+z=6$ and $2x+3y+4z+5=0$ and the point (1, 1, 1). *(Uttrakhand B. 2015)*

44. Find the equation of the plane through the line of intersection of the planes :

$x + y + z = 1$ and $2x + 3y + 4z = 5$,

which is perpendicular to the plane $x - y + z = 0$.

(Kashmir B, 2017; H.P.B. 2016, 13)

45. (*i*) Find the equation of the plane passing through the line of intersection of the planes :

$$\vec{r}.(\hat{i}+\hat{j}+\hat{k})=1 \quad\text{and}\quad \vec{r}.(2\hat{i}+3\hat{j}-\hat{k})+4=0$$

and parallel to x-axis. *(N.C.E.R.T.)*

(*ii*) Find the equation of a plane through the intersection of the planes :

$$\vec{r}.\left(2\hat{i}+\hat{j}+3\hat{k}\right)=7 \text{ and } \vec{r}.\left(2\hat{i}+3\hat{j}+3\hat{k}\right)=9$$

and passing through the point (2, 1, 3). *(H.P.B. 2011)*

46. Find the equation of the plane passing through the line of intersection of the planes :

$$\vec{r}.(\hat{i}+3\hat{j})-6=0 \text{ and } \vec{r}.(3\hat{i}-\hat{j}-4\hat{k})=0,$$

whose perpendicular distance form origin is unity.

(Type : Assam B. 2017; A.I.C.B.S.E. 2013)

47. Find the equation of the plane passing through the line of intersection of the planes :

$2x + y - z = 3$ and $5x - 3y + 4z = 9$

and parallel to the line $\dfrac{x-1}{2}=\dfrac{y-3}{4}=\dfrac{z-5}{5}$.

(A.I.C.B.S.E. 2011)

48. (*i*) Find the equation of the plane through the line of intersection of the planes :

$2x + 3y - z + 1 = 0$ and $x + y - 2z + 3 = 0$

and perpendicular to the plane $3x - y - 2z - 4 = 0$.

(N.C.E.R.T.; Meghalaya B. 2018, 14; H.B. 2010)

(*ii*) Find the vector equation of the plane passing through the intersection of the planes :

$\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=2$ and $\vec{r}\cdot(3\hat{i}+\hat{j}-2\hat{k})=-2$ and perpendicular to the vector $\vec{a}=5\hat{i}-2\hat{j}+3\hat{k}$.

(Assam B. 2016)

(*iii*) Find the equation of the plane through the points (2, 2, 1), (9, 3, 6) and perpendicular to the plane $2x + 6y + 6z = 1$. *(P.B. 2014 S)*

49. (*i*) Find the equation of the plane passing through (1, – 1, 2) and perpendicular to the planes :

$2x + 3y - 2z = 5,\ x + 2y - 3z = 8.$

(N.C.E.R.T.; Jammu B. 2017; H.P.B. 2015; H.B. 2013)

(*ii*) Find the equation of the plane passing through the point (1, 1, –1) and perpendicular to each of the planes :

$x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0$.

(C.B.S.E. (F) 2011)

(*iii*) Find the equation of the plane passing through the point (– 1, – 1, 2) and perpendicular to the planes :

$3x + 2y - 3z = 1$ and $5x - 4y + z = 5$. *(H.B. 2013)*

Long Answer Type Questions

LATQ

50. (*i*) Find the distance of the point (– 2, 3, – 4) from the line :

$$\frac{x+2}{3}=\frac{2y+3}{4}=\frac{3z+4}{5},$$

measured parallel to the plane $4x + 12y - 3z + 1 = 0$.

(Assam B. 2017)

(*ii*) Find the distance of the point $-2\hat{i}+3\hat{j}-4\hat{k}$ from the line :

$$\vec{r}=\hat{i}+2\hat{j}-\hat{k}+\lambda(\hat{i}+3\hat{j}-9\hat{k}),$$

measured parallel to the plane $x - y + 2z - 3 = 0$.

(C.B.S.E. Sample Paper 2018)

51. Find the ratio in which the line-segment joining the points :

(*i*) (2, 1, 5) and (3, 4, 3) is divided by the plane :

$x + y - z = \dfrac{1}{2}$ *(H.B. 2011)*

(*ii*) (1, 2, 3) and (– 3, 4, – 5) is divided by the xy-plane

(H.B. 2011)

52. Find the equation of the plane passing through the point (1, 2, 1) and perpendicular to the line joining the points (1, 4, 2) and (2, 3, 5). *(C.B.S.E. 2010 C)*

Find also the perpendicular distance of the origin from the plane.

53. Find the image of the point :

(*i*) (2, – 3, 2) in the plane $2x + y - 3z = 10$ (*P.B. 2017*)

(*ii*) (1, 2, 3) in the plane $x + 2y + 4z = 38$ (*Meghalaya B. 2018*)

(*iii*) (2, – 1, 3) in the plane $3x - 2y - z = 9$. (*H.B. 2016; J. & K. B. 2011*)

54. (*i*) Find the co-ordinates of foot of perpendicular drawn from the point (2, 3, 5) on the plane given by the equation :

$2x - 3y + 4z + 10 = 0$. (*P.B. 2014*)

(*ii*) Find the distance between the point (2, 3, – 1) and foot of perpendicular drawn from (3, 1, – 1) to the plane $x - y + 3z = 10$. (*P.B. 2018*)

55. The foot of the perpendicular drawn from origin to a plane is (4, – 2, 5).

(*a*) How far is the plane from the origin ?

(*b*) Find a unit vector perpendicular to that plane.

(*c*) Obtain the equation of the plane in general form. (*Kerala B. 2014*)

56. Find the co-ordinates of the foot of the perpendicular and the perpendicular distance of the point P (3, 2, 1) from the plane $2x - y + z + 1 = 0$, Find also, the image of the point in the plane. (*A.I.C.B.S.E. 2010*)

57. Find the length and the foot of the perpendicular from the point P (7, 14, 5) to the plane $2x + 4y - z = 2$. Also find the image of point P in the plane. (*A.I.C.B.S.E. 2012*)

58. Find the distance of the point P (1, 2, 3) from its image in the plane $x + 2y + 4z = 38$. (*Nagaland B. 2016*)

59. Find the co-ordinates of the point where the line through (3, – 4, – 5) and (2, – 3, 1) crosses the plane, passing through the points (2, 2, 1), (3, 0, 1) and (4, –1, 0). (*C.B.S.E. 2013*)

60. (*i*) A variable plane, which remains at a constant distance '3*p*' from the origin cuts the co–ordinate axes at A, B, C. Show that the locus of the centroid of the triangle ABC is :

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}.$$ (*A.I.C.B.S.E. 2017*)

(*ii*) A variable plane is at a constant distance '*p*' from the origin and meets the axes in A, B, C respectively, then show that locus of the centroid of the triangle ABC is :

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{9}{p^2}.$$

61. (*i*) A variable plane which remains at a constant distance '4*p*' from the origin and cuts axes at A, B and C. Show that the centroid of the tetrahedron OABC lies on :

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}.$$

(*ii*) A variable plane is at a constant distance '*p*' from the origin cuts the co-ordinate axes in A, B, C. Through A, B, C planes are drawn parallel to the co-ordinate planes. Show that the locus of their point of intersection is :

$$x^{-2}+y^{-2}+z^{-2}=p^{-2}.$$

62. A variable plane passes through a fixed point (*a, b, c*) and meets the co–ordinate axes in A, B, C. Show that the locus of the point common to the planes through A, B, C parallel to the co–ordinate planes is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$.

63. A variable plane moves so that the sum of the reciprocals of its intercepts on the three co–ordinate axes is constant, show that it passes through a fixed point.

64. Show that the sum of the reciprocals of the intercepts on rectangular axes made by a fixed plane is same for all systems of rectangular axes, with a given origin.

65. Find the equations of the bisector planes of the angles between the planes :

$3x - 2y + 6z + 8 = 0$ and $2x - y + 2z + 3 = 0$.

66. In the following, determine whether the given planes are parallel or perpendicular and in case they are neither, find the angles between them :

(*i*) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

(*ii*) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

(*iii*) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

(*iv*) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

(*v*) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$. (*N.C.E.R.T.*)

Answers

1. (*i*) $6x + 3y + 2z = 6$ (*ii*) $6x + 3y + 2z = 18$

(*iii*) $y = 3$ (*iv*) $z = 4$ (*v*) $\frac{5}{7}, 5, -5$.

2. (*i*) $\vec{r}\cdot(3\hat{i}+5\hat{j}-6\hat{k})=7\sqrt{70}$

(*ii*) $\vec{r}\cdot(2\hat{i}-3\hat{j}+6\hat{k})=35$.

3. (*i*) $\vec{r}\cdot(5\hat{i}-7\hat{j}+2\hat{k})=3$

(*ii*) $\vec{r}\cdot(\hat{i}-2\hat{j}+3\hat{k})=-1$.

4. (*i*) $x + y - z = 2$

(*ii*) $2x + 3y - 4z = 1$

(*iii*) $(s - 2t)x + (3 - t)y + (2s + t)z = 15$.

5. (*a*) $<\frac{3}{\sqrt{22}},\frac{2}{\sqrt{22}},\frac{-3}{\sqrt{22}}>$ (*b*) $<\frac{1}{3},\frac{2}{3},-\frac{2}{3}>$.

6. $\vec{r}=\lambda(\hat{i}-2\hat{j}+\hat{k})$.

7. (*i*) 1 unit (*ii*) $\frac{13}{7}$ units.

8. (*i*) $\frac{7}{\sqrt{14}}$ units (*ii*) 3 units (*iii*) $\frac{3}{13}$ unit.

9. $\frac{1}{7}(2\hat{i} - 3\hat{j} + 6\hat{k})$.

10. (*i*) $\cos^{-1}\left(\frac{4}{21}\right)$ (*ii*) $\frac{\pi}{4}$ (*iii*) $\frac{\pi}{3}$.

11. (*i*) $\frac{\pi}{3}$ (*ii*) $\cos^{-1}\left(\frac{-15}{\sqrt{17 \times 43}}\right)$.

12. $k = 0$.

13. (*i*) $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 6\hat{k}) + 28 = 0$

(*ii*) $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 3\hat{k}) - 1 = 0$.

14. $x - y + 3z - 2 = 0$; $\frac{2\sqrt{11}}{11}$ units.

15. $[\vec{r} - (5\hat{i} + 2\hat{j} - 4\hat{k})] \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0$; $2x + 3y - z = 20$.

16. (*i*) $\vec{r} - (5\hat{i} + 2\hat{j} - 4\hat{k}) \cdot (2\hat{i} + 3\hat{j} - \hat{k}) = 0$; $2x + 3y - z = 20$

(*ii*) $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 3; x + y - z = 3$

(*iii*) $\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) + 1 = 0 ; x - 2y + z + 1 = 0$.

17. $\sqrt{11}$; (5, 2, 6).

18. (*i*) $\frac{1}{3}$ (*ii*) $\frac{13}{3}$ (*iii*) 2 (*iv*) 3.

19. (*i*) $< 0, 0, 1 >$; 2 (*ii*) $< 0, 1, 0 >$; $\frac{8}{5}$.

20. $p = 1 ; \frac{7}{3}$.

21. (*i*) $\left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29}\right)$ (*ii*) $\left(0, \frac{18}{25}, \frac{24}{25}\right)$ (*iii*) $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

(*iv*) $\left(0, -\frac{8}{5}, 0\right)$.

22. $\sqrt{189}$; (13, 26, 2).

23. (*i*) $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-3\hat{i} + 5\hat{j} + 4\hat{k})$

(*ii*) $\vec{r} = (1 + \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 - 5\lambda)\hat{k}$.

24. (*i*) $\vec{r} \cdot (\hat{i} + \hat{j} + \vec{k}) = a + b + c$

(*ii*) $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3$.

25. $\hat{r} \cdot (10\hat{i} + 5\hat{j} - 4\hat{k}) = 37 ; 10x + 5y - 4z - 37 = 0$.

26. $\cos^{-1}\left(\frac{8}{\sqrt{29}\sqrt{14}}\right)$.

28. $\frac{x-1}{-3} = \frac{y+2}{7} = \frac{z-3}{5}$.

29. $4x - 7y + 3z - 28 = 0$.

30. (*a*) $7x - 5y + 4z - 8 = 0$

(*b*) $\vec{r} \cdot (20\hat{i} + 23\hat{j} + 26\hat{k}) = 69$.

31. $\hat{r} \cdot (5\hat{i} - 2\hat{j} - 3\hat{k}) = 7$.

32. (*a*) (*i*) $5x - 7y + 11z + 4 = 0$

(*ii*) $3x + 3y - 3z - 5 = 0$

(*iii*) $2x + 3y + 4z - 7 = 0$

(*iv*) $4x - 3y + 2z = 3$

(*v*) $3x - 4y + 3z - 19 = 0$

(*vi*) $5x + 2y - 3z - 17 = 0$

(*vii*) $3x - 4y + 4z + 2 = 0$

(*viii*) $7x + 3y - z = 17$.

(*c*) $x + y - z - 1 = 0$;

$\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$, $\pi - \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

(*d*) $[\vec{r} - (2\hat{i} + 5\hat{j} - 3\hat{k})]$. $[(-4\hat{i} - 8\hat{j} + 8\hat{k}) \times (3\hat{i} - 2\hat{j})] = 0$.

33. $z = 0$, $2x + 2y + z = 0$, $2x - 4y + z = 0$, $2x - z = 0$.

34. (*i*) $\frac{6}{\sqrt{34}}$ (*ii*) $\sqrt{29}$ units.

35. (*i*) $x - y - z - 4 = 0$

(*ii*) $\vec{r} \cdot (18\hat{i} + 17\hat{j} + 4\hat{k}) = 49$.

36. $x - 19y - 11z = 0$.

37. (*b*) $5x - 7y + 11z + 4 = 0$.

38. $2x - 3y - 4z = 29$.

39. (*i*) (5, –1, 8) ; $\sqrt{38}$ units

(*ii*) 13·7 units; $\left(-\frac{35}{13}, \frac{70}{13}, \frac{128}{13}\right)$.

40. (1, –2, 7). **41.** $\frac{2}{29}\sqrt{29}$ units.

42. (*i*) $15x - 47y + 28z - 7 = 0$ (*ii*) $7x - 5y + 4z - 8 = 0$.

43. (*i*) $\vec{r} \cdot (20\hat{i} + 23\hat{j} + 26\hat{k}) = 69$

(*ii*) $\vec{r} \cdot (53\hat{i} + 55\hat{j} + 40\hat{k}) + 11 = 0$

(*iii*) $20x + 23y + 26z = 69$.

44. $x - z + 2 = 0$.

45. (*i*) $\vec{r} \cdot (\hat{j} - 3\hat{k}) + 6 = 0$

(*ii*) $\vec{r} \cdot \left(2\hat{i} - 13\hat{j} + 3\hat{k}\right) = 0$.

46. $\vec{r} \cdot (2\hat{i} + \hat{j} - 2\hat{k}) = 3$ *or* $\vec{r} \cdot (\hat{i} - 2\hat{j} - 2\hat{k}) = -3$.

47. $7x + 9y - 10z - 9 = 0$.

48. (*i*) $7x + 13y + 4z - 9 = 0$

(*ii*) $\vec{r} \cdot (8\hat{i} + 5\hat{j} - 10\hat{k}) + 10 = 0$

(*iii*) $3x + 4y - 5z - 9 = 0$.

49. (*i*) $5x - 4y - z = 7$

(*ii*) $17x + 2y - 7z - 26 = 0$

(*iii*) $5x + 9y + 11z - 8 = 0$.

50. (*i*) $\frac{27}{2}$ (*ii*) $\frac{\sqrt{59}}{2}$.

51. (*i*) 5 : 7 (*ii*) 3 : 5.

52. $x - y + 3z - 2 = 0$; $\frac{2}{11}\sqrt{11}$ units.

53. (*i*) $\left(\frac{44}{7}, \frac{-6}{7}, \frac{-41}{7}\right)$ (*ii*) $(-3, 6, 11)$

(*iii*) $\left(\frac{26}{7}, -\frac{15}{7}, \frac{17}{7}\right)$.

54. (*i*) $\left(\frac{8}{29}, \frac{162}{29}, \frac{45}{29}\right)$ (*ii*) $\sqrt{22}$ units.

55. (*a*) $3\sqrt{5}$ (*b*) $\frac{4}{3\sqrt{5}}\hat{i} - \frac{2}{3\sqrt{5}}\hat{j} + \frac{5}{3\sqrt{5}}\hat{k}$

(*c*) $4x - 2y + 5z = 35$.

56. (1, 3, 0) ; $\sqrt{6}$; (– 1, 4, – 1).

57. 17 units ; (1, 2, 8); (– 5, – 10, – 11).

58. $2\sqrt{21}$ units.

59. (1, –2, 7).

65. $23x - 13y + 32z + 45 = 0$, $5x - y - 4z - 3 = 0$.

66. (*i*) Neither ; $\cos^{-1}\left(-\frac{2}{5}\right)$ (*ii*) perpendicular

(*iii*) parallel (*iv*) parallel (*v*) neither ; 45°.

Hints to Selected Questions

12. The planes are perpendicular if
$(3)(2) + (-6)(1) + (-2)(-k) = 0 \Rightarrow k = 0$.

31. Given equation is :

$$\vec{r} = \hat{i} - \hat{j} + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k}),$$

which contains lines with *d*-ratios < 1, 1, 1 > and < 1, – 2, 3 >.

33. Let A, B, C and D be the vertices of the tetrahedron. Find the equations of the faces ABC, ACD, ABD and BCD.

37. (*a*) (*i*) – (*ii*) Find the equation of the plane through any three given points. Show that the remaining fourth point satisfies the equation.

38. Direction-ratios of the normal to the plane are :
$< 2 - 0, -3 - 0, -4 - 0 >$ *i.e.* $< 2, -3, -4 >$.

∴ The equation of the plane is :
$2(x - 2) - 3(y + 3) - 4(z + 4) = 0$.

50. The line is parallel to the plane
⇒ the line is perpendicular to the normal to the plane.

60. (*i*) Distance of $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ from the origin $= 3p$. Find the locus of $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$.

65. Equations of the bisectors are :

$$\frac{3x - 2y + 6z + 8}{\sqrt{9 + 4 + 36}} = \pm\frac{2x - y + 2y + 3}{\sqrt{4 + 1 + 4}}.$$

SUB CHAPTER 11.4 Line and Plane

11.26. ANGLE BETWEEN A LINE AND A PLANE

Let α be the plane and L (not parallel to α) be the line. Then the angle between L and α is defined as *the complement of the acute angle between the normal to the plane α and the line L.*

Thus if $\theta \left(0 < \theta < \frac{\pi}{2}\right)$ is the angle between α and L, then $\left(\frac{\pi}{2} - \theta\right)$ is the acute angle between the normal to the plane α and the line L.

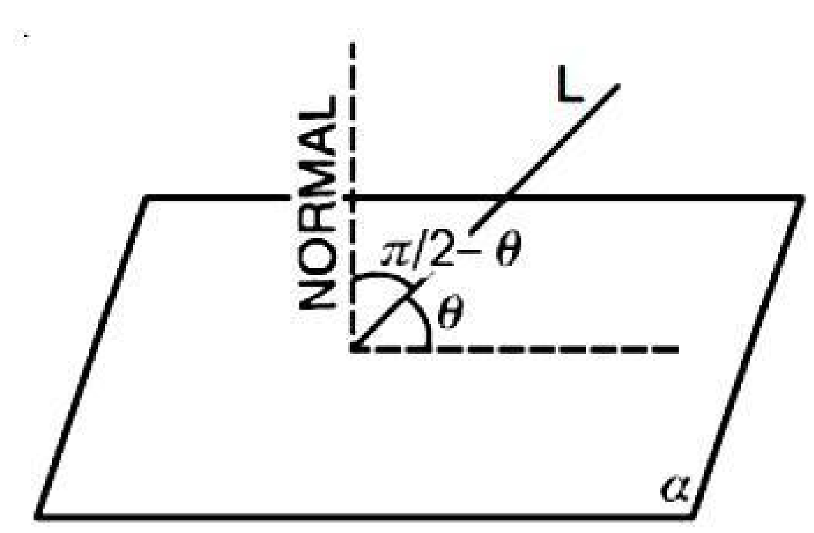


Fig.

KEY POINT

If L is normal to α, then $\theta = \frac{\pi}{2}$.

To find the angle between the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane $ax + by + cz + d = 0$.

The given line is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$.

Direction-ratios of the line are $< l, m, n >$.

The given plane is $ax + by + cz + d = 0$.

Direction-ratios of the normal to the plane are $< a, b, c >$.

