7 Integrals

❝ *There are many applications of integration in fields such as Physics, Engineering, Business, Economics etc. One of the important applications of integration is finding the profit function of producing a certain number of cars if the marginal cost and revenue functions are known. Companies can thus determine the maximum profit that can be earned and in this way, they can plan their production, labour and other infrastructure accordingly.*

Topic Notes

- *Indefinite Integrals*
- *Methods of Integration*
- *Special Methods of Integration*
- *Definite Integrals*
- *Properties of Definite Integrals*

INDEFINITE INTEGRALS 1

TOPIC 1

INTEGRATION AS THE INVERSE PROCESS OF DIFFERENTIATION

In earlier chapters, we have learnt that $f'(x)$ exists if $f(x)$ is differentiable on the interval I.

The converse of the above is "Given $f'(x)$ at each point of the interval I, can we determine the function $f(x)$?

Yes, it is possible to find $f(x)$. This process of finding the function whose derivative is given, is called integration or anti-differentiation.

This leads to the following definition:

A function F(x) is called anti-derivative or primitive or integral of a given function $f(x)$, with respect to x if $\frac{d}{dx}[F(x)] = f(x)$. The anti-derivative of a function is not unique *i.e.* if F(x) is the anti-derivative of $f(x)$, then F(x) + C is also anti-derivative of $f(x)$.

Integration is also known as Inverse process of Differentiation.

Integration is generally finding the anti-derivative for the given function. Integration is the inverse process of differentiation. Instead of differentiating a function, we are given the derivative of a function and asked to find its primitive, *i.e.*, the original function. Such a process is called Integration or Anti-Differentiation.

Illustration: Consider the following:

$\frac{d}{dx}(x^4) = 4x^3; \frac{d}{dx}(x^4+1) = 4x^3; \frac{d}{dx}(x^4-1) = 4x^3$......, and so on.

In fact, we have $\frac{d}{dx}(x^4+C) = 4x^3$, for any real number C.

If we assume $f(x) = 4x^3$ and $F(x) = x^4$, then $\frac{d}{dx}(F(x)+C) = f(x)$

So, F(x) + C represents the anti-derivative of $f(x)$, for every real number C.

Thus, there are infinitely many anti-derivatives of $f(x)$.

We represent the anti-derivative of $f(x)$ by the symbol $\int f(x)\,dx$, and in this case, we have

$$\int f(x)\,dx = F(x) + C \qquad ...(i)$$

Relation (i) is called indefinite integral and the process of finding the anti-derivatives is called indefinite integration (or simply integration). The constant C is an arbitrary constant and it is called constant of integration.

Important

➥ *The representation $\int f(x)\,dx$ is read as integration of $f(x)$ with respect to x. The symbol $\int$ is called integral symbol, $f(x)$ is called integrand and x is called variable of integration.*

So, integration is the inverse of differentiation in the following sense:

$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x) + C;$$

$$\int f'(x)dx = f(x) + C$$

Standard Formulae

We already know the formulae for the derivative of many important functions. From these formulae, we can write down immediately the corresponding formulae for the integrals of these functions, as listed below which will be used to find the integrals of other functions.

Formulae of Derivaties

(1) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, n \neq -1$

Particularly,

$\frac{d}{dx}(kx) = k$, where k is constant.

(2) $\frac{d}{dx}(e^x) = e^x$

(3) $\frac{d}{dx}(\log|x|) = \frac{1}{x}$

Formulae of Integrals (anti-derivatives)

(1) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

$\int k.dx = kx + C$, where k is a constant

(2) $\int e^x\,dx = e^x + C$

(3) $\int \frac{1}{x}dx = \log|x| + C$

(4) $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x$

(5) $\frac{d}{dx}(\sin x) = \cos x$

(6) $\frac{d}{dx}(-\cos x) = \sin x$

(7) $\frac{d}{dx}(\tan x) = \sec^2 x$

(8) $\frac{d}{dx}(-\cot x) = \text{cosec}^2 x$

(9) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(10) $\frac{d}{dx}(-\text{cosec}\, x) = \text{cosec}\, x \cot x$

(11) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(12) $\frac{d}{dx}(-\cos^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(13) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(14) $\frac{d}{dx}(-\cot^{-1} x) = \frac{1}{1+x^2}$

(15) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(16) $\frac{d}{dx}(-\text{cosec}^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(4) $\int a^x\, dx = \frac{a^x}{\log a} + C$

(5) $\int \cos x\, dx = \sin x + C$

(6) $\int \sin x\, dx = -\cos x + C$

(7) $\int \sec^2 x\, dx = \tan x + C$

(8) $\int \text{cosec}^2 x\, dx = -\cot x + C$

(9) $\int \sec x \tan x\, dx = \sec x + C$

(10) $\int \text{cosec}\, x \cot x\, dx = -\text{cosec}\, x + C$

(11) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

(12) $\int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1} x + C$

(13) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$

(14) $\int \frac{1}{1+x^2} dx = -\cot^{-1} x + C$

(15) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + C$

(16) $\int \frac{1}{x\sqrt{x^2-1}} dx = -\text{cosec}^{-1} x + C$

TOPIC 2

PROPERTIES OF INDEFINITE INTEGRALS

(1) Two indefinite integrals with the same derivative lead to the same family of curves and so they are equivalent.

(2) If the functions f and g have integrals on I, then $f \pm g$ has an integral on I and $\int (f \pm g) dx = \int f\, dx \pm \int g\, dx$

(3) If the function f has integral on I and k is a real number, then kf has an integral on I and

$$\int (kf)\, dx = k\int f\, dx$$

Important

Properties (2) and (3) can be generalised to a finite number of functions $f_1, f_2, \ldots, f_n$ and the real numbers $k_1, k_2, \ldots, k_n$ giving $\int (k_1 f_1 \pm k_2 f_2 \pm \ldots \pm k_n f_n) dx = k_1 \int f_1\, dx \pm k_2 \int f_2\, dx \pm \ldots \pm k_n \int f_n\, dx$

Example 1.1: Find the anti-derivative (or integral) of the following functions by the method of inspection:

(A) $\cos 3x$ **(B) e^{2x}**

(C) $(ax + b)^2$ **[NCERT]**

Ans. (A) We know that $\frac{d}{dx}(\sin 3x) = 3\cos 3x$

$$\Rightarrow \quad \cos 3x = \frac{1}{3}\frac{d}{dx}(\sin 3x)$$

$$= \frac{d}{dx}\left(\frac{1}{3}\sin 3x\right)$$

Thus, the anti-derivative of $\cos 3x$ is $\frac{1}{3}\sin 3x$.

(B) We know that $\frac{d}{dx}(e^{2x}) = 2e^{2x}$

$$\Rightarrow \qquad e^{2x} = \frac{1}{2}\frac{d}{dx}(e^{2x})$$

$$= \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right)$$

Thus, the anti-derivative of e^{2x} is $\frac{1}{2}e^{2x}$.

(C) We know that $\frac{d}{dx}((ax+b)^3) = 3a(ax+b)^2$

$$\Rightarrow \quad (ax+b)^2 = \frac{1}{3a}\frac{d}{dx}((ax+b)^3)$$

$$= \frac{d}{dx}\left(\frac{1}{3a}(ax+b)^3\right)$$

Thus, the anti-derivative of $(ax + b)^2$ is $\frac{1}{3a}(ax+b)^3$.

Example 1.2: Find the following integrals:

(A) $\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

(B) $\int(2x - 3\cos x + e^x)\, dx$

(C) $\int \frac{2 - 3\sin x}{\cos^2 x} dx$ **[NCERT]**

Ans. (A) $\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx = \int\left(x - 2 + \frac{1}{x}\right)dx$

$$= \int x\,dx - \int 2\,dx + \int \frac{1}{x}dx$$

$$= \frac{x^2}{2} - 2x + \log|x| + C$$

(B) $\int(2x - 3\cos x + e^x)dx$

$$= \int 2x\,dx - 3\int \cos x\,dx + \int e^x\,dx$$

$$= x^2 - 3\sin x + e^x + C$$

(C) $\int\frac{2 - 3\sin x}{\cos^2 x}dx = \int\frac{2}{\cos^2 x}dx - 3\int\frac{\sin x}{\cos^2 x}dx$

$$= 2\int \sec^2 x\,dx - 3\int \tan x \sec x\,dx$$

$$= 2\tan x - 3\sec x + C$$

Caution

- *Throughout, we shall write only one constant for integration in the final answer.*

OBJECTIVE Type Questions

[**1** mark]

Multiple Choice Questions

1. $(1 - x)\sqrt{x}\,dx$ **is equal to:**

(a) $\frac{2}{5}x^{2/3} - \frac{2}{3}x^{2/3} + C$

(b) $\frac{2}{5}x^{2/3} - \frac{2}{3}x^{2/3} + C$

(c) $x^{3/2} - \frac{2}{5}x^{5/2} + C$

(d) $\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$

Ans. *(d)* $\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$

Explanation:

$$\int(1-x)\sqrt{x}\,dx = \int\sqrt{x}\,dx - \int x\sqrt{x}\,dx$$

$$= \int x^{1/2}dx - \int x^{3/2}dx$$

$$= \frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2} + C$$

2. $\int(e^x + 2x - 3\cos x)dx$ **is equal to:**

(a) $x^2 - 3\sin x + e^x + C$

(b) $x^2 - 3\sin x - e^x + C$

(c) $x^2 + 3\sin x + e^x + C$

(d) $-x^2 - 3\sin x + e^x + C$

3. The anti-derivative of $\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$ **equals:**

(a) $\frac{2}{3}x^{3/2} + 2x^{1/2} + C$

(b) $\frac{2}{3}x^{2/3} + \frac{1}{2}x^2 + C$

(c) $\frac{3}{2}x^{3/2} + \frac{1}{2}x^{1/2} + C$

(d) $\frac{3}{2}x^{2/3} + \frac{1}{2}x^{1/2} + C$

Ans. *(a)* $\frac{2}{3}x^{3/2} + 2x^{1/2} + C$

Explanation:

$$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx = \int\sqrt{x}\,dx+\int\frac{1}{\sqrt{x}}\,dx$$

$$= \int x^{1/2}\,dx+\int x^{-1/2}\,dx$$

$$= \frac{2}{3}x^{3/2}+2x^{1/2}+C$$

4. Integrate $\frac{2^x}{3^x}\,dx$.

(a) $x\log 2-\log 3+C$

(b) $\frac{\left(\frac{2}{3}\right)^x}{\log\left(\frac{2}{3}\right)}+C$

(c) $\frac{\left(\frac{3}{2}\right)^x}{\log\left(\frac{3}{2}\right)}+C$

(d) None of these [DIKSHA]

5. If $\frac{d}{dx}f(x)=4x^3-\frac{3}{x^4}$ such that $f(2)=0$. Then, $f(x)$ is:

(a) $x^3+\frac{1}{x^3}-\frac{129}{8}$ (b) $x^3+\frac{1}{x^4}+\frac{129}{8}$

(c) $x^4+\frac{1}{x^3}+\frac{129}{8}$ (d) $x^3+\frac{1}{x^4}-\frac{129}{8}$

Ans. *(d)* $x^3+\frac{1}{x^4}-\frac{129}{8}$

Explanation: Here,

$$\frac{d}{dx}f(x) = 4x^3-\frac{3}{x^4}$$

Integrating both sides, we have

$$f(x) = \int\left(4x^3-\frac{3}{x^4}\right)dx$$

$$= \int 4x^3dx-\int\frac{3}{x^4}\,dx$$

$$= 4\int x^3\,dx-3\int x^{-4}\,dx$$

$$= x^4+x^{-3}+C$$

Since $f(2)=0$, we have $(2)^4+(2)^{-3}+C=0$

$$\Rightarrow \qquad C = -\frac{129}{8}$$

Hence, $$f(x) = x^4+\frac{1}{x^3}-\frac{129}{8}$$

6. $\int\frac{\cos 2x-\cos 2\theta}{\cos x-\cos\theta}\,dx$ is equal to:

(a) $2(\sin x+x\cos\theta)+C$

(b) $2(\sin x-x\cos\theta)+C$

(c) $2(\sin x+2x\cos\theta)+C$

(d) $2(\sin x-2x\cos\theta)+C$

[NCERT Exemplar]

7. The value of $\int\frac{x^2+5x-3}{\sqrt{x}}\,dx$ is:

(a) $\left[\frac{x^{5/2}}{5}+\frac{5}{3}x^{3/2}-3x^{1/2}\right]+C$

(b) $2\left[\frac{x^{5/2}}{5}+\frac{5x^{3/2}}{3}-3x^{1/2}\right]+C$

(c) $\left[\frac{x^{5/2}}{5}-\frac{5x^{3/2}}{3}+3x^{1/2}\right]$

(d) None of these

Ans. *(b)* $2\left[\frac{x^{5/2}}{5}+\frac{5}{3}x^{3/2}-3x^{1/2}\right]+C$

Explanation: Let

$$I = \int\frac{x^2+5x-3}{\sqrt{x}}\,dx$$

$$= \int\left(x^{2-\frac{1}{2}}+5x^{1-\frac{1}{2}}-3x^{-\frac{1}{2}}\right)dx$$

$$= \int\left(x^{\frac{3}{2}}+5x^{\frac{1}{2}}-3x^{-\frac{1}{2}}\right)dx$$

$$= \frac{x^{5/2}}{5/2}+\frac{5x^{3/2}}{3/2}-\frac{3x^{1/2}}{1/2}+C$$

$$= 2\left[\frac{x^{5/2}}{5}+\frac{5x^{3/2}}{3}-3x^{1/2}\right]+C$$

8. $\int\frac{x^3}{x+1}\,dx$ is equal to:

(a) $x+\frac{x^2}{2}+\frac{x^3}{3}-\log|1-x|+C$

(b) $x+\frac{x^2}{2}-\frac{x^3}{3}-\log|1-x|+C$

(c) $x-\frac{x^2}{2}-\frac{x^3}{3}-\log|1+x|+C$

(d) $x-\frac{x^2}{2}+\frac{x^3}{3}-\log|1+x|+C$

[NCERT Exemplar]

Ans. (d) $x-\frac{x^2}{2}+\frac{x^3}{3}-\log|1+x|+C$

Explanation: Let $I = \int \frac{x^3}{x+1}dx$

$$= \int \frac{x^3+1-1}{x+1}dx$$

$$= \int \frac{x^3+1}{x+1}dx - \int \frac{1}{x+1}dx$$

$$= \int \frac{(x+1)(x^2-x+1)}{x+1}dx - \int \frac{1}{x+1}dx$$

$$[a^3 \pm b^3 = (a \pm b)(a^2 \pm ab + b^2)]$$

$$= \int (x^2-x+1)\,dx - \int \frac{1}{x+1}dx$$

$$= \frac{x^3}{3}-\frac{x^2}{2}+x-\log|x+1|+C$$

9. The value of $\int \left(x^2 - \frac{1}{x^2}\right)^2 dx$ is:

(a) $\frac{x^5}{5}+\frac{1}{3x^3}+2x+C$

(b) $\frac{x^5}{5}-\frac{1}{3x^3}-x+C$

(c) $\frac{x^5}{5}-\frac{1}{3x^3}-2x+C$

(d) $\frac{x^5}{5}-\frac{1}{x^3}-2x+C$

10. Evaluate $\int \sin^{-1}\left(\cos\left(\frac{\pi}{2}-3x\right)\right)dx$.

(a) $\frac{x^2}{2}+C$ (b) $\frac{3x^2}{2}+C$

(c) $\frac{\sin 3x^2}{2}+C$ (d) None of these

Ans. (b) $\frac{3x^2}{2}+C$

Explanation: Let

$$I = \int \sin^{-1}\left(\cos\left(\frac{\pi}{2}-3x\right)\right)dx$$

$$= \int \sin^{-1}(\sin 3x)\,dx$$

$$= \int 3x\,dx$$

$$= \frac{3x^2}{2}+C$$

11. The value of $\int \frac{1}{\sqrt{x+3}+\sqrt{x}}dx$ is:

(a) $\frac{2}{3}[(x+3)^{3/2}-x^{3/2}]+C$

(b) $\frac{2}{9}[(x+3)^{3/2}-x^{3/2}]+C$

(c) $[(x+3)^{3/2}-x^{3/2}]+C$

(d) None of these

Ans. (b) $\frac{2}{9}[(x+3)^{3/2}-x^{3/2}]+C$

Explanation: Let

$$I = \int \frac{1}{\sqrt{x+3}+\sqrt{x}} \times \frac{(\sqrt{x+3}-\sqrt{x})}{(\sqrt{x+3}-\sqrt{x})}dx$$

$$= \int \frac{\sqrt{x+3}-\sqrt{x}}{(\sqrt{x+3})^2-(\sqrt{x})^2}dx$$

$$= \int \frac{(\sqrt{x+3}-\sqrt{x})}{x+3-x}dx$$

$$= \frac{1}{3}\int (\sqrt{x+3}-\sqrt{x})\,dx$$

$$= \frac{1}{3}\left[\frac{(x+3)^{3/2}}{3/2}-\frac{(x)^{3/2}}{3/2}\right]+C$$

$$= \frac{1}{3}\left[\frac{2}{3}(x+3)^{3/2}-\frac{2}{3}(x)^{3/2}\right]+C$$

$$= \frac{2}{9}[(x+3)^{3/2}-(x)^{3/2}]+C$$

12. The value of $\int (3^x + \operatorname{cosec} x \cot x)\,dx$ is:

(a) $\frac{3^x}{\log 3}+\operatorname{cosec} x+C$

(b) $\frac{3^x}{\log 3}-\operatorname{cosec} x+C$

(c) $3^x \log 3+\operatorname{cosec} x+C$

(d) None of these

Ans. (b) $\frac{3^x}{\log 3}-\operatorname{cosec} x+C$

Explanation: Let

$$I = \int (3^x+\operatorname{cosec} x \cot x)\,dx$$

$$= \frac{3^x}{\log 3}-\operatorname{cosec} x+C$$

13. $\int \frac{(x^3+8)(x-1)}{x^2-2x+4}dx$ is:

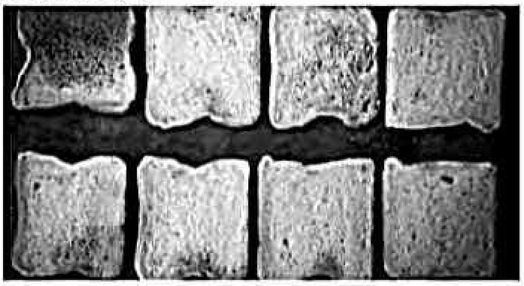
(a) $\frac{x^3}{3}+\frac{x^2}{2}+2x+C$

(b) $\frac{x^3}{3}+\frac{x^2}{2}-2x+C$

(c) $\frac{x^3}{3}+x^2-2x+C$

(d) None of these

14. When you leave bread out for a long time, discolouration on bread occurs which is known as bread mould. This bread mould is a microorganism which grows when the bread is kept at normal room temperature under moist conditions. The growth of bread mould is an exponential function. Let us assume that the function describing the total number of bread moulds at an instant is given by $4e^{3x} + 1$.



