CHAPTER

03

Sequences and Series

The word "Sequence" in Mathematics has same meaning as in ordinary English. A collection of objects listed in a sequence means it has identified first member, second member, third member and so on. The most common examples are depreciate values of certain commodity like car, machinery and amount deposits in the bank for a number of years.

Session 1

Sequence, Series, Progression

Sequence

A succession of numbers arranged in a definite order or arrangement according to some well-defined law is called a sequence.

Or

A sequence is a function of natural numbers (N) with codomain is the set of real numbers (R) [complex numbers (C)]. If range is subset of real numbers (complex numbers), it is called a real sequence (complex sequence).

Or

A mapping $f : N \to C$, then $f(n) = t_n, n \in N$ is called a sequence to be denoted it by $\{f(1), f(2), f(3), \ldots\} = \{t_1, t_2, t_3, \ldots\} = \{t_n\}$.

The nth term of a sequence is denoted by $T_n, t_n, a_n, a(n), u_n$, etc.

Remark

The sequence $a_1, a_2, a_3, \ldots$ is generally written as $\{a_n\}$.

For example ,

(i) 1, 3, 5, 7, ... is a sequence, because each term (except first) is obtained by adding 2 to the previous term and $T_n = 2n - 1, n \in N$.

Or

If $T_1 = 1, T_{n+1} = T_{n+2}, n \geq 1$

(ii) 1, 2, 3, 5, 8, 13, ... is a sequence, because each term (except first two) is obtained by taking the sum of preceding two terms.

Or

If $T_1 = 1, T_2 = 2, T_{n+2} = T_n + T_{n+1}, n \geq 1$

(iii) 2, 3, 5, 7, 11, 13, 17, 19, ... is a sequence.

Here, we cannot express $T_n, n \in N$ by an algebraic formula.

Recursive Formula

A formula to determine the other terms of the sequence in terms of its preceding terms is known as recursive formula.

For example,

If $T_1 = 1$ and $T_{n+1} = 6T_n, n \in N$.

Then, $T_2 = 6T_1 = 6 \cdot 1 = 6$

$T_3 = 6T_2 = 6 \cdot 6 = 36$

$T_4 = 6T_3 = 6 \cdot 36 = 216 \ldots$

Then, sequence is 1, 6, 36, 216,...

Types of Sequences

There are two types of sequences

1. Finite Sequence

A sequence is said to be finite sequence, if it has finite number of terms. A finite sequence is described by $a_1, a_2, a_3, \ldots, a_n$ or $T_1, T_2, T_3, \ldots, T_n$, where $n \in N$.

For example

(i) 3, 5, 7, 9, ..., 37

(ii) 2, 6, 18, 54, ..., 4374

2. Infinite Sequence

A sequence is said to be an infinite sequence, if it has infinite number of terms. An infinite sequence is described by $a_1, a_2, a_3, \ldots$ or $T_1, T_2, T_3, \ldots$

For example,

(i) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

(ii) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Series

In a sequence, the sum of the directed terms is called a series.

For example, If 1, 4, 7, 10, 13, 16,... is a sequence, then its sum i.e., $1+4+7+10+13+16+\dots$ is a series.

In general, if $T_1, T_2, T_3, \dots, T_n, \dots$ denote a sequence, then the symbolic expression $T_1+T_2+T_3+\dots+T_n+\dots$ is called a series associated with the given sequence.

Each member of the series is called its term.

In a series $T_1+T_2+T_3+\dots+T_r+\dots$, the sum of first n terms is denoted by S_n. Thus,

$$S_n = T_1+T_2+T_3+\dots+T_n = \sum_{r=1}^{n} T_r = \sum T_n$$

If S_n denotes the sum of n terms of a sequence.

Then, $$S_n - S_{n-1} = (T_1+T_2+T_3+\dots+T_n) - (T_1+T_2+\dots+T_{n-1}) = T_n$$

Thus, $$T_n = S_n - S_{n-1}$$

Types of Series

There are two types of series

1. Finite Series

A series having finite number of terms is called a finite series.

For example,

(i) $3+5+7+9+\dots+21$

(ii) $2+6+18+54+\dots+4374$

2. Infinite Series

A series having an infinite number of terms is called an infinite series.

For example,

(i) $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots$

(ii) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$

Progression

If the terms of a sequence can be described by an explicit formula, then the sequence is called a progression.

Or

A sequence is said to be progression, if its terms increases (respectively decreases) numerically.

For example, The following sequences are progression :

(i) $1, 3, 5, 7, \dots$

(ii) $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{54}, \dots$

(iii) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \dots$

(iv) $1, 8, 27, 256, \dots$

(v) $8, -4, 2, -1, \frac{1}{2}, \dots$

The sequences (iii) and (v) are progressions, because

$$|1| > \left|-\frac{1}{3}\right| > \left|\frac{1}{9}\right| > \left|-\frac{1}{27}\right| > \dots$$

i.e. $$1 > \frac{1}{3} > \frac{1}{9} > \frac{1}{27} > \dots$$

and $$|8| > |-4| > |2| > |-1| > \left|\frac{1}{2}\right| > \dots$$

i.e. $$8 > 4 > 2 > 1 > \frac{1}{2} > \dots$$

Remark

All the definitions and formulae are valid for complex numbers in the theory of progressions but it should be assumed (if not otherwise stated) that the terms of the progressions are real numbers.