If 'θ' $\left(0 < \theta < \frac{\pi}{2}\right)$ is the angle between the given line and plane, then $\left(\frac{\pi}{2} - \theta\right)$ is the acute angle between the given line and the normal to the given plane.

$$\therefore \quad \cos\left(\frac{\pi}{2} - \theta\right) = \frac{|al + bm + cn|}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}}$$

$$\Rightarrow \quad \sin\theta = \frac{|al + bm + cn|}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\frac{|al + bm + cn|}{\sqrt{l^2 + m^2 + n^2}\sqrt{a^2 + b^2 + c^2}}, 0 < \theta < \frac{\pi}{2},$$

which is the reqd. angle between the line and the plane.

VECTORIALLY :

Let the given plane be $\vec{r} \cdot \vec{n} = p$ ...(1)

and the given line be parallel to $\vec{b}$.

If 'ϕ' be the angle between the normal to the plane *i.e.* $\vec{n}$ and the given line (parallel to $\vec{b}$), then $\vec{n} \cdot \vec{b} = nb \cos\phi$, where $|\vec{n}| = n$ and $|\vec{b}| = b$.

$$\therefore \cos\phi = \frac{\vec{n} \cdot \vec{b}}{nb} \qquad \Rightarrow \phi = \cos^{-1}\left(\frac{\vec{n} \cdot \vec{b}}{nb}\right).$$

If 'θ' be the angle between the line and the plane, then :

$$\theta = \frac{\pi}{2} - \phi = \frac{\pi}{2} - \cos^{-1}\left(\frac{\vec{n} \cdot \vec{b}}{nb}\right) = \sin^{-1}\left(\frac{\vec{n} \cdot \vec{b}}{nb}\right).$$

11.27. INTERSECTION OF A LINE AND A PLANE

(*a*) To find the point of intersection of the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ and the plane ax + by + cz + d = 0.

The given line L is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ ...(1)

The given plane α is $ax + by + cz + d = 0$ ...(2)

Any point on the line (1) is :

$(x_1 + lr, y_1 + mr, z_1 + nr)$ ...(3)

This lies on plane (2)

iff $a(x_1 + lr) + b(y_1 + mr) + c(z_1 + nr) + d = 0$

iff $r(al + bm + cn) + (ax_1 + by_1 + cz_1 + d) = 0$

iff $r = -\frac{ax_1 + by_1 + cz_1 + d}{al + bm + cn}$*.

Putting this value of r in (3), we get the required point of intersection.

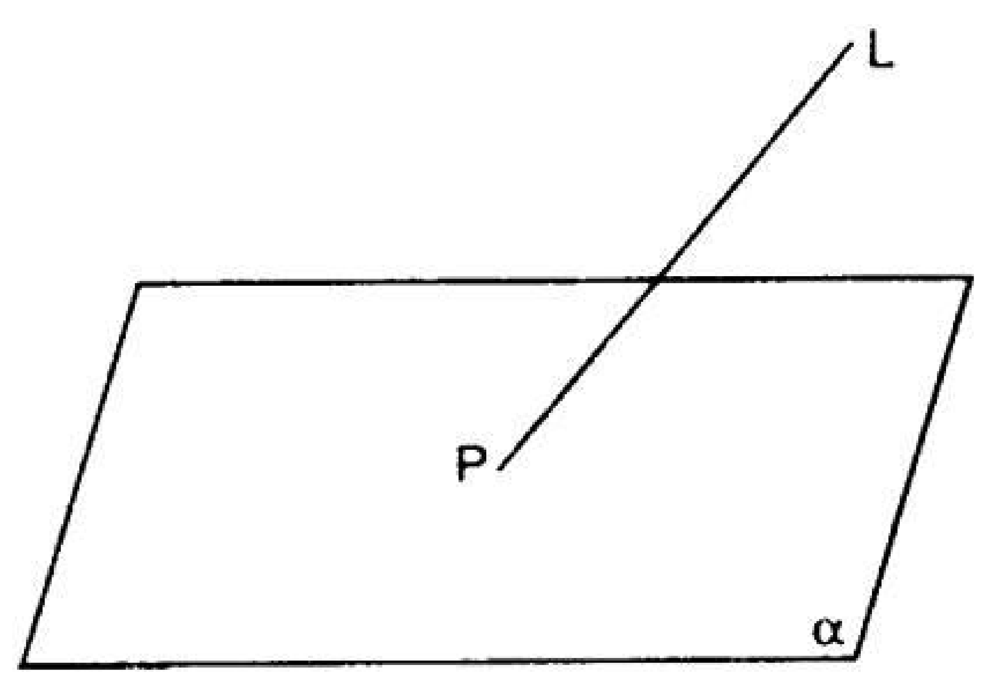


Fig.

* r is infinite if $al + bm + cn = 0$ *i.e.* if L is parallel to α.

(*b*) Conditions of Perpendicularity.

To find the conditions that the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ **is perpendicular to the plane :**

$$ax + by + cz + d = 0.$$

The given line L is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ ...(1)

The given plane α is $ax + by + cz + d = 0$ ...(2)

Direction-ratios of L are $< l, m, n >$.

Direction-ratios of normal to the plane α are $< a, b, c >$.

The line L is perpendicular to plane α

iff L is parallel to the normal to the plane

iff $l : m : n = a : b : c$, which are the reqd. conditions.

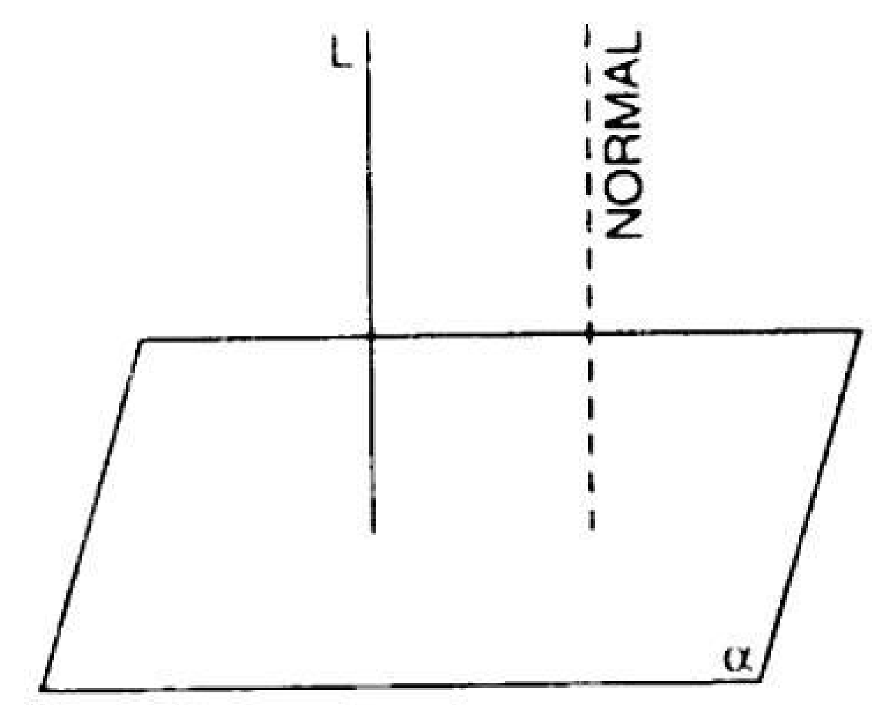


Fig.

(*c*) Conditions of Parallelism.

To find the conditions that the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ **is parallel to the plane :**

$$ax + by + cz + d = 0.$$

The given line L is :

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad ...(1)$$

The given plane α is :

$$ax + by + cz + d = 0 \quad ...(2)$$

Direction-ratios of L are $< l, m, n >$.

Direction-ratios of normal to the plane α are $< a, b, c >$.

The line L is parallel to plane α

iff L is perpendicular to normal to plane (1) and (x_1, y_1, z_1) does not lie in plane (1)

iff $al + bm + cn = 0$ and $ax_1 + by_1 + cz_1 + d \neq 0$,

which are the reqd. conditions.

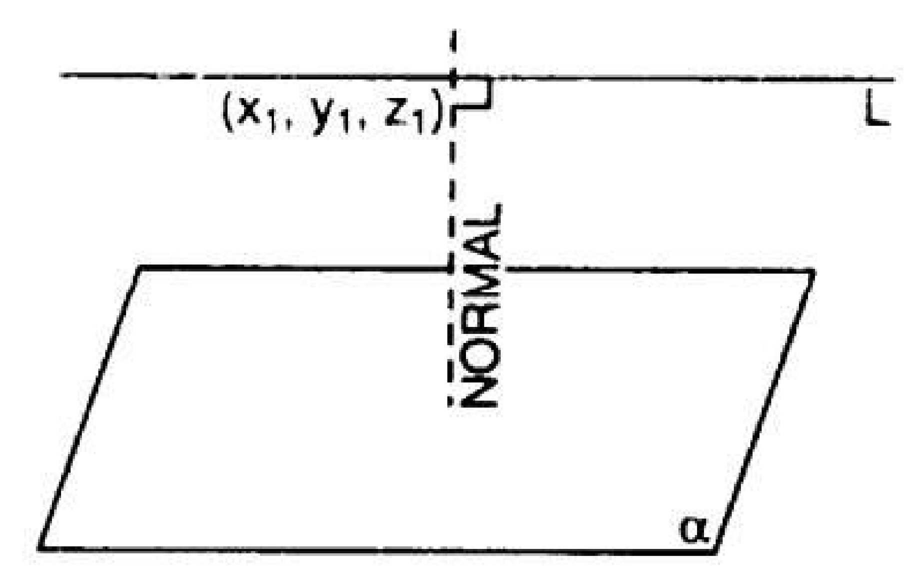


Fig.

(*d*) Conditions that a line lies in the plane.

To find the conditions that the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ **may lie in the plane :**

$$ax + by + cz + d = 0.$$

The given line L is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ ...(1)

The given plane α is $ax + by + cz + d = 0$ ...(2)

Direction-ratios of L are $< l, m, n >$.

Direction-ratios of normal to the plane α are $< a, b, c >$.

The line L lies in the plane α

iff L is perpendicular to normal to plane (1) and (x_1, y_1, z_1) lies on the plane

iff $al + bm + cn = 0$ and $ax_1 + by_1 + cz_1 + d = 0$,

which are the reqd. conditions.

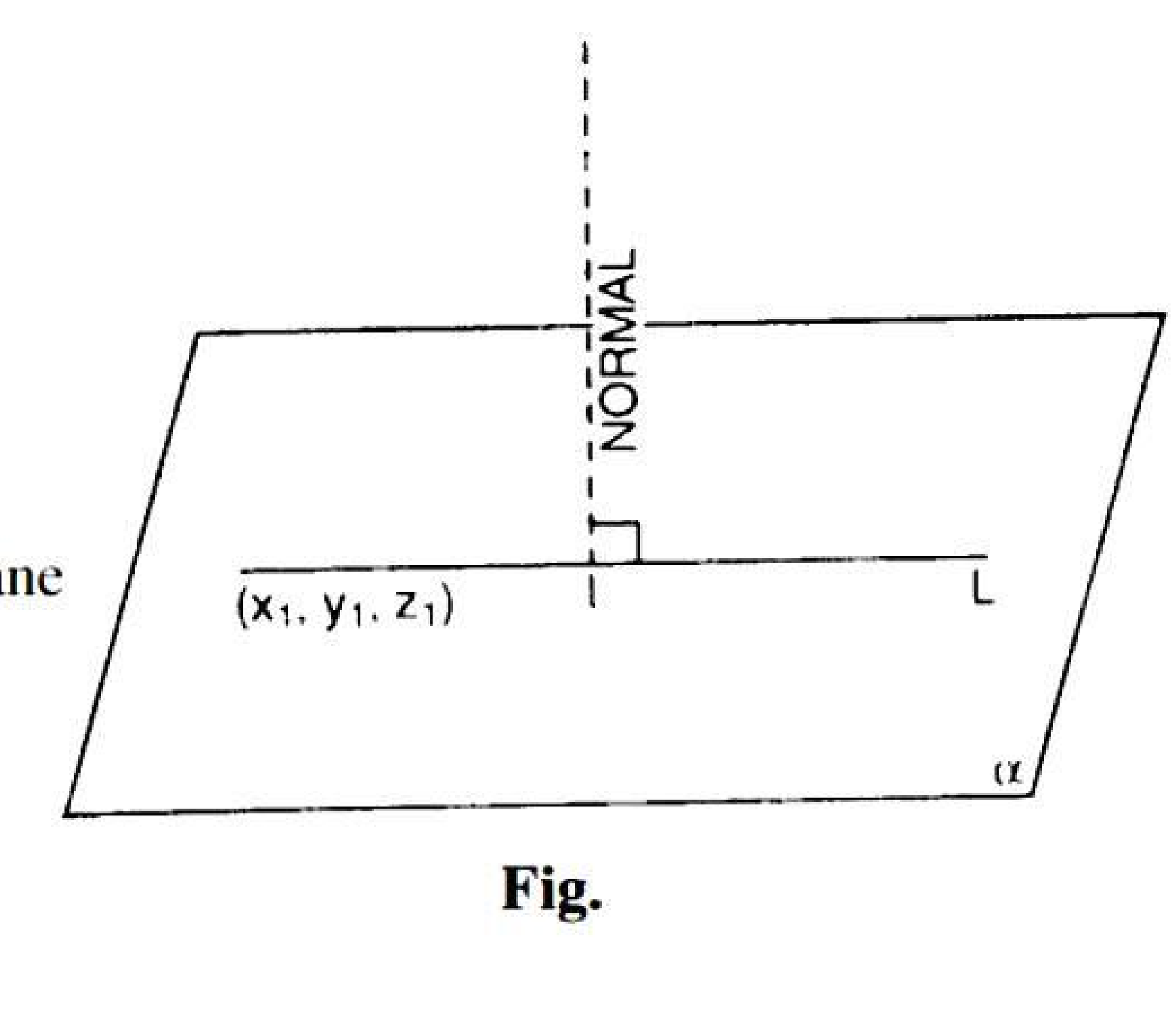


Fig.

ILLUSTRATIVE EXAMPLES

Example 1. Show that the plane whose vector equation is $\vec{r}.(\hat{i}+2\hat{j}-\hat{k})=3$ contains the line whose vector equation is $\vec{r}=\hat{i}+\hat{j}+\lambda(2\hat{i}+\hat{j}+4\hat{k})$.

Solution. The given plane is $\vec{r}.(\hat{i}+2\hat{j}-\hat{k})=3$ ...(1)

i.e. $(x\hat{i}+y\hat{j}+z\hat{k}).(\hat{i}+2\hat{j}-\hat{k})=3$

$\Rightarrow$ $x+2y-z=3$ ...(2)

To prove that the line $\vec{r}=\hat{i}+\hat{j}+\lambda(2\hat{i}+\hat{j}+4\hat{k})$ lies in the plane (1), we are to prove that :

(*i*) the point (1, 1, 0) lies in the plane

(*ii*) the normal to the plane is perpendicular to the line.

For (*i*) : (1, 1, 0) lies on (2) if $1+2(1)-0=3$, which is true.

For (*ii*) : Direction–ratios of the normal to the plane are $<1, 2, -1>$.

Direction–ratios of the line are $<2, 1, 4>$.

Since $(1)(2)+(2)(1)+(-1)(4)=0$ *i.e.* $0=0$.

Hence, the result.

Example 2. Find the angle between the line :

$$\vec{r}=(\hat{i}-\hat{j}+\hat{k})+\lambda(2\hat{i}-\hat{j}+3\hat{k})$$

and the plane $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})=4$.

Also, find whether the line is parallel to the plane or not.

Solution. The given line is :

$$\vec{r}=(\hat{i}-\hat{j}+\hat{k})+\lambda(2\hat{i}-\hat{j}+3\hat{k})$$

and the given plane is $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})=4$.

Now the line is parallel to $2\hat{i}-\hat{j}+3\hat{k}$ and normal to the plane $2\hat{i}+\hat{j}-\hat{k}$.

If 'θ' is the angle between the line and the plane, then $\left(\frac{\pi}{2}-\theta\right)$ is the angle between the line and normal to the plane.

Then $\cos\left(\frac{\pi}{2}-\theta\right)=\frac{(2\hat{i}-\hat{j}+3\hat{k}).(2\hat{i}+\hat{j}-\hat{k})}{\sqrt{4+1+9}\sqrt{4+1+1}}$

$\Rightarrow$ $\sin\theta=\frac{4-1-3}{\sqrt{14}\sqrt{6}}=0 \quad \Rightarrow \theta=0^{o}$.

Hence, the line is parallel to the plane.

Example 3. Find the point of intersection of the line : $\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(2\hat{i}+\hat{j}+2\hat{k})$ and the plane $\vec{r}.(2\hat{i}-6\hat{j}+3\hat{k})+5=0$.

Solution. The given line is $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$ ...(1)

and the given plane is $2x-6y+3z+5=0$ ...(2)

Any point on given line is $(1+2k, 2+k, 3+2k)$ ...(3)

If the line intersects the plane, then this point must lie on the plane for some value of k.

$\therefore$ $2(1+2k)-6(2+k)+3(3+2k)+5=0$

$\Rightarrow$ $2+4k-12-6k+9+6k+5=0$

$\Rightarrow$ $4k+4=0 \quad \Rightarrow \quad k=-1$.

Putting in (3), the point of intersection is

$(1-2, 2-1, 3-2)$ *i.e.* $(-1, 1, 1)$.

Example 4. Show that the line whose vector equation is $\vec{r}=(\hat{i}+\hat{j})+\lambda(2\hat{i}+\hat{j}+4\hat{k})$ is parallel to the plane π whose vector equation is $\vec{r}.(\hat{i}+2\hat{j}-\hat{k})=3$; and find the distance between them. Also, state whether the line lies in the plane.

Solution. The line will be parallel to the plane if it is perpendicular to the normal to the plane.

Here the line is parallel to $\vec{b}=2\vec{i}+\hat{j}+4\hat{k}$ and $\vec{n}=\hat{i}+2\hat{j}-\hat{k}$.

Now $\vec{b}\cdot\vec{n}=\ 2\hat{i}+\hat{j}+4\hat{k}\ .\ \hat{i}+2\hat{j}-\hat{k}$

$=(2)(1)+(1)(2)+(4)(-1)$

$=2+2-4=0$

$\Rightarrow$ the line is parallel to π-plane.

(*ii*) Any point of the line is (1, 1, 0).

Equation of the plane in cartesian form is :

$x+2y-z-3=0$.

$\therefore$ Distance between the line and the plane

$$=\left|\frac{1+2(1)-0-3}{\sqrt{1+4+1}}\right|=\left|\frac{3-3}{\sqrt{6}}\right|=0.$$

Hence, the line lies in the plane.

Check Up : (1, 1, 0) lies on $x+2y-z-3=0$ because $1+2-0-3=0$ *i.e.* $0=0$, which is true.

Example 5. Find the distance from the point (3, 4, 5) to the point, where the line :

$$\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$$

meets the plane x + y + z = 2.

Solution. The given line is $\frac{x-3}{1}=\frac{y-4}{2}=\frac{z-5}{2}$ ...(1)

The given plane is $x + y + z = 2$ ... (2)

Any point on line (1) is $(3 + k, 4 + 2k, 5 + 2k)$ ... (3),

where k is any constant.

This lies on plane (2)

if $(3 + k) + (4 + 2k) + (5 + 2k) = 2$

if $5k = -10$ if $k = -2$.

Putting in (3), $(3 - 2, 4 - 4, 5 - 4)$

i.e. A (1, 0, 1) is the point, where the line (1) meets the plane (2).

If P (3, 4, 5) be the given point,

then $|AP| = \sqrt{(3-1)^2+(4-0)^2+(5-1)^2}$

$= \sqrt{4+16+16} = \sqrt{36} = 6$ units.

Example 6. Find the angle between the plane 2x + 3y – 5z = 10 and the line passing through the points (2, 3, – 1) and (1, 2, 1). *(P.B. 2018)*

Solution. The given plane is $2x + 3y - 5z - 10 = 0$ ...(1)

The given line is $\frac{x-2}{1-2}=\frac{y-3}{2-3}=\frac{z+1}{1+1}$

$\Rightarrow \frac{x-2}{-1}=\frac{y-3}{-1}=\frac{z+1}{2}$

$\Rightarrow \frac{x-2}{1}=\frac{y-3}{1}=\frac{z+1}{-2}$ ...(2)

If the line (2) makes an angle 'θ' with the plane (1), then the line (1) will make angle $(90° - \theta)$ with the normal to the plane (1).

Now, direction-ratios of line (2) are < 1, 1, – 2 >

and direction-ratios of normal to plane (1) are < 2, 3, – 5 >.

$\therefore \cos(90° - \theta) = \left|\frac{(1)(2)+(1)(3)+(-2)(-5)}{\sqrt{1+1+4}\sqrt{4+9+25}}\right|$

$\Rightarrow \sin\theta = \left|\frac{2+3+10}{\sqrt{6}\sqrt{38}}\right| = \frac{15}{\sqrt{6}\sqrt{38}}$.

Hence, $\theta = \sin^{-1}\left(\frac{15}{\sqrt{6}\sqrt{38}}\right)$.