$\int(4e^{3x} + 1)\,dx$ is equal to:

(a) $\frac{4e^{3x}}{3} - x + C$ (b) $12e^{3x} + x + C$

(c) $12e^{3x} - x + C$ (d) $\frac{4e^{3x}}{3} + x + C$

Ans. *(d)* $\frac{4e^{3x}}{3} + x + C$

Explanation:

$$\int(4e^{3x} + 1)dx = \int 4e^{3x}dx + \int 1\,dx$$
$$= 4\int e^{3x}\,dx + \int dx$$
$$= \frac{4e^{3x}}{3} + x + C$$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

15. Find the integral of the following

(A) $\int 5\sin 5x\,dx$ (B) $\int e^{5x}\,dx$

Ans. (A) Let $I = \int 5\sin 5x\,dx$

$$= 5\frac{\cos 5x}{5} + C$$
$$= \cos 5x + C$$

(B) Let $I = \int e^{5x}\,dx$

$$= \frac{e^{5x}}{5} + C$$

16. Find the integral of $\int x^5\left(1 + \frac{1}{x^5}\right)dx$.

17. Find the integral $\int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$.

Ans. Let $I = \int\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx$

$$= \int\left(x + \frac{1}{x} - 2\right)dx$$
$$= \frac{x^2}{2} + \log|x| - 2x + C$$

18. Find the anti-derivative of $\frac{3x^3 + 4x^2 - 4}{x^2}dx$.

19. Evaluate $\int \frac{x^4+x^2+1}{x^2-x+1}dx$.

Ans. Let $I = \int \frac{x^4+x^2+1}{x^2-x+1}dx$

$$= \int \frac{(x^2+1)^2 - x^2}{(x^2 - x + 1)}dx$$
$$= \int \frac{(x^2+1+x)(x^2+1-x)}{x^2 - x + 1}dx$$
$$= \int (x^2 + 1 + x)dx$$
$$= \frac{x^3}{3} + x + \frac{x^2}{2} + C$$

20. Find the value of $\int (2-3x)\sqrt{x}\,dx$.

21. Find the value of $\int\left(5\sqrt{x} + \frac{1}{2\sqrt{x}}\right)dx$.

22. Evaluate $\int \frac{x^3-x^2+x-1}{(x-1)}dx$. [CBSE 2011]

Ans. Let $$I = \int \frac{x^3 - x^2 + x - 1}{(x-1)} dx$$
$$= \int \frac{x^2(x-1) + 1(x-1)}{(x-1)} dx$$
$$= \int \frac{(x^2+1)(x-1)}{(x-1)} dx$$
$$= \int x^2\, dx + \int 1 \cdot dx$$
$$= \frac{x^3}{3} + x + C$$

23. Find the anti-derivative of $\cos 2x - e^{3x}$.

24. Evaluate $\int \cos^{-1}(\sin x)\, dx$. [CBSE 2014]

Ans. $$\int \cos^{-1}(\sin x)\, dx = \int \cos^{-1}\left[\cos\left(\frac{\pi}{2} - x\right)\right] dx$$
$$= \int \left(\frac{\pi}{2} - x\right) dx$$
$$= \frac{\pi}{2}x - \frac{x^2}{2} + C$$

25. Write the value of $\int \sec x(\sec x + \tan x)\, dx$. [CBSE 2011]

26. If $\frac{d}{dx} f(x) = 5x^2 - \frac{1}{x}$ such that $f(3) = 0$, then find $f(x)$.

Ans. We have, $\frac{d}{dx} f(x) = 5x^2 - \frac{1}{x}$

Antiderivative of $\left(5x^2 - \frac{1}{x}\right) = f(x)$

$$\therefore \quad f(x) = \int \left(5x^2 - \frac{1}{x}\right) dx$$
$$= 5\int x^2 dx - \int \frac{1}{x} dx$$
$$= 5\left[\frac{x^3}{3}\right] - \log|x| + C$$
$$\Rightarrow \quad f(x) = \frac{5x^3}{3} - \log|x| + C$$

Given $f(3) = 0$

$$\therefore \quad 0 = \frac{5}{3}(3)^3 - \log|3| + C$$
$$C = \log 3 - 45$$
$$\therefore \quad f(x) = \frac{5x^3}{3} - \log|x| + \log 3 - 45$$

27. Evaluate $\int (3x^2 - 3\cos x + e^x)\, dx$.

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

28. Evaluate $\int \sqrt{x}(5x^2 + 3x + 5)\, dx$.

Ans. Let $$I = \int \sqrt{x}(5x^2 + 3x + 5)\, dx$$
$$= \int (5x^{2+\frac{1}{2}} + 3x^{\frac{1}{2}+1} + 5x^{\frac{1}{2}}) dx$$
$$= \int 5x^{5/2} dx + 3\int x^{3/2} dx + \int 5x^{1/2} dx$$
$$= \frac{5x^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{3x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{5x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$
$$= \frac{10}{7}x^{7/2} + \frac{6}{5}x^{5/2} + \frac{10}{3}x^{3/2} + C$$

29. Verify that: $\int \frac{2x-1}{2x+3} dx = x - \log|(2x+3)^2| + C$ [NCERT Exemplar]

Ans. Let $$\text{L.H.S.} = \int \frac{2x-1}{2x+3} dx$$
$$= \int \frac{2x+3-4}{2x+3} dx$$
$$= \int \frac{2x+3}{2x+3} dx - \int \frac{4}{2x+3} dx$$
$$= \int 1.dx - 4\int \frac{1}{2x+3} dx$$
$$= x - \frac{4}{2}\log|(2x+3)| + C$$
$$= x - \log|(2x+3)^2| + C$$
$$= \text{R.H.S.}$$

Hence, verified.

30. Find the integration of $\frac{\operatorname{cosec}^2 x}{\sec^2 x}$.

Ans. Let $$I = \int \frac{\operatorname{cosec}^2 x}{\sec^2 x} dx$$

$$= \int \frac{\cos^2 x}{\sin^2 x} dx$$

$$= \int \cot^2 x \, dx$$

$$= \int (\text{cosec}^2 x - 1) dx$$

$$= -\cot x - x + C$$

31. Evaluate $\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}} dx$

[NCERT Exemplar]

32. Evaluate $\int \frac{\sec x}{(\sec x + \tan x)} dx$

Ans. Let $I = \int \frac{\sec x}{(\sec x + \tan x)} dx$

$$= \int \frac{\sec x}{\sec x + \tan x} \times \frac{\sec x - \tan x}{\sec x - \tan x} dx$$

$$= \int \frac{\sec^2 x - \sec x \tan x}{\sec^2 x - \tan^2 x} dx$$

$$= \int \frac{\sec^2 x - \sec x \tan x}{1} dx$$

$$= \int \sec^2 x \, dx - \int \sec x \tan x \, dx$$

$$= \tan x - \sec x + C$$

33. Evaluate $\int (e^{x\log a} + e^{a\log x} + e^{b\log b}) \, dx$

Ans. $\int (e^{x\log a} + e^{a\log x} + e^{b\log b}) dx$

$$= \int (e^{\log a^x} + e^{\log x^a} + e^{\log b^b}) dx$$

$$= \int a^x dx + \int x^a dx + \int b^b \, dx$$

$$= \frac{a^x}{\log a} + \frac{x^{a+1}}{a+1} + b^b x + C$$

34. Evaluate $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx$

[NCERT Exemplar]

35. Find $\int \frac{1}{\sqrt{x+2} - \sqrt{x}} dx.$

Ans. Let $I = \int \frac{1}{\sqrt{x+2} - \sqrt{x}} dx$

$$= \int \frac{1}{\sqrt{x+2} - \sqrt{x}} \times \frac{\sqrt{x+2} + \sqrt{x}}{\sqrt{x+2} + \sqrt{x}} dx$$

$$= \int \frac{\sqrt{x+2} + \sqrt{x}}{(\sqrt{x+2})^2 - (\sqrt{x})^2} dx$$

$$= \int \frac{(\sqrt{x+2} + \sqrt{x})}{x+2-x} dx$$

$$= \int \frac{(\sqrt{x+2} + \sqrt{x})}{2} dx$$

$$= \frac{1}{2}\left[\frac{(x+2)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2}\right] + C$$

$$= \frac{2}{3 \times 2}[(x+2)^{3/2} + x^{3/2}] + C$$

$$= \frac{1}{3}[(x+2)^{3/2} + x^{3/2}] + C$$

36. Evaluate $\int \left(x + \frac{1}{x}\right)^3 dx.$

SHORT ANSWER Type-II Questions (SA-II)

[**3** marks]

37. Evaloute the integral

$$\int \left(3\cos x - 5\sin x + \frac{6}{\cos^2 x} - \frac{8}{\sin^2 x} + 3\tan^2 x\right) dx$$

Ans. $\int \left(3\cos x - 5\sin x + \frac{6}{\cos^2 x} - \frac{8}{\sin^2 x} + 3\tan^2 x\right) dx$

$$= \int 3\cos x dx - 5\int \sin x \, dx + 6\int \sec^2 x \, dx$$

$$- 8\int \text{cosec}^2 dx + 3\int (\sec^2 x - 1) dx$$

$$= 3 \sin x + 5 \cos x + 6 \tan x + 8 \cot x + 3 (\tan x - x) + C$$

$$= 3 \sin x + 5 \cos x + 9 \tan x + 8 \cot x - 3x + C$$

38. Evaluate $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} dx$

[NCERT Exemplar]

39. $f'(x) = a \sin x + b \cos x$ and $f'(0) = 4$, $f(0) = 3$, $f\left(\frac{\pi}{2}\right) = 8$, find $f(x)$.

Ans. Given $f'(x) = a \sin x + b \cos x$ ...(i)

Put $x = 0$, we get

$$f'(0) = a \sin 0 + b \cos 0$$

$\Rightarrow \quad 4 = 0 + b$

$\Rightarrow \quad b = 4$

On integrating Eq. (i), we get

$$\int f'(x)\,dx = \int (a\sin x + b\cos x)\,dx$$

$\Rightarrow \quad f(x) = -a\cos x + b\sin x + C \quad$ ...(ii)

Put $x = 0$ and $x = \frac{\pi}{2}$ in Eq. (ii), we get

$f(0) = -a\cos 0 + b\sin 0 + C$

$\Rightarrow \quad 3 = -a + 0 + C$

$\Rightarrow \quad 3 = -a + C \quad$ ...(iii)

and $f\left(\frac{\pi}{2}\right) = -a\cos\left(\frac{\pi}{2}\right) + b\sin\left(\frac{\pi}{2}\right) + C$

$\Rightarrow \quad 8 = 0 + b + C$

$\Rightarrow \quad 8 = b + C$

$\Rightarrow \quad 8 = 4 + C$

$\Rightarrow \quad C = 4$

From eq. (iii),

$3 = -a + 4$

$\Rightarrow \quad a = 4 - 3$

$\Rightarrow \quad a = 1$

Put the values of a, b and C in Eq. (ii), we get

$f(x) = -\cos x + 4\sin x + 4$

or $\quad f(x) = 4\sin x - \cos x + 4$

40. Evaluate $\int \left[x^3 + e^{\log_e x^2} + \left(\frac{e}{3}\right)^x\right] dx$

Ans. Let $I = \int \left[x^3 + e^{\log_e x^2} + \left(\frac{e}{3}\right)^x\right] dx$

$$= \int \left[x^3 + x^2 + \left(\frac{e}{3}\right)^x\right] dx$$

$$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{\left(\frac{e}{3}\right)^x}{\log_e \frac{e}{3}} + C$$

$$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{\left(\frac{e}{3}\right)^x}{(\log_e e - \log_e 3)} + C$$

$$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{\left(\frac{e}{3}\right)^x}{1 - \log_e 3} + C$$

41. Evaluate $\int \frac{2x^4 + 7x^3 + 6x^2}{x^2 + 2x}\,dx.$

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

42. Solve the following integrals

(A) $\int \frac{x^3}{(x+3)^4}\,dx$

(B) $\int \frac{x^6 + 8}{x^2 + 2}\,dx$

Ans. (A) Let,

$$I = \int \frac{x^3}{(x+3)^4}\,dx$$

$$= \int \frac{[(x+3) - 3]^3}{(x+3)^4}\,dx$$

$$= \int \frac{(x+3)^3 - 27 - 9(x+3)^2 + 27(x+3)}{(x+3)^4}\,dx$$

$$= \int \left[\frac{1}{x+3} - \frac{27}{(x+3)^4} - \frac{9}{(x+3)^2} + \frac{27}{(x+3)^3}\right] dx$$

$$= \log|x+3| + \frac{27}{3(x+3)^3} + \frac{9}{(x+3)} - \frac{27}{2(x+3)^2} + C$$

$$= \log|x+3| + \frac{9}{(x+3)^3} + \frac{9}{(x+3)} - \frac{27}{2(x+3)^2} + C$$

(B) Let $\quad I = \int \frac{x^6 + 8}{x^2 + 2}\,dx$

$$= \int \frac{(x^2)^3 + 2^3}{x^2 + 2}\,dx$$

$$= \int \frac{(x^2+2)(x^4 + 4 - 2x^2)}{x^2 + 2}\,dx$$

$$= \int (x^4 + 4 - 2x^2)\,dx$$

$$= \frac{x^5}{5} + 4x - \frac{2x^3}{3} + C$$

METHODS OF INTEGRATION 2

TOPIC 1

INTEGRATION BY SUBSTITUTION

Finding the integral of functions by the method of inspection is not suitable for many functions. So, we need to develop additional techniques or methods of finding the integrals by reducing them into standard forms. One such method is Integration by Substitution.

In this method, we transform the given integral $\int f(x)\,dx$ into another form by changing the independent variable x to t by substituting $x = g(t)$.

It is important to guess what will be the useful substitution. Usually, we make a substitution for a function whose derivative also occurs in the integrand as illustrated below:

Illustration: Integrate each of the following w.r.t. x:

(A) $\dfrac{1}{x + x\log x}$

$$\int \frac{1}{x + x\log x}\,dx = \int \frac{1}{x(1+\log x)}\,dx$$

Put $(1 + \log x) = t$

Differentiating both sides, we get

$$\frac{1}{x}\,dx = dt$$

Substituting the values of $(1 + \log x)$ and dx, we get

$$\int \frac{1}{x(1+\log x)}\,dx = \int \frac{1}{t}\,dt$$

$$= \log|t| + C$$

$$= \log|1 + \log x| + C$$

(B) $\displaystyle\int \frac{x^2}{(2+3x^3)^3}\,dx$

Put $(2 + 3x^3) = t$

Differentiating both sides, we get

$$\Rightarrow 9x^2\,dx = dt \Rightarrow x^2\,dx = \frac{1}{9}dt$$

Substituting the values of $(2 + 3x^3)$ and dx, we get

$$\int \frac{x^2}{(2+3x^3)^3}\,dx = \int \frac{1}{9}\frac{1}{t^3}\,dt$$

$$= \frac{1}{9}\int t^{-3}dt$$

$$= \frac{1}{9}\left(\frac{t^{-3+1}}{-3+1}\right) + C$$

$$= \frac{1}{9}\left(\frac{t^{-2}}{-2}\right) + C$$

$$= -\frac{1}{18}\cdot\frac{1}{t^2} + C$$

$$= -\frac{1}{18}\cdot\frac{1}{(2+3x^3)^2} + C$$

Example 2.1: Find: (A) $\int x\sqrt{1+2x^2}\,dx$

(B) $\displaystyle\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$ [NCERT]

Ans. (A) $\int x\sqrt{1+2x^2}\,dx$

Put $(1 + 2x^2) = t \Rightarrow 4x\,dx = dt \Rightarrow x\,dx = \dfrac{1}{4}dt$

$$\therefore \int x\sqrt{1+2x^2}\,dx = \int \frac{1}{4}\sqrt{t}\,dt$$

$$= \frac{1}{4}\int t^{1/2}\,dt$$

$$= \frac{1}{4}\left(\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) + C$$

$$= \frac{1}{4}\frac{t^{3/2}}{3/2} + C$$

$$= \frac{1}{6}\cdot t^{3/2} + C$$

$$= \frac{1}{6}\cdot(1+2x^2)^{3/2} + C$$

(B) $\displaystyle\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

Put $\tan^{-1}x = t \Rightarrow \dfrac{1}{1+x^2}\,dx = dt$

$$\therefore \int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx = \int e^t\,dt = e^t + C = e^{\tan^{-1}x} + C$$

Example 2.2: Find: (A) $\int \cos 2x\sqrt{\sin 2x}\,dx$

(B) $\displaystyle\int \frac{(1+\log x)^2}{x}\,dx$ [NCERT]

Ans. (A) $\int \cos 2x\sqrt{\sin 2x}\, dx$

Put $\sin 2x = t$

$\Rightarrow \quad 2\cos 2x\, dx = dt$

$\Rightarrow \quad \cos 2x\, dx = \frac{1}{2}dt$

$$\therefore \int \cos 2x\sqrt{\sin 2x}\, dx = \int \frac{1}{2}\sqrt{t}\; dt$$

$$= \frac{1}{2}\int t^{1/2}dt$$

$$= \frac{1}{2}\frac{t^{3/2}}{3/2} + C$$

$$= \frac{1}{3}\cdot t^{3/2} + C$$

$$= \frac{1}{3}\cdot(\sin 2x)^{3/2} + C$$

(B) $\int \frac{(1+\log x)^2}{x}dx$

Put $(1 + \log x) = t \Rightarrow \frac{1}{x}dx = dt$

$$\therefore \int \frac{(1+\log x)^2}{x}dx = \int t^2\, dt$$

$$= \frac{t^{2+1}}{2+1} + C$$

$$= \frac{t^3}{3} + C$$

$$= \frac{(1+\log x)^2}{3} + C$$

Using substitution method, we can easily find the integral of the remaining four trigonometric functions, namely $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$.