Example 1. ***If $f: N \to R$, where $f(n) = a_n = \frac{n}{(2n+1)^2}$, write the sequence in ordered pair form.***

Sol. Here, $a_n = \frac{n}{(2n+1)^2}$

On putting $n = 1, 2, 3, 4, \dots$ successively, we get

$$a_1 = \frac{1}{(2\cdot 1+1)^2} = \frac{1}{9}, \quad a_2 = \frac{2}{(2\cdot 2+1)^2} = \frac{2}{25}$$

$$a_3 = \frac{3}{(2\cdot 3+1)^2} = \frac{3}{49}, \quad a_4 = \frac{4}{(2\cdot 4+1)^2} = \frac{4}{81}$$

$$\vdots \qquad \vdots \qquad \vdots$$

Hence, we obtain the sequence $\frac{1}{9}, \frac{2}{25}, \frac{3}{49}, \frac{4}{81}, \dots$

Now, the sequence in ordered pair form is

$$\left\{\left(1, \frac{1}{9}\right), \left(2, \frac{2}{25}\right), \left(3, \frac{3}{49}\right), \left(4, \frac{4}{81}\right), \dots\right\}$$

Example 2. ***The Fibonacci sequence is defined by*** $a_1 = 1 = a_2, a_n = a_{n-1} + a_{n-2}, n > 2$. ***Find*** $\frac{a_{n+1}}{a_n}$ ***for*** $n = 1, 2, 3, 4, 5$.

Sol. $\because \quad a_1 = 1 = a_2$

$\therefore \quad a_3 = a_2 + a_1 = 1 + 1 = 2,$

$a_4 = a_3 + a_2 = 2 + 1 = 3$

$a_5 = a_4 + a_3 = 3 + 2 = 5$

and $\quad a_6 = a_5 + a_4 = 5 + 3 = 8$

$\therefore \quad \frac{a_2}{a_1} = 1, \frac{a_3}{a_2} = \frac{2}{1} = 2, \frac{a_4}{a_3} = \frac{3}{2}, \frac{a_5}{a_4} = \frac{5}{3}$ and $\frac{a_6}{a_5} = \frac{8}{5}$

Example 3. ***If the sum of*** n ***terms of a series is*** $2n^2 + 5n$ ***for all values of*** n***, find its 7th term.***

Sol. Given, $\quad S_n = 2n^2 + 5n$

$\Rightarrow \quad S_{n-1} = 2(n-1)^2 + 5(n-1) = 2n^2 + n - 3$

$\therefore \quad T_n = S_n - S_{n-1} = (2n^2 + 5n) - (2n^2 + n - 3) = 4n + 3$

Hence, $\quad T_7 = 4 \times 7 + 3 = 31$

Example 4.

(i) Write $\sum_{r=1}^{n} (r^2 + 2)$ in expanded form.

(ii) Write the series $\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \ldots + \frac{n}{n+2}$ in sigma form.

Sol. (i) On putting $r = 1, 2, 3, 4, \ldots, n$ in $(r^2 + 2)$,

we get $3, 6, 11, 18, \ldots, (n^2 + 2)$

Hence, $\sum_{r=1}^{n} (r^2 + 2) = 3 + 6 + 11 + 18 + \ldots + (n^2 + 2)$

(ii) The rth term of series $= \frac{r}{r+2}$.

Hence, the given series can be written as

$$\frac{1}{3} + \frac{2}{4} + \frac{3}{5} + \frac{4}{6} + \ldots + \frac{n}{n+2} = \sum_{r=1}^{n} \left(\frac{r}{r+2} \right)$$

Exercise for Session 1

1 First term of a sequence is 1 and the $(n+1)$ th term is obtained by adding $(n+1)$ to the nth term for all natural numbers n, the 6th term of the sequence is

(a) 7 (b) 13
(c) 21 (d) 27

2. The first three terms of a sequence are 3, 3, 6 and each term after the second is the sum of two terms preceding it, the 8th term of the sequence is

(a) 15 (b) 24
(c) 39 (d) 63

3. If $a_n = \sin\left(\frac{n\pi}{6}\right)$, the value of $\sum_{n=1}^{6} a_n^2$ is

(a) 2 (b) 3
(c) 4 (d) 7

4. If for a sequence $\{a_n\}$, $S_n = 2n^2 + 9n$, where S_n is the sum of n terms, the value of a_{20} is

(a) 65 (b) 75
(c) 87 (d) 97

5. If $a_1 = 2, a_2 = 3 + a_1$ and $a_n = 2a_{n-1} + 5$ for $n > 1$, the value of $\sum_{r=2}^{5} a_r$ is

(a) 130 (b) 160
(c) 190 (d) 220