Example 7. State when the line $\vec{r} = \vec{a} + \lambda\vec{b}$ is parallel to the plane $\vec{r}.\vec{n} = d$. Show that the line $\vec{r} = \hat{i}+\hat{j}+\lambda(2\hat{i}+\hat{j}+4\hat{k})$ is parallel to the plane $\vec{r}.(-2\hat{i}+\hat{k}) = 5$. Also, find the distance between the line and the plane.

Solution. (*i*) A line is parallel to the plane if it is perpendicular to the normal to the plane.

The given line is $\vec{r} = \vec{a} + \lambda\vec{b}$

$\Rightarrow \vec{b}$ is parallel to the line.

The given plane is $\vec{r}.\vec{n} = d$

$\Rightarrow \vec{n}$ is normal to the plane.

Thus the line is parallel to the plane when $\vec{b}.\vec{n} = 0$.

(*ii*) Here $\vec{b} = 2\hat{i}+\hat{j}+4\hat{k}$ and $\vec{n} = -2\hat{i}+\hat{k}$.

Now $\vec{b}.\vec{n} = (2)(-2) + (1)(0) + (4)(1)$

$= -4 + 0 + 4 = 0$.

Hence, the given line is parallel to the given plane.

(*iii*) (1, 1, 0) is a point on the given line.

Equation of the plane is $-2x + z - 5 = 0$.

$\therefore$ Reqd. distance

$= \left|\frac{-2(1)+0-5}{\sqrt{4+0+1}}\right| = \frac{7}{\sqrt{5}} = \frac{7\sqrt{5}}{5}$ units.

Example 8. Find the equation to the plane through the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ and parallel to the line

$$\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}.$$

Solution. The given lines are :

$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ ...(1)

and $\frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}$ ...(2)

The plane through line (1) will also contain the point (α, β, γ).

Any plane through (α, β, γ) is :

$$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0 \quad ...(3)$$

Since plane (3) contains line (1),

$\therefore$ normal to plane (3) is perpendicular to line (1)

$$\Rightarrow \quad al+bm+cn=0 \quad ...(4)$$

Again, since plane (3) is parallel to line (2),

$\therefore$ normal to plane (3) is perpendicular to line (2)

$$\Rightarrow \quad al'+bm'+cn'=0 \quad ...(5)$$

Solving (4) and (5),

$$\frac{a}{mn'-m'n}=\frac{b}{nl'-n'l}=\frac{c}{lm'-l'm}=k \text{ (say)}$$

$\therefore a=k(mn'-m'n), b=k(nl'-n'l),$

$c=k(lm'-l'm)$; $k \neq 0$.

Putting in (3),

$$k(mn'-m'n)(x-\alpha)+k(nl'-n'l)(y-\beta)$$
$$+k(lm'-l'm)(z-\gamma)=0$$

$$\Rightarrow (mn'-m'n)(x-\alpha)+(nl'-n'l)(y-\beta)$$
$$+(lm'-l'm)(z-\gamma)=0, [\because k \neq 0]$$

which is the reqd. equation.

Example 9. If lines : $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1}$ intersect, then find the value of 'k' and hence find the equation of the plane containing these lines. *(C.B.S.E. 2015)*

Solution. The given lines are :

$$L_1: \frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4} \quad ...(1)$$

and $$L_2: \frac{x-3}{1}=\frac{y-k}{2}=\frac{z}{1} \quad ...(2)$$

Any point on L_1 is $(2\lambda+1, 3\lambda-1, 4\lambda+1)$ ...(3)

Any point on L_2 is $(\mu+3, 2\mu+k, \mu)$ ...(4)

The lines (1) and (2) intersect if (3) and (4) coincide

$$\Rightarrow \quad 2\lambda+1=\mu+3,$$
$$3\lambda-1=2\mu+k$$
and $$4\lambda+1=\mu$$

Taking first two,

$$2\lambda-\mu=2 \quad ...(5)$$

Taking middle two,

$$3\lambda-2\mu=k+1 \quad ...(6)$$

Taking last two,

$$4\lambda-\mu=-1 \quad ...(7)$$

Solving (5) and (7), $\lambda=-\frac{3}{2}, \mu=-5.$

Putting in (6),

$$3\left(-\frac{3}{2}\right)-2(-5)=k+1$$

$$\Rightarrow \quad -\frac{9}{2}+10=k+1.$$

Hence, $$k=9-\frac{9}{2}=\frac{9}{2}.$$

The equation of the plane containing given lines is :

$$\begin{vmatrix} x-1 & y+1 & z-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}=0$$

$$\Rightarrow (x-1)(3-8)-(y+1)(2-4)+(z-1)(4-3)=0$$
$$\Rightarrow -5x+5+2y+2+z-1=0$$
$$\Rightarrow \quad 5x-2y-z-6=0.$$

Example 10. Find the equation of the plane parallel to the line $\frac{x-2}{1}=\frac{y-1}{3}=\frac{z-3}{2}$, which contains the point (5, 2, – 1) and passes through the origin.

Solution. Let $ax+by+cz=0$ ...(1) be the equation of the plane, which passes through the origin. $[\because d=0]$

Since it passes through (5, 2, – 1),

$$\therefore \quad (a)(5)+b(2)+c(-1)=0$$
$$\Rightarrow \quad 5a+2b-c=0 \quad ...(2)$$

Again since the plane (1) is parallel to the line :

$$\frac{x-2}{1}=\frac{y-1}{3}=\frac{z-3}{2} \quad ...(3)$$

$\therefore$ the normal to the plane (1) is perp. to the line (3)

$$\Rightarrow \quad (1)(a)+(3)(b)+(2)(c)=0$$
$$\Rightarrow \quad a+3b+2c=0 \quad ...(4)$$

Solving (2) and (4), $\frac{a}{4+3}=\frac{b}{-1-10}=\frac{c}{15-2}=k$

(say), where $k \neq 0$

$$\therefore \quad a=7k, b=-11k, c=13k.$$

Putting in (1), $7kx-11ky+13kz=0$

$$\Rightarrow \quad 7x-11y+13z=0, \quad [\because k \neq 0]$$

which is the reqd. equation.

Example 11. Find the equation of the plane containing the line :

$$\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z-3}{4}$$

and perpendicular to the plane x + 2y + z – 2 = 0.

(P.B. 2011)

Solution. Any plane containing $\frac{x-1}{2}=\frac{y-2}{-1}=\frac{z-3}{4}$ is :

$a(x-1)+b(y-2)+c(z-3)=0$ ...(1),

where $2a-b+4c=0$ ...(2)

Also (1) is perpendicular to $x+2y+z-2=0$.

$\therefore\ a+2b+c=0$ ...(3)

Solving (2) and (3),

$$\frac{a}{-1-8}=\frac{b}{4-2}=\frac{c}{4+1}$$

$$\Rightarrow \quad \frac{a}{-9}=\frac{b}{2}=\frac{c}{5}=k \text{ (say)}$$

$\therefore$ $a=-9k,\ b=2k,\ c=5k$, where $k\neq 0$.

Putting in (1),

$-9k(x-1)+2k(y-2)+5k(z-3)=0$

$\Rightarrow -9(x-1)+2(y-2)+5(z-3)=0$ $[\because k\neq 0]$

$\Rightarrow -9x+2y+5z-10=0$

$\Rightarrow \quad 9x-2y-5z+10=0$,

which is the reqd. equation.

Example 12. Find the vector and cartesian equations of the plane containing the lines :

$$\vec{r}=2\hat{i}+\hat{j}-3\hat{k}+\lambda(\hat{i}+2\hat{j}+5\hat{k}) \text{ and}$$
$$\vec{r}=3\hat{i}+3\hat{j}+2\hat{k}+\mu(3\hat{i}-2\hat{j}+5\hat{k}).$$

Solution. The given lines are :

$$\vec{r}=2\hat{i}+\hat{j}-3\hat{k}+\lambda(\hat{i}+2\hat{j}+5\hat{k})$$

and $$\vec{r}=3\hat{i}+3\hat{j}+2\hat{k}+\mu(3\hat{i}-2\hat{j}+5\hat{k}).$$

Here $\vec{a_1}=2\hat{i}+\hat{j}-3\hat{k}$ and $\vec{a_2}=3\hat{i}+3\hat{j}+2\hat{k}$

$\vec{b_1}=\hat{i}+2\hat{j}+5\hat{k}$ and $\vec{b_2}=3\hat{i}-2\hat{j}+5\hat{k}$.

$$\therefore \quad \vec{n}=\vec{b_1}\times\vec{b_2}$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 5\\ 3 & -2 & 5\end{vmatrix}$$

$$=\hat{i}(10+10)-\hat{j}(5-15)+\hat{k}(-2-6)$$

$$=20\hat{i}+10\hat{j}-8\hat{k}.$$

$\therefore$ Vector equation of the reqd. plane is :

$$(\vec{r}-\vec{a_1}).\vec{n}=0$$

$$\Rightarrow \quad \vec{r}.\vec{n}=\vec{a_1}.\vec{n}$$

$$\Rightarrow \vec{r}.(20\hat{i}+10\hat{j}-8\hat{k})=(2\hat{i}+\hat{j}-3\hat{k}).(20\hat{i}+10\hat{j}-8\hat{k})$$

$$=(2)(20)+(1)(10)+(-3)(-8)$$

$$=40+10+24=74$$

$$\Rightarrow \vec{r}.(10\hat{i}+5\hat{j}-4\hat{k})=37.$$

Cartesian equation is :

$$(x\hat{i}+y\hat{j}+z\hat{k}).(10\hat{i}+5\hat{j}-4\hat{k})=37$$

i.e. $\quad 10x+5y-4z=37.$

Example 13. Find the equation of the plane through the line :

$$\frac{x-1}{3}=\frac{y-4}{2}=\frac{z-4}{-2}$$

and parallel to the line :

$$\frac{x+1}{2}=\frac{1-y}{4}=\frac{z+2}{1}.$$

Hence, find the shortest distance betwen the lines.

(C.B.S.E. Sample Paper 2019)

Solution. (*i*) The given lines are :

$$\frac{x-1}{3}=\frac{y-4}{2}=\frac{z-4}{-2} \quad ...(1)$$

and $$\frac{x+1}{2}=\frac{y-1}{-4}=\frac{z+2}{1} \quad ..(2)$$

Let <a, b, c> be the direction-ratios of the normal to the plane, which contains line (1).

$\therefore$ The equation of the plane is :

$a(x-1)+b(y-4)+c(z-4)=0$ ...(3),

where $3a+2b-2c=0$ ...(4)

Alos, $2a-4b+c=0$ ...(5)

[$\because$ *line (2) is parallel to plane (3)*]

Solving (4) and (5),

$$\frac{a}{2-8}=\frac{b}{-4-3}=\frac{c}{-12-4}$$

$\Rightarrow \quad \frac{a}{6} = \frac{b}{7} = \frac{c}{16} = k$ (say) $\quad (k \neq 0)$

$\therefore \; a = 6k, b = 7k$ and $c = 16k$.

Putting in (3), we get :

$6k(x-1) + 7k(y-4) + 16k(z-4) = 0$

$\Rightarrow 6(x-1) + 7(y-4) + 16(z-4) = 0 \quad [\because k \neq 0]$

$\Rightarrow 6x + 7y + 16z - 98 = 0,$

which is the reqd. equation the plane.

(*ii*) S.D. between two lines

= Perpendicular distance of the point (– 1, 1, – 2) from the plane

$$= \left| \frac{6(-1)+7(1)+16(-2)-98}{\sqrt{(6)^2+(7)^2+(16)^2}} \right|$$

$$= \left| \frac{-6+7-32-98}{\sqrt{36+49+256}} \right| = \left| \frac{-129}{\sqrt{341}} \right| = \frac{129}{\sqrt{341}} \text{ units.}$$

EXERCISE 11 (f)

Very Short Answer Type Questions

VSATQ

1. What is the point of intersection of the line $x = y = z$ with the plane $x + 2y + 3z = 6$? ***(Bihar B. 2012)***

2. Find the angle between the lines in which the planes : $3x - 7y - 5z = 1$, $5x - 13y + 3z + 2 = 0$ cut the plane $8x - 11y + 2z = 0$.

3. (*a*) (*i*) Show that the line :

$$\vec{r} = 2\hat{i} - 3\hat{j} + 5\hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$$

lies in the plane $\vec{r} \cdot (3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$.

(*ii*) Show that the line :

$$\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} + \hat{j} + 4\hat{k})$$

lies in the plane $\vec{r} \cdot (\hat{i} + 2\hat{j} - \hat{k}) = 3$.

(*b*) Find the value of '*m*' for which the line $\vec{r} = (\hat{i} + 2\hat{k}) + \lambda(2\hat{i} - m\hat{j} - 3\hat{k})$ is parallel to the plane $\vec{r} \cdot (m\hat{i} + 3\hat{j} + \hat{k}) = 4$. ***(Mizoram B. 2018)***

Short Answer Type Questions

SATQ

4. Find the vector equation of the line passing through (3, 1, 2) and perpendicular to the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 4$. Find also the point of intersection of this line and the plane. ***(N.C.E.R.T.)***

5. Find the point where the line $\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4}$ meets the plane $x + y + 4z = 6$.

6. (*i*) Find the angle between the line :

$(2\hat{i} + 3\hat{j} + 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$

and the plane : $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 5$. ***(H.B. 2016)***

(*ii*) Find the angle between the line joining (3, – 4, – 2) and (12, 2, 0) and the plane $\vec{r} \cdot \left(\hat{i} + \hat{j} + \hat{k}\right) = 4$. ***(J. & K. B. 2010)***

7. (*i*) Find the angle between the line $\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6}$ and the plane $10x + 2y - 11z = 3$

(N.C.E.R.T. ; Jammu B. 2015; Kashmir B. 2012)

(*ii*) Find the angle between the line :

$\frac{x+1}{2} = \frac{y-1}{2} = \frac{z-2}{4}$ and the plane $2x + y - 3z + 4 = 0$. ***(H.B. 2016)***

(*iii*) Find the angle between the plane $2x + 4y - z = 8$ and line $\frac{x-1}{2} = \frac{2-y}{7} = \frac{3z+6}{12}$. ***(P.B. 2017)***

8. Find the distance of the point (– 1, – 5, – 10) from the point of intersection of the line :

$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$. ***(C.B.S.E. 2015; P.B. 2012)***

9. (*i*) Find the distance of the point (– 1, – 5, – 10) from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$. ***(N.C.E.R.T.; C.B.S.E. 2018; H.P.B. 2015; A.I.C.B.S.E. 2011)***

(*ii*) Find the distance of the point with position vector $-\hat{i} - 5\hat{j} - 10\hat{k}$ from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 12\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

(*iii*) Find the distance of the point (2, 12, 5) from the point of intersection of the line :

$$\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$$

and the plane $\vec{r} \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0$. ***(A.I.C.B.S.E. 2014)***

10. Find the distance between the point with position vector $-\hat{i}-5\hat{j}-10\hat{k}$ and the point of intersection of the line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{12}$ with the plane $x-y+z=5$.

11. Find the vector and cartesian equations of the line passing through the point P (1, 2, 3) and parallel to the planes:

$$\vec{r}.(\hat{i}-\hat{j}+2\hat{k})=5 \text{ and } \vec{r}.(3\hat{i}+\hat{j}+\hat{k})=6.$$

(N.C.E.R.T.; C.B.S.E. (F) 2012)

12. Find the vector equation of the line passing through (1, 2, 3) and perpendicular to the plane :

$$\vec{r}.\left(\hat{i}+2\hat{j}-5\hat{k}\right)+9=0.$$

(H.P.B. 2015)

13. Find the equation of the plane which is parallel to the line $\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$ and passes through the points (0, 0, 0) and (3, – 1, 2).

14. Find the equations of the plane through the points : (1, 0, –1), (3, 2, 2) and parallel to the line $\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}$.

15. Find the equation of the plane containing the line :

$\frac{x+2}{2}=\frac{y+3}{3}=\frac{z-4}{-2}$ and the point (0, 6, 0).

16. Find the equation of the plane, which contains two lines :

$$\frac{x-4}{1}=\frac{y-3}{-4}=\frac{z-2}{5} \text{ and } \frac{x-3}{1}=\frac{y+2}{-4}=\frac{z}{5}.$$

17. Find the vector and cartesian equations of the plane containing the lines :

$$\vec{r}=\hat{i}+2\hat{j}-4\hat{k}+\lambda(2\hat{i}+3\hat{j}+6\hat{k}) \text{ and}$$

$$\vec{r}=3\hat{i}+3\hat{j}-5\hat{k}+\mu(-2\hat{i}+3\hat{j}+8\hat{k}).$$

18. Find the equation of the plane through the point (1, 1, 1) and perpendicular to the line :

$$x-2y+z=2,\ 4x+3y-z+1=0.$$

19. If the line drawn from (4, – 1, 2) to the point (–3, 2, 3) meets a plane at right angles, at the point (– 10, 5, 4), then find the equation of the plane.

20. (*a*) Find the length and the foot of the perpendicular from :

P (1, 1, 2) to the plane $2x-2y+4z+5=0$.

(*b*) Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane $2x-3y+4z-6=0$.

(N.C.E.R.T.)

21. Find the co-ordinates of the foot of the perpendicular from the point (2, 3, 7) to the plane $3x-y-z=7$. Also find the length of the perpendicular.

Long Answer Type Questions

LATQ

22. Find the equation of the plane containing the line :

$\frac{x-1}{3}=\frac{y+2}{1}=\frac{z-3}{2}$ and perpendicular to the plane $2x-y+2z-3=0$. *(P.B. 2011)*

23. Show that the line L whose vector equation is $\vec{r}=2\hat{i}-2\hat{j}+3\hat{k}+\lambda(\hat{i}-\hat{j}+4\hat{k})$ is parallel to the plane π whose vector equation is $\vec{r}.(\hat{i}+5\hat{j}+\hat{k})=5$ and find the distance between them.

24. State when the line $\vec{r}=\vec{a}+\lambda\vec{b}$ is parallel to the plane $\vec{r}.\vec{n}=d$. Show that the line $\vec{r}=\hat{i}+\hat{j}+\lambda(3\hat{i}-\hat{j}+2\hat{k})$ is parallel to the plane $\vec{r}.(2\hat{i}+\hat{k})=3$. Also, find the distance between the line and the plane.

25. Find the equations of the line through (– 1, 3, 2) and perpendicular to the plane $x+2y+2z=3$, the length of the perpendicular and co-ordinates of its foot.

26. Find the vector equation of the line passing through the point (3, 1, 2) and perpendicular to the plane $\vec{r}.(2\hat{i}-\hat{j}+\hat{k})=4$. Also find the point of intersection of this line and the plane.

27. Find the vector equation of a line passing through the point with position vector $(2\hat{i}-3\hat{j}-5\hat{k})$ and perpendicular to the plane $\vec{r}.(6\hat{i}-3\hat{j}-5\hat{k})+2=0$. Also, find the point of intersection of this line and the plane.

28. Find the co-ordinates of the point, where the line :

$$\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-2}{2}$$

intersects the plane $x-y+z-5=0$.

Also find the angle between the line and the plane.

(C.B.S.E. 2013)

29. Find the foot of the perpendicular from P (1, 2, 3) on the line :

$$\frac{x-6}{3}=\frac{y-7}{2}=\frac{z-7}{-2}.$$

Also obtain the equation of the plane containing the line and the point (1, 2, 3).

30. Find the point, where the line joining the points (1, 3, 4) and (– 3, 5, 2) intersects the plane $\vec{r}.(2\hat{i}+\hat{j}+\hat{k})+3=0$.

Is the point equidistant from the given points ?

31. Find the co-ordinates of the point where the line joining the points (1, – 2, 3) and (2, – 1, 5) cuts the plane $x-2y+3z=19$. Hence, find the distance of this point from the point (5, 4, 1).

32. Find the equation of the plane passing through the point (1, 1, 1) and containing the line :

$$\vec{r} = (-3\hat{i} + \hat{j} + 5\hat{k}) + \lambda(3\hat{i} - \hat{j} + 5\hat{k}).$$

Also, show that the plane contains the line :

$$\vec{r} = (-\hat{i} + 2\hat{j} + 5\hat{k}) + \lambda(\hat{i} - 2\hat{j} - 5\hat{k}).$$

33. Find the equation of the plane passing through the point A (1, 2, 1) and perpendicular to the line joining the points P (1, 4, 2) and Q (2, 3, 5). Also, find the distance of this plane from the line :

$$\frac{x+3}{2} = \frac{y-5}{-1} = \frac{z-7}{-1}.$$ *(C.B.S.E. 2010 C)*

34. Find the vector equation of the plane passing through three points with position vectors $\hat{i}+\hat{j}-2\hat{k}, 2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+2\hat{j}+\hat{k}$. Also find the co-ordinates of the point of intersection of this plane and the line $\vec{r} = 3\hat{i}-\hat{j}-\hat{k}+\lambda(2\hat{i}-2\hat{j}+\hat{k})$.