(1) $\int \tan x\, dx = \int \frac{\sin x}{\cos x}dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

Thus, $\int \tan x\, dx = -\int \frac{1}{t}dt$

$= -\log|t| + C$

$= -\log|\cos x| + C$, or

$\log|\sec x| + C$

(2) $\int \cot x\, dx = \int \frac{\cos x}{\sin x}dx$

Put $\cos x = t \Rightarrow \cos x\, dx = dt$

Thus, $\int \cot x\, dx = \int \frac{1}{t}dt$

$= \log|t| + C$

$= \log|\sin x| + C$

(3) $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}dx$

Put $(\sec x + \tan x) = t$

$\Rightarrow (\sec x\tan x + \sec^2 x)dx = dt$

or $[\sec x(\sec x + \tan x)]dx = dt$

Thus, $\int \sec x\, dx = \int \frac{1}{t}\, dt$

$= \log|t| + C$

$= \log|\sec x + \tan x| + C$

(4) $\int \operatorname{cosec} x\, dx$

$$= \int \frac{\operatorname{cosec} x(\operatorname{cosec} x + \cot x)}{\operatorname{cosec} x + \cot x}dx$$

Put $(\operatorname{cosec} x + \cot x) = t$

$\Rightarrow (-\operatorname{cosec} x\cot x - \operatorname{cosec}^2 x)dx = dt$

i.e., $[\operatorname{cosec} x(\operatorname{cosec} x + \cot x)]\, dx = -dt$

Thus, $\int \operatorname{cosec} x\, dx = -\int \frac{1}{t}dt$

$= -\log|t| + C$

$= -\log|\operatorname{cosec} x + \cot x| + C$

$= \log|\operatorname{cosec} x - \cot x| + C$

When the integrand involves some trigonometric functions, we also use some known trigonometric identities to find the integral.

TOPIC 2

IMPORTANT TRIGONOMETRIC IDENTITIES

(1) $\sin 2x = 2\sin x\cos x$

$= \frac{2\tan x}{1+\tan^2 x}$

(2) $\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$= 2\cos^2 x - 1$

$= \frac{1-\tan^2 x}{1+\tan^2 x}$

(3) $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

$\cot 2x = \frac{\cot^2 x - 1}{2\cot x}$

(4) $\sin 3x = 3\sin x - 4\sin^3 x$;

$\cos 3x = 4\cos^3 x - 3\cos x$;

$\tan 3x = \frac{3\tan x - \tan^3 x}{1-\tan^2 x}$

(5) $\tan\frac{x}{2} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$

$\cot\frac{x}{2} = \frac{\sin x}{1-\cos x} = \frac{1+\cos x}{\sin x}$

(6) $\sin(x+y) = \sin x\cos y + \cos x\sin y$

$\sin(x-y) = \sin x\cos y - \cos x\sin y$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

$\cos(x-y) = \cos x\cos y + \sin x\sin y$

(7) $\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x\tan y}$

$\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x\tan y}$

(8) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$;

$2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$;

$2\sin A\sin B = \cos(A-B) - \cos(A+B)$

(9) $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$;

$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$

$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$;

$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

(10) $\sin^{-1}(\sin x) = x$; $\cos^{-1}(\cos x) = x$;

$\tan^{-1}(\tan x) = x$; $\operatorname{cosec}^{-1}(\operatorname{cosec} x) = x$;

$\sec^{-1}(\sec x) = x$; $\cot^{-1}(\cot x) = x$

Example 2.3: Find:

(A) $\int \frac{\sin^2 x}{1+\cos x}dx$

(B) $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$ **[NCERT]**

Ans. (A) $\int \frac{\sin^2 x}{1+\cos x}dx = \int \frac{1-\cos^2 x}{1+\cos x}dx$

$= \int \frac{(1-\cos x)(1+\cos x)}{1+\cos x}dx$

$= \int (1-\cos x)\,dx$

$= x - \sin x + C$

(B) $\int \frac{1}{\cos(x-a)\cos(x-b)}dx$

$= \frac{1}{\sin(a-b)}\int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)}dx$

$= \frac{1}{\sin(a-b)}\int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cos(x-b)}dx$

$= \frac{1}{\sin(a-b)}\int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}$

$= \frac{1}{\sin(a-b)}\int \left[\frac{\sin(x-b)\cos(x-a)}{\cos(x-a)\cos(x-b)} - \frac{\cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)}\right]dx$

$= \frac{1}{\sin(a-b)}\int [\tan(x-b) - \tan(x-a)]\,dx$

$= \frac{1}{\sin(a-b)}\{\log|\sec(x-b)| - \log|\sec(x-a)|\} + C$

$= \frac{1}{\sin(a-b)}\log\left|\frac{\sec(x-b)}{\sec(x-a)}\right| + C,$

or $\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right| + C$

TOPIC 3

SOME ELEMENTARY STANDARD INTEGRALS

When the integrand involves some particular functions, we also use substitution to find the integral.

(1) $\int \frac{1}{x^2-a^2}dx = \int \frac{1}{(x-a)(x+a)}dx$

$= \frac{1}{2a}\int \frac{(x+a)-(x-a)}{(x-a)(x+a)}dx$

$= \frac{1}{2a}\left[\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx\right]$

$= \frac{1}{2a}[\log|x-a| - \log|x+a|] + C$

$= \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

(2) $\int \frac{1}{a^2-x^2}dx = \int \frac{1}{(a-x)(a+x)}dx$

$= \frac{1}{2a}\int \frac{(a+x)+(a-x)}{(a-x)(a+x)}dx$

$$= \frac{1}{2a}\left[\int \frac{1}{a-x}dx + \int \frac{1}{a+x}dx\right]$$

$$= \frac{1}{2a}\,[-\log|a-x| + \log|a+x|] + C$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

(3) $\int \frac{1}{x^2+a^2}dx$

Put $x = a\tan t$. Then, $dx = a\sec^2 t\,dt$

$$\therefore \int \frac{1}{x^2+a^2}dx = \int \frac{1}{a^2\tan^2 t + a^2}\,[a\sec^2 t]dt$$

$$= \int \frac{1}{a^2(\tan^2 t + 1)}\,[a\sec^2 t]\,dt$$

$$= \int \frac{1}{a^2\sec^2 t}[a\sec^2 t]\,dt$$

$$= \frac{1}{a}\int 1.dt$$

$$= \frac{t}{a} + C, \text{ or } \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(4) $\int \frac{1}{\sqrt{a^2-x^2}}dx$

Put $x = a\sin t$. Then, $dx = a\cos t\,dt$

$$\therefore \int \frac{1}{\sqrt{a^2-x^2}}dx = \int \frac{1}{\sqrt{a^2 - a^2\sin^2 t}}[a\cos t]\,dt$$

$$= \int \frac{1}{\sqrt{a^2(1-\sin^2 t)}}[a\cos t]\,dt$$

$$= \int \frac{1}{\sqrt{a^2\cos^2 t}}[a\cos t]dt$$

$$= \int 1.dt$$

$$= t + C$$

$$= \sin^{-1}\frac{x}{a} + C$$

(5) $\int \frac{1}{\sqrt{x^2-a^2}}dx$

Put $x = a\sec t$. Then, $dx = a\sec t\tan t\,dt$

$$\therefore \int \frac{1}{\sqrt{x^2-a^2}}dx$$

$$= \int \frac{1}{\sqrt{a^2\sec^2 t - a^2}}\,[a\sec t\tan t]\,dt$$

$$= \int \frac{1}{\sqrt{a^2(\sec^2 t - 1)}}[a\sec t\tan t]dt$$

$$= \int \frac{1}{\sqrt{a^2\tan^2 t}}[a\sec t\tan t]dt$$

$$= \int \sec t.dt$$

$$= \log|\sec t + \tan t| + C$$

$$= \log\left|\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right| + C$$

$$= \log|x + \sqrt{x^2-a^2}| - \log|a| + C$$

$$= \log|x + \sqrt{x^2-a^2}| + C_1$$

where $C_1 = C - \log|a|$

(6) $\int \frac{1}{\sqrt{x^2+a^2}}dx$

Put $x = a\tan t$. Then, $dx = a\sec^2 t\,dt$

$$\therefore \int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{\sqrt{a^2\tan^2 t + a^2}}[a\sec^2 t]\,dt$$

$$= \int \frac{1}{\sqrt{a^2\sec^2 t}}[a\sec^2 t]dt$$

$$= \int \sec t\,dt$$

$$= \log|\sec t + \tan t| + C$$

$$= \log|\sqrt{1+\tan^2 t} + \tan t| + C$$

$$= \log\left|\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right| + C$$

$$= \log|x + \sqrt{x^2+a^2}| - \log|a| + C$$

$$= \log|x + \sqrt{x^2+a^2}| + C_1$$

where $C_1 = C - \log|a|$.

Example 2.4: Find

(A) $\int \frac{3x^2}{x^6+1}dx$

(B) $\int \frac{x^2}{1-x^6}dx$

(C) $\int \frac{1}{9x^2+12x+3}dx$ **[NCERT]**

Ans. (A) $\int \frac{3x^2}{x^6+1}dx$

Put $x^3 = t$. Then, $3x^2\,dx = dt$

$$\therefore \int \frac{3x^2}{x^6+1}dx = \int \frac{dt}{t^2+1} = \int \frac{dt}{t^2+1^2}$$

$$\{\because x^6 = (x^3)^2 = t^2\}$$

$$= \tan^{-1} t + C, \text{ or } \tan^{-1}(x^3) + C$$

(B) $\int \frac{x^2}{1-x^6}dx$

Put $x^3 = t$. Then, $3x^2\,dx = dt \Rightarrow x^2\,dx = \frac{1}{3}dt$

$$\therefore \int \frac{x^2}{1-x^6}dx = \frac{1}{3}\int \frac{dt}{1-t^2} \quad \{\because x^6 = (x^3)^2 = t^2\}$$

$$= \frac{1}{3}\int \frac{dt}{1^2-t^2}$$

$$= \frac{1}{3}\left\{\frac{1}{2\times 1}\log\left|\frac{1+t}{1-t}\right|\right\} + C,$$

$$\text{or } \frac{1}{6}\log\left|\frac{1+x^3}{1-x^3}\right| + C$$

(C) $\int \frac{1}{9x^2+12x+3}dx = \int \frac{1}{(3x+2)^2-(1)^2}dx$

Put $(3x + 2) = t$. Then, $3dx = dt \Rightarrow dx = \frac{1}{3}dt$

$$\therefore \int \frac{1}{9x^2+12x+3}dx$$

$$= \int \frac{1}{(3x+2)^2-(1)^2}dx$$

$$= \frac{1}{3}\int \frac{dt}{t^2-(1)^2}$$

$$= \frac{1}{3}\left\{\frac{1}{2\times 1}\log\left|\frac{t-1}{t+1}\right|\right\} + C$$

$$= \frac{1}{6}\log\left|\frac{(3x+2)-1}{(3x+2)+1}\right| + C$$

$$= \frac{1}{6}\log\left|\frac{3x+1}{3x+3}\right| + C$$

Example 2.5: Find (A) $\int \frac{1}{\sqrt{9-25x^2}}dx$ (B) $\int \frac{1}{\sqrt{1+4x^2}}dx$ (C) $\int \frac{1}{\sqrt{5x^2-2x}}dx$ [NCERT]

Ans. (A) $\int \frac{1}{\sqrt{9-25x^2}}dx = \int \frac{1}{\sqrt{3^2-(5x)^2}}dx$

Put $5x = t$. Then, $5dx = dt \Rightarrow dx = \frac{1}{5}dt$

$$\therefore \int \frac{1}{\sqrt{9-25x^2}}dx = \int \frac{1}{\sqrt{3^2-(5x)^2}}dx$$

$$= \frac{1}{5}\int \frac{1}{\sqrt{3^2-t^2}}dt$$

$$= \frac{1}{5}\sin^{-1}\frac{t}{3} + C,$$

$$\text{or } \frac{1}{5}\sin^{-1}\frac{5x}{3} + C$$

(B) $\int \frac{1}{\sqrt{1+4x^2}}dx = \int \frac{1}{\sqrt{1^2+(2x)^2}}dx$

Put $2x = t$. Then, $2dx = dt \Rightarrow dx = \frac{1}{2}dt$

$$\therefore \int \frac{1}{\sqrt{1+4x^2}}dx = \int \frac{1}{\sqrt{1^2+(2x)^2}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1^2+t^2}}dt$$

$$= \frac{1}{2}\log|t + \sqrt{1+t^2}| + C$$

$$= \frac{1}{2}\log|2x + \sqrt{1+4x^2}| + C$$

(C) $\int \frac{1}{\sqrt{5x^2-2x}}dx$

$$= \int \frac{1}{\sqrt{5\left(x^2-\frac{2x}{5}\right)}}dx$$

$$= \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{\left(x-\frac{1}{5}\right)^2-\left(\frac{1}{5}\right)^2}}dx$$

$$= \frac{1}{\sqrt{5}}\int \frac{1}{\sqrt{t^2-\left(\frac{1}{5}\right)^2}}dt,$$

$$\text{where } x - \frac{1}{5} = t$$

$$= \frac{1}{\sqrt{5}}\log\left|t + \sqrt{t^2-\left(\frac{1}{5}\right)^2}\right| + C$$

$$= \frac{1}{\sqrt{5}}\log\left|x - \frac{1}{5} + \sqrt{x^2-\frac{2x}{5}}\right| + C$$

(1) Integral of the Type $\int \frac{1}{ax^2+bx+c}dx$

To evaluate such integral, we use the following steps:

Step 1: Write $ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right]$

$$= a\left[\left(x+\frac{b}{2a}\right)^2 + \left(\frac{c}{a}-\frac{b^2}{4a^2}\right)\right]$$

Step 2: Put $\left(x+\frac{b}{2a}\right)=t$ and get $dx = dt$

Step 3: Write $\left(\frac{c}{a}-\frac{b^2}{4a^2}\right)=\pm k^2$ and reduce the given integral to the form $\frac{1}{a}\int\frac{1}{t^2\pm k^2}dt$, which can be evaluated easily.

(2) Integral of the Type $\int\frac{1}{\sqrt{ax^2+bx+c}}dx$

To evaluate such integral, we do the same steps as in (1). The only difference is, here the given integral is reduced to $\frac{1}{a}\int\frac{1}{\sqrt{t^2\pm k^2}}dx$, which can be evaluated easily.

(3) Integral of the Type $\int\frac{px+q}{ax^2+bx+c}dx$

To evaluate such integral, we use the following steps:

Step 1: Express $px+q = A\left\{\frac{d}{dx}(ax^2+bx+c)\right\}+B$

$= A(2ax+b)+B$

Step 2: Find the values of A and B by comparing the coefficients of like powers of x from both sides.

Step 3: Write the given integral as

$$\int\frac{A(2ax+b)+B}{ax^2+bx+c}dx$$

$$= A\int\frac{2ax+b}{ax^2+bx+c}dx + B\int\frac{1}{ax^2+bx+c}dx$$

And hence evaluate the two integrals.

(4) Integral of the Type $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$

To evaluate such integral, we do the same steps as in (3). The only difference is, here the given integral is reduced to $A\int\frac{2ax+b}{\sqrt{ax^2+bx+c}}dx$ $+B\int\frac{1}{\sqrt{ax^2+bx+c}}dx$, which can be evaluated easily.

Example 2.6: Find: (A) $\int\frac{1}{9x^2+6x+5}dx$

(B) $\frac{1}{\sqrt{7-6x-x^2}}dx$

(C) $\int\frac{x+3}{x^2-2x-5}dx$

(D) $\int\frac{x+2}{\sqrt{x^2+2x+3}}dx$ **[NCERT]**

Ans. (A) Let $I = \int\frac{1}{9x^2+6x+5}dx$

Here, $9x^2+6x+5 = 9\left\{x^2+\frac{2}{3}x+\frac{5}{9}\right\}$

$$= 9\left\{\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2\right\}$$

Thus, $I = \frac{1}{9}\int\frac{1}{\left(x+\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2}dx$

$$= \frac{1}{9}\left\{\frac{1}{2/3}\tan^{-1}\frac{x+\frac{1}{3}}{2/3}\right\}+C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right)+C$$

(B) Let $I = \int\frac{1}{\sqrt{7-6x-x^2}}dx$

Here,

$$7-6x-x^2 = 16-(9+6x+x^2)$$
$$= (4)^2-(x+3)^2$$

$\therefore$ $I = \int\frac{1}{\sqrt{(4)^2-(x+3)^2}}dx$

$$= \sin^{-1}\left(\frac{x+3}{4}\right)+C$$

(C) Let $I = \int\frac{x+3}{x^2-2x-5}dx$

Let $x+3 = A\frac{d}{dx}(x^2-2x-5)+B$

$$= A(2x-2)+B$$

Equating the coefficients of x and the constant terms from both sides, we have

$$1 = 2A \text{ and } 3 = B-2A$$

$$\Rightarrow A = \frac{1}{2}, B = 4$$

Thus, $I = \int\frac{A(2x-2)+B}{x^2-2x-5}dx$

$$= \int\frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int\frac{2x-2}{x^2-2x-5}dx + 4\int\frac{1}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int\frac{2x-2}{x^2-2x-5}dx + 4\int\frac{1}{(x-1)^2-(\sqrt{6})^2}dx$$

Put $x^2 - 2x - 5 = t$

$\Rightarrow \quad (2x-2)dx = dt$

$$\therefore \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}dx$$

$$= \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| + C_1$$

Thus, $I = \frac{1}{2}\log|x^2-2x-5| + C_1 + 4$

$$\int \frac{1}{(x-1)^2+(\sqrt{3})^2}dx$$

$$= \frac{1}{2}\log|x^2-2x-5|$$

$$+\frac{4}{2\sqrt{6}}\log\left|\frac{(x-1)-\sqrt{6}}{(x-1)+\sqrt{6}}\right| + C_1 + C_2$$

$$= \frac{1}{2}\log|x^2-2x-5|$$

$$+\frac{2}{\sqrt{6}}\log\left|\frac{(x-1)-\sqrt{6}}{(x-1)+\sqrt{6}}\right| + C$$

where, $C = C_1 + C_2$.

(D) Let $I = \int \frac{x+2}{\sqrt{x^2+2x+3}}dx$

Let $\quad x + 2 = A\frac{d}{dx}(x^2+2x+3) + B$

$= A(2x+2) + B$

Equating the coefficients of x and the constant terms from both sides, we have

$$1 = 2A \text{ and } 2 = B + 2A$$

$$\Rightarrow A = \frac{1}{2}, B = 1$$

Thus,

$$I = \int \frac{A(2x+2)+B}{\sqrt{x^2+2x+3}}dx$$

$$= \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}dx + \int \frac{1}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{t}}dt + \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}dx$$

where, $t = x^2 + 2x + 3 \Rightarrow dt = (2x+2)dx$

$$= \frac{1}{2}\left(\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right)$$

$$+\log\left|(x+1)+\sqrt{(x+1)^2+(\sqrt{2})^2}\right| + C_1 + C_2$$

$$= \sqrt{x^2+2x+3}$$

$$+\log|(x+1)+\sqrt{x^2+2x+3}| + C$$

where, $C = C_1 + C_2$.

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. The value of $\int \frac{2x}{(2x+1)^2}dx$ is:

(a) $\frac{1}{2}\left[\log|2x+1| + \frac{1}{(2x+1)}\right] + C$

(b) $\frac{1}{2}\left[\log|2x+1| - \frac{1}{(2x+1)}\right] + C$

(c) $\left[\log|2x+1| + \frac{1}{(2x+1)}\right] + C$

(d) None of these

Ans. *(a)* $\frac{1}{2}\left[\log|2x+1| + \frac{1}{(2x+1)}\right] + C$

Explanation: Let $I = \int \frac{2x}{(2x+1)^2}dx$

Put $2x + 1 = t \Rightarrow 2dx = dt$

$$\therefore \quad I = \int \frac{(t-1)}{t^2} \times \frac{dt}{2}$$

$$= \frac{1}{2}\int \left(\frac{1}{t} - \frac{1}{t^2}\right)dt$$

$$= \frac{1}{2}\left[\log t + \frac{1}{t}\right] + C$$

$$= \frac{1}{2}\left[\log|2x+1| + \frac{1}{(2x+1)}\right] + C$$

2. $\int \frac{1}{\sqrt{1+\cos 2x}}dx$ is:

(a) $\frac{1}{\sqrt{2}}\log|\sec x + \tan x| + C$

(b) $\frac{1}{\sqrt{2}}\log|\sec x - \tan x| + C$

(c) $\log|\sec x + \tan x| + C$

(d) None of these

Ans. *(a)* $\frac{1}{\sqrt{2}}\log|\sec x + \tan x| + C$

Explanation: Let

$$I = \int \frac{1}{\sqrt{1+\cos 2x}}dx$$

$$= \int \frac{1}{\sqrt{2\cos^2 x}}dx$$

$$= \frac{1}{\sqrt{2}}\int \sec x\,dx$$

$$= \frac{1}{\sqrt{2}}\log|\sec x + \tan x| + C$$

3. $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}dx$ **is:**

(a) $10^x - x^{10} + C$ (b) $10^x + x^{10} + C$

(c) $(10^x - x^{10})^{-1} + C$ (d) $\log(10^x + x^{10}) + C$

Ans. *(d)* $\log(10^x + x^{10}) + C$

Explanation: Let $I = \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}dx$

Put $x^{10} + 10^x = t.$

Then, $(10x^9 + 10^x \times \log_e 10)dx = dt$

Now, $I = \int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x}dx$

$$= \int \frac{1}{t}dt = \log|t| + C,$$

or $\log(x^{10} + 10^x) + C$

4. $\int \frac{1}{\sin^2 x \cos^2 x}dx$ **equals:**

(a) $\tan x + \cot x + C$

(b) $\tan x - \cot x + C$

(c) $\tan x \cot x + C$

(d) $\tan x - \cot 2x + C$

5. $\int \frac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}dx$ **is:**

(a) $\tan x + \cot x + C$

(b) $\tan x - \cot x + C$

(c) $\tan x \cot x + C$

(d) $\tan x - \cot 2x + C$

6. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$ **is equal to:**

(a) $\sin(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$

(b) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$

(c) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$

(d) $\sin(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right| + C$

[NCERT Exemplar]

7. $\int \frac{(x+1)e^x}{\cos^2(xe^x)}dx$ **equals:**

(a) $-\cot(xe^x) + C$ (b) $\tan(xe^x) + C$

(c) $\tan(e^x) + C$ (d) $\cot(e^x) + C$

Ans. *(b)* $\tan(xe^x) + C$

Explanation: Let $I = \int \frac{(x+1)e^x}{\cos^2(xe^x)}dx$

Put $xe^x = t.$

Then, $(xe^x + e^x)dx = dt$

or $(x+1)e^x dx = dt$

Now, $I = \int \frac{(x+1)e^x}{\cos^2(xe^x)}dx$

$$= \int \frac{1}{\cos^2 t}dt$$

$$= \int \sec^2 t\,dt$$

$$= \tan t + C \text{ or } \tan(xe^x) + C$$

8. $\int \frac{1}{x^2 + 2x + 2}dx$ **is:**

(a) $x\tan^{-1}(x+1) + C$

(b) $\tan^{-1}(x+1) + C$

(c) $(x+1)\tan^{-1}(x+1) + C$

(d) $\tan^{-1}x + C$

9. $\int \frac{1}{\sqrt{9x - 4x^2}}dx$ **is:**

(a) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$

(b) $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right) + C$

(c) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$

(d) $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{8}\right) + C$

Ans. *(b)* $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right) + C$

Explanation: $\int \frac{1}{\sqrt{9x-4x^2}}dx$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{9}{4}x - x^2}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{81}{64} - \left(x^2 - \frac{9}{4}x + \frac{81}{64}\right)}}dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2}}dx$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{x-\frac{9}{8}}{\frac{9}{8}}\right) + C \text{ or } \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right) + C$$

10. $\int \frac{\sec^2 x}{3+\tan x}dx$ is:

(a) $\log|3+\tan x| + C$ (b) $\log|3-\tan x| + C$
(c) $\log|\sec x| + C$ (d) $\log|\tan x| + C$

11. $\int \frac{1}{4x^2+12x+13}dx$ is

(a) $\frac{1}{2}\tan^{-1}\left(\frac{2x+3}{2}\right) + C$

(b) $\frac{1}{4}\tan^{-1}\left(\frac{2x+3}{2}\right) + C$

(c) $\frac{1}{4}\tan^{-1}(2x+3) + C$

(d) $\frac{1}{2}\tan^{-1}(2x+3) + C$

Ans. *(b)* $\frac{1}{4}\tan^{-1}\left(\frac{2x+3}{2}\right) + C$

Explanation: Let

$$I = \int \frac{dx}{4x^2+12x+13}$$

$$= \int \frac{dx}{(2x)^2 + 2\times 2\times 3x + (3)^2 + (2)^2}$$

$$= \int \frac{dx}{(2x+3)^2 + (2)^2}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{2x+3}{2}\right)\right] \times \frac{1}{2} + C$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{2x+3}{2}\right) + C$$

12. $\int \frac{dx}{\sqrt{16-6x-x^2}}dx$ is equal to:

(a) $\sin^{-1}(x+3) + C$

(b) $\frac{1}{5}\sin^{-1}\left(\frac{x+3}{5}\right) + C$

(c) $\sin^{-1}\left(\frac{x+3}{5}\right) + C$

(d) $\sin^{-1}\left(\frac{x-3}{5}\right) + C$

13. $\int x^2 e^{x^3} dx$ equals:

(a) $\frac{1}{3}e^{x^3} + C$ (b) $\frac{1}{3}e^{x^4} + C$

(c) $\frac{1}{2}e^{x^3} + C$ (d) $\frac{1}{2}e^{x^2} + C$

[CBSE 2020]

Ans. *(a)* $\frac{1}{3}e^{x^3} + C$

Explanation: Let $I = \int x^2 e^{x^3} dx$

Put $x^3 = t \Rightarrow 3x^2dx = dt \Rightarrow x^2\,dx = \frac{dt}{3}$

$$\therefore \quad I = \frac{1}{3}\int e^t\,dt$$

$$= \frac{1}{3}e^t + C$$

$$= \frac{1}{3}e^{x^3} + C$$

14. $\int \frac{1}{16-9x^2}dx$ is

(a) $\frac{1}{12}\log\left|\frac{4+3x}{4-3x}\right| + C$

(b) $\frac{1}{24}\log\left|\frac{3x-4}{3x+4}\right| + C$

(c) $\frac{1}{24}\log\left|\frac{4+3x}{4-3x}\right| + C$

(d) $\frac{1}{12}\log\left|\frac{3x+4}{3x-4}\right| + C$

Ans. *(c)* $\frac{1}{24}\log\left|\frac{4+3x}{4-3x}\right| + C$

Explanation: Let

$$I = \int \frac{1}{16-9x^2}dx$$

$$= \int \frac{1}{(4)^2 - (3x)^2} dx$$

$$= \frac{1}{3^2} \int \frac{dx}{\left(\frac{4}{3}\right)^2 - (x)^2}$$

$$= \frac{1}{9} \times \frac{1}{2 \times \frac{4}{3}} \log \left| \frac{\frac{4}{3} + x}{\frac{4}{3} - x} \right| + C$$

$$= \frac{1}{24} \log \left| \frac{4 + 3x}{4 - 3x} \right| + C$$

VERY SHORT ANSWER Type Questions (VSA)

[**1** mark]

15. Evalaute $\int \sin x \sin(\cos x) dx$.

Ans. Let $\quad I = \int \sin x \sin(\cos x)\, dx$

Put $\cos x = t$

$\Rightarrow \quad -\sin x\, dx = dt$

$\therefore \quad I = \int \sin(t)(-dt)$

$= -\int \sin t\, dt = \cos t + C$

$= \cos(\cos x) + C$

16. Simplify $\int x\sqrt{3 + 4x^2}\, dx$.

17. Evaluate $\int \frac{e^{\tan^{-1} x}}{1 + x^2} dx$. **[CBSE 2011]**

Ans. Let $\quad I = \int \frac{e^{\tan^{-1} x}}{1 + x^2} dx$

Put $\quad \tan^{-1} x = t$

$\Rightarrow \quad \frac{1}{1 + x^2} dx = dt$

$\therefore \quad I = \int e^t\, dt$

$= e^t + C$

$= e^{\tan^{-1} x} + C$

18. Find the value of $\int \frac{3x^2 + 7}{x^3 + 7x + 5} dx$.

Ans. Let $\quad I = \int \frac{3x^2 + 7}{x^3 + 7x + 5} dx$

Put $x^3 + 7x + 5 = t$

$\Rightarrow \quad (3x^2 + 7)dx = dt$

$\therefore \quad I = \int \frac{dt}{t}$

$= \log |t| + C$

$= \log |x^3 + 7x + 5| + C$

19. Evaluate $\int \frac{1 - \sin x}{x + \cos x} dx$.

20. Evaluate $\int \frac{2}{x(5 + \log x)} dx$.

Ans. Let $\quad I = \int \frac{2}{x(5 + \log x)} dx$

Put $5 + \log x = t$

$\Rightarrow \quad 0 + \frac{1}{x} dx = dt$

$\Rightarrow \quad \frac{1}{x} dx = dt$

$\therefore \quad I = \int \frac{2}{t} dt$

$= 2 \log |t| + C$

$= 2 \log |5 + \log x| + C$

21. Find $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$. **[CBSE 2017]**

22. Find $\int \frac{\sin^6 x}{\cos^8 x} dx$. **[CBSE 2014]**

Ans. Let $\quad I = \int \frac{\sin^6 x}{\cos^8 x} dx$

$= \int \tan^6 x \sec^2 x\, dx$

Put $\tan x = t$

$\Rightarrow \quad \sec^2 x\, dx = dt$

$\therefore \quad I = \int t^6\, dt$

$= \frac{t^7}{7} + C$

$= \frac{(\tan x)^7}{7} + C$

23. Integrate $\int f'(ax + b)[f(ax + b)]^n dx$.

Ans. Let $\quad I = \int f'(ax + b)[f(ax + b)]^n\, dx$

Put $f(ax + b) = t$

$\Rightarrow \quad f'(ax + b) \times a\, dx = dt$

$\therefore \quad I = \int t^n \times \frac{dt}{a}$

$= \frac{1}{a}\frac{t^{n+1}}{n+1} + C$

$= \frac{[f(ax+b)]^{n+1}}{a(n+1)} + C$

24. Evaluate $\int \frac{dx}{x^2+2x+5}$

25. Evaluate $\int \frac{1}{\sqrt{4+3x-x^2}}dx$

26. Evaluate $\int \frac{1-\cos 2x}{1+\cos 2x}dx$

Ans. Let $I = \int \frac{1-\cos 2x}{1+\cos 2x}dx$

$= \int \frac{2\sin^2 x}{2\cos^2 x}dx$

$= \int \tan^2 x\, dx$

$= \int (\sec^2 x - 1)\, dx$

$= \tan x - x + C$

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

27. Solve $\int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x}dx.$

Ans. Let $I = \int \frac{\operatorname{cosec} x}{\operatorname{cosec} x - \cot x}dx$

$= \int \frac{\frac{1}{\sin x}}{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}dx$

$= \int \frac{1}{1-\cos x}dx$

$= \int \frac{1}{2\sin^2 \frac{x}{2}}dx$

$= \frac{1}{2}\int \operatorname{cosec}^2 \frac{x}{2}dx$

$= -\frac{1}{2}\cot\frac{x}{2} \times 2 + C$

$= -\cot\frac{x}{2} + C$

28. Evaluate $\int \frac{x^2}{\sqrt{x^6+b^6}}dx.$

29. Evaluate $\int \frac{\sin x}{\sin(x-a)}dx.$

Ans. Let $I = \int \frac{\sin x}{\sin(x-a)}dx$

$= \int \frac{\sin[(x-a)+a]\,dx}{\sin(x-a)}$

$= \int \frac{\sin(x-a)\cos a + \cos(x-a)\sin a}{\sin(x-a)}dx$

$= \cos a \int 1\, dx + \sin a \int \cot(x-a)\, dx$

$= (\cos a)\, x + \sin a \log|\sin(x-a)| + C$

30. Find $\int \frac{\sin 2x}{\sqrt{9-\cos^4 x}}dx.$

[CBSE Term-2 SQP 2022]

Ans. Put $\cos^2 x = t$

$\Rightarrow -2\cos x \sin x dx = dt$

$\Rightarrow \quad \sin 2x\, dx = -dt$

The given integral $= -\int \frac{dt}{\sqrt{3^2-t^2}} = -\sin^{-1}\frac{t}{3} + c$

$= -\sin^{-1}\frac{\cos^2 x}{3} + c$

[CBSE Marking Scheme Term-2 SQP 2022]

31. Evaluate $\int \tan^2 x \sec^4 x\, dx.$

[NCERT Exemplar]

Ans. Let $I = \int \tan^2 x \sec^4 x\, dx$

$= \int \tan^2 x . \sec^2 x . \sec^2 x . dx$

$= \int \tan^2 x(1+\tan^2 x)\sec^2 x . dx$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$\therefore \quad I = \int t^2(1+t^2)\, dt$

$= \int (t^2 + t^4)\, dt$

$= \frac{t^3}{3} + \frac{t^5}{5} + C$

$= \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$

Caution

➥ *Remember that $\sec^2 x = 1 + \tan^2 x$.*

32. Find the integral of $\int \frac{\sec x}{\sec 2x} dx$.

33. Evaluate $\int \frac{dx}{x\sqrt{x^4 - 1}}$. **[NCERT Exemplar]**

Ans. Let $I = \int \frac{dx}{x\sqrt{x^4-1}}$

Put $x^2 = \sec\theta$, where $\theta = \sec^{-1} x^2$

$\Rightarrow \quad 2x\, dx = \sec\theta \tan\theta\, d\theta$

$\Rightarrow \quad dx = \frac{\sec\theta\tan\theta}{2x} d\theta$

$\therefore \quad I = \int \frac{\sec\theta\tan\theta}{2x^2\sqrt{x^4-1}} d\theta$

$= \int \frac{\sec\theta\tan\theta}{2\sec\theta\sqrt{\sec^2\theta-1}} d\theta$

$= \frac{1}{2}\int \frac{\tan\theta}{\tan\theta} d\theta$

$= \frac{1}{2}\theta + C$

$= \frac{1}{2}\sec^{-1} x^2 + C$

34. Evaluate $\int 3^{3^{3^x}} 3^{3^x} 3^x dx$

Ans. Let $I = \int 3^{3^{3^x}} 3^{3^x} 3^x dx$

Put $3^{3^{3^x}} = t$

$\Rightarrow 3^{3^{3^x}} \times 3^{3^x} \times 3^x \times (\log 3)^3 dx = dt$

$\Rightarrow 3^{3^{3^x}} \times 3^{3^x} \times 3^x dx = \frac{1}{(\log 3)^3} dt$

$\therefore \quad I = \int \frac{dt}{(\log 3)^3} = \frac{t}{(\log 3)^3} + C$

$= \frac{3^{3^{3^x}}}{(\log 3)^3} + C$

35. Find $\int \sqrt{1 - \sin 2x}\, dx, \frac{\pi}{4} < x < \frac{\pi}{2}$

[CBSE 2019]

36. Integrate:

$\int \frac{\sin a \cos a}{2\cos^2 a - 1} da$

Show your steps.

[CBSE Question Bank 2022]

Ans. Let $I = \int \frac{\sin a \cos a}{2\cos^2 a - 1} dt$

Put $2\cos^2 a - 1 = t$

$\Rightarrow -4\cos a \sin a\, da = dt$

$\Rightarrow \quad \sin a \cos a\, da = \frac{-dt}{4}$

$\therefore \quad I = \frac{-1}{4}\int\frac{dt}{t} = \frac{-1}{4}\log t + C$

$I = -\frac{1}{4}\log|2\cos^2 a - 1| + C$

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

37. Evaluate the following:

(A) $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} dx, 0 \le x \le \frac{\pi}{2}$

(B) $\int \frac{1+\cos 4x}{\cot x - \tan x} dx$

Ans. (A) $\int \tan^{-1}\left\{\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right\} dx, 0 < x < \frac{\pi}{2}$

$= \int \tan^{-1}\left\{\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right\} dx$

$= \int \tan^{-1}\{\sqrt{\tan^2 x}\} dx$

$= \int x\, dx$

$= \frac{x^2}{2} + C$

(B) $\int \frac{1+\cos 4x}{\cot x - \tan x} dx$

$$= \int \frac{2\cos^2 2x}{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}} dx$$

$$= \int \frac{2\cos^2 2x \sin x \cos x}{\cos^2 x - \sin^2 x} dx$$

$$= \int \frac{\cos^2 2x \sin 2x}{\cos 2x} dx$$

$$= \int \frac{2}{2} \cos 2x \sin 2x \, dx$$

$$= \frac{1}{2} \int \sin 4x \, dx$$

$$= -\frac{1}{8} \cos 4x + C$$

38. Find $\int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$. **[CBSE 2016]**

Ans. Let $I = \int \frac{(3\sin\theta - 2)\cos\theta}{5 - \cos^2\theta - 4\sin\theta} d\theta$

$$= \int \frac{(3\sin\theta - 2)\cos\theta}{5 - (1 - \sin^2\theta) - 4\sin\theta} d\theta$$

$$= \int \frac{3\sin\theta\cos\theta}{4 + \sin^2\theta - 4\sin\theta} d\theta - \int \frac{2\cos\theta}{(4 + \sin^2\theta - 4\sin\theta)} d\theta$$

$$= 3I_1 - 2I_2 \quad \text{...(i)}$$

where $I_1 = \int \frac{\sin\theta\cos\theta}{4 + \sin^2\theta - 4\sin\theta} d\theta$

and $I_2 = \int \frac{\cos\theta}{(4 + \sin^2\theta - 4\sin\theta)} d\theta$

Consider $I_1 = \int \frac{\sin\theta\cos\theta}{4 + \sin^2\theta - 4\sin\theta} d\theta$

Put $\sin^2\theta = t$

$\Rightarrow$ $2\sin\theta\cos\theta \, d\theta = dt$

$$\therefore \quad I_1 = \frac{1}{2} \int \frac{dt}{4 + t - 4\sqrt{t}}$$

$$= \frac{1}{2} \int \frac{dt}{(\sqrt{t} - 2)^2}$$

Put $(\sqrt{t} - 2) = u \Rightarrow \frac{1}{2\sqrt{t}} dt = du$

$\Rightarrow \quad \frac{1}{2\sqrt{t}} dt = du$

$\Rightarrow dt = 2(u + 2) du$

$\therefore \quad I_1 = \int \frac{u+2}{u^2} du$

$$= \int \frac{du}{u} + 2\int \frac{du}{u^2}$$

$$= \log |u| - \frac{2}{u} + C_1$$

$$= \log |\sqrt{t} - 2| - \frac{2}{\sqrt{t} - 2} + C_1$$

$$= \log |\sin\theta - 2| - \frac{2}{\sin\theta - 2} + C_1 \quad \text{...(ii)}$$

Consider $I_2 = \int \frac{\cos\theta}{4 + \sin^2\theta - 4\sin\theta} d\theta$

Put $\sin\theta = v$

$\Rightarrow$ $\cos\theta \, d\theta = dv$

$$I_2 = \int \frac{dv}{4 + u^2 - 4v} = \int \frac{dv}{(v-2)^2}$$

$$= \frac{-1}{(v-2)} + C_2 = -\frac{1}{\sin\theta - 2} + C_2 \quad \text{...(iii)}$$

From Eqs. (i), (ii) and (iii), we get

$$I = 3\log|\sin\theta - 2| - \frac{6}{(\sin\theta - 2)} + \frac{2}{(\sin\theta - 2)} + C_1 + C_2$$

$$= 3\log|\sin\theta - 2| - \frac{4}{(\sin\theta - 2)} + C$$

where $C = C_1 + C_2$

39. **Evaluate** $\int \frac{\cos 2x - \cos 4x}{1 - \cos 2x} dx$.