Answers

1. (1, 1, 1). **2.** 90°. **3.** (*b*) $m = -3$.

4. $2x - y + z - 4 = 0; \left(2, \frac{3}{2}, \frac{3}{2}\right)$.

5. (1, 1, 1).

6. (*i*) $\sin^{-1}\left(\frac{9}{\sqrt{87}}\right)$ (*ii*) $\sin^{-1}\left(\frac{17}{11\sqrt{3}}\right)$.

7. (*i*) $\sin^{-1}\left(\frac{8}{21}\right)$ (*ii*) $\sin^{-1}\left(\frac{1}{2}\cdot\frac{\sqrt{3}}{7}\right)$ (*iii*) $\sin^{-1}\left(\frac{1}{6}\right)$.

8. 13 units. **9.** (*i*)-(*ii*) 13 units (*iii*) 13 units.

10. $\sqrt{221}$ units.

11. $\vec{r} = (\hat{i}+2\hat{j}+3\hat{k})+\lambda(-3\hat{i}+5\hat{j}+4\hat{k})$;

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}.$$

12. $\vec{r} = (1+\lambda)\hat{i}+2(1+\lambda)\hat{j}+(3-5\lambda)\hat{k}$.

13. $x - 19y - 11z = 0$.

14. $4x - y - 2z - 6 = 0$.

15. $3x + 2y + 6z - 12 = 0$.

16. $11x - y - 3z - 35 = 0$.

17. $[\hat{r} - (\hat{i} + 2\hat{j} - 4\hat{k})].[(2\hat{i} + 3\hat{j} + 6\hat{k}) \times (-2\hat{i} + 3\hat{j} + 8\hat{k})] = 0,$

$3x - 14y + 6z + 49 = 0$.

18. $x - 5y - 11z + 15 = 0$.

19. $7x - 3y - z + 89 = 0$.

20. (*a*) $\frac{13}{2\sqrt{6}}; \left(-\frac{1}{12}, \frac{25}{12}, -\frac{1}{6}\right)$ (*b*) $\frac{12}{\sqrt{29}}, \frac{-18}{\sqrt{29}}, \frac{24}{\sqrt{29}}$.

21. (5, 2, 6) ; $\sqrt{11}$.

22. $4x - 2y - 5z + 7 = 0$.

23. $\frac{10}{\sqrt{27}}$.

24. $\vec{b}\cdot\vec{n} = 0; \frac{1}{\sqrt{5}}$.

25. $\frac{x+1}{1} = \frac{y-3}{2} = \frac{z-2}{2}; 2; \left(-\frac{5}{3}, \frac{5}{3}, \frac{2}{3}\right)$.

26. $\vec{r} = (3\hat{i} + \hat{j} + 2\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k}); \left(2, \frac{3}{2}, \frac{3}{2}\right)$.

27. $\vec{r} = (2\hat{i} - 3\hat{j} - 5\hat{k}) + \lambda(6\hat{i} - 3\hat{j} - 5\hat{k});$

$$\left(\frac{41}{7}, \frac{-1}{14}, \frac{-25}{14}\right).$$

28. (2, –1, 2); $\sin^{-1}\left(\frac{1}{\sqrt{87}}\right)$.

29. (3, 5, 9) ; $18x - 22y + 5z + 11 = 0$.

30. (– 5, 6, 1); No.

31. (2, – 1, 5); $5\sqrt{2}$ units.

32. $x - 2y + z = 0$

33. $x - y + 3z - 2 = 0; \sqrt{11}$.

34. $\vec{r}.(9\hat{i}+3\hat{j}-\hat{k}) = 14; (1, 1, -2)$.

Hints to Selected Questions

9. (*i*) Any point on the given line :

$$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 12\hat{k})$$

is $(3\mu + 2, 4\mu - 1, 12\mu + 2)$ ...(1)

This lies on $\vec{r}.(\hat{i} - \hat{j} - \hat{k}) = 5$ *i.e.* $x - y - z = 5$

$\Rightarrow (3\mu + 2) - (4\mu - 1) - (12\mu + 2) = 5$

$\Rightarrow \mu = -\frac{4}{13}$.

Putting in (1), the reqd. point is

$$\left(\frac{-12}{13} + 2, \frac{-16}{13} - 1, \frac{-48}{13} + 2\right) \text{ i.e. } \left(\frac{14}{3}, \frac{-29}{13}, \frac{-22}{13}\right)$$

25. The line through (– 1, 3, 2) and perpendicular to $x + 2y + 2z = 3$ is $\frac{x+1}{1} = \frac{y-3}{2} = \frac{z-2}{2}$.

NCERT-FILE

Questions from NCERT Book

(For each unsolved question, refer : "Solution of Modern's abc of Mathematics")

Exercise 11.1

1. If a line makes angles 90°, 135°, 45° with the x, y and z-axes respectively, find its direction cosines.

Solution : Direction angles are 90°, 135°, 45°.

$\therefore$ Direction cosines are < cos 90°, cos 135°, cos 45° >

i.e. $< 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$.

2. Find the direction cosines of a line which makes equal angles with the coordinate axes. *(Karnataka B. 2017)*

Solution : Let each directional angle be 'α'.

$\therefore$ Direction–cosines are < cos α, cos α, cos α >.

But $l^2 + m^2 + n^2 = 1$

$\Rightarrow \cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1 \Rightarrow 3 \cos^2 \alpha = 1$

$\Rightarrow \cos^2 \alpha = \frac{1}{3} \Rightarrow \cos \alpha = \pm \frac{1}{\sqrt{3}}$.

Hence, the direction–cosines of the line are :

$< \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}} >$.

3. If a line has the direction ratios < –18, 12, – 4>, then what are its direction cosines ?

Solution. Direction-ratios of the line are :

< – 18, 12, – 4 >

Dividing each by

$$\sqrt{(-18)^2 + (12)^2 + (-4)^2} = \sqrt{489}$$

$$= \sqrt{2 \times 2 \times 11 \times 11} = 2 \times 11 = 22,$$

$\therefore$ Direction–cosines are $< -\frac{18}{22}, \frac{12}{22}, -\frac{4}{22} >$

i.e. $< -\frac{9}{11}, \frac{6}{11}, -\frac{2}{11} >$.

4. Show that the points (2, 3, 4), (–1, –2, 1), (5, 8, 7) are collinear.

Solution : Let A (2, 3, 4); B (– 1, – 2, 1) and C (5, 8, 7) be the given points.

$\therefore$ Direction-ratios of AB are :

< – 1 – 2, – 2 – 3, 1 – 4 >

i.e. < – 3, – 5, – 3 > *i.e.* < 3, 5, 3 >

Direction-ratios of BC are :

< 5 – (– 1), 8 – (– 2), 7 – 1 >

i.e. < 6, 10, 6 > *i.e.* < 3, 5, 5 >

$\Rightarrow$ AB and BC have same direction-ratios

$\Rightarrow$ AB || BC.

But B is a common point.

Hence, A, B, C are collinear.

5. Find the direction cosines of the sides of the triangle whose vertices are (3, 5, –4), (–1, 1, 2) and (–5, –5, –2).

[Solution. Refer Q. 12; Ex. 11(a)]

Exercise 11.2

1. Show that the three lines wih direction cosines :

$\left\langle \frac{12}{13}, \frac{-3}{13}, \frac{-4}{13} \right\rangle; \left\langle \frac{4}{13}, \frac{12}{13}, \frac{3}{13} \right\rangle; \left\langle \frac{3}{13}, \frac{-4}{13}, \frac{12}{13} \right\rangle$ are mutually perpendicular.

[Solution. Refer Q. 2; Ex. 11(b)]

2. Show that the line through the points (1, –1, 2), (3, 4, –2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

3. Show that the line through the points (4, 7, 8), (2, 3, 4) is parallel to the line through the points (–1, –2, 1), (1, 2, 5).

[Solutions : (2–3) Refer Q. 16; Ex. 11(b)]

4. Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

Solution : The equation of the line through the point $\vec{a}$ and parallel to vector $\vec{m}$ is $\vec{r} = \vec{a} + \lambda \vec{m}$.

Here $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{m} = 3\hat{i} + 2\hat{j} - 2\hat{k}$.

$\therefore$ The equation of the required line is :

$$\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda \left(3\hat{i} + 2\hat{j} - 2\hat{k}\right) ; \lambda \in \mathbf{R}.$$

5. Find the equation of the line in vector and in cartesian form that passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.

[Solution. Refer Q. 7; Ex. 11(b)]

6. Find the cartesian equation of the line which passes through the point (–2, 4, –5) and parallel to the line given by $\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6}$.

Solution : The given line is :

$$\frac{x+3}{3}=\frac{y-4}{5}=\frac{z+8}{6} \quad ...(1)$$

∴ The direction ratios of the given line are < 3, 5, 6 >

∴ The vector equation of the required line is :

$$\vec{r} = \left(-2\hat{i}+4\hat{j}-5\hat{k}\right)+\lambda\left(3\hat{i}+5\hat{j}+6\hat{k}\right)$$

$$\Rightarrow \left(x\hat{i}+y\hat{j}+z\hat{k}\right)=(3\lambda-2)\hat{i}+(5\lambda+4)\hat{j}+(6\lambda-5)\hat{k}.$$

Comparing, $x = 3\lambda - 2$, $y = 5\lambda + 4$, $z = 6\lambda - 5$.

Eliminating λ,

$$\frac{x+2}{3}=\frac{y-4}{5}=\frac{z+5}{6} \ (=\lambda),$$

which is the reqd. cartesian equation of the line.

7. The cartesian equation of a line is $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$.

Write its vector form.

[Solution. Refer Q. 4(*b*) (*i*); Ex. 11(*b*)]

8. Find the vector and the cartesian equations of the line that passes through the origin and (5, –2, 3).

[Solution. Refer Q. 9(*i*); Ex. 11(*b*)]

9. Find the vector and the cartesian equations of the line that passes through the point (3, –2, –5), (3, –2, 6).

Solution : Let $\vec{a}$ and $\vec{b}$ be the position vectors of the points A (3, – 2, – 5) and B (3, – 2, 6).

Then $\vec{a} = 3\hat{i}-2\hat{j}-5\hat{k}$

and $\vec{b} = 3\hat{i}-2\hat{j}+6\hat{k}$.

∴ $\vec{b}-\vec{a} = 11\hat{k}$.

Let $\vec{r}$ be the position vector of any point on the line. Then the vector equation of the line is :

$$\vec{r} = \vec{a}+\lambda\left(\vec{b}-\vec{a}\right)$$

i.e. $\vec{r} = 3\hat{i}-2\hat{j}-5\hat{k}+\lambda\left(11\hat{k}\right)$

$$\Rightarrow x\hat{i}+y\hat{j}+z\hat{k} = 3\hat{i}-2\hat{j}+(11\lambda-5)\hat{k}.$$

Comparing, $x = 3$, $y = -2$ and $z = 11\lambda - 5$.

Eliminating λ, $\frac{x-3}{0}=\frac{y+2}{0}=\frac{z+5}{11} \ (=\lambda)$,

which is the cartesian from of the equation of the line.

10. Find the angle between the following pairs of lines :

(*i*) $\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$ and

$\vec{r}=7\hat{i}-6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$

(*ii*) $\vec{r}=3\hat{i}+\hat{j}-2\hat{k}+\lambda(\hat{i}-\hat{j}-2\hat{k})$ and

$\vec{r}=2\hat{i}-\hat{j}-56\hat{k}+\mu(3\hat{i}-5\hat{j}-4\hat{k})$.

Solution : (*i*) The given lines are parallel to the vectors :

$$\vec{b_1} = 3\hat{i}+2\hat{j}+6\hat{k}$$

and $\vec{b_2} = \hat{i}+2\hat{j}+2\hat{k}$.

If 'θ' be the angle between the given lines,

then $\cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|}$

$$= \frac{\left(3\hat{i}+2\hat{j}+6\hat{k}\right)\cdot\left(\hat{i}+2\hat{j}+2\hat{k}\right)}{\left|3\hat{i}+2\hat{j}+6\hat{k}\right|\left|\hat{i}+2\hat{j}+2\hat{k}\right|}$$

$$= \frac{(3)(1)+(2)(2)+(6)(2)}{\sqrt{9+4+36}\sqrt{1+4+4}}$$

$$= \frac{3+4+12}{\sqrt{49}\sqrt{9}} = \frac{19}{(7)(3)} = \frac{19}{21}.$$

Hence, $\theta = \cos^{-1}\left(\frac{19}{21}\right)$.

(*ii*) The given lines are parrallel to the vectors :

$$\vec{b_1} = \hat{i}-\hat{j}-2\hat{k}$$

and $\vec{b_2} = 3\hat{i}-5\hat{j}-4\hat{k}$.

If 'θ' be the angle between the given lines,

then $\cos\theta = \frac{\vec{b_1}\cdot\vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|}$

$$= \frac{(\hat{i} - \hat{j} - 2\hat{k}) \cdot (3\hat{i} - 5\hat{j} - 4\hat{k})}{|\hat{i} - \hat{j} - 2\hat{k}||3\hat{i} - 5\hat{j} - 4\hat{k}|}$$

$$= \frac{(1)(3) + (-1)(-5) + (-2)(-4)}{\sqrt{1+1+4}\sqrt{9+25+16}}$$

$$= \frac{3+5+8}{\sqrt{6}\sqrt{50}} = \frac{16}{\sqrt{2}\sqrt{3}.5\sqrt{2}} = \frac{8}{5\sqrt{3}}.$$

Hence, $\theta = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$.

11. Find the angle between the following pair of lines :

(*i*) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(*ii*) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$.

Solution : (*i*) The given lines are parallel to the vectors :

$$\vec{b_1} = 2\hat{i} + 5\hat{j} - 3\hat{k}$$

and $\vec{b_2} = -\hat{i} + 8\hat{j} + 4\hat{k}$.

If 'θ' be the angle between the given lines,

then $\cos\theta = \frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$

$$= \frac{(2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (-\hat{i} + 8\hat{j} + 4\hat{k})}{|2\hat{i} + 5\hat{j} - 3\hat{k}||-\hat{i} + 8\hat{j} + 4\hat{k}|}$$

$$= \frac{(2)(-1) + (5)(8) + (-3)(4)}{\sqrt{4+25+9}\sqrt{1+64+16}}$$

$$= \frac{-2+40-12}{\sqrt{38}\sqrt{81}} = \frac{26}{\sqrt{38}(9)} = \frac{26}{9\sqrt{38}}.$$

Hence, $\theta = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$.

(*ii*) The given lines are parrallel to the vectors :

$$\vec{b_1} = 2\hat{i} + 2\hat{j} + \hat{k}$$

and $\vec{b_2} = 4\hat{i} + \hat{j} + 8\hat{k}$.

If 'θ' be the angle between the given lines,

then $\cos\theta = \frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}$

$$= \frac{(2\hat{i} + 2\hat{j} + \hat{k}) \cdot (4\hat{i} + \hat{j} + 8\hat{k})}{\sqrt{4+4+1}\sqrt{16+1+64}}$$

$$= \frac{(2)(4) + (2)(1) + (1)(8)}{\sqrt{9}\sqrt{81}}$$

$$= \frac{8+2+8}{(3)(9)} = \frac{18}{(3)(9)} = \frac{18}{27} = \frac{2}{3}.$$

Hence, $\theta = \cos^{-1}\left(\frac{2}{3}\right)$.

12. Find the values of 'p' so that the lines :

$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$

are at right angles.

[Solution. Refer Q. 15(*i*); Ex. 11(*b*)]

13. Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

are perpendicular to each other.

[Solution. Refer Q. 14(*i*) ; Ex. 11(*b*)]

14. Find the shortest distance between the lines :

$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and

$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

Solution : The given lines are :

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$$

and $\vec{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

The shortest distance between the lines :

$$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$$

and $\vec{r} = \vec{a_2} + \lambda\vec{b_2}$ is given by :

$$d = \left|\frac{(\vec{a_2} - \vec{a_1}) \cdot (\vec{b_1} \times \vec{b_2})}{|\vec{b_1} \times \vec{b_2}|}\right|.$$

Here $\vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}$,

$\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}$

and $\vec{b_1} = \hat{i} - \hat{j} + \hat{k}$,

$\vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$.

$$\therefore \vec{a_2} - \vec{a_1} = \left(2\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right)$$

$$= \hat{i} - 3\hat{j} - 2\hat{k}$$

and $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$

$$= \hat{i}(-2-1) - \hat{j}(2-2) + \hat{k}(1+2)$$

$$= -3\hat{i} + 3\hat{k}.$$

$\therefore$ d, the shortest distance between the given lines is given by :

$$d = \left| \frac{\left(\hat{i} - 3\hat{j} - 2\hat{k}\right) \cdot \left(-3\hat{i} + 3\hat{k}\right)}{\left|-3\hat{i} + 3\hat{k}\right|} \right|$$

$$= \left| \frac{(1)(-3) + (-3)(0) + (-2)(3)}{\sqrt{9+0+9}} \right|$$

$$= \left| \frac{-3-0-6}{\sqrt{18}} \right| = \left| \frac{-9}{3\sqrt{2}} \right| = \left| \frac{-3}{\sqrt{2}} \right|$$

$$= \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \text{ units.}$$

15. Find the shortest distance between the lines :

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \text{ and } \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}.$$

(H.P.B. 2016, 15, 13, 12, 11)

Solution : The given lines are :

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \quad ...(1)$$

and $$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \quad ....(2)$$

The line (1) passes through the point (– 1, – 1, – 1).

The line (2) passes through the point (3, 5, 7).

Now d, the shortest distance between the given lines is given by :

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}.$$

Here $x_2 - x_1 = 3 - (-1) = 4$, $y_2 - y_1 = 5 - (-1) = 6$, $z_2 - z_1 = 7 - (-1) = 8$;

$a_1 = 7$, $b_1 = -6$, $c_1 = 1$ and $a_2 = 1$, $b_2 = -2$, $c_2 = 1$.

Now, $N = \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$

$$= \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 4(-6+2) - 6(7-1) + 8(-14+6)$$

$$= -16 - 36 - 64 = -116.$$

and

$$D = \sqrt{((-6)(1) - (-2)(1))^2 + ((1)(1) - (1)(7))^2 + ((7)(-2) - (1)(-6))^2}$$

$$= \sqrt{(-6+2)^2 + (1-7)^2 + (-14+6)^2}$$

$$= \sqrt{16+36+64} = \sqrt{116}.$$

$\therefore$ d, the shortest distance $= \dfrac{-116}{\sqrt{116}} = -\sqrt{116}$

$= -2\sqrt{29} = 2\sqrt{29}$ units, in magnitude.

16. Find the shortest distance between the lines whose vector equations are :

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$ and

$\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$. *(H.P.B. 2016)*

Solution : The given lines are :

$$\vec{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(\hat{i} - 3\hat{j} + 2\hat{k}\right)$$

and $$\vec{r} = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) + \mu\left(2\hat{i} + 3\hat{j} + \hat{k}\right).$$

The shortest distance between the lines :

$\vec{r} = \vec{a_1} + \lambda \vec{b_1}$ and $\vec{r} = \vec{a_2} + \lambda \vec{b_2}$ is given by :

$$d = \left| \frac{\left(\vec{a_2} - \vec{a_1}\right) \cdot \left(\vec{b_1} \times \vec{b_2}\right)}{\left|\vec{b_1} \times \vec{b_2}\right|} \right|.$$

Here $\vec{a_1} = \hat{i} + 2\hat{j} + 3\hat{k}$,

$\vec{a_2} = 4\hat{i} + 5\hat{j} + 6\hat{k}$

and $\vec{b_1} = \hat{i} - 3\hat{j} + 2\hat{k}$,

$\vec{b_2} = 2\hat{i} + 3\hat{j} + \hat{k}$.

$\therefore \quad \vec{a_2} - \vec{a_1} = \left(4\hat{i} + 5\hat{j} + 6\hat{k}\right) - \left(\hat{i} + 2\hat{j} + 3\hat{k}\right)$

$= 3\hat{i} + 3\hat{j} + 3\hat{k}$

and $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$

$= \hat{i}(-3-6) - \hat{j}(1-4) + \hat{k}(3+6)$

$= -9\hat{i} + 3\hat{j} + 9\hat{k}$.

$\therefore$ d, the shortest-distance between the given lines is given by :

$$d = \left| \frac{\left(3\hat{i} + 3\hat{j} + 3\hat{k}\right) \cdot \left(-9\hat{i} + 3\hat{j} + 9\hat{k}\right)}{\left|-9\hat{i} + 3\hat{j} + 9\hat{k}\right|} \right|$$

$$= \left| \frac{(3)(-9) + (3)(3) + (3)(9)}{\sqrt{(-9)^2 + 3^2 + 9^2}} \right|$$

$$= \left| \frac{-27 + 9 + 27}{\sqrt{81 + 9 + 81}} \right|$$

$$= \left| \frac{9}{\sqrt{171}} \right| = \frac{9}{3\sqrt{19}} = \frac{3}{\sqrt{19}} \text{ units.}$$

17. Find the shortest distance between the lines whose vector equations are :

$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and

$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$.