40. **Evaluate:** $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$.

[NCERT Exemplar]

Ans. Let $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$

$$= \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{(\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{(1)(\sin^4 x - \sin^2 x \cos^2 x + \cos^4 x)}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^4 x}{\sin^2 x \cos^2 x} dx - \int \frac{\sin^2 x \cos^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^4 x}{\sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin^2 x}{\cos^2 x} dx - \int 1 \, dx + \int \frac{\cos^2 x}{\sin^2 x} dx$$

$$= \int \tan^2 x \, dx - \int 1.dx + \int \cot^2 x \, dx$$

$$= \int(\sec^2 x - 1)\,dx - \int 1.\,dx + \int(\text{cosec}^2 x - 1)dx$$

$$= \int \sec^2 x\,dx + \int \text{cosec}^2 x\,dx - 3\int dx$$

$$= \tan x - \cot x - 3x + C$$

Caution

- *Use $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ for simplifying the integral.*

41. **Evaluate**

$$\int \tan^{-1}(\sec x + \tan x)\,dx, -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

Ans. Let $I = \int \tan^{-1}(\sec x + \tan x)\,dx$

$$= \int \tan^{-1}\left(\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right)dx$$

$$= \int \tan^{-1}\left(\frac{1 + \sin x}{\cos x}\right)dx$$

$$= \int \tan^{-1}\left(\frac{1 + \cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right)}\right)$$

$$= \int \tan^{-1}\left(\frac{2\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)}{2\sin\left(\frac{\pi}{4} - \frac{x}{2}\right)\cos\left(\frac{\pi}{4} - \frac{x}{2}\right)}\right)dx$$

$$= \int \tan^{-1}\left(\cot\left(\frac{\pi}{4} - \frac{x}{2}\right)\right)dx$$

$$= \int \tan^{-1}\left[\tan\left\{\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{x}{2}\right)\right\}\right]dx$$

$$= \int \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)dx$$

$$= \int\left(\frac{\pi}{4} + \frac{x}{2}\right)dx$$

$$= \frac{\pi}{4}x + \frac{x^2}{4} + C$$

42. **Evaluate the following integral:** $\int \frac{\sin x}{1 + \sin x}\,dx$

43. **Integrate the function $\frac{\cos(x + a)}{\sin(x + b)}$ with respect to x.** **[CBSE 2019]**

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

44. **Evaluate the following integrals:**

(A) $\int \cos 2x \cos 4x \cos 6x\,dx$

(B) $\int \sin^4 x \cos^4 x\,dx$

Ans. (A) Let

$$I = \int(\cos 2x \cos 4x \cos 6x)\,dx$$

$$= \frac{1}{2}\int(2\cos 2x \cos 4x)\cos 6x\,dx$$

$$= \frac{1}{2}\int[\cos(2x + 4x) + \cos(2x - 4x)]\cos 6x\,dx$$

$$= \frac{1}{2}\int[\cos 6x + \cos 2x]\cos 6x\,dx$$

$$= \frac{1}{4}\int[2\cos^2 6x + 2\cos 6x \cos 2x]\,dx$$

$$= \frac{1}{4}\int(1 + \cos 12x + \cos 8x + \cos 4x)\,dx$$

$$= \frac{1}{4}\left[x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4}\right] + C$$

(B) Let

$$I = \int \sin^4 x \cos^4 x\,dx$$

$$= \frac{1}{16}(2\sin x \cos x)^4\,dx$$

$$= \frac{1}{16}\int(\sin 2x)^4\,dx$$

$$= \frac{1}{16}\int\left(\frac{1 - \cos 4x}{2}\right)^2 dx$$

$$= \frac{1}{16 \times 4}\int(1 + \cos^2 4x - 2\cos 4x)dx$$

$$= \frac{1}{64}\int\left[1 + \left(\frac{1 + \cos 8x}{2}\right) - 2\cos 4x\right]dx$$

$$= \frac{1}{64}\left[\frac{3}{2}x + \frac{1}{2} \times \frac{\sin 8x}{8} - \frac{2 \times \sin 4x}{4}\right] + C$$

$$= \frac{1}{1024}[24x + \sin 8x - 8\sin 4x] + C$$

45. Evaluate $\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$

Ans. Let $I = \int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$

Dividing numerator and denominator of integrand by $\cos^4 x$, we get

$$I = \int \frac{\frac{1}{\cos^4 x}}{\tan^4 x + \tan^2 x + 1} dx$$

$$= \int \frac{(\sec^2 x)(\sec^2 x)}{\tan^4 x + \tan^2 x + 1} dx$$

$$= \int \frac{(\tan^2 x + 1)\sec^2 x}{\tan^4 x + \tan^2 x + 1} dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$\therefore \quad I = \int \frac{t^2 + 1}{t^4 + t^2 + 1} dt$$

Dividing numerator and denominator of integrand by t^2, we get

$$I = \int \frac{1 + \frac{1}{t^2}}{t^2 + 1 + \frac{1}{t^2}} dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 3} dt$$

Again put $t - \frac{1}{t} = v \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dv$

$$\therefore \quad I = \int \frac{dv}{v^2 + (\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{v}{\sqrt{3}}\right) + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{\left(t - \frac{1}{t}\right)}{\sqrt{3}} + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{3}t}\right) + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{3}\tan x}\right) + C$$

SPECIAL METHODS OF INTEGRATION 3

TOPIC 1

INTEGRATION OF RATIONAL ALGEBRAIC FUNCTIONS BY USING PARTIAL FRACTIONS

A rational fraction is defined as the ratio of two polynomials in the form $\frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$. If the degree of $P(x)$ is less than the degree of $Q(x)$, then the rational function is called proper fraction, otherwise is called improper fraction. The improper rational fractions can be reduced to the proper fractions by long division. Thus, if $\frac{P(x)}{Q(x)}$ is improper, we can write $\frac{P(x)}{Q(x)} = A(x) + \frac{B(x)}{Q(x)}$, where $A(x)$ is a polynomial and $\frac{B(x)}{Q(x)}$ is a proper rational function.

In this section, we shall consider proper rational functions whose denominatros can be factorized into linear and/or quadratic factors. Writing a proper rational function as a sum of simpler rational functions is called *partial fraction*. Integration of simpler rational functions can be carried out easily by the methods learnt so far.

Here is a table that indicates the types of simpler partial fractions that are to be associated with various kinds of proper rational functions.

	Form of the proper rational	Form of the partial fractions
(1)	$\frac{px+q}{(x-a)(x-b)}$	$\frac{A}{x-a}+\frac{B}{x-b}$
(2)	$\frac{px+q}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
(3)	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$
(4)	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-b}$
(5)	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$

where $x^2 + bx + c$ cannot be factorised further.

⚠ Caution

➥ *A, B and C are real numbers to be determined suitably.*

We shall now explain how to integrate a rational function form-wise, with the help of an example.

Form 1: $\frac{px+q}{(x-a)(x-b)}$

Example 3.1: **Integrate the following rational functions:**

(A) $\frac{x}{(x+1)(x+2)}$

(B) $\frac{1}{x^2-9}$ **[NCERT]**

Ans. **(A)** Let $\quad I = \int \frac{x}{(x+1)(x+2)}dx \quad$ ...(i)

Write $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1}+\frac{B}{x+2} \quad$ ...(ii)

$$\Rightarrow \quad \frac{x}{(x+1)(x+2)} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

$$\Rightarrow \quad x = A(x+2) + B(x+1)$$

$$= (A + B)x + (2A + B)$$

On comparing coefficients of x and constant terms on both sides, we have

$$1 = A + B \text{ and } 0 = 2A + B$$

On solving these equations, we get

$A = -1, B = 2$

Putting these values of A and B in (ii), we get

$$\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1}+\frac{2}{x+2} \quad ...(iii)$$

From (i) and (iii), we have

$$I = \int \frac{x}{(x+1)(x+2)}dx$$

$$= \int\left\{\frac{-1}{x+1}+\frac{2}{x+2}\right\}dx$$

$$= -\log|x+1| + 2\log|x+2| + C$$

$$= \log(x+2)^2 - \log|x+1| + C$$

$$= \log\frac{(x+2)^2}{|x+1|} + C$$

(B) Let $\quad I = \int \frac{1}{x^2-9}dx$

$$= \int \frac{1}{(x+3)(x-3)}dx \quad ...(i)$$

Write $\dfrac{1}{(x+3)(x-3)} = \dfrac{A}{x+3} + \dfrac{B}{x-3}$ ...(ii)

$\Rightarrow \dfrac{1}{(x+3)(x-3)} = \dfrac{A(x-3)+B(x+3)}{(x+3)(x-3)}$

$\Rightarrow 1 = A(x-3) + B(x+3)$

$= (A+B)x + (-3A+3B)$

On comparing coefficients of x and constant terms on both sides, we have

$0 = A + B$

and $1 = -3A + 3B$

On solving these equations, we get $A = -\dfrac{1}{6}$, $B = \dfrac{1}{6}$

Putting these values of A and B in (ii), we get

$\dfrac{1}{(x+3)(x-3)} = \dfrac{-1/6}{x+3} + \dfrac{1/6}{x-3}$ ...(iii)

From (i) and (iii), we have

$$I = \int \frac{1}{(x+3)(x-3)}dx$$

$$= \int\left\{\frac{-1/6}{x+3} + \frac{1/6}{x-3}\right\}dx$$

$$= \int\frac{-1/6}{x+3}dx + \int\frac{1/6}{x-3}dx$$

$$= -\frac{1}{6}\log|x+3| + \frac{1}{6}\log|x-3| + C$$

$$= \frac{1}{6}\log\left|\frac{x-3}{x+3}\right| + C$$

Caution

Alternatively, part (B) could also be solved by directly applying the formula $\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

Form 2: $\dfrac{px+q}{(x-a)^2(x-b)}$

Example 3.2: Integrate the following rational functions:

(A) $\dfrac{3x-1}{(x+2)^2}$

(B) $\dfrac{x}{(x-1)^2(x+2)}$

Ans. (A) Let $I = \int \dfrac{3x-1}{(x+2)^2}dx$ ...(i)

Write $\dfrac{3x-1}{(x+2)^2} = \dfrac{A}{x+2} + \dfrac{B}{(x+2)^2}$ ...(ii)

$\Rightarrow \dfrac{3x-1}{(x+2)^2} = \dfrac{A(x+2)+B}{(x+2)^2}$

$\Rightarrow 3x - 1 = A(x+2) + B$

$= Ax + (2A + B)$

On comparing coefficients of x and constant terms on both sides, we have

$3 = A; -1 = 2A + B$

On solving these equations, we get

$A = 3$ and $B = -7$

Putting these values of A and B in (ii), we get

$\dfrac{3x-1}{(x+2)^2} = \dfrac{3}{x+2} + \dfrac{-7}{(x+2)^2}$ ...(iii)

From (i) and (iii), we have

$$I = \int\frac{3x-1}{(x+2)^2}dx$$

$$= \int\left\{\frac{3}{x+2} + \frac{-7}{(x+2)^2}\right\}dx$$

$$= \int\frac{3}{x+2}dx + \int\frac{-7}{(x+2)^2}dx$$

$$= 3\log|x+2| + (-7)\left[\frac{(x+2)^{-2+1}}{-2+1}\right] + C$$

$$= 3\log|x+2| + \frac{7}{x+2} + C$$

(B) Let $I = \int \dfrac{x}{(x-1)^2(x+2)}dx$ ...(i)

Write $\dfrac{x}{(x-1)^2(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{x+2}$ ...(ii)

$\Rightarrow \dfrac{x}{(x-1)^2(x+2)}$

$= \dfrac{A(x-1)(x+2) + B(x+2) + C(x-1)^2}{(x-1)^2(x+2)}$

$\Rightarrow x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$

$\Rightarrow x = (A+C)x^2 + (A+B-2C)x$

$+ (-2A + 2B + C)$

On comparing coefficients of x^2, x and constant terms on both sides, we have

$0 = A + C;$

$1 = A + B - 2C$

and $0 = -2A + 2B + C$

On solving these equations, we get

$A = \dfrac{2}{9}$, $B = \dfrac{1}{3}$ and $C = \dfrac{-2}{9}$

Putting these values of A, B and C in (ii), we get

$\dfrac{x}{(x-1)^2(x+2)} = \dfrac{2/9}{x-1} + \dfrac{1/3}{(x-1)^2} + \dfrac{-2/9}{x+2}$ ...(iii)

From (i) and (iii), we have

$$I = \int \frac{x}{(x-1)^2(x+2)}dx$$

$$= \int \left\{\frac{2/9}{x-1} + \frac{1/3}{(x-1)^2} + \frac{-2/9}{x+2}\right\}dx$$

$$= \frac{2}{9}\log|x-1| + \frac{1}{3}\left[\frac{(x-1)^{-2+1}}{-2+1}\right] - \frac{2}{9}\log|x+2| + C$$

$$= \frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$$

Form 3: $\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$

Example 3.3: Find: $\int \dfrac{x^2+6x-3}{x^3-4x}dx$

Ans. Let $$I = \int \frac{x^2+6x-3}{x^3-4x}dx$$

Expressing the denominator as a product of linear factors we have

$$I = \int \frac{x^2+6x-3}{x^3-4x}dx$$

$$= \int \frac{x^2+6x-3}{x(x-2)(x+2)}dx \quad \text{...(i)}$$

Write $\dfrac{x^2+6x-3}{x(x-2)(x+2)} = \dfrac{A}{x} + \dfrac{B}{x-2} + \dfrac{C}{x+2}$ ...(ii)

$$\Rightarrow \frac{x^2+6x-3}{x(x-2)(x+2)} = \frac{A(x-2)(x+2)+Bx(x+2)+Cx(x-2)}{x(x-2)(x+2)}$$

$$\Rightarrow \quad x^2+6x-3 = A(x-2)(x+2) + Bx(x+2) + Cx(x-2)$$

$$\Rightarrow \quad x^2+6x-3 = (A+B+C)x^2 + (2B-2C)x + (-4A)$$

On comparing coefficients of x^2, x and constant terms on both sides, we have

$$1 = A+B+C;$$
$$6 = 2B-2C$$
and $$-3 = -4A$$

On solving these equations, we get

$$A = \frac{3}{4}, B = \frac{13}{8} \text{ and } C = \frac{-11}{8}$$

Putting these values of A, B and C in (ii), we get

$$\frac{x^2+6x-3}{x(x-2)(x+2)} = \frac{3/4}{x} + \frac{13/8}{x-2} + \frac{-11/8}{x+2} \quad \text{...(iii)}$$

From (i) and (iii), we have

$$I = \int \frac{x^2+6x-3}{x(x-2)(x+2)}dx$$

$$= \int \left\{\frac{3/4}{x} + \frac{13/8}{x-2} + \frac{-11/8}{x+2}\right\}dx$$

$$= \frac{3}{4}\int \frac{1}{x}dx + \frac{13}{8}\int \frac{1}{x-2}dx - \frac{11}{8}\int \frac{1}{x+2}dx$$

$$= \frac{3}{4}\log|x| + \frac{13}{8}\log|x-2| - \frac{11}{8}\log|x+2| + C$$

Form 4: $\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$

Example 3.4: Find: $\int \dfrac{5x^2+6}{x^2(2x+1)}dx$

Ans. Let $$I = \int \frac{5x^2+6}{x^2(2x+1)}dx \quad \text{...(i)}$$

Write $\dfrac{5x^2+6}{x^2(2x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{2x+1}$ ...(ii)

$$\Rightarrow \quad \frac{5x^2+6}{x^2(2x+1)} = \frac{Ax(2x+1)+B(2x+1)+Cx^2}{x^2(2x+1)}$$

$$\Rightarrow \quad 5x^2+6 = Ax(2x+1) + B(2x+1) + Cx^2$$

$$\Rightarrow \quad 5x^2+6 = (2A+C)x^2 + (A+2B)x + B$$

On comparing coefficients of x^2, x and constant terms on both sides, we have

$$5 = 2A + C; 0 = A + 2B \text{ and } 6 = B$$

On solving these equations, we get

$$A = -12, B = 6 \text{ and } C = 29$$

Putting these values of A, B and C in (ii), we get

$$\frac{5x^2+6}{x^2(2x+1)} = \frac{-12}{x} + \frac{6}{x^2} + \frac{29}{2x+1} \quad \text{...(iii)}$$

From (i) and (iii), we have

$$I = \int \frac{5x^2+6}{x^2(2x+1)}dx$$

$$= \int \left\{\frac{-12}{x} + \frac{6}{x^2} + \frac{29}{2x+1}\right\}dx$$

$$= -12\int \frac{1}{x}dx + 6\int \frac{1}{x^2}dx + 29\int \frac{1}{2x+1}dx$$

$$= -12\log|x| - \frac{6}{x} + \frac{29}{2}\log|2x+1| + C$$

Form 5: $\dfrac{px^2 + qx + r}{(x - a)(x^2 + bx + c)}$

Example 3.5: Find: $\int \frac{2}{(1-x)(1+x^2)}dx$ **[NCERT]**

Ans. Let $$I = \int \frac{2}{(1-x)(1+x^2)}dx \quad ...(i)$$

Write $$\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2} \quad ...(ii)$$

$$\Rightarrow \frac{2}{(1-x)(1+x^2)} = \frac{A(1+x^2)+(Bx+C)(1-x)}{(1-x)(1+x^2)}$$

$$\Rightarrow 2 = A(1+x^2) + (Bx + C)(1 - x)$$

$$\Rightarrow 2 = (A - B)x^2 + (B - C)x + A + C$$

On comparing coefficients of x^2, x and constant terms on both sides, we have

$0 = A - B;$

$0 = B - C$

and $2 = A + C$

On solving these equations, we get

$A = 1, B = 1$ and $C = 1$

Putting these values of A, B and C in (ii), we get

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{(1+x^2)} \quad ...(iii)$$

From (i) and (iii), we have

$$I = \int \frac{2}{(1-x)(1+x^2)}dx$$

$$= \int \left\{\frac{1}{1-x} + \frac{x+1}{1+x^2}\right\}dx$$

$$= \int \frac{1}{1-x}dx + \int \frac{x+1}{1+x^2}dx$$

$$= \int \frac{1}{1-x}dx + \int \frac{x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= -\log|1-x| + \frac{1}{2}\log(1+x^2) + \tan^{-1}x + C$$

TOPIC 2

INTEGRATION USING BY PARTS

In this section, we describe one more method of integration, which is quite useful in integrating product of functions.

Let u and v be two differentiable functions of a single variable x(say). Then, by the product rule of differentiation, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides, we have

$$uv = \int \left\{u\frac{dv}{dx} + v\frac{du}{dx}\right\}dx$$

$$= \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$\Rightarrow \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx \quad ...(i)$$

Let $u = f(x)$ and $\frac{dv}{dx} = g(x)$.

Then, $\frac{du}{dx} = f'(x)$ and $v = \int g(x)dx$

So, the expression (i) becomes

$$\int \{f(x)g(x)\}dx = f(x)\int g(x)dx - \int\{f'(x)\int g(x)dx\}dx$$

If we take $f(x)$ as the first function and $g(x)$ as the second function, then the above result can be remembered as follows:

"The integral of the product of two functions = (first function) × (integral of second function) – integral of {(derivative of first function) × (integral of second function)}"

Selection of First Function

The function whose initial appears first in the word ILATE can be chosen as the first function, where

(i) I stands for *Inverse trigonometric functions*

(ii) L stands for *Logarithmic functions*

(iii) A stands for *Algebraic functions*

(iv) T stands for *Trigonometric functions*

(v) E stands for *Exponential functions*

Illustrations: Choose the first function from the following:

(A) $\int x\sin x\,dx$ (B) $\int x^2e^x\,dx$

(C) $\int x\log x\,dx$ (D) $\int x\tan^{-1}x\,dx$

[NCERT]

(A) x(as A precedes T)

(B) x^2(as A precedes E)

(C) $\log x$ (as L precedes A)

(D) $\tan^{-1}x$ (as I precedes A)

Example 3.6: Find: (A) $\int x\sin x\,dx$

(B) $\int x^2e^x\,dx$

(C) $\int x\log x\,dx$

(D) $\int x\tan^{-1}x\,dx$ **[NCERT]**

Ans. (A) $\int x\sin x\,dx$

$= x\int \sin x\,dx - \int\left\{\frac{d}{dx}(x)\int \sin x\,dx\right\}dx$

$= x(-\cos x) - \int\{1.(-\cos x)\}dx$

$= -x\cos x + \sin x + C$

(B) $\int x^2e^x\,dx$

$= x^2\int e^x\,dx - \int\left\{\frac{d}{dx}(x^2)\int e^x\,dx\right\}dx$

$= x^2e^x - \int 2xe^x\,dx$

$= x^2e^x - 2\int x\,e^x\,dx$ ...(i)

Further,

$\int x\,e^x\,dx = x\int e^x\,dx - \int\left\{\frac{d}{dx}(x)\int e^x\,dx\right\}dx$

$= xe^x - \int 1.e^x\,dx$

$= xe^x - e^x$

Substituting in (i), we have

$\int x^2e^x\,dx = x^2e^x - 2\{xe^x - e^x\}$

$= x^2e^x - 2\,xe^x + 2\,e^x + C,$

or $(x^2 - 2x + 2)\,e^x + C$

(C) $\int x\log x\,dx$

$= \log x\int x\,dx - \int\left\{\frac{d}{dx}(\log x)\int x\,dx\right\}dx$

$= \log x\frac{x^2}{2} - \int\frac{1}{x}\cdot\frac{x^2}{2}dx$

$= \frac{x^2}{2}\log x - \frac{1}{2}\int x\,dx$

$= \frac{x^2}{2}\log x - \frac{x^2}{4} + C$

(D) $\int x\tan^{-1}x\,dx$

$= \tan^{-1}x\int x\,dx - \int\left\{\frac{d}{dx}(\tan^{-1}x)\int x\,dx\right\}dx$

$= \tan^{-1}x\frac{x^2}{2} - \int\frac{1}{1+x^2}\cdot\frac{x^2}{2}dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2+1-1}{1+x^2}dx$

$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left\{\int 1\,dx - \int\frac{1}{1+x^2}dx\right\}$

$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + C$

Integral of the type

$\int e^x[f(x) + f'(x)]dx = e^xf(x) + C$

Let $I = \int e^x[f(x) + f'(x)]dx$

$= \int e^x.f(x)dx + \int e^x.f'(x)\,dx$

$= I_1 + I_2$ (say) ...(i)

Now, $I_1 = \int e^x.f(x)dx$

$= f(x)\int e^x\,dx - \int\left\{\frac{d}{dx}(f(x))\int e^xdx\right\}dx$

$= f(x)\int e^xdx - \int\{f'(x)\int e^xdx\}\,dx$

$= f(x).e^x - \int e^x\,f'(x)\,dx$

$= f(x).e^x - I_2$

$\Rightarrow \quad I_1 + I_2 = f(x).e^x + C$

i.e., $\int e^x[f(x) + f'(x)]dx = f(x).e^x + C$ ...(A)

Important

This formula is useful in evaluating integrals of the form $\int e^xg(x)dx$, when g is of the form $f + f'$ for some differentiable function.

Example 3.7: Find:

(A) $\int e^x\left[\frac{1}{x} - \frac{1}{x^2}\right]dx$

(B) $\int\frac{xe^x}{(1+x)^2}dx$

(C) $\int e^x\left[\frac{1+\sin x}{1+\cos x}\right]dx$ **[NCERT]**

Ans. (A) Take $\frac{1}{x} = f(x)$.

Then, $f'(x) = -\frac{1}{x^2}$

So, $\int e^x\left[\frac{1}{x} - \frac{1}{x^2}\right]dx = \int e^x[f(x) + f'(x)]\,dx$

Using formula (A), we have

$\int e^x\left[\frac{1}{x} - \frac{1}{x^2}\right]dx = f(x).e^x + C$

i.e., $\int e^x\left[\frac{1}{x} - \frac{1}{x^2}\right]dx = \frac{1}{x}\cdot e^x + C$

(B) Using partial fraction,

$\frac{x}{(1+x)^2} = \frac{1}{1+x} - \frac{1}{(1+x)^2}$

Take $\frac{1}{1+x} = f(x)$.

Then, $f'(x) = -\frac{1}{(1+x)^2}$

So, $\int \frac{xe^x}{(1+x)^2}dx = \int e^x\left[\frac{1}{1+x} - \frac{1}{(1+x)^2}\right]dx$

$= \int e^x[f(x) + f'(x)]dx$

Using formula (A), we have

$$\int \frac{xe^x}{(1+x)^2}dx = f(x).e^x + C$$

So, $\int \frac{xe^x}{(1+x)^2}dx = \frac{1}{1+x}\cdot e^x + C$

(C) Here, $\frac{1+\sin x}{1+\cos x} = \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\left(\frac{x}{2}\right)}$

$= \frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}$

$= \tan\frac{x}{2} + \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

Take $\tan\frac{x}{2} = f(x)$.

Then, $f'(x) = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

So, $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

$= \int e^x\left[\tan\frac{x}{2} + \frac{1}{2}\sec^2\left(\frac{x}{2}\right)\right]dx$

$= \int e^x[f(x) + f'(x)]dx$

Using formula (A), we have

$\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx = f(x).e^x + C,$

$= \tan\frac{x}{2}\cdot e^x + C$

TOPIC 3

SPECIAL INTEGRALS

Here we shall discuss some special types of standard integrals based on the technique of integration by parts.

Form 1: $\int\sqrt{x^2 - a^2}\,dx$

Let $I = \int\sqrt{x^2-a^2}\,dx$

$= \int 1.\sqrt{x^2-a^2}\,dx$

Taking constant function 1 as the second function, we have

$I = \sqrt{x^2-a^2}\int 1\,dx - \int\left\{\frac{d}{dx}(\sqrt{x^2-a^2})\int 1\,dx\right\}dx$

$= x(\sqrt{x^2-a^2}) - \int\left\{\frac{1}{2}\frac{2x}{\sqrt{x^2-a^2}}\cdot(x)\right\}dx$

$= x(\sqrt{x^2-a^2}) - \int\frac{x^2}{\sqrt{x^2-a^2}}dx$

$= x(\sqrt{x^2-a^2}) - \int\frac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}dx$

$= x(\sqrt{x^2-a^2}) - \int\sqrt{x^2-a^2}dx - a^2\int\frac{1}{\sqrt{x^2-a^2}}dx$

$= x(\sqrt{x^2-a^2}) - I - a^2\int\frac{1}{\sqrt{x^2-a^2}}dx$

$\Rightarrow 2I = x(\sqrt{x^2-a^2}) - a^2\int\frac{1}{\sqrt{x^2-a^2}}dx$

$\Rightarrow 2I = x(\sqrt{x^2-a^2}) - a^2\log|x+\sqrt{x^2-a^2}| + C$

$\Rightarrow I = \frac{x}{2}(\sqrt{x^2-a^2}) - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$

Form 2: $\int\sqrt{x^2 + a^2}\,dx$

Let $I = \int\sqrt{x^2+a^2}\,dx$

$= \int 1.\sqrt{x^2+a^2}\,dx$

Taking constant function 1 as the second function, we have

$I = \sqrt{x^2+a^2}\int 1\,dx - \int\left\{\frac{d}{dx}(\sqrt{x^2+a^2})\int 1\,dx\right\}dx$

$= x(\sqrt{x^2+a^2}) - \int\left\{\frac{1}{2}\frac{2x}{\sqrt{x^2+a^2}}\cdot(x)\right\}dx$

$= x(\sqrt{x^2+a^2}) - \int\frac{x^2}{\sqrt{x^2+a^2}}dx$

$= x(\sqrt{x^2+a^2}) - \int\frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}dx$

$$= x(\sqrt{x^2+a^2}) - \int \sqrt{x^2+a^2}\,dx + a^2\int \frac{1}{\sqrt{x^2+a^2}}\,dx$$

$$= x(\sqrt{x^2+a^2}) - I + a^2\int \frac{1}{\sqrt{x^2+a^2}}\,dx$$

$$2I = x(\sqrt{x^2+a^2}) + a^2\int \frac{1}{\sqrt{x^2+a^2}}\,dx$$

$$2I = x(\sqrt{x^2+a^2}) + a^2\log|x+\sqrt{x^2+a^2}| + C$$

$$\Rightarrow \quad I = \frac{x}{2}(\sqrt{x^2+a^2}) + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$$

Form 3: $\int \sqrt{a^2 - x^2}\,dx$

Let $I = \int \sqrt{a^2-x^2}\,dx = \int 1.\sqrt{a^2-x^2}\,dx$

Taking constant function 1 as the second function, we have

$$I = \sqrt{a^2-x^2}\int 1\,dx - \int\left\{\frac{d}{dx}(\sqrt{a^2-x^2})\int 1\,dx\right\}dx$$

$$= x(\sqrt{a^2-x^2}) - \int\left\{\frac{1}{2}\frac{-2x}{\sqrt{a^2-x^2}}\cdot(x)\right\}dx$$

$$= x(\sqrt{a^2-x^2}) - \int \frac{-x^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x(\sqrt{a^2-x^2}) - \int \frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$= x(\sqrt{a^2-x^2}) - \int \sqrt{a^2-x^2}\,dx + a^2\int \frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$= x(\sqrt{a^2-x^2}) - I + a^2\int \frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$\Rightarrow 2I = x(\sqrt{a^2-x^2}) + a^2\int \frac{1}{\sqrt{a^2-x^2}}\,dx$$

$$= x(\sqrt{a^2-x^2}) + a^2\sin^{-1}\frac{x}{a} + C$$

$$\Rightarrow \quad I = \frac{x}{2}(\sqrt{a^2-x^2}) + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

Example 3.9: Find: (A) $\int \sqrt{4-x^2}\,dx$ (B) $\int \sqrt{x^2+4x+6}\,dx$ (C) $\int \sqrt{x^2+4x-5}\,dx$

[NCERT]

Ans. (A) Let $I = \int \sqrt{4-x^2}\,dx$

$$= \int \sqrt{(2)^2 - x^2}\,dx$$

We know,

$$\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

$$\therefore \quad I = \frac{x}{2}(\sqrt{(2)^2-x^2}) + \frac{2^2}{2}\sin^{-1}\frac{x}{a} + C$$

$$= \frac{x}{2}(\sqrt{4-x^2}) + 2\sin^{-1}\frac{x}{2} + C$$

(B) Let $I = \int \sqrt{x^2+4x+6}\,dx$

$$= \int \sqrt{(x+2)^2+2}\,dx$$

$$= \int \sqrt{(x+2)^2+(\sqrt{2})^2}\,dx$$

We know,

$$\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log|x+\sqrt{x^2+a^2}| + C$$

$$\therefore \quad I = \frac{x+2}{2}(\sqrt{(x+2)^2+\sqrt{2}})^2 + \frac{(\sqrt{2})^2}{2}\log|(x+2)+\sqrt{(x+2)^2+(\sqrt{2})^2}| + C$$

$$= \frac{x+2}{2}(\sqrt{x^2+4x+6}) + \log|(x+2)+\sqrt{x^2+4x+6}| + C$$

(C) Let $I = \int \sqrt{x^2+4x-5}\,dx$

$$= \int \sqrt{(x+2)^2-9}\,dx$$

$$= \int \sqrt{(x+2)^2-(3)^2}\,dx$$

We know,

$$\sqrt{x^2-a^2} = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log|x+\sqrt{x^2-a^2}| + C$$

$$\therefore \quad I = \frac{x+2}{2}(\sqrt{(x+2)^2-(3)^2}) - \frac{(3)^2}{2}\log|(x+2)+\sqrt{(x+2)^2-(3)^2}| + C$$

$$= \frac{x+2}{2}(\sqrt{x^2+4x-5}) - \frac{9}{2}\log|(x+2)+\sqrt{x^2+4x-5}| + C$$

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. $\int \sqrt{1-25x^2}\,dx$ is equal to:

(a) $\frac{x}{2}\sqrt{1-25x^2}-\frac{1}{10}\sin^{-1}5x+C$

(b) $\frac{x}{2}\sqrt{1-25x^2}+\frac{1}{10}\sin^{-1}5x+C$

(c) $\frac{x}{2}\sqrt{25-x^2}+\frac{1}{10}\sin^{-1}5x+C$

(d) $\frac{x}{2}\sqrt{25-x^2}-\frac{1}{10}\sin^{-1}5x+C$

Ans. (b) $\frac{x}{2}\sqrt{1-25x^2}+\frac{1}{10}\sin^{-1}5x+C$

Explanation: Let

$$I = \int \sqrt{1-25x^2}\,dx$$

$$= \int 5\sqrt{\left(\frac{1}{5}\right)^2 - x^2}\,dx$$

$$= 5\left[\frac{x}{2}\sqrt{\left(\frac{1}{5}\right)^2 - x^2} + \frac{1/25}{2}\sin^{-1}\frac{x}{1/5}\right]+C$$

$$= 5\left[\frac{x}{10}\sqrt{1-25x^2}+\frac{1}{50}\sin^{-1}5x\right]+C$$

$$= \frac{x}{2}\sqrt{1-25x^2}+\frac{1}{10}\sin^{-1}5x+C$$

2. $\int x\cos 2x\,dx$ is equal to:

(a) $\frac{x}{2}\sin 2x-\frac{1}{2}\cos 2x+C$

(b) $\frac{x}{2}\sin 2x+\frac{1}{2}\cos 2x+C$

(c) $\frac{x}{2}\sin 2x+\frac{1}{4}\cos 2x+C$

(d) $\frac{x}{2}\sin 2x-\frac{1}{4}\cos 2x+C$

Ans. (c) $\frac{x}{2}\sin 2x+\frac{1}{4}\cos 2x+C$

Explanation: Let

$$I = \int \underset{\text{I}}{x}\,\underset{\text{II}}{\cos 2x}\,dx$$

By using integration by parts, we get

$$I = x\int\cos 2x\,dx - \int\left\{\frac{d}{dx}(x)\int\cos 2x\,dx\right\}dx$$

$$= x\frac{\sin 2x}{2} - \int 1\cdot\frac{\sin 2x}{2}dx$$

$$= \frac{x}{2}\sin 2x - \frac{1}{2}\int\sin 2x\,dx$$

$$= \frac{x}{2}\sin 2x + \frac{1}{4}\cos 2x + C$$

3. Ⓐ $\int x\log x\,dx$ is equal to:

(a) $x^2\log x-\frac{x^2}{4}+C$

(b) $\frac{x^2}{2}\log x+\frac{x^2}{2}+C$

(c) $\frac{x^2}{2}\log x-\frac{x}{2}+C$

(d) $\frac{x^2}{2}\log x-\frac{x^2}{4}+C$

4. $\int \frac{x}{(x-1)(x-2)}dx$ is:

(a) $\log\left|\frac{(x-1)^2}{x-2}\right|+C$

(b) $\log\left|\frac{(x-2)^2}{x-1}\right|+C$

(c) $\log\left|\left(\frac{x-1}{x-2}\right)^2\right|+C$

(d) $\log|(x-1)(x-2)|+C$ [NCERT]

Ans. (b) $\log\left|\frac{(x-2)^2}{x-1}\right|+C$

Explanation: Let $I = \int\frac{x}{(x-1)(x-2)}dx$ ...(i)