[**Solution.** Refer Q. 6(*i*); Ex. 11(*d*)]

Exercise 11.3

1. In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(*a*) $z = 2$ (*b*) $x + y + z = 1$

(*c*) $2x + 3y - z = 5$ (*d*) $5y + 8 = 0$.

Solution : (*i*) The given plane is $z = 2$ ...(1)

Direction-cosines of the normal to the plane are $< 0, 0, 1 >$. And distance of (1) from the origin

$$= \frac{|0-2|}{\sqrt{0+0+1}} = \frac{2}{1} = 2.$$

(*ii*) The given plane is $x + y + z = 1$ ...(1)

Direction-ratios of the normal to the plane are : $< 1, 1, 1 >$

$\therefore$Direction-cosines of the normal to the plane are

$$< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$$

And distance of (1) from the origin

$$= \frac{|0+0+0-1|}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}}.$$

(*iii*) The given plane is $2x + 3y - z = 5$ ...(1)

Direction–ratios of the normal to the plane are $< 2, 3, -1 >$.

$\therefore$ Direction–cosines of the normal to the plane are :

$$< \frac{2}{\sqrt{4+9+1}}, \frac{3}{\sqrt{4+9+1}}, \frac{-1}{\sqrt{4+9+1}} >$$

i.e. $< \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}} >$.

And distance of (1) from the origin

$$= \frac{|0+0-0-5|}{\sqrt{4+9+1}} = \frac{5}{\sqrt{14}}.$$

(*iv*) The given plane is $5y + 8 = 0$ ...(1)

Direction–ratios of the normal to the plane are $< 0, 5, 0 >$ *i.e.* $< 0, 1, 0 >$

$\therefore$ Direction–cosines of the normal to the plane are $< 0, 1, 0 >$.

And distance of (1) from the origin

$$= \frac{|0+8|}{\sqrt{0+25+0}} = \frac{8}{5}.$$

2. Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.

[Solution. Refer Q. 2; Ex. 11(*e*)]

3. Find the Cartesian equation of the following planes :

(*a*) $\vec{r}.(\hat{i}+\hat{j}-\hat{k}) = 2$

(*b*) $\vec{r}.(2\hat{i}+3\hat{j}-4\hat{k}) = 1$

(*c*) $\vec{r}.[(s-2t)\hat{i}+(3-t)\hat{j}+(2s+t)\hat{k}] = 15.$

[Solution. Refer Q. 4(*iii*); Ex. 11(*e*)]

4. In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

(*a*) $2x + 3y + 4z - 12 = 0$ (*b*) $3y + 4z - 6 = 0$

(*c*) $x + y + z = 1$ (*d*) $5y + 8 = 0$.

[Solution : Refer Q. 21; Ex. 11(*e*)]

5. Find the vector and cartesian equations of the planes :

(*a*) that passes through the point (1, 0, –2) and the normal to the plane is $\hat{i} + \hat{j} - \hat{k}$

(*b*) that passes through the point (1, 4, 6) and the normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.

[Solution. Refer Q. 16; Ex. 11(*e*)]

6. Find the equations of the planes that passes through three points :

(*a*) (1, 1, –1), (6, 4, –5), (– 4, –2, 3)

(*b*) (1, 1, 0), (1, 2, 1), (–2, 2, –1).

Solution : (*a*) Any plane through (1, 1, – 1) is :

$a(x-1) + b(y-1) + c(z+1) = 0$ ....(1)

Since the plane passes through the points (6, 4, – 5) and (– 4, – 2, 3),

$\therefore \quad a(6-1) + b(4-1) + c(-5+1) = 0$

and $a(-4-1) + b(-2-1) + c(3+1) = 0$

$\Rightarrow \quad 5a + 3b - 4c = 0$ ....(2)

and $\quad -5a - 3b + 4c = 0$ ....(3)

Solving (2) and (3),

$$\frac{a}{12-12} = \frac{b}{20-20} = \frac{c}{-15+15}$$

$$\Rightarrow \quad \frac{a}{0} = \frac{b}{0} = \frac{c}{0}$$

$\Rightarrow$ *a, b, c* can't be found.

Since the points are collinear,

$\therefore$ an infinite number of planes can be found through the given points.

(*b*) Any plane through (1, 1, 0) is :

$a(x-1) + b(y-1) + cz = 0$ ....(1)

Since the plane passes through the points (1, 2, 1) and (– 2, 2, – 1),

$\therefore a(1-1) + b(2-1) + c(1) = 0$

and $a(-2-1) + b(2-1) + c(-1) = 0$

$\Rightarrow \quad 0.a + b + c = 0$ ....(2)

and $\quad -3a + b - c = 0$ ....(3)

Solving (2) and (3),

$$\frac{a}{-1-1} = \frac{b}{-3+0} = \frac{c}{0+3}$$

$$\Rightarrow \frac{a}{-2} = \frac{b}{-3} = \frac{c}{3} = k \text{ (say), where } k \neq 0$$

$\therefore a = -2k,\ b = -3k,\ c = 3k.$

Putting these values of *a, b, c* in (1), we get :

$-2k(x-1) - 3k(y-1) + 3kz = 0$

$\Rightarrow \quad -2(x-1) - 3(y-1) + 3z = 0 \quad [\because k \neq 0]$

$\Rightarrow \quad -2x + 2 - 3y + 3 + 3z = 0$

$\Rightarrow \quad 2x + 3y - 3z = 5,$

which is the reqd. equation.

7. Find the intercepts cut off by the plane $2x + y - z = 5$.

Solution : The given equation of the plane is :

$$2x + y - z = 5$$

$$\Rightarrow \quad \frac{x}{5/2} + \frac{y}{5} + \frac{z}{-5} = 1.$$

Comparing with $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$

the intercepts on the axes are $\frac{5}{2}, 5$ and -5.

8. Find the equation of the plane with intercept 3 on the *y*-axis and parallel to ZOX plane.

[Solution. Refer Q. 1(*iii*); Ex. 11(*e*)]

9. Find the equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ and the point (2, 2, 1).

(Meghalaya B. 2017)

Solution : The given planes are :

$3x - y + 2z - 4 = 0$ ...(1)

and $\quad x + y + z - 2 = 0$ ...(2)

Any plane through the intersection of (1) and (2) is :

$(3x - y + 2z - 4) + k\ (x + y + z - 2) = 0$ ...(3)

Since it passes thro' (2, 2, 1).

$\therefore\ (6 - 2 + 2 - 4) + k\ (2 + 2 + 1 - 2) = 0$

$\Rightarrow \quad 2 + 3k = 0 \Rightarrow k = -\dfrac{2}{3}.$

Putting in (1),

$(3x - y + 2z - 4) - \dfrac{2}{3}\ (x + y + z - 2) = 0$

$\Rightarrow 9x - 3y + 6z - 12 - 2x - 2y - 2z + 4 = 0$

$\Rightarrow \quad 7x - 5y + 4z - 8 = 0,$

which is the reqd. equation.

10. Find the vector equation of the plane passing through the intersection of the planes

$\vec{r}.(2\hat{i} + 2\hat{j} - 3\hat{k}) = 7, \vec{r}.(2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$

and through the point (2, 1, 3).

Solution : The given planes are :

$$\vec{r}.\left(2\hat{i} + 2\hat{j} - 3\hat{k}\right) = 7 \quad ....(1)$$

and $$\vec{r}.\left(2\hat{i} + 5\hat{j} + 3\hat{k}\right) = 9 \quad ....(2)$$

i.e. $\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(2\hat{i} + 2\hat{j} - 3\hat{k}\right) = 7$

and $\left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(2\hat{i} + 5\hat{j} + 3\hat{k}\right) = 9$

i.e. $2x + 2y - 3z - 7 = 0$ ...(1)′

and $2x + 5y + 3z - 9 = 0$ ...(2)′

Any plane through the intesection of (1)′ and (2)′ is :

$(2x + 2y - 3z - 7) + k\ (2x + 5y + 3z - 9) = 9$ ...(3)

Since it passes thro' (2, 1, 3),

$\therefore (4 + 2 - 9 - 7) + k\ (4 + 5 + 9 - 9) = 0$

$\Rightarrow \quad -10 + k\ (9) = 0 \Rightarrow k = \dfrac{10}{9}.$

Putting in (3),

$(2x + 2y - 3z - 7) + \dfrac{10}{9}\ (2x + 5y + 3z - 9) = 0$

$\Rightarrow 18x + 18y - 27z - 63 + 20x + 50y + 30z - 90 = 0$

$\Rightarrow 38x + 68y + 3z - 153 = 0$

$\Rightarrow \left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(38\hat{i} + 68\hat{j} + 3\hat{k}\right) = 153$

$\Rightarrow \vec{r}.\left(38\hat{i} + 68\hat{j} + 3\hat{k}\right) = 153,$

which is the reqd. equation.

11. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$.

(Meghalaya B. 2017)

Solution : Any plane through the line of intersection of the planes :

$x + y + z - 1 = 0$ and $2x + 3y + 4z - 5 = 0$ is :

$(x + y + z - 1) + k\ (2x + 3y + 4z - 5) = 0$

i.e. $(1 + 2k)\ x + (1 + 3k)\ y + (1 + 4k)\ z - (1 + 5k) = 0$ ...(1)

Since it is perpendicular to the plane :

$x - y + z = 0$ ...(2)

$\therefore$ their normals are perpendicular

$\Rightarrow (1 + 2k)\ (1) + (1 + 3k)\ (-1) + (1 + 4k)\ (1) = 0$

$\Rightarrow 1 + 2k - 1 - 3k + 1 + 4k = 0$

$\Rightarrow \quad 3k = -1 \Rightarrow k = -\dfrac{1}{3}.$

Putting in (1), we get :

$\left(1 - \dfrac{2}{3}\right) x + (1 - 1)\ y + \left(1 - \dfrac{4}{3}\right) z - \left(1 - \dfrac{5}{3}\right) = 0$

$\Rightarrow \quad \dfrac{1}{3}x - \dfrac{1}{3}z + \dfrac{2}{3} = 0$

$\Rightarrow \quad x - z + 2 = 0,$

which is the reqd. equation.

12. Find the angle between the planes whose vector equations are $\vec{r}.(2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$ and $\vec{r}.(3\hat{i} - 3\hat{j} + 5\hat{k}) = 3.$

[Solution. Refer Q. 11(*ii*); Ex. 11(*e*)]

13. In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them :

(*a*) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$
(*b*) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$
(*c*) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$
(*d*) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$
(*e*) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$.

[Solution. Refer Q. 66; Ex. 11(*e*)]

14. In the following cases, find the distance of each of the given points from the corresponding given plane.

Point	**Plane**
(*a*) (0, 0, 0)	$3x - 4y + 12z = 3$
(*b*) (3, –2, 1)	$2x - y + 2z + 3 = 0$
(*c*) (2, 3, –5)	$x + 2y - 2z = 9$
(*d*) (–6, 0, 0)	$2x - 3y + 6z - 2 = 0.$

[Solution. Refer Q. 18; Ex. 11(*e*)]

Miscellaneous Exercise on Chapter 11

1. Show that the line joining the origin to the point (2, 1, 1) is perpendicular to the line determined by the points (3, 5, –1), (4, 3, –1).

[Solution. Refer Q. 27; Ex. 11(*b*)]

2. If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both the these are : $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$.

Solution : Let $< l, m, n >$ be the direction-cosines of the line, which is perpendicular to two lines whose direction-cosines are :

$< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$.

$\therefore \quad ll_1 + mm_1 + nn_1 = 0 \quad$ ...(1)

and $\quad ll_2 + mm_2 + nn_2 = 0 \quad$ ...(2)

Solving (1) and (2), by cross-multiplication, we have :

$$\frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

$$= \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_1l_1)^2+(l_1m_2-l_2m_1)^2}}$$

$$= \frac{1}{\sin\theta} = \frac{1}{\sin 90°} = \frac{1}{1} = 1.$$

Hence, $\quad l = m_1n_2 - m_2n_1,$

$m = n_1l_2 - n_2l_1$

and $\quad n = l_1m_2 - l_2m_1.$

3. Find the angle between the lines whose direction ratios are *a, b, c* and $b - c, c - a, a - b$.

[Solution. Refer Q. 11(*ii*); Ex. 11(*b*)]

4. Find the equation of a line parallel to *x*-axis and passing through the origin.

[Solution. Refer Q. 5(*a*); Ex. 11(*b*)]

5. If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (– 4, 3, – 6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD.

Solution : Direction-ratios of AB are :

$< 4 - 1, 5 - 2, 7 - 3 >$

i.e. $< 3, 3, 4 >$ ...(1)

Direction-ratios of CD are :

$< 2 - (- 4), 9 - 3, 2 - (- 6) >$

i.e. $< 6, 6, 8 >$ ...(2)

From (1) and (2), Direction-ratios of AB and CD are proportional

$\Rightarrow$ the lines AB and CD are parallel.

Hence, the angle between AB and CD is 0°.

6. If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$

and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of '*k*'.

Solution : The given lines are :

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{y-3}{2} \quad ...(1)$$

and
$$\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5} \quad ...(2)$$

The direction-ratios of line (1) are $< - 3, 2k, 2 >$.

The direction-ratios of line (2) are $< 3k, 1, - 5 >$.

The lines (1) and (2) are perpendicular

if $(- 3)(3k) + (2k)(1) + (2)(- 5) = 0$

$[a_1a_2 + b_1b_2 + c_1c_2 = 0]$

if $\quad - 9k + 2k - 10 = 0$ if $7k = - 10$

if $\quad k = -\frac{10}{7}$.

7. Find the vector equation of the line passing through (1, 2, 3) and perpendicular to the plane

$\vec{r}.(\hat{i}+2\hat{j}-5\hat{k})+9=0.$

[Solution. Refer Q. 23(*ii*); Ex. 11(*e*)]

8. Find the equation of the plane passing through (*a, b, c*)

and parallel to the plane $\vec{r}.(\hat{i}+\hat{j}+\hat{k})=2.$

Solution : Any plane parallel to :

$$\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = 2$$

is $\quad \vec{r}.(\hat{i}+\hat{j}+\hat{k}) = k \quad$ ...(1)

This passes through the point (*a, b, c*) *i.e.* $(a\hat{i}+b\hat{j}+c\hat{k})$.

$\therefore (a\hat{i}+b\hat{j}+c\hat{k}).(\hat{i}+\hat{j}+\hat{k}) = k$

$\Rightarrow (a)(1) + (b)(1) + (c)(1) = k$

$\Rightarrow \quad k = a + b + c.$

Putting in (1), the required equation of the plane is :

$$\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = a + b + c$$

i.e. $(x\hat{i}+y\hat{j}+z\hat{k}).(\hat{i}+\hat{j}+\hat{k}) = a + b + c$

i.e. $\quad x + y + z = a + b + c.$

9. Find the shortest distance between lines :

$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$ and

$\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$.

[Solution. Refer Q. 3(*i*); Ex. 11(*d*)]

10. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the YZ-plane.

Solution : The equations of line through (5, 1, 6) and (3, 4, 1) are :

$$\frac{x-5}{3-5} = \frac{y-1}{4-1} = \frac{z-6}{1-6}$$

$$\Rightarrow \quad \frac{x-5}{-2} = \frac{y-1}{3} = \frac{z-6}{-5} \quad ...(1)$$

Any point on (1) is $(5 - 2k, 1 + 3k, 6 - 5k)$ ...(2)

This lies on YZ-plane ($x = 0$).

$\therefore \quad 5 - 2k = 0$

$\Rightarrow \quad k = \frac{5}{2}$.

Putting in (2), $\left(5-5, 1+\frac{15}{2}, 6-\frac{25}{2}\right)$

i.e. $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$, which is the reqd. point.

11. Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the ZX-plane.

Solution : As in Q. 10, any point on (1) is :

$(5, - 2k, 1 + 3k, 6 - 5k)$ ...(1)

This lies on ZX plane ($y = 0$)

$\therefore \quad 1 + 3k = 0$

$\Rightarrow \quad k = -\frac{1}{3}$.

Putting in (1), $\left(5+\frac{2}{3}, 1-1, 6+\frac{5}{3}\right)$

i.e. $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$, which is the reqd. point.

12. Find the coordinates of the point where the line through (3, – 4, – 5) and (2, – 3, 1) crosses the plane $2x + y + z = 7$.

[Solution. Refer Q. 40 ; Ex. 11(*e*)]

13. Find the equation of the plane passing through the point (– 1, 3, 2) and perpendicular to each of the planes : $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

Solution : Any plane through (– 1, 3, 2) is :

$a(x + 1) + b(y - 3) + c(z - 2) = 0$ ...(1)

Since plane (1) is perpendicular to the planes :

$x + 2y + 3z = 5$ ...(2)

and $3x + 3y + z = 0$ ...(3)

$\therefore (a)(1) + b(2) + c(3) = 0$

and $a(3) + b(3) + c(1) = 0$

$\Rightarrow \quad a + 2b + 3c = 0$ ...(4)

and $3a + 3b + c = 0$ ...(5)

Solving (4) and (5),

$$\frac{a}{2-9} = \frac{b}{9-1} = \frac{c}{3-6}$$

$$\Rightarrow \quad \frac{a}{-7} = \frac{b}{8} = \frac{c}{-3}.$$

Putting in (1),

$-7(x + 1) + 8(y - 3) - 3(z - 2) = 0$

$\Rightarrow \quad -7x - 7 + 8y - 24 - 3z + 6 = 0$

$\Rightarrow \quad 7x - 8y + 3z + 25 = 0,$

which is the reqd. equation.

14. If the points (1, 1, *p*) and (– 3, 0, 1) be equidistant from the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$, then find the value of '*p*'.

[Solution. Refer Q. 20; Ex. 11(*e*)]

15. Find the equation of the plane passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to *x*-axis.

[Solution. Refer Q. 45(*i*); Ex. 11(*e*)]

16. If O be the origin and the coordinates of P be (1, 2, –3), then find the equation of the plane passing through P and perpendicular to OP.

[Solution. Refer Q. 17; Rev. Ex.]

17. Find the equation of the plane which contains the line of intersection of the planes :

$$\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0, \vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$$

and which is perpendicular to the plane :

$$\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0.$$

Solution. The given planes (in cartesian form) are :

$x + 2y + 3z - 4 = 0$ ...(1)

$2x + y - z + 5 = 0$ ...(2)

and $5x + 3y - 6z + 8 = 0$ ...(3)

Any plane thro' the intersection of (1) and (2) is :

$(x + 2y + 3z - 4) + k(2x + y + z + 5) = 0$

$\Rightarrow (1 + 2k)x + (2 + k)y + (3 + k)z + (-4 + 5k) = 0$ ...(4)

Now plane (4) is perp. to plane (3).

$\therefore 5(1 + 2k) + 3(2 + k) - 6(3 + k) = 0$

$\Rightarrow 5 + 10k + 6 + 3k - 18 - 6k = 0$

$\Rightarrow \quad 7k = 7 \Rightarrow k = 1.$

Putting in (4), $3x + 3y + 4z + 1 = 0$,

which is the reqd. equation of the plane.

18. Find the distance of the point (–1, –5, –10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$.

[Solution. Refer Q. 9(*i*); Ex. 11(*f*)]

19. Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes $\vec{r}.(\hat{i} - \hat{j} + 2\hat{k}) = 5$ and $\vec{r}.(3\hat{i} + \hat{j} + \hat{k}) = 6$.

[Solution. Refer Q. 11; Ex. 11(*f*)]

20. Find the vector equation of the line passing through the point (1, 2, –4) and perpendicular to the two lines :

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \text{ and } \frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}.$$

[Solution. Refer Q. 28(*ii*); Ex. 11(*b*)]

21. Prove that if a plane has the intercepts *a, b, c* and is at a distance of *p* units from the origin, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$.

Solution : The equation of the plane is :

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

i.e. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$ ...(1)

By the question , $\frac{10+0+0-1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = p$

$$\Rightarrow \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} = p^2 \Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2},$$

which is true.

Choose the correct answer in Exercises 22 and 23.

22. Distance between the two planes $2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$ is :

(A) 2 units (B) 4 units

(C) 8 units (D) $\frac{2}{\sqrt{29}}$ units. **[Ans.** (D)]

23. The planes : $2x - y + 4z = 5$ and $5x - 2{\cdot}5y + 10z = 6$ are :

(A) Perpendicular (B) Parallel

(C) Intersect *y*-axis (D) Passes through $\left(0, 0, \frac{5}{4}\right)$.

[Ans. (B).]

Questions From NCERT Exemplar

Example 1. The x-coordinate of a point on the line joining the points Q (2, 2, 1) and R(5, 1, –2) is 4. Find its z-coordinate.

Solution. Let P divide the line segment [QR] in the ratio $\lambda : 1$.