Write $\frac{x}{(x-1)(x-2)} = \frac{A}{x-1}+\frac{B}{x-2}$ ...(ii)

$$\Rightarrow \frac{x}{(x-1)(x-2)} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$\Rightarrow x = A(x-2)+B(x-1)$$

$$= (A+B)x+(-2A-B)$$

On comparing coefficients of x and constant terms on both sides, we have

$$1 = A + B \text{ and } 0 = 2A + B$$

On solving these equations, we get

$A = -1, B = 2$

Putting these values of A and B in (ii), we get

$$\frac{x}{(x-1)(x-2)} = \frac{-1}{x-1} + \frac{2}{x-2} \quad ...(iii)$$

From (i) and (iii), we have

$$I = \int \frac{x}{(x-1)(x-2)}dx$$

$$= \int \left\{\frac{-1}{x-1} + \frac{2}{x-2}\right\}dx$$

$$= \int \frac{-1}{x-1}dx + \int \frac{2}{x-2}dx$$

$$= -\log|x-1| + 2\log|x-2| + C$$

$$= \log\frac{(x-2)^2}{|x-1|} + C$$

5. $\int \frac{1}{x(x^2+1)}dx.$

(a) $\log|x| - \frac{1}{2}\log(x^2+1) + C$

(b) $\log|x| + \frac{1}{2}\log(x^2+1) + C$

(c) $-\log|x| + \frac{1}{2}\log(x^2+1) + C$

(d) $\frac{1}{2}\log|x| + \frac{1}{2}\log(x^2+1) + C$ **[NCERT]**

6. If $\int\frac{dx}{(x+2)(x^2+1)} = a\log|1+x^2| + b\tan^{-1}x + \frac{1}{5}\log|x+2| + C$, then:

(a) $a = \frac{-1}{10}, b = \frac{-2}{5}$ (b) $a = \frac{1}{10}, b = \frac{-2}{5}$

(c) $a = \frac{-1}{10}, b = \frac{2}{5}$ (d) $a = \frac{1}{10}, b = \frac{2}{5}$

Ans. (c) $a = \frac{-1}{10}, b = \frac{2}{5}$

Explanation:

Given that,

$$\int\frac{dx}{(x+2)(x^2+1)} = a\log|1+x^2| + b\tan^{-1}x + \frac{1}{5}\log(x+2) + C$$

Let, $I = \int\frac{dx}{(x+2)(x^2+1)}$

By partial fraction, we have

$$\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$\Rightarrow 1 = A(x^2+1) + (Bx+C)(x+2)$

$\Rightarrow 1 = (A+B)x^2 + (2B+C)x + A + 2C$

On comparing coefficients of x^2, x and constant terms both sides we get

$A + B = 0, 2B + C = 0, A + 2C = 1$

$\therefore \quad A = \frac{1}{5}, B = \frac{-1}{5}$ and $C = \frac{2}{5}$

$$\therefore \int\frac{dx}{(x+2)(x^2+1)}$$

$$= \frac{1}{5}\int\frac{1}{x+2}dx + \int\frac{-\frac{1}{5}x+\frac{2}{5}}{x^2+1}dx$$

$$= \frac{1}{5}\log|x+2| - \frac{1}{10}\int\frac{2x}{1+x^2}dx + \frac{2}{5}\int\frac{1}{1+x^2}dx$$

$$= \frac{1}{5}\log|x+2| - \frac{1}{10}\log|1+x^2| + \frac{2}{5}\tan^{-1}x + C$$

$\therefore a = \frac{-1}{10}$ and $b = \frac{2}{5}$

⚠ Caution

- *We need to use the method of partial fraction for solving this problem.*

7. $\int\frac{x+\sin x}{1+\cos x}dx$ is equal to:

(a) $\log|1+\cos x| + C$ (b) $\log|1+\sin x| + C$

(c) $x - \tan\frac{x}{2} + C$ (d) $x\tan\frac{x}{2} + C$

[NCERT Exemplar]

8. $\int e^x \sec x(1+\tan x)\,dx$ equals:

(a) $e^x\cos x + C$ (b) $e^x\sec x + C$

(c) $e^x\sin x + C$ (d) $e^x\tan x + C$

Ans. (b) $e^x\sec x + C$

Explanation: Let

$$I = \int e^x \sec x(1+\tan x)dx$$

$$= \int e^x[\sec x + \sec x\tan x]dx$$

Let $\sec x = f(x)$. Then, $\sec x\tan x = f'(x)$

So, $I = \int e^x[f(x) + f'(x)]dx$

$$= e^x f(x) + C \text{ or } e^x\sec x + C$$

9. ⓐ $\int \sqrt{x^2-8x+7}\,dx$ equals

(a) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+9\log|(x-4)+\sqrt{x^2-8x+7}|+C$

(b) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}+\frac{9}{2}\log|(x-4)+\sqrt{x^2-8x+7}|+C$

(c) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-9\log|(x-4)+\sqrt{x^2-8x+7}|+C$

(d) $\frac{1}{2}(x-4)\sqrt{x^2-8x+7}-\frac{9}{2}\log|(x-4)+\sqrt{x^2-8x+7}|+C$

10. $\int e^x\left(\frac{x\log x+1}{x}\right)dx$ is equal to:

(a) $\log(e^x\log x)+C$

(b) $\frac{e^x}{x}+C$

(c) $x\log x+e^x+C$

(d) $e^x\log x+C$

Ans. *(d)* $e^x\log x+C$

Explanation: Let $I=\int e^x\left(\frac{x\log x+1}{x}\right)dx$

$$=\int e^x\left[\log x+\frac{1}{x}\right]dx$$

Let $f(x)=\log x$. Then, $f'(x)=\frac{1}{x}$

Thus, given integral is of the form

$\int e^x[f(x)+f'(x)]\,dx$

Thus, $I=\int e^x\left[\log x+\frac{1}{x}\right]dx$

$$=e^x\log x+C$$

11. ⓐ $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx$ is equal to:

(a) $\frac{x^2}{2}e^x+C$ (b) $-\frac{e^x}{x}+C$

(c) $e^x\frac{1}{x}+C$ (d) xe^x+C

12. ⓐ $\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx$ is equal to:

(a) $\frac{e^x}{1+x^2}+C$ (b) $\frac{-e^x}{1+x^2}+C$

(c) $\frac{e^x}{(1+x^2)^2}+C$ (d) $\frac{-e^x}{(1+x^2)^2}+C$

[NCERT Exemplar]

13. $\int \cos x\sqrt{9-\sin^2 x}\,dx$ is:

(a) $\frac{\cos x}{2}\sqrt{9-\sin^2 x}+\frac{9}{2}\sin^{-1}\left(\frac{\cos x}{3}\right)+C$

(b) $\frac{\sin x}{2}\sqrt{9-\sin^2 x}+\frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right)+C$

(c) $\frac{\sin x}{2}\sqrt{9-\sin^2 x}+\frac{3}{2}\sin^{-1}\left(\frac{\sin x}{3}\right)+C$

(d) $\frac{\sin x}{2}\sqrt{9-\sin^2 x}+\frac{3}{2}\sin^{-1}\left(\frac{\cos x}{3}\right)+C$

Ans. *(c)* $\frac{\sin x}{2}\sqrt{9-\sin^2 x}+\frac{3}{2}\sin^{-1}\left(\frac{\sin x}{3}\right)+C$

Explanation: Let $I=\int\cos x\sqrt{9-\sin^2 x}\,dx$

Put $\sin x=t$

$\Rightarrow \cos x\,dx=dt$

$\therefore\ I=\int\sqrt{(3)^2-t^2}\,dt$

$$=\frac{t}{2}\sqrt{(3)^2-t^2}+\frac{3^2}{2}\sin^{-1}\frac{t}{3}+C$$

$$=\frac{\sin x}{2}\sqrt{9-\sin^2 x}+\frac{9}{2}\sin^{-1}\left(\frac{\sin x}{3}\right)+C$$

14. The Metro rail network has come as a boon for the people as it offers a comfortable and fast ride. Currently, metro rail is operating in thirteen Indian cities. The speed of the train at any instant can be found if the instantaneous position function is known. If the function describing the position of the train at any instant is given by $\sqrt{1+x^2}$, its speed can easily be calculated.

$\int\sqrt{1+x^2}\,dx$ equals:

(a) $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log|x+\sqrt{1+x^2}|+C$

(b) $\frac{2}{3}(1+x^2)^{3/2}+C$

(c) $\frac{2}{3}x(1+x^2)^{3/2}+C$

(d) $\frac{x^2}{2}\sqrt{1+x^2}+\frac{1}{2}x^2\log|x+\sqrt{1+x^2}|+C$

Ans. *(a)* $\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log|x+\sqrt{1+x^2}|+C$

Explanation: Let

$$I=\int\sqrt{1+x^2}\,dx$$

$$=\int\sqrt{(1)^2+x^2}\,dx$$

$$=\frac{x}{2}\sqrt{(1)^2+x^2}+\frac{1}{2}\log|x+\sqrt{(1)^2+x^2}|+C$$

$$=\frac{x}{2}\sqrt{1+x^2}+\frac{1}{2}\log|x+\sqrt{1+x^2}|+C$$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

15. Integrate $\frac{1}{x^2-16}dx$ by using partial fraction.

Ans. Let $I=\int\frac{1}{x^2-16}dx$

$$=\int\frac{1}{x^2-4^2}dx$$

$$=\int\frac{1}{(x-4)(x+4)}dx$$

$$=\frac{1}{8}\int\left[\frac{1}{x-4}-\frac{1}{x+4}\right]dx$$

$$=\frac{1}{8}[\log|x-4|-\log|x+4|]+C$$

$$=\frac{1}{8}\log\left|\frac{x-4}{x+4}\right|+C$$

16. Find $\int\frac{1}{(x-3)(x+4)}dx$

17. Find $\int x^4\log x\,dx$

Ans. Let $I=\int \underset{\text{I}}{x^4}\underset{\text{II}}{\log x}\,dx$

By using integration by parts, we get

$$I=\log x\int x^4\,dx-\int\left\{\frac{d}{dx}(\log x)\int x^4dx\right\}dx$$

$$=(\log x)\times\frac{x^5}{2}-\int\frac{1}{x}\times\frac{x^5}{5}dx$$

$$=\frac{x^5\log x}{5}-\int\frac{x^4}{5}dx$$

$$=\frac{x^5\log x}{5}-\frac{x^5}{5\times5}+C$$

$$=\frac{x^5\log x}{5}-\frac{x^5}{25}+C$$

18. Evaluate $\int e^x(\cos x-\sin x)\,dx$

19. Given $\int e^x(\tan x+1)\sec x\,dx=e^xf(x)+C$.

Write $f(x)$ satisfying above. **[CBSE 2012]**

Ans. We have $\int e^x(\tan x+1)\sec x\,dx=e^x f(x)+C$

$\Rightarrow e^x[\tan x\sec x+\sec x]\,dx=e^x f(x)+C$

or $e^x[\sec x+\tan x\sec x]\,dx=e^x f(x)+C$ ...(i)

Consider $g(x)=\sec x$. Then, $g'(x)=\sec x\tan x$

We know, $\int e^x[g(x)+g'(x)]\,dx=e^xg(x)+C$

$\therefore\int e^x[\sec x+\sec x\tan x]\,dx=e^x\sec x+C$ ...(ii)

From (i) and (ii), we get

$$e^x f(x)+C=e^x\sec x+C$$

$$\Rightarrow\quad f(x)=\sec x$$

20. Find $\int x\,e^x\,dx$.

21. Evaluate $\int\log 2x\,dx$.

Ans. Let $I=\int\underset{\text{I}}{1}.\underset{\text{II}}{\log 2x}\,dx$

By using integration by parts,

$$I=\log 2x\int 1\,dx-\int\left\{\frac{d}{dx}(\log 2x)\int 1\,dx\right\}dx$$

$$=(\log 2x)x-\int\frac{1}{2x}\times2\times x\,dx$$

$$=x(\log 2x)-\int 1\,dx$$

$$=x\log 2x-x+C$$

22. Evaluate $\int\sqrt{16-4x^2}\,dx$.

Ans. Let $I = \int \sqrt{16-4x^2}\,dx$

$$= \int 2\sqrt{4-x^2}\,dx$$

$$= 2\int \sqrt{(2)^2-(x)^2}\,dx$$

$$= 2\left[\frac{x}{2}\sqrt{2^2-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]+C$$

$$= x\sqrt{4-x^2}+4\sin^{-1}\frac{x}{2}+C$$

23. **Evaluate** $\int \sqrt{x^2-6x+4}\,dx$

24. **Integrate the function** $f(x)=\sqrt{x^2+4x+10}$.

Ans. Let $I = \int f(x)\,dx$

$$= \int \sqrt{x^2+4x+10}\,dx$$

$$= \int \sqrt{x^2+2\times 2x+(2)^2+(6)^2}\,dx$$

$$= \int \sqrt{(x+2)^2+(\sqrt{6})^2}\,dx$$

$$= \frac{x+2}{2}\sqrt{x^2+4x+6}$$
$$+\frac{6}{2}\log\left|(x+2)+\sqrt{(x+2)^2+(\sqrt{6})^2}\right|+C$$

$$= \frac{x+2}{2}\sqrt{x^2+4x+6}$$
$$+3\log\left|(x+2)+\sqrt{x^2+4x+10}\right|+C$$

25. **Evaluate** $\int \sec^2 x\sqrt{25-\tan^2 x}\,dx$

Ans. Let $I = \int \sec^2 x\sqrt{25-\tan^2 x}\,dx$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$\therefore I = \int \sqrt{25-t^2}\,dt$

$$= \frac{t}{2}\sqrt{25-t^2}+\frac{25}{2}\sin^{-1}\frac{t}{5}+C$$

$$= \frac{\tan x}{2}\sqrt{25-\tan^2 x}+\frac{25}{2}\sin^{-1}\left(\frac{\tan x}{2}\right)+C$$

26. **Find the integral** $\int x^2\sqrt{x^6-a^4}\,dx$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

27. **Evaluate** $\int \frac{x^2+3}{x^2-1}\,dx$

Ans. Let $I = \int \frac{x^2+3}{x^2-1}\,dx$

$$= \int \frac{x^2-1+4}{x^2-1}\,dx$$

$$= \int \left(1+\frac{4}{x^2-1}\right)dx$$

$$= \int \left(1+\frac{4}{(x-1)(x+1)}\right)dx$$

$$= \int dx + 4\int \frac{1}{2}\left[\frac{1}{x-1}-\frac{1}{x+1}\right]dx$$

$$= x + 2[\log|x-1| - \log|x+1|] + C$$

$$= x+2\log\left|\frac{x-1}{x+1}\right|+C$$

28. **Find** $\int \frac{\log x}{(1+\log x)^2}\,dx$

[CBSE Term-2 SQP 2022]

Ans.

$$\int \frac{\log x}{(1+\log x)^2}\,dx = \int \frac{\log x+1-1}{(1+\log x)^2}\,dx$$

$$= \int \frac{1}{1+\log x}\,dx - \int \frac{1}{(1+\log x)^2}\,dx$$

$$= \frac{1}{1+\log x}\times x - \int \frac{-1}{(1+\log x)^2}\times\frac{1}{x}\times x\,dx$$

$$- \int \frac{1}{(1+\log x)^2}\,dx = \frac{x}{1+\log x}+c$$

[CBSE Marking Scheme Term-2 SQP 2022]

Explanation:

Let,

$$I = \int \frac{\log x}{(1+\log x)^2}\,dx$$

$$= \int \frac{1+\log x-1}{(1+\log x)^2}\,dx$$

$$= \int \frac{1+\log x}{(1+\log x)^2}\,dx - \int \frac{1}{(1+\log x)^2}.\,dx$$

$$= \int \frac{1}{1+\log x}.\,dx - \int \frac{1}{(1+\log x)^2}.\,dx$$

On integrating first integral by parts, we get

$$I = \frac{1}{1+\log x} \times x - \int \frac{-1}{(1+\log x)^2} \times \frac{1}{x} \times x \, dx$$

$$+ C - \int \frac{1}{(1+\log x)^2} dx$$

$$= \frac{x}{1+\log x} + \int \frac{1}{(1+\log x)^2} dx$$

$$- \int \frac{1}{(1+\log x)^2} dx + C$$

$$= \frac{x}{1+\log x} + C$$

Caution

- *The tricky part of the question lies where, one need to break the given integral into two parts. So integrate it wisely.*

29. Evaluate $\int \sqrt{5-2x+x^2}\, dx$

[NCERT Exemplar]

30. Find the integral $\int \frac{\sqrt{36-(\log x)^2}}{x} dx$

Ans. Let $I = \int \frac{\sqrt{36-(\log x)^2}}{x} dx$

Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\therefore I = \int \sqrt{(6)^2 - t^2}\, dt$$

$$= \frac{t}{2}\sqrt{6^2-t^2} + \frac{36}{2}\sin^{-1}\left(\frac{t}{6}\right) + C$$

$$= \frac{\log x}{2}\sqrt{6^2-(\log x)^2} + 18\sin^{-1}\left(\frac{\log x}{6}\right) + C$$

31. Evaluate $\int \sqrt{2ax - x^2}\, dx$ **[NCERT Exemplar]**

32. Evaluate $\int \frac{x-1}{(x+1)(x-2)} dx$

Ans. Let $I = \int \frac{x-1}{(x+1)(x-2)} dx$

Consider $\frac{x-1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$ ...(i)

$\Rightarrow \quad (x-1) = A(x-2) + B(x+1)$

Put $x = -1$, we get

$-1-1 = A(-1-2) + 0 \Rightarrow A = \frac{2}{3}$

Put $x = 2$, we get

$2-1 = 0 + B(2+1) \Rightarrow B = \frac{1}{3}$

Put the values of A and B in Eq. (i), we get

$$\frac{x-1}{(x+1)(x-2)} = \frac{2}{3(x+1)} + \frac{1}{3(x-2)}$$

$$\therefore \quad I = \int \frac{2}{3(x+1)} dx + \int \frac{1}{3(x-2)} dx$$

$$= \frac{2}{3}\log|x+1| + \frac{1}{3}\log|x-2| + C$$

33. Find $\int \sin^{-1}(2x)\, dx$ **[CBSE 2019]**

34. Find $\int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}} dx$

Ans. Let $I = \int \frac{\cos x}{\sqrt{\sin^2 x - 2\sin x - 3}} dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\therefore I = \int \frac{dt}{\sqrt{t^2-2t-3}}$$

$$= \int \frac{dt}{\sqrt{t^2-2t+1-4}} = \int \frac{dt}{\sqrt{(t-1)^2-2^2}}$$

$$= \log\left|(t-1) + \sqrt{(t-1)^2-2^2}\right| + C$$

$$= \log\left|(t-1) + \sqrt{t^2-2t-3}\right| + C$$

$$= \log|(\sin x - 1) + \sqrt{\sin^2 x - 2\sin x - 3}| + C$$

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

35. Find $\int \frac{2x}{(x^2+1)(x^4+4)} dx$ **[CBSE 2017]**

Ans. Let $I = \int \frac{2x}{(x^2+1)(x^4+4)} dx$

Put $x^2 = v$

$\Rightarrow \quad 2x\, dx = dv$

$$\therefore \quad I = \int \frac{dv}{(v+1)(v^2+4)}$$

Consider $\dfrac{1}{(v+1)(v^2+4)} = \dfrac{A}{v+1} + \dfrac{Bv+C}{v^2+4}$ ...(i)