$\therefore$ Co-ordinates of P are $\left(\frac{5\lambda+2}{\lambda+1}, \frac{\lambda+2}{\lambda+1}, \frac{-2\lambda+1}{\lambda+1}\right)$.

By the question, $\frac{5\lambda+2}{\lambda+1} = 4 \Rightarrow 5\lambda + 2 = 4\lambda + 4$

$\Rightarrow \lambda = 2$.

Hence, the *z*-co-ordinate of P is $\frac{-2\lambda+1}{\lambda+1}$

i.e. $\frac{-4+1}{2+1}$ *i.e.* – 1.

Example 2. Find the distance of the point (– 1, – 5, – 10) from the point of intersection of the line : $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$.

Solution. We have :

$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$.

Solving, we get :

$$[(2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})].(\hat{i} - \hat{j} + \hat{k}) = 5$$

$\Rightarrow (2 + 3\lambda)(1) + (-1 + 4\lambda)(-1) + (2 + 2\lambda)(1) = 5$

$\Rightarrow 2 + 3\lambda + 1 - 4\lambda + 2 + 2\lambda = 5 \Rightarrow \lambda = 0$.

$\therefore$ Point of intersection of the line and plane is (2, –1, 2). The other given point is (–1, –5, –10).

Hence, the reqd. distance

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$$

$$= \sqrt{9+16+144} = \sqrt{169} = 13 \text{ units.}$$

Exercise

1. Find the equation of a plane which is at a distance $3\sqrt{3}$ units from origin and the normal to which is equally inclined to co-ordinate axes.

2. Find the equations of two lines through the origin which intersect the line $\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.

3. The plane $ax + by = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle α. Prove that the equation of the plane in its new position is :

$$ax + by \pm (\sqrt{a^2 + b^2} \tan\alpha) z = 0.$$

Answers

1. $x + y + z = 9$.

2. $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$.

Revision Exercise

1. A variable line in two adjacent positions has direction-cosines $< l, m, n >$ and $< l + \delta l, m + \delta m, n + \delta n >$. Show that the small angle $\delta\theta$ between two positions is given by :

$$(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2.$$

Solution. Since $< l, m, n >$ and $< l + \delta l, m + \delta m, n + \delta n >$ are direction-cosines,

$$\therefore \quad l^2 + m^2 + n^2 = 1 \quad ...(1)$$

and $(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$

$\Rightarrow (l^2 + m^2 + n^2) + ((\delta l)^2 + (\delta m)^2 + (\delta n)^2) + 2(l(\delta l) + m(\delta m) + n(\delta n)) = 1$

$\Rightarrow (\delta l)^2 + (\delta m)^2 + (\delta n)^2 = -2(l(\delta l) + m(\delta m) + n(\delta n))$ ...(2)

[*Using (1)*]

Now $\cos \delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$

$\Rightarrow 1 - 2\sin^2\frac{\delta\theta}{2} = l^2 + m^2 + n^2 + (l(\delta l) + m(\delta m) + n(\delta n))$

$\Rightarrow \quad -2\sin^2\frac{\delta\theta}{2} = l(\delta l) + m(\delta m) + n(\delta n)$

[*Using (1)*]

$\Rightarrow \quad -2\left(\frac{\delta\theta}{2}\right)^2 = l(\delta l) + m(\delta m) + n(\delta n)$

$\left[\because \delta\theta \text{ is small} \Rightarrow \frac{\delta\theta}{2} \text{ is small} \Rightarrow \sin\frac{\delta\theta}{2} = \frac{\delta\theta}{2}\right]$

$\Rightarrow \quad (\delta\theta)^2 = -2((l(\delta l) + n(\delta m) + n(\delta n))$

$\Rightarrow \quad (\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2,$

[*Using (2)*]

which is true.

2. Prove that the lines, whose direction-cosines are given by al + bm + cn = 0, fmn + gnl + hlm = 0 are :

(i) perpendicular if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

(ii) parallel if $a^2f^2 + b^2g^2 + c^2h^2 - 2(bcgh + cahf + abfg) = 0$.

Solution. We have : $al + bm + cn = 0$ ...(1)

and $fmn + gnl + hlm = 0$ ...(2)

From (1), $n = -\frac{al + bm}{c}$ ...(3)

Putting in (2),

$$(fm + gl)\left(-\frac{al + bm}{c}\right) + hlm = 0$$

$\Rightarrow l^2 ag + (af + bg - ch) lm + bfm^2 = 0$ ...(4)

Clearly both l and m can't be zero.

[$\because$ *If $l = m = 0$, then from (3), $n = 0$, then $l^2 + m^2 + n^2 = 1$ is not true*]

So let $m \neq 0$. [*Of-course, we can also take $l \neq 0$*]

Dividing (4) by m^2, $(ag)\frac{l^2}{m^2} + (af + bg - ch)\frac{l}{m} + (bf) = 0$ ...(5),

which is quadratic in $\frac{l}{m}$, giving two values of $\frac{l}{m}$.

If $< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$ be the direction-cosines of the two lines,

then $\left(\frac{l_1}{m_1}\right)\left(\frac{l_2}{m_2}\right) = \frac{bf}{ag} \Rightarrow \frac{l_1l_2}{f/a} = \frac{m_1m_2}{g/b}$ ...(6)

Similarly, $\frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c}$ ...(7)

Combining (6) and (7), $\frac{l_1l_2}{f/a} = \frac{m_1m_2}{g/b} = \frac{n_1n_2}{h/c} = k$, where $k \neq 0$

$\therefore l_1l_2 = \frac{f}{a}k,\ m_1m_2 = \frac{g}{b}k,\ n_1n_2 = \frac{h}{c}k.$

(*i*) The two lines are perpendicular if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

i.e. if $\frac{f}{a}k + \frac{g}{b}k + \frac{h}{c}k = 0$ *i.e.* if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$,

[$\because k \neq 0$]

which is true.

(*ii*) The two lines are parallel if (5) has equal roots

i.e. if $(af + bg - ch)^2 - 4\,(ag)\,(bf) = 0$

i.e. if $a^2f^2 + b^2g^2 + c^2h^2 + 2afbg - 2bgch - 2chaf - 4afbg = 0$

i.e. if $a^2f^2 + b^2g^2 + c^2h^2 - 2\,(bcgh + cahf + abfg) = 0$, which is true.

3. Show that the line joining the mid-points of the two sides of the triangle is parallel to the third side and is half of its length.

Solution. Let A (x_1, y_1, z_1), B (x_2, y_2, z_2) and C (x_3, y_3, z_3) be the vertices of ΔABC. If D, E are mid-points of [AB] and [AC] respectively, then their co-ordinates are :

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

and $\left(\frac{x_1+x_3}{2}, \frac{y_1+y_3}{2}, \frac{z_1+z_3}{2}\right)$ respectively.

Now direction-ratios of BC are :

$< x_3 - x_2, y_3 - y_2, z_3 - z_2 >$

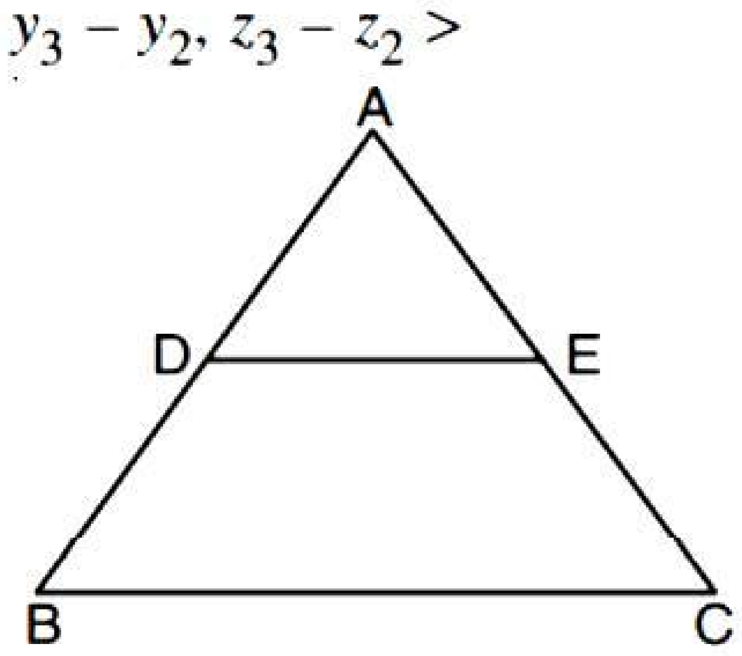


Fig.

and direction-ratios of DE are :

$$< \frac{x_1+x_3}{2} - \frac{x_1+x_2}{2}, \frac{y_1+y_3}{2} - \frac{y_1+y_2}{2}, \frac{z_1+z_3}{2} - \frac{z_1+z_2}{2} >$$

i.e. $< \frac{x_3-x_2}{2}, \frac{y_3-y_2}{2}, \frac{z_3-z_2}{2} >$

i.e. $< x_3 - x_2, y_3 - y_2, z_3 - z_2 >$.

Thus DE || BC.

Now

$$|\,DE\,| = \sqrt{\left(\frac{x_1+x_3}{2}-\frac{x_1+x_2}{2}\right)^2+\left(\frac{y_1+y_3}{2}-\frac{y_1+y_2}{2}\right)^2+\left(\frac{z_1+z_3}{2}-\frac{z_1+z_2}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{(x_3-x_2)^2+(y_3-y_2)^2+(z_3-z_2)^2}$$

$$= \frac{1}{2}\,|\,BC\,|.$$

Hence, DE || BC and $|\,DE\,| = \frac{1}{2}\,|\,BC\,|$.

4. Verify that $< \frac{l_1+l_2+l_3}{\sqrt{3}}, \frac{m_1+m_2+m_3}{\sqrt{3}}, \frac{n_1+n_2+n_3}{\sqrt{3}} >$ can be taken as the direction-cosines of a line L equally inclined to three mutually perpendicular lines with direction-cosines :

$$< l_1, m_1, n_1 > ; < l_2, m_2, n_2 > ; < l_3, m_3, n_3 >.$$

5. Find the vector equation of the straight line passing through (1, 2, 3) and perpendicular to the plane :

$$\vec{r}.(\hat{i}+2\hat{j}-5\hat{k})+9=0.$$

6. Show that the lines :

$x = ay + b,\ z = cy + d$ and $x = a'y + b',\ z = c'y + d'$ are perpendicular to each other if $aa' + cc' + 1 = 0$.

7. Prove that the line joining the points $\vec{6a} - \vec{4b} + \vec{4c}$ and $-\vec{4c}$ and the line joining the points $-\vec{a} - \vec{2b} - \vec{3c}, \vec{a} + \vec{2b} - \vec{5c}$ intersect at $-\vec{4c}$.

8. Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes :

$$\vec{r}.(\hat{i}-\hat{j}+2\hat{k})=5 \text{ and } \vec{r}.(3\hat{i}+\hat{j}+\hat{k})=6.$$

9. Find the vector equation of the line passing through the point (1, 2, – 4) and perpendicular to the two lines :

$$\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$$

and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. *(N.C.E.R.T.)*

10. Find the co-ordinates of the point, where the line through (3, – 4, – 5) and (2, – 3, 1) crosses the plane $2x + y + z = 7$.

11. Show that the equation of the plane passing through a point having position vector $\vec{a}$ and parallel to $\vec{b}$ and $\vec{c}$ is $\vec{r} = \vec{a} + \lambda\vec{b} + \mu\vec{c}$.

12. Find the distance of the point (2, 3, 4) from the plane 3x + 2y + 2z + 5 = 0, measured parallel to the line $\frac{x+3}{3}=\frac{y-2}{6}=\frac{z}{2}$.

Solution. The line through A (2, 3, 4) and parallel to $\frac{x+3}{3}=\frac{y-2}{6}=\frac{z}{2}$ is :

$$\frac{x-2}{3}=\frac{y-3}{6}=\frac{z-4}{2} \quad ...(1)$$

Any point on (1) is : $(2 + 3k, 3 + 6k, 4 + 2k)$, where k is arbitrary real number.

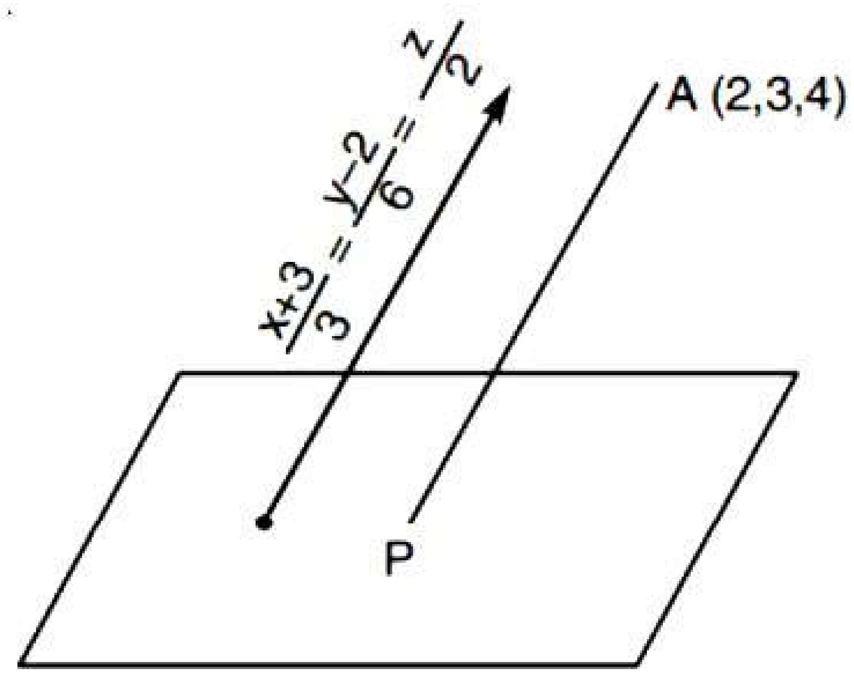


Fig.

This point is P *i.e.* it lies on the plane

$$3x + 2y + 2z + 5 = 0 \quad ...(2)$$

iff $3(2 + 3k) + 2(3 + 6k) + 2(4 + 2k) + 5 = 0$

iff $25k = -25$ iff $k = -1$.

$\therefore$ P is $(2 - 3, 3 - 6, 4 - 2)$ *i.e.* $(-1, -3, 2)$.

Hence, the reqd. distance = $|AP|$

$$= \sqrt{(2+1)^2 + (3+3)^2 + (4-2)^2}$$

$$= \sqrt{9 + 36 + 4} = \sqrt{49} = 7 \text{ units.}$$

13. Find the distance of the point with position vector $-\hat{i} - 5\hat{j} - 10\hat{k}$ from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 12\hat{k})$ with the plane $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$.

14. Find the point R, where the line joining P (1, 3, 4) and Q (– 3, 5, 2) cuts the plane $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$. Is $|\overleftrightarrow{PR}| = |\overleftrightarrow{QR}|$?

15. Find the equation of the plane passing through the line of intersection of the planes $2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$ and parallel to the line :

$$\frac{x-1}{2}=\frac{y-3}{4}=\frac{z-5}{5}.$$

16. If from a point P (a, b, c) perpendiculars PA and PB are drawn to yz and zx-planes, find the vector equation of the plane OAB.

17. If O be the origin and the co-ordinates of P be (1, 2, – 3), then find the equation of the plane passing through P and perpendicular to OP. ***(N.C.E.R.T.)***

18. Find the equation of the plane, which contain, the line of intersection of the planes :

$$\vec{r}.(\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0 \text{ and } \vec{r}.(2\hat{i} + \hat{j} + \hat{k}) + 5 = 0$$

and which is perpendicular to the plane :

$$\vec{r}.(5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0.$$ ***(N.C.E.R.T.)***

19. Prove that the S.D. between a diagonal of a rectangular parallelopiped and its edges not meeting it are :

$$\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}},$$

where a, b, c are lengths of the edges.

20. A variable plane is at a constant distance 'p' from the origin and meets the co-ordinate axes in A, B and C. Show that the locus of the centroid of the tetrahedra OABC is :

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}.$$

(W. Bengal B. 2017)

Answers

5. $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} + 2\hat{j} - 5\hat{k})$.

8. $\vec{r} = (\hat{i} + \hat{j} + 3\hat{k}) + \lambda(-3\hat{i} + 5\hat{j} + 4\hat{k})$.

9. $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$.

10. (1, – 2, 7).

13. 13 units.

14. (– 1, 4, 3) ; Yes.

15. $7x + 9y - 10z - 27 = 0$.

16. $\vec{r}.\left(\frac{\hat{i}}{a}+\frac{\hat{j}}{b}-\frac{\hat{k}}{c}\right)=0$.

17. $x + 2y - 3z - 14 = 0$.

18. $\vec{r}.(3\hat{i} + 3\hat{j} + 4\hat{k}) + 1 = 0$.

CHECK YOUR UNDERSTANDING

1. What is the equation of the xy-plane ?

   [**Ans.** $z = 0$] *(Assam B. 2016)*

2. If a line makes an angle α, β, γ with x-axis, y-axis and z-axis, then : $\sin^2\alpha + \sin^2\beta + \sin^2\gamma =$ ....

   [**Ans.** 2]

3. Write the direction-cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

   [**Ans.** $< \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} >$]

4. Find the vector equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{6-z}{2}$.

   [**Ans.** $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} - 2\hat{k})$]

5. Find the equation of a st. line through $(-1, 2, 3)$ and equally inclined to the axes.

   [**Ans.** $x + 1 = y - 2 = z - 3$]

6. The distance of the point $(2, 3, -5)$ from the plane $x + 2y - 2z - 9 = 0$ is .... *(Jammu B. 2016)*

   [**Ans.** 3.]

7. Find the distance of the plane $2x - 3y + 4z - 6 = 0$ from the origin.

   [**Ans.** $\frac{6\sqrt{29}}{29}$ units]

8. Find the intercepts cut off by the plane $2x + y - z = 5$ with the axes.

   [**Ans.** $\frac{5}{2}$, 5 and -5]

9. Find the equation of the plane with intercept 3 as the y-axis and parallel to ZOX plane.

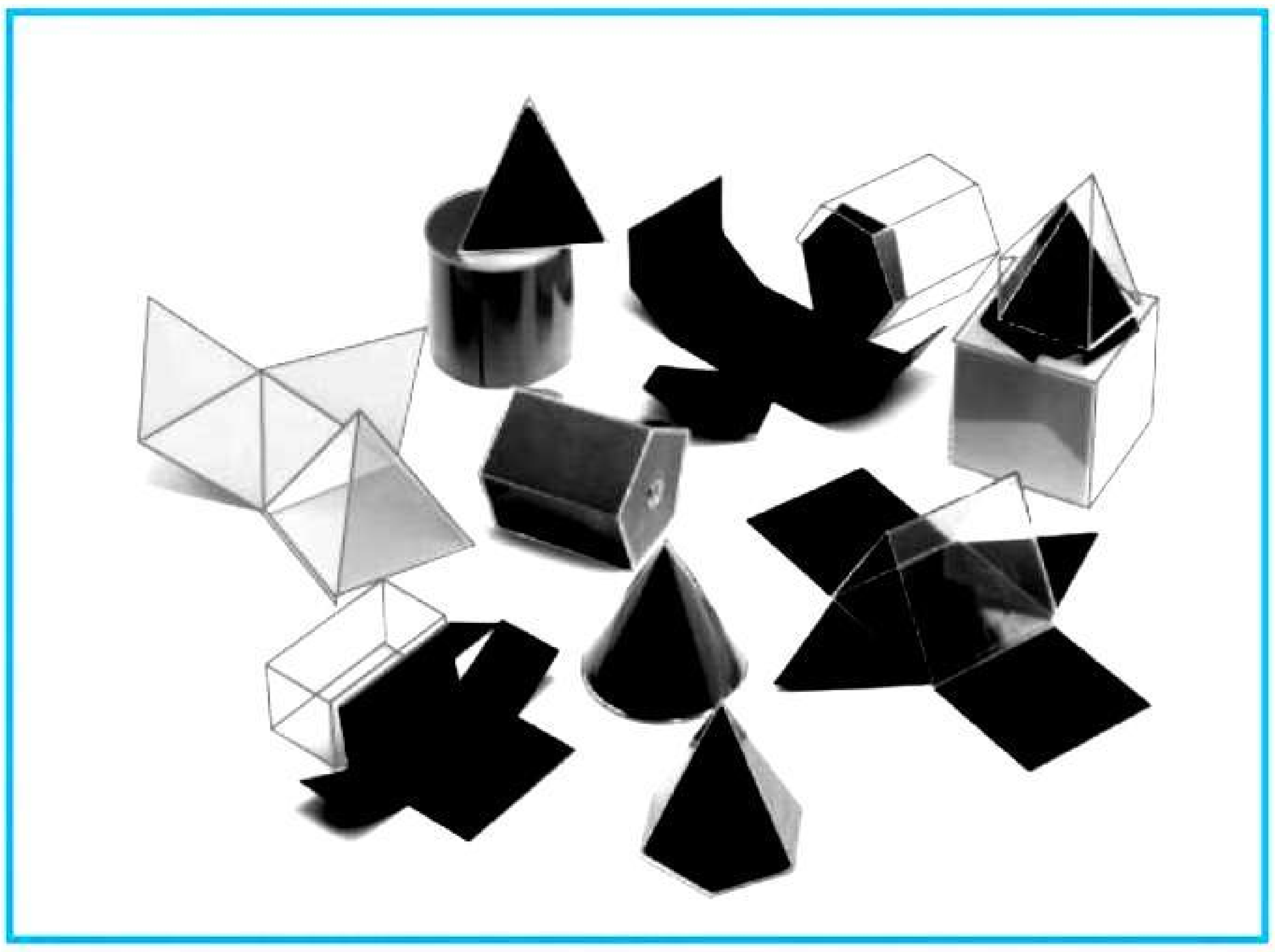
   [**Ans.** $y = 3$]

10. What is the point of intersection of the line $x = y = z$ with the plane $x + 2y + 3z = 6$?

    [**Ans.** $(1, 1, 1)$]

SUMMARY

THREE-DIMENSIONAL GEOMETRY



DEFINITIONS AND IMPORTANT RESULTS

1. IMPORTANT RESULT

$\mathbf{l^2 + m^2 + n^2 = 1}$, where $< l, m, n >$ are direction-cosines of a st. line.

2. DIRECTION-RATIOS

The direction-ratios of the line joining of the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are :

$$< x_2 - x_1, y_2 - y_1, z_2 - z_1 >.$$

3. (a) Angle between two lines. The angle between two lines having direction-cosines

$< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$ is given by :

$$\cos\theta = | l_1 l_2 + m_1 m_2 + n_1 n_2 |.$$

(b) The lines are :

(i) perpendicular iff $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.

(ii) parallel iff $l_1 = l_2$, $m_1 = m_2$, $n_1 = n_2$.

4. SHORTEST DISTANCE

The shortest distance between two lines :

$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is :

$$\left| \frac{(\vec{b_1} \times \vec{b_2}) . (\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|.$$

5. EQUATIONS OF PLANES

(*i*) Equation of a plane, which is at a distance '*p*' from the origin and perpendicular to the unit vector $\hat{n}$ is $\vec{r}\,.\,\hat{n} = p$.

(*ii*) **General Form.** The general equation of first degree *i.e.* $ax + by + cz + d = 0$ represents a plane.

(*iii*) **One-point Form.** The equation of a plane through (x_1, y_1, z_1) and having $< a, b, c >$ as direction–ratios of the normal is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

(*iv*) **Three–point Form.** The equation of the plane through (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is :

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \\ x - x_3 & y - y_3 & z - z_3 \end{vmatrix} = 0.$$

(*v*) **Intercept Form.** Equation of the plane, which cuts off intercepts *a, b, c* on the axes, is :

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

6. ANGLE BETWEEN TWO PLANES

The angle between the planes :

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is

given by : $\cos\theta = \dfrac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

7. BISECTING PLANES

The equations of the planes bisecting the planes :

$a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

are : $\dfrac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \dfrac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$.

MULTIPLE CHOICE QUESTIONS

For Board Examinations

1. Distance between plane $3x + 4y - 20 = 0$ and point $(0, 0, -7)$ is :

(A) 4 units (B) 3 units

(C) 2 units (D) 1 unit. ***(P. B. 2018)***

2. If a line makes an angle of $\frac{\pi}{4}$ with each of *y*-and *z*-axis, then the angle which it makes with *x*-axis is :

(A) $\frac{3\pi}{2}$ (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{2}$ (D) $\frac{3\pi}{2}$. ***(H.P. B. 2018)***

3. If a line makes angles α, β, γ with positive directions of co-ordinate axes, then value of :

$\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is :

(A) -1 (B) 2

(C) 1 (D) -2. ***(H.P. B. 2018)***

4. If a line makes angles $\frac{\pi}{2}, \frac{3\pi}{4}$ and $\frac{\pi}{4}$ with *x, y, z*-axis respectively, then direction cosines of this line are :

(A) $\pm(1, 1, 1)$ (B) $\pm\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(C) $\pm\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ (D) $\pm\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

(H.P. B. 2018)

5. If direction-cosines of two lines are proportional to 4, 3, 2 and 1, –2, 1, then the angle between the lines is :

(A) 90° (B) 60°

(C) 45° (D) None of these.

(H.B. 2018)

6. The direction-cosines of a line equally inclined to the co-ordinate axes are :

(A) <1, 1, 1> (B) $<\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}>$

(C) $<\pm\frac{1}{3}, \pm\frac{1}{3}, \pm\frac{1}{3}>$ (D) $<\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}>$.

(H.B. 2018)

7. The line $\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-3}{4}$ meets the plane :

$2x + 3y - z = 14$ in the point :

(A) (3, 5, 7) (B) (5, 7, 3)

(C) (6, 5, 3) (D) (2, 5, 7).

(Mizoram B. 2018)

8. Direction-ratios of line given by :

$\frac{x-1}{3} = \frac{2y+6}{10} = \frac{1-z}{-7}$ are :

(A) <3, 10, – 7> (B) <3, – 5, 7>

(C) <3, 5, 7> (D) <3, 5, – 7>. ***(P. B. 2017)***

9. The distance of the plane $3x - 4y + 12z = 3$ from the point (0, 0, 0) is :

(A) $\frac{3}{13}$ (B) $\frac{13}{3}$

(C) – 2 (D) 3. ***(H.P.B. 2017)***

10. Angle between pair of lines :

$\frac{x+3}{3} = \frac{y-1}{5} = \frac{z+3}{4}$ and $\frac{x+1}{1} = \frac{y-4}{1} = \frac{z-5}{2}$ is :

(A) $\cos^{-1}\left(\frac{8\sqrt{3}}{15}\right)$ (B) $\cos^{-1}\left(\frac{5\sqrt{7}}{15}\right)$

(C) $\cos^{-1}\left(\frac{15}{8\sqrt{3}}\right)$ (D) $\cos^{-1}\left(\frac{3\sqrt{8}}{15}\right)$.

(H.B. 2017)

11. If the lines $\frac{x-1}{-3}=\frac{x-2}{2k}=\frac{z-3}{2}$ and

$$\frac{x-1}{3k}=\frac{y-1}{1}=\frac{z-6}{-5}$$

are perpendicular to each other, then the value of k is :

(A) $-\frac{1}{7}$ (B) $-\frac{1}{10}$

(C) $\frac{7}{10}$ (D) $-\frac{10}{7}$. ***(H.B. 2017)***

12. The direction-cosines of the vector $\vec{a}=\hat{i}-\hat{j}-2\hat{k}$ are :

(A) <1, – 1, – 2> (B) $\left\langle\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}}\right\rangle$

(C) $\left\langle\frac{1}{4},-\frac{1}{4},\frac{-2}{4}\right\rangle$ (D) $\left\langle\sqrt{\frac{1}{6}},-\sqrt{\frac{1}{6}},-\sqrt{\frac{2}{6}}\right\rangle$.

(Nagaland B. 2017)

13. The angle between the vector $\vec{r}=4\hat{i}+8\hat{j}+\hat{k}$ makes with the x-axis is :

(A) $\cos^{-1}\left(\frac{13}{9}\right)$ (B) $\cos^{-1}\left(\frac{13}{3}\right)$

(C) $\cos^{-1}\left(\frac{\sqrt{13}}{4}\right)$ (D) $\cos^{-1}\left(\frac{4}{9}\right)$.

(Mizoram B. 2017)

14. The length of perpendicular from the origin to the plane :

$\vec{r}.(3\hat{i}-12\hat{j}-4\hat{k})+39=0$ is :

(A) 19 (B) 3

(C) 13 (D) 12. ***(Mizoram B. 2017)***

15. Distance between the point (0, 1, 7) and the plane $3x + 4y + 1 = 0$ is :

(A) 1 Unit (B) 2 Units

(C) 3 Units (D) 4 Units. ***(P. B. 2016)***

16. If a line makes angle 90º, 60º and 30º with the positive direction of x, y and z–axis respectively, then its direction-cosines will be :

(A) $<0,\frac{1}{2},\frac{\sqrt{3}}{2}>$ (B) $<1,\frac{1}{2},\frac{\sqrt{3}}{2}>$

(C) $<0,-\frac{1}{2},\frac{\sqrt{3}}{2}>$ (D) None of these.

(H.P.B. 2016)

17. The direction-cosines of the line whose direction ratios are :
6, – 2, – 3 are :

(A) $\frac{6}{7},\frac{2}{7},\frac{-3}{7}$ (B) $\frac{6}{7},\frac{-2}{7},\frac{-3}{7}$

(C) $\frac{6}{7},\frac{2}{7},\frac{3}{7}$ (D) None of these.

(H.B. 2016)

18. The equation of line passing through the point :

(2, –1, 4) and in the direction of $\hat{i}+\hat{j}-2\hat{k}$ in cartesian form is :

(A) $\frac{x-2}{1}=\frac{y+1}{1}=\frac{z-4}{-2}$

(B) $\frac{x-2}{1}=\frac{y+1}{1}=\frac{z+4}{-2}$

(C) $\frac{x+2}{1}=\frac{y-1}{1}=\frac{z-4}{-2}$

(D) None of these. ***(H.B. 2016)***

19. Equation of the plane with intercepts 2, 3, 4 on the x, y, and z axis respectively is :

(A) $2x + 3y + 4z = 1$ (B) $2x + 3y + 4z = 12$

(C) $6x + 4y + 3z = 1$ (D) $6x + 4y + 3z = 12$.

(Kerala B. 2016)

20. The distance of the point (2, 1, –1) from the plane $x - 2y + 4z = 9$ is :

(A) $-\frac{13}{\sqrt{21}}$ (B) $\frac{9}{\sqrt{6}}$

(C) $\frac{13}{\sqrt{21}}$ (D) $\frac{-9}{\sqrt{6}}$. ***(Mizoram B. 2016)***

RCQ Pocket

(Single Correct Answer Type)

(JEE–Main and Advanced)

21. The line passing through the points (5, 1, a) and (3, b, 1) crosses the yz-plane at the point $\left(0,\frac{17}{2},-\frac{13}{2}\right)$

Then :

(A) $a = 8, b = 2$ (B) $a = 2, b = 8$

(C) $a = 4, b = 6$ (D) $a = 6, b = 4$.

(A.I.E.E.E. 2008)

22. If the straight lines :

$$\frac{x-1}{k}=\frac{y-2}{2}=\frac{z-3}{3} \text{ and } \frac{x-2}{3}=\frac{y-3}{k}=\frac{z-1}{2}$$

intersect at a point, then the integer k is equal to :

(A) – 2 (B) – 5

(C) 5 (D) 2. ***(A.I.E.E.E. 2008)***

23. Let the line $\frac{x-2}{3}=\frac{y-1}{-5}=\frac{z-2}{2}$ lie in the plane $x + 3y - \alpha z + \beta = 0$. Then (α, β) equals :

(A) (– 6, – 17) (B) (5, – 15)

(C) (– 5, 5) (D) (6, – 17).

(A.I.E.E.E. 2009)

24. The projections of a vector on the three co-ordinate axes are 6, – 3, 2 respectively. The direction-cosines of the vector are :

(A) $\frac{6}{5}, -\frac{3}{5}, \frac{2}{5}$ (B) $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$

(C) $\frac{-6}{7}, \frac{-3}{7}, \frac{2}{7}$ (D) 6, – 3, 2.

(A.I.E.E.E. 2009)

25. A line AB in three-dimensional space makes angles 45° and 120° with the positive x-axis and the positive y-axis respectively. If AB makes an acute angle θ with the positive z-axis, then θ equals :

(A) 30° (B) 45°

(C) 60° (D) 75°. ***(A.I.E.E.E. 2010)***

26. If the angle between the lines $x = \frac{y-1}{2} = \frac{z-3}{\lambda}$ and the plane $x + 2y + 3z = 4$ is $\cos^{-1}\left(\sqrt{\frac{5}{14}}\right)$, then λ equals :

(A) $\frac{2}{3}$ (B) $\frac{3}{2}$

(C) $\frac{2}{5}$ (D) $\frac{5}{3}$. ***(A.I.E.E.E 2011)***

27. The length of the perpendicular drawn from the point (3, – 1, 11) to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is :

(A) $\sqrt{29}$ (B) $\sqrt{33}$

(C) $\sqrt{53}$ (D) $\sqrt{65}$.

(A.I.E.E.E. 2011 S)

28. The distance of the point (1, – 5, 9) from the plane $x - y + z = 5$, measued along a straight line : $x = y = z$ is :

(A) $10\sqrt{3}$ (B) $5\sqrt{3}$

(C) $3\sqrt{10}$ (D) $3\sqrt{5}$.

(A.I.E.E.E. 2011 S)

29. If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then k is equal to :

(A) –1 (B) $\frac{2}{9}$

(C) $\frac{9}{2}$ (D) 0. ***(A.I.E.E.E. 2012)***

30. An equation of a plane parallel to the plane $x - 2y + 2z - 5 = 0$ and at a unit distance from the origin is :

(A) $x - 2y + 2z - 3 = 0$ (B) $x - 2y + 2z + 1 = 0$

(C) $x - 2y + 2z - 1 = 0$ (D) $x - 2y + 2z + 5 = 0$.

(A.I.E.E.E. 2012)

31. If the lines :

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k} \text{ and } \frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$$

are coplanar, then k can have :

(A) exactly one value

(B) exactly two values

(C) exactly three values

(D) any value. ***(J.E.E. (Main) 2013)***

32. Distance between two parallel planes :
$2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is :

(A) $\frac{5}{2}$ (B) $\frac{7}{2}$

(C) $\frac{9}{2}$ (D) $\frac{3}{2}$.

(J.E.E. (Main) 2013)

33. The image of the line $\frac{x-1}{3} = \frac{y-3}{1} = \frac{z-4}{-5}$ in the plane :
$2x - y + z + 3 = 0$ is the line :

(A) $\frac{x+3}{-3} = \frac{y-5}{-1} = \frac{z+2}{5}$

(B) $\frac{x-3}{3} = \frac{y+5}{1} = \frac{z-2}{-5}$

(C) $\frac{x-3}{-3} = \frac{y+5}{-1} = \frac{z-2}{5}$

(D) $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$. ***(J.E.E. (Main) 2014)***

34. The angle between the lines whose direction-cosines satisfy the equations :
$l + m + n = 0$ and $l^2 = m^2 + n^2$ is :

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$

(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{3}$.

(J.E.E. (Main) 2014)

35. The distance of the point (1, 0, 2) from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 16$ is :

(A) $2\sqrt{14}$ (B) 8

(C) $3\sqrt{21}$ (D) 13.

(J.E.E. (Main) 2015)

36. The equation of the plane containing the line :
$2x - 5y + z = 3; x + y + 4z = 5$ and parallel to the plane :
$x + 3y + 6z = 1$ is :

(A) $2x + 6y + 12z = 13$ (B) $x + 3y + 6z = -7$

(C) $x + 3y + 6z = 7$ (D) $2x + 6y - 12z = -13$.

(J.E.E. (Main) 2015)

37. If the line $\frac{x-3}{2}=\frac{y+2}{-1}=\frac{z+4}{3}$ lies in the plane $lx+my-z=9$, then l^2+m^2 is equal to :

(A) 18 (B) 5

(C) 2 (D) 26.

(J.E.E. (Main.) 2016)

38. The distance of the point (1, –5, 9) from the plane $x-y+z=5$ measured along $x=y=z$ is :

(A) $10\sqrt{3}$ (B) $\frac{10}{\sqrt{3}}$

(C) $\frac{20}{3}$ (D) $3\sqrt{10}$.

(J.E.E. (Main.) 2016)

39. If the image of the point P(1, – 2, 3) in the plane $2x+3y-4z+22=0$, measured parallel to the line $\frac{x}{1}=\frac{y}{4}=\frac{z}{5}$ is Q, then PQ is equal to :

(A) $\sqrt{42}$ (B) $6\sqrt{5}$

(C) $3\sqrt{5}$ (D) $2\sqrt{42}$.

(J.E.E. (Main) 2017)

40. The distance of the point (1, 3, – 7) from the plane passing through the point (1, – 1, – 1) having normal perpendicular to both the lines :

$\frac{x-1}{1}=\frac{y+2}{-2}=\frac{z-4}{3}$ and $\frac{x-2}{2}=\frac{y+1}{-1}=\frac{z+7}{-1}$ is :

(A) $\frac{5}{\sqrt{83}}$ (B) $\frac{10}{\sqrt{74}}$

(C) $\frac{20}{\sqrt{74}}$ (D) $\frac{10}{\sqrt{83}}$.

(J.E.E. (Main) 2017)

41. The equation of the plane passing through the point (1, 1, 1) and perpendicular to the planes :

$2x+y-2z=5$ and $3x-6y-2z=7$ is :

(A) $14x+2y-15z=1$

(B) $14x-2y+15z=27$

(C) $14x+2y+15z=31$

(D) $-14x+2y+15z=3$.

(J.E.E. (Advanced) 2017)

42. If L_1 is the line of intersection of the planes :

$2x-2y+3z-2=0$ and $x-y+z+1=0$ is

and L_2 is the line of intersection of the planes :

$x+2y-z-3=0,\ 3x-y+2z-1=0$,

then the distance of the origin rom the plane containing L_1 and L_2 is :

(A) $\frac{1}{4\sqrt{2}}$ (B) $\frac{1}{3\sqrt{2}}$

(C) $\frac{1}{2\sqrt{2}}$ (D) $-\frac{1}{\sqrt{2}}$.

(J.E.E. (Main) 2018)

43. The length of the projection of the line segment joining the points (5, –1, 4) and (4, –1, 3) on the plane :

$x+y+z=7$ is :

(A) $\frac{2}{\sqrt{3}}$ (B) $\frac{2}{3}$

(C) $\frac{1}{3}$ (D) $\sqrt{\frac{2}{3}}$.

(J.E.E. (Main) 2018)

Answers

1. (A)	**2.** (C)	**3.** (B)	**4.** (D)	**5.** (A)	**6.** (D)	**7.** (A)	**8.** (C)	**9.** (A)	**10.** (A)
11. (D)	**12.** (B)	**13.** (D)	**14.** (B)	**15.** (A)	**16.** (A)	**17.** (B)	**18.** (A)	**19.** (D)	**20.** (C)
21. (D)	**22.** (B)	**23.** (A)	**24.** (B)	**25.** (C)	**26.** (A)	**27.** (C)	**28.** (A)	**29.** (B)	**30.** (A)
31. (B)	**32.** (B)	**33.** (D)	**34.** (D)	**35.** (D)	**36.** (C)	**37.** (C)	**38.** (A)	**39.** (D)	**40.** (D)
41. (C)	**42.** (B)	**43.** (D).							

Hints/Solutions

RCQ Pocket

21. (D) The equations of the line passing thro' (3, b, 1) and (5, 1, a) are :

$$\frac{x-5}{3-5}=\frac{y-1}{b-1}=\frac{z-a}{1-a}$$

$$\Rightarrow \frac{x-5}{2}=\frac{y-1}{1-b}=\frac{z-a}{a-1} = k \text{ (say)}$$

The line crosses the yz- plane ($x = 0$), where

$$\frac{-5}{2}=k \Rightarrow k = \frac{-5}{2}.$$

Also $y = k(1-b) + 1 = \frac{17}{2}$

$$\Rightarrow \frac{-5}{2}(1-b)+1=\frac{17}{2}$$

$$\Rightarrow \frac{-5}{2}(1-b) = \frac{15}{2}$$

$$\Rightarrow 1-b = -3$$

$$\Rightarrow b = 4.$$

Again $z = k(a-1) + a = -\frac{13}{2}$

$$\Rightarrow \frac{-5}{2}(a-1)+a=\frac{13}{2}$$

$$\Rightarrow \frac{-3}{2}a+\frac{5}{2} = -\frac{13}{2}$$

$$\Rightarrow -\frac{3}{2}a = -9$$

$$\Rightarrow a = 6.$$

Hence, $a = 6, b = 4$.

22. (B) Since $\frac{x-1}{k} = \frac{y-2}{2} = \frac{z-3}{3}$

and $\frac{x-2}{3}=\frac{y-3}{k} = \frac{z-1}{2}$ intersect,

$$\therefore \begin{vmatrix} k & 2 & 3 \\ 3 & k & 2 \\ 1 & 1 & -2 \end{vmatrix} = 0$$

$\Rightarrow k(-2k-2) - 2(-6-2) + 3(3-k) = 0$

$\Rightarrow -2k^2 - 2k + 16 + 9 - 3k = 0$

$\Rightarrow 2k^2 + 5k - 25 = 0$

$\Rightarrow (2k-5)(k+5) = 0 \Rightarrow k = \frac{5}{2}, -5.$

Hence, $k = -5$. [$\because$ *k is an integer*]

23. (A) The line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z-2}{2}$ lies in the plane

$x + 3y - \alpha z + \beta = 0$,

$\therefore$ $2 + 3(1) - \alpha(2) + \beta = 0 \Rightarrow 2\alpha - \beta = 5$ ...(1)

and $(1)(3) + (-5)(3) + (2)(-\alpha) = 0$

$\Rightarrow 3 - 15 - 2\alpha = 0 \Rightarrow 2\alpha = -12 \Rightarrow \alpha = -6.$

Putting in (1), $2(-6) - \beta = 5 \Rightarrow \beta = -12 - 5 = -17.$

Hence, (α, β) is $(-6, -17)$.

24. (B) Direction-cosines are :

$$< \frac{6}{\sqrt{36+9+4}}, \frac{-3}{\sqrt{36+9+4}}, \frac{2}{\sqrt{36+9+4}} >$$

i.e., $< \frac{6}{7}, \frac{-3}{7}, \frac{2}{7} >$.

25. (C) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Here $\alpha = 45°, \beta = 120°, \gamma = \theta.$

$\therefore$ $\cos^2 45° + \cos^2 120° + \cos^2\theta = 1$

$$\Rightarrow \frac{1}{2}+\frac{1}{4}+\cos^2\theta = 1$$

$$\Rightarrow 1-\cos^2\theta = \frac{3}{4} \Rightarrow \sin^2\theta = \frac{3}{4} = \sin^2 60°$$

$$\Rightarrow \theta = 60°.$$

26. (A) The given line is $\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-3}{\lambda}$

and the plane is $x + 2y + 3z = 4$.

$\therefore$ Angle between the line and the plane is :

$$\theta = \sin^{-1}\left(\frac{(1)(1)+(2)(2)+(\lambda)(3)}{\sqrt{1+4+\lambda^2}\sqrt{1+4+9}}\right)$$

$$= \sin^{-1}\left(\frac{5+3\lambda}{\sqrt{14}\sqrt{5+\lambda^2}}\right) = \cos^{-1}\left(\sqrt{1-\frac{(5+3\lambda)^2}{14(5+\lambda^2)}}\right)$$

$$= \cos^{-1}\left(\sqrt{\frac{5}{14}}\right) \quad \text{[Given]}$$

$$\Rightarrow 1-\frac{(5+3\lambda)^2}{14(5+\lambda^2)} = \frac{5}{14}$$

$\Rightarrow 14(5+\lambda^2) - (25 + 9\lambda^2 + 30\lambda) = 5(5+\lambda^2)$

$\Rightarrow 45 + 5\lambda^2 - 30\lambda = 25 + 5\lambda^2 \Rightarrow 30\lambda = 20 \Rightarrow \lambda = \frac{2}{3}.$

27. (C) Let any point on the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ be P $(2k, 2+3k, 3+4k)$.

If P be the foot of perpendicular,

then direction-ratios of the perpendicular are :

$< 2k-3, 2+3k+1, 3+4k-11 >$

i.e. $< 2k-3, 3k+3, 4k-8 >$.

And direction-ratios of the line are $< 2, 3, 4 >$.

$\therefore 2(2k-3) + 3(3k+3) + 4(4k-8) = 0$

$\Rightarrow 29k - 29 = 0 \Rightarrow k = 1.$

$\therefore$ P is $(2, 2+3, 3+4)$ *i.e.* $(2, 5, 7)$.

Also Q is $(3, -1, 11)$.

$\therefore$ Length of perpendicular

$= \sqrt{(2-3)^2 + (5+1)^2 + (7-11)^2}$

$= \sqrt{1+36+16} = \sqrt{53}$.

28. (A) The line thro' P (1, – 5, 9) parallel to $x = y = z$ is :

$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1}$ ....(1)

Any point on (1) is Q $(1 + \lambda, -5 + \lambda, 9 + \lambda)$.

This lies on $x - y + z = 5$

$\Rightarrow 1 + \lambda + 5 - \lambda + 9 + \lambda = 5$

$\Rightarrow \lambda = -10$.

$\therefore$ Q is (– 9, – 15, – 1).

$\therefore |PQ| = \sqrt{(-9-1)^2 + (-15+5)^2 + (-1-9)^2}$

$= \sqrt{100+100+100} = 10\sqrt{3}$.

29. (C) Any point on $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ $(= t)$ is $(2t + 1, 3t - 1, 4t + 1)$.

Any point on $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1} (= s)$ is $(s + 3, 2s + k, s)$.

Since the given lines are intersecting,

$\therefore 2t + 1 = s + 3,\ 3t - 1 = 2s + k$ and $4t + 1 = s$.

Solving (1) and (2), $t = -\frac{3}{2}$ and $s = -5$.

Putting in (2) and (3), $3\left(-\frac{3}{2}\right) - 1 = 2(-5) + k$

$\Rightarrow -\frac{9}{2} - 1 = -10 + k$

$\Rightarrow k = 10 - \frac{11}{2} = \frac{9}{2}$.

30. (A) Any plane parallel to $x - 2y + 2z - 5 = 0$ is :

$x - 2y + 2z + k = 0$ ....(1)

Its distance from the origin = 1

$\Rightarrow \frac{|0-0+0+k|}{\sqrt{1+4+4}} = 1 \Rightarrow |k| = 3 \Rightarrow k = \pm 3$.

Putting in (1), $x - 2y + 2z - 3 = 0$.

31. (B) The given lines are coplanar if

$$\begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

if $(1)(1 + 2k) + (1)(1 + k^2) + (-1)(2 - k) = 0$

if $1 + 2k + 1 + k^2 - 2 + k = 0$

if $k^2 + 3k = 0$ if $k(k + 3) = 0$

if $k = 0, -3$.

Hence, k can have exactly two values.

32. (B) The given parallel planes are :

$2x + y + 2z - 8 = 0$

and $2x + y + 2z + \frac{5}{2} = 0$.

$\therefore$ Distance between the planes

$= \frac{|8 + 5/2|}{\sqrt{4+1+4}} = \frac{21}{6} = \frac{7}{2}$.

33. (D) Since $(3)(2) + (1)(-1) + (-5)(1) = 0$,

$\therefore$ the line is parallel to the plane.

Image of (1, 3, 4) is (–3, 5, 2).

$\therefore$ The required. image is $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$.

34. (D) $l + m + n = 0 \Rightarrow l = -m - n$

$\therefore l^2 = m^2 + n^2 \Rightarrow (-m - n)^2 = m^2 + n^2$

$\Rightarrow 2mn = 0 \Rightarrow mn = 0$.

Either $m = 0$ or $n = 0$.

$\therefore$ Possibilities are $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ or $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$.

$\therefore \cos\theta = \frac{1}{2} + 0 + 0 \Rightarrow \theta = \frac{\pi}{3}$.

35. (D) Any point on the line is $(3k + 2, 4k - 1, 12k + 2)$.

This lies on the plane

$\Rightarrow 3k + 2 - 4k + 1 + 12k + 2 = 16$

$\Rightarrow 11k = 11 \Rightarrow k = 1$.

$\therefore$ Point of intersection is (5, 3, 14).

$\therefore$ Its distance from (1, 0, 2)

$= \sqrt{(5-1)^2 + (3-0)^2 + (14-2)^2}$

$= \sqrt{16+9+144} = \sqrt{169} = 13$.

36. (C) Putting $z = 0$, $2x - 5y = 3$ and $x + y = 5$.

Solving, $x = 4$, $y = 1$.

Let $x + 3y + 6z = k$ be a plane parallel to given plane.

$\therefore 4 + 3 + 0 = k \Rightarrow k = 7$

$\therefore$ Required. equation of the plane is $x + 3y + 6z = 7$.

37. (C) Since the line $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane $lx + my - z = 9$,

$\therefore 3l - 2m + 4 = 9$ and $2l - m - 3 = 0$

Solving for l and m, me get :

$l = 1$ and $m = -1$.

$\therefore l^2 + m^2 = 1 + 1 = 2$.

38. (A) Equation of PM is :

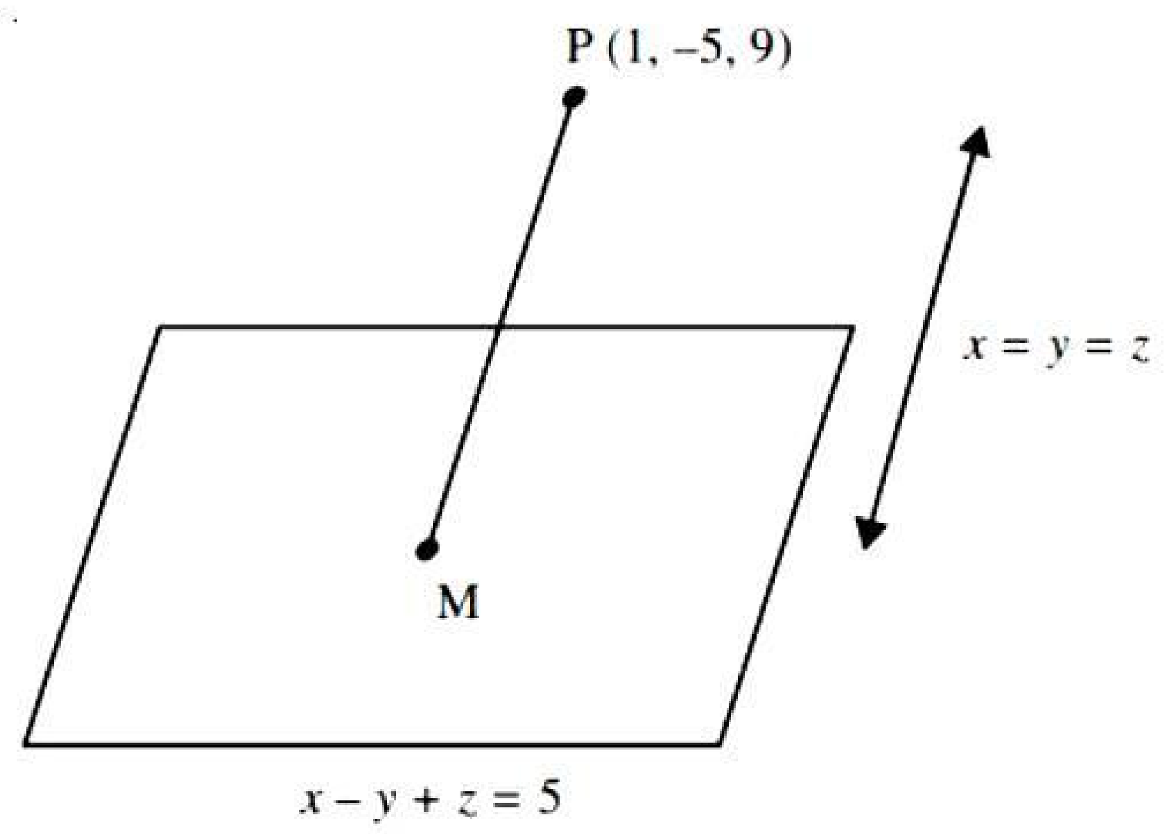
$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda$ (say)

Let M be $(1 + \lambda, -5 + \lambda, 9 + \lambda)$.

This lies on $x - y + z = 5$.

$\therefore 1 + \lambda + 5 - \lambda + 9 + \lambda = 5 \Rightarrow \lambda = -10$.

$\therefore$ M is $(1 - 10, -5 - 10, 9 - 10)$



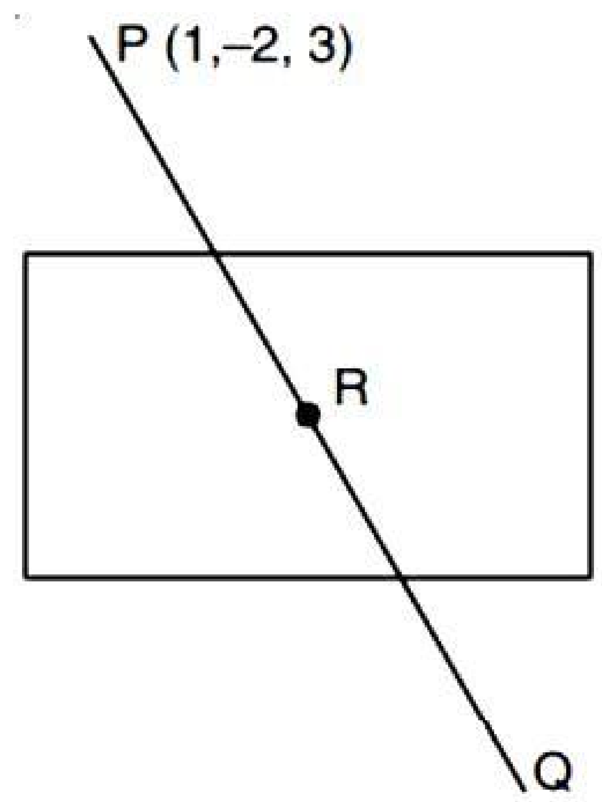
i.e., (–9, –15, –1).

Also, $|PM| = \sqrt{(1+9)^2 + (-5+15)^2 + (9+1)^2}$

$= \sqrt{100+100+100} = 10\sqrt{3}$.

39. (D) The line PQ is

$$\frac{x-1}{1} = \frac{y+2}{4} = \frac{y-3}{5}.$$



Let Q be $(k + 1, 4k - 2, 5k + 3)$.

Q lies on $2x + 3y - 4z + 22 = 0$.

$\therefore 2(k + 1) + 3(4k - 2) - 4(5k + 3) + 22 = 0$

$\Rightarrow 2k + 2 + 12k - 6 - 20k - 12 + 22 = 0$

$\Rightarrow -6k + 6 = 0 \Rightarrow k = 1$.

$\therefore$ R is (2, 2, 8).

Hence, $PQ = 2\sqrt{(2-1)^2 + (2+2)^2 + (8-3)^2}$

$= 2\sqrt{1+16+25} = 2\sqrt{42}$.

40. (D) Normal vector is :

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix} = 5\hat{i} + 7\hat{j} + 3\hat{k}.$$

Then the plane is $5(x - 1) + 7(y + 1) + 3(z + 1) = 0$

$\Rightarrow \quad 5x + 7y + 3z + 5 = 0$.

$\therefore$ Required distance $= \dfrac{|5(1) + 7(3) + 3(-7) + 5|}{\sqrt{25+49+9}}$

$= \dfrac{5+21-21+5}{\sqrt{83}} = \dfrac{10}{\sqrt{83}}$.

41. (C) The normal vector of the required plane is parallel to the vector :

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 3 & -6 & -2 \end{vmatrix} = -14\hat{i} - 2\hat{j} - 15\hat{k}.$$

$\therefore$ The equation of the required plane is :

$-14(x - 1) - 2(y - 1) - 15(z - 1) = 0$

$\Rightarrow 14x + 2y + 15z = 31$.

42. (B) Plane passing through line of intersection of first two planes is :

$(2x - 2y + 3z - 2) + \lambda(x - y + z + 1) = 0$

$\Rightarrow (\lambda + 2)x - (\lambda + 2)y + (\lambda + 3)z + (\lambda - 2) = 0$ ...(1)

(1) has infinite number of solutions with

$x + 2y - z - 3 = 0$ and $3x - y + 2z - 1 = 0$,

Then,

$$\begin{vmatrix} \lambda+2 & -(\lambda+2) & \lambda+3 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 0$$

$\Rightarrow (\lambda + 2)(4 - 1) + (\lambda + 2)(2 + 3) + (\lambda + 3)(-1 - 6) = 0$

$\Rightarrow 3\lambda + 6 + 5\lambda + 10 - 7\lambda - 21 = 0$

$\Rightarrow \quad \lambda = 5$.

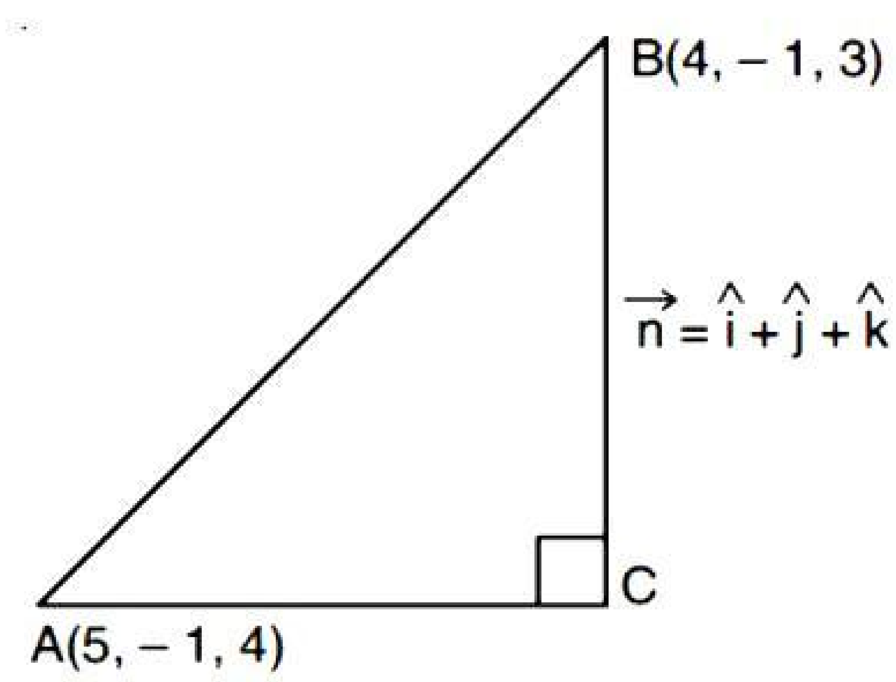
$\therefore$ (1) becomes : $7x - 7y + 8y + 3 = 0$.

Hence, the distance of the plane from (0, 0, 0)

$$= \left|\frac{3}{\sqrt{49+49+64}}\right| = \frac{3}{9\sqrt{2}} = \frac{1}{3\sqrt{2}}.$$

43. (D) Normal to the plane $x + y + z = 7$ is $\vec{n} = \hat{i} + \hat{j} + \hat{k}$.

Now, $\quad \overrightarrow{AB} = -\hat{i} - \hat{k}$.



$\therefore \quad |\overrightarrow{AB}| = AB = \sqrt{1+1} = \sqrt{2}$

And $\quad BC$ = Length of projection of $\overrightarrow{AB}$ on $\vec{n}$

$= |\overrightarrow{AB} \cdot \vec{n}|$

$= \left|(-\hat{i} - \hat{k}) \cdot \dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}\right|$

$= \left|\dfrac{-1-1}{\sqrt{3}}\right| = \dfrac{2}{\sqrt{3}}$.

$\therefore$ Length of projection of the line segment on the plance = AC.

Then, $\quad AC^2 = AB^2 - BC^2 = 2 - \dfrac{4}{3} = \dfrac{2}{3}$.

Hence, $\quad AC = \sqrt{\dfrac{2}{3}}$.

CHAPTER TEST 11

Time Allowed : 1 Hour **Max. Marks : 34**

Notes : 1. *All questions are compulsory.*

2. *Marks have been indicated against each question.*

1. If a line has direction-ratio <2, –1, –2>, determine its direction-cosines. **(1)**

2. Find the cartesian equation of the plane $\vec{r}.(\hat{i}+\hat{j}-\hat{k})=2$. **(1)**

3. Show that the line $\vec{r}=\hat{i}+\hat{j}+\lambda(2\hat{i}+\hat{j}+4\hat{k})$ lies in the plane $\vec{r}.(\hat{i}+2\hat{j}-\hat{k})=3$. **(2)**

4. Find the area of the triangle whose vertices are :
A(1, 2, 3) ; B(2, – 1, 4) and C(4, 5, – 1). **(2)**

5. Find the equations of the straight line passing through the point (2, 3, –1) and is perpendicular to the lines :

$$\frac{x-2}{2}=\frac{y+1}{1}=\frac{z-3}{-3} \text{ and } \frac{x-3}{1}=\frac{y+2}{1}=\frac{z-1}{1}.$$ **(4)**

6. Find the image of the point (1, 6, 3) in the line :

$$\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}.$$ **(4)**

7. Find the equation of the plane passing through the point (–1, 3, 2) and is perpendicular to the planes :
$x + 2y + 3z = 5$ and $3x + 3y + z = 0$. **(4)**

8. Prove that if a plane has the intercepts *a, b, c* and is at distance of '*p*' units from the origin, then :

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2}.$$ **(4)**

9. A line makes angels $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$. **(6)**

10. Show that the lines :

$$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta} \quad \text{and} \quad \frac{x-b-c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma} \text{ are coplanar.}$$ **(6)**

Answers

1. $<\frac{2}{3},\frac{-1}{3},\frac{-2}{3}>$. **2.** $x+y-z=2$. **4.** $\sqrt{\frac{137}{2}}$ sq. units.

5. $\frac{x-2}{4}=\frac{y-3}{-5}=\frac{z+1}{1}$. **6.** (1, 0, 7). **7.** $7x-8y+3z+25=0$.