$\Rightarrow \quad 1 = A(v^2+4) + (Bv+C)(v+1)$

$\Rightarrow \quad 1 = (A+B)v^2 + (B+C)v + 4A + C$

On comparing the coefficients of v^2, v and constant terms both sides, we get

$A + B = 0$, $B + C = 0$ and $4A + C = 1$

On solving, we get

$$A = \frac{1}{5}, B = -\frac{1}{5} \text{ and } C = \frac{1}{5}$$

Put the values of A, B and C in Eq. (i), we get

$$\frac{1}{(v+1)(v^2+4)} = \frac{1}{5(v+1)} + \frac{-\frac{1}{5}v + \frac{1}{5}}{v^2+4}$$

$$\therefore \quad I = \int \frac{dv}{5(v+1)} + \int \frac{\left(-\frac{1}{5}v + \frac{1}{5}\right)}{v^2+4} dv$$

$$= \frac{1}{5}\log|v+1| + \frac{1}{5}\int \frac{-v+1}{v^2+4} dv$$

$$= \frac{1}{5}\log|v+1| - \frac{1}{10}\int \frac{2v}{v^2+4} dv + \frac{1}{5}\int \frac{1}{v^2+4} dv$$

$$= \frac{1}{5}\log|v+1| - \frac{1}{10}\log|v^2+4| + \frac{1}{5} \times \frac{1}{2}\tan^{-1}\frac{v}{2} + C$$

$$= \frac{1}{5}\log\left|\frac{(v+1)}{\sqrt{v^2+4}}\right| + \frac{1}{10}\tan^{-1}\frac{v}{2} + C$$

$$= \frac{1}{5}\log\left|\frac{x^2+1}{\sqrt{x^4+4}}\right| + \frac{1}{10}\tan^{-1}\left(\frac{x^2}{2}\right) + C$$

36. Find : $\dfrac{x+1}{(x^2+1)x}dx$

[CBSE Term-2 SQP 2022]

Ans. Let $\dfrac{x+1}{(x^2+1)x} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x} = \dfrac{(Ax+B)x + C(x^2+1)}{(x^2+1)x}$

$\Rightarrow x + 1 = (Ax + B)x + C(x^2 + 1)$ (An identity)

Equating the coefficients, we get

$B = 1, C = 1, A + C = 0$

Hence, $A = -1, B = 1, C = 1$

The given integral $= \int \dfrac{-x+1}{x^2+1}dx + \int \dfrac{1}{x}dx$

$= \dfrac{-1}{2}\int \dfrac{2x-2}{x^2+1}dx + \int \dfrac{1}{x}dx$

$= \dfrac{-1}{2}\int \dfrac{2x}{x^2+1}dx + \int \dfrac{1}{x^2+1}dx + \int \dfrac{1}{x}dx$

$= \dfrac{-1}{2}\log(x^2+1) + \tan^{-1}x + \log|x| + c$

[CBSE Marking Scheme Term-2 SQP 2022]

Explanation:

Let, $\quad I = \int \dfrac{x+1}{(x^2+1)x}dx$

By partial fraction

Let

$$\frac{x+1}{(x^2+1)x} = \frac{Ax+B}{x^2+1} + \frac{C}{x}$$

$$\Rightarrow \quad \frac{x+1}{(x^2+1)x} = \frac{(Ax+B)x + C(x^2+1)}{x(x^2+1)}$$

$\Rightarrow \quad x + 1 = Ax^2 + Bx + Cx^2 + C$

$\Rightarrow \quad x + 1 = (A + C)x^2 + Bx + C$

On equating coefficients to both sides, we get

$A + C = 0$

$B = 1$ and $C = 1$

$\therefore \quad A = -C = -1$

Then, $\quad \dfrac{x+1}{(x^2+1)x} = \dfrac{-x+1}{x^2+1} + \dfrac{1}{x}$

$$\therefore \int \frac{x+1}{(x^2+1)x}dx = \int\left(\frac{-x+1}{x^2+1}\right)dx + \int \frac{1}{x}dx$$

$$= \int \frac{1}{1+x^2}dx - \int \frac{x}{x^2+1}dx + \int \frac{1}{x}dx$$

$$= \tan^{-1}x - \frac{1}{2}\log(x^2+1) + \log x + C$$

37. Find $\int \sec^3 x\, dx$ **[CBSE 2020]**

38. Find $\int \dfrac{2\cos x}{(1-\sin x)(2-\cos^2 x)}dx$ **[CBSE 2019]**

Ans. Let $I = \int \dfrac{2\cos x}{(1-\sin x)(2-\cos^2 x)}dx$

$$= 2\int \frac{\cos x}{(1-\sin x)(1+\sin^2 x)}dx$$

Put $\sin x = v$

$\Rightarrow \cos x\, dx = dv$

$$\therefore \quad I = \int \frac{2dv}{(1-v)(1+v^2)}$$

Consider $\dfrac{x}{(1-v)(1+v^2)} = \dfrac{A}{1-v} + \dfrac{Bv+C}{1+v^2}$ ...(i)

$$2 = A(1+v^2) + (Bv+C)(1-v) \quad \text{...(ii)}$$

Put $v = 1$ in Eq. (ii)

$2 = A(1+1) + 0$

$\Rightarrow \quad A = 1$

Put $v = 0$ in Eq. (ii),

$2 = A(1+0) + (0+C)(1-0)$

$2 = A + C$

$\Rightarrow 2 - 1 = C$

$C = 1$

Put $v = -1$ in Eq. (ii), we get

$2 = A(1+1) + (-B+C)(1+1)$

$2 = 1 \times 2 + 2(-B+1)$

$2B = 2$

$\Rightarrow \quad B = 1$

Put the values of A, B and C in Eq. (i), we get

$$\frac{2}{(1-v)(1+v^2)} = \frac{1}{1-v} + \frac{v+1}{1+v^2}$$

$$\therefore \quad I = \int \left(\frac{1}{1-v} + \frac{v+1}{1+v^2}\right)dv$$

$$= \int \frac{1}{1-v}dv + \int \frac{v}{1+v^2}dv + \int \frac{1}{1+v^2}dv$$

$$= -\log|1-v| + \frac{1}{2}\log|1+v^2| + \tan^{-1}v + C$$

$$= -\log|1-\sin x| + \frac{1}{2}\log|1+\sin^2 x| + \tan^{-1}(\sin x) + C$$

39. Evaluate $\displaystyle\int \frac{5x-2}{3x^2+2x+1}dx$. **[NCERT]**

Ans. Let $I = \displaystyle\int \frac{5x-2}{3x^2+2x+1}dx$

Consider $\quad 5x - 2 = A\dfrac{d}{dx}(3x^2+2x+1) + B$

$\Rightarrow \quad 5x - 2 = A(6x+2) + B$ ...(i)

$\Rightarrow \quad 5x - 2 = 6Ax + (2A + B)$

On equating the coefficients of x and constant terms both sides, we get

$5 = 6A$

$\Rightarrow \quad A = \dfrac{5}{6}$

and $\quad -2 = 2A + B$

$\Rightarrow \quad -2 = 2 \times \dfrac{5}{6} + B$

$\Rightarrow \quad B = -2 - \dfrac{5}{3} = -\dfrac{11}{3}$

Putting the values of A and B in Eq. (i), we get

$$5x - 2 = \frac{5}{6}(5x+2) - \frac{11}{3}$$

$$\therefore I = \int \frac{5x-2}{3x^2+2x+1}dx$$

$$= \int \frac{5(6x+2)}{6(3x^2+2x+1)}dx - \frac{11}{3}\int \frac{1}{(3x^2+2x+1)}dx$$

$$= \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{3}\int \frac{1}{1+3\left(x^2+\frac{2}{3}x+\frac{1}{9}-\frac{1}{9}\right)}dx$$

$$= \frac{5}{6}\log|3x^2+2x+1) - \frac{11}{9}\int \frac{dx}{3\left(x+\frac{1}{3}\right)^2+1-\frac{1}{3}}$$

$$= \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}$$

$$= \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{9}\times\frac{1}{\frac{\sqrt{2}}{3}}\tan^{-1}\frac{\left(x+\frac{1}{3}\right)}{\frac{\sqrt{2}}{3}} + C$$

$$= \frac{5}{6}\log|3x^2+2x+1| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

40. Evaluate $\displaystyle\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}dx$. [NCERT Exemplar]

41. Evalaute $\displaystyle\int e^{-x}\cos x\, dx$.

Ans. Let $I = \int e^{-x}\cos x\, dx$. Then,

$$I = \int \underset{\text{I}}{e^{-x}}\ \underset{\text{II}}{\cos x}\, dx$$

$$\Rightarrow \quad I = e^{-x}\sin x - \int(-e^{-x})\sin x\,dx$$

$$\Rightarrow \quad I = e^{-x}\sin x + \int \underset{I}{e^{-x}} \underset{II}{\sin x}\,dx$$

$$\Rightarrow \quad I = e^{-x}\sin x + e^{-x}(-\cos x)$$

$$- \int(-e^{-x})(-\cos x)\,dx$$

$$\Rightarrow \quad I = e^{-x}\sin x - e^{-x}\cos x - \int e^{-x}\cos x\,dx$$

$$\Rightarrow \quad I = e^{-x}\sin x - e^{-x}\cos x - I$$

$$\Rightarrow 2I = e^{-x}\sin x - e^{-x}\cos x$$

$$\Rightarrow \quad I = \frac{e^{-x}}{2}(\sin x - \cos x) + C$$

42. Find the integral of $e^{3x}\sin x$.

43. Find $\int x^2\tan^{-1}x\,dx$

LONG ANSWER Type Questions (LA)

[**4** & **5** marks]

44. Find $\int \frac{x^2}{(x^2+1)(x^2+4)}dx$ **[CBSE 2014]**

Ans. Let $\quad I = \int \frac{x^2}{(x^2+1)(x^2+4)}dx$

Put $x^2 = t$, then

$$\frac{x^2}{(x^2+1)(x^2+4)} = \frac{t}{(t+1)(t+4)}$$

Let $\quad \frac{t}{(t+1)(t+4)} = \frac{A}{(t+1)} + \frac{B}{(t+4)}$

$$\Rightarrow \quad t = A(t+4) + B(t+1)$$

$$\Rightarrow \quad t = t(A+B) + 4A + B$$

On equating the coefficient of t and constant terms, we get

$1 = A + B$ and $0 = 4A + B$

$\therefore \; A = -\frac{1}{3}, B = \frac{4}{3}$

$$\therefore \quad \frac{t}{(t+1)(t+4)} = \frac{-1}{3(t+1)} + \frac{4}{3(t+4)}$$

$$\therefore \quad I = \int -\frac{1}{3(x^2+1)}dx + \int\frac{4}{3(x^2+4)}dx$$

$$= -\frac{1}{3}\times\frac{1}{1}\tan^{-1}\frac{x}{1} + \frac{4}{3}\times\frac{1}{2}\tan^{-1}\frac{x}{2} + C$$

$$= -\frac{1}{3}\tan^{-1}x + \frac{2}{3}\tan^{-1}\frac{x}{2} + C$$

45. Evaluate $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}dx$

Ans. Let $I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}dx$

Consider $3\sin x + 2\cos x = \lambda_1 \frac{d}{dx}$

$$(3\cos x + 2\sin x) + \lambda_2(3\cos x + 2\sin x)$$

$$\Rightarrow \; 3\sin x + 2\cos x = \lambda_1(-3\sin x + 2\cos x) + \lambda_2(3\cos x + 2\sin x)$$

On comparing the coefficients of $\sin x$ and $\cos x$ on both sides, we get

$$-3\lambda_1 + 2\lambda_2 = 3$$

and $\quad 2\lambda_1 + 3\lambda_2 = 2$

On solving, we get

$$\lambda_1 = -\frac{5}{13}$$

and $\quad \lambda_2 = \frac{12}{13}$

$\therefore \; I =$

$$\int \frac{\lambda_1(-3\sin x + 2\cos x) + \lambda_2(3\cos x + 2\sin x)}{3\cos x + 2\sin x}dx$$

$$= \lambda_2\int 1\,dx + \lambda_1\int\frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x}dx$$

Put $3\cos x + 2\sin x = t$

$\Rightarrow (-3\sin x + 2\cos x)\,dx = dt$

$$\therefore \quad I = \lambda_2 x + \lambda_1\int\frac{1}{t}dt$$

$$= \lambda_2 x + \lambda_1(\log t) + C$$

$$= \lambda_2 x + \lambda_1 \log|3\cos x + 2\sin x| + C$$

$$= \frac{12}{13}x - \frac{5}{13}\log|3\cos x + 2\sin x| + C$$

46. Evalaute $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx$

47. Find $\int \frac{1}{1+\cot x}dx$

48. @Evaluate $\int \frac{x^2}{x^4 - x^2 - 12} dx$

[NCERT Exemplar]

49. Evaluate the integral

$$\int \left[\log(\log x) + \frac{1}{(\log x)^2} \right] dx$$

Ans. Let $I = \int \left[\log(\log x) + \frac{1}{(\log x)^2} \right] dx$

$$= \int (\log(\log x)) dx + \int \frac{1}{(\log x)^2} dx$$

Consider $I_1 = \int \log(\log x)\, dx$

and $I_2 = \int \frac{1}{(\log x)^2} dx$

Then $I = I_1 + I_2$ ...(i)

Now, $I_1 = \int \underset{II}{1} . \underset{I}{\log(\log x)}\, dx$

By using integration by parts,

$$I_1 = \log(\log x) \int 1\, dx - \int \left[\frac{d}{dx} \log(\log x) \int 1\, dx \right] dx$$

$$= [\log(\log x)]x - \int \left[\frac{1}{\log x} \times \frac{1}{x} \times x \right] dx$$

$$= x \log(\log x) - \int \underset{II}{1} . \underset{I}{(\log x)^{-1}}\, dx$$

Again using integration by parts

$I_1 = x \log(\log x) -$

$$\left[(\log x)^{-1} \int 1\, dx - \int \left[\frac{d}{dx} (\log x)^{-1} \int 1\, dx \right] \right]$$

$= x \log(\log x)$

$$- \left[(\log x)^{-1} . x - \int - \frac{1}{(\log x)^2 x} \times x\, dx \right]$$

$$= x \log(\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2} dx$$

$$= x \log(\log x) - \frac{x}{\log x} - I_2 + C$$

Put the value of I_1 in Eq. (i), we get

$$I = x \log(\log x) - \frac{x}{\log x} - I_2 + C + I_2$$

$$= x \log(\log x) - \frac{x}{\log x} + C$$

50. Evaluate $\int \sin^{-1} \sqrt{\frac{x}{b+x}}\, dx$.

Ans. Let $I = \int \sin^{-1} \sqrt{\frac{x}{b+x}}\, dx$

Put $x = b \tan^2 \theta \Rightarrow dx = 2b \tan \theta \sec^2 \theta\, d\theta$

$$\therefore \quad I = \int \sin^{-1} \sqrt{\frac{b \tan^2 \theta}{b + b \tan^2 \theta}} \times (2b \tan\theta \sec^2 \theta)\, d\theta$$

$$= 2b \int \sin^{-1} \sqrt{\frac{\tan^2 \theta}{1 + \tan^2 \theta}} \times (\tan\theta \sec^2 \theta)\, d\theta$$

$$= 2b \int \sin^{-1} \sqrt{\frac{\tan^2 \theta}{\sec^2 \theta}} \times (\tan\theta \sec^2 \theta)\, d\theta$$

$$= 2b \int \sin^{-1} \left(\frac{\tan\theta}{\sec\theta} \right) \times (\tan\theta \sec^2 \theta)\, d\theta$$

$$= 2b \int \sin^{-1}(\sin\theta) \times \tan\theta \sec^2 \theta\, d\theta$$

$$= 2b \int \underset{I}{\theta} (\underset{II}{\tan\theta \sec^2 \theta})\, d\theta$$

By using integration by parts,

$$I = 2b \left[\theta \int \tan\theta \sec^2 \theta\, d\theta - \int \left\{ \frac{d}{d\theta}(\theta)(\tan\theta \sec^2 \theta) d\theta \right] d\theta \right\}$$

$$= 2b \left[\theta \times \frac{\tan^2 \theta}{2} - \int 1 \cdot \frac{\tan^2 \theta}{2} d\theta \right]$$

$$= b\theta \tan^2 \theta - b \int (\sec^2 \theta - 1)\, d\theta$$

$$= b\theta \tan^2 \theta - b[\tan \theta - \theta] + C$$

$$= b \left[\frac{x}{b} \tan^{-1} \sqrt{\frac{x}{b}} - \sqrt{\frac{x}{b}} + \tan^{-1} \sqrt{\frac{x}{b}} \right] + C$$

51. Evaluate $\int (\sqrt{\cot x} + \sqrt{\tan x})\, dx$. [CBSE 2014]

Ans. Let $I = \int (\sqrt{\cot x} + \sqrt{\tan x})\, dx$

$$= \int \sqrt{\tan x}\, (1 + \cot x) dx$$

Put $\tan x = t^2$

$\Rightarrow \sec^2 x\, dx = 2t\, dt$

$$\Rightarrow \quad dx = \frac{2t}{1 + \tan^2 x} dt$$

$$= \frac{2t}{1 + t^4} dt$$

$$\therefore \quad I = \int t \left(1 + \frac{1}{t^2} \right) \times \frac{2t}{1 + t^4} dt$$

$$= 2\int \frac{t^2+1}{1+t^4}dt$$

$$= 2\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-2+2}dt$$

$$= 2\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+(\sqrt{2})^2}dt$$

Now, put $t-\frac{1}{t}=v$

$$\Rightarrow \left(1+\frac{1}{t^2}\right)dt = dv$$

$$\therefore \qquad I = 2\int \frac{dv}{v^2+(\sqrt{2})^2}$$

$$= 2\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{v}{\sqrt{2}}\right)+C$$

$$= \sqrt{2}\tan^{-1}\frac{\left(t-\frac{1}{t}\right)}{\sqrt{2}}+C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)+C$$

$$= \sqrt{2}\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)+C$$

DEFINITE INTEGRALS 4

TOPIC 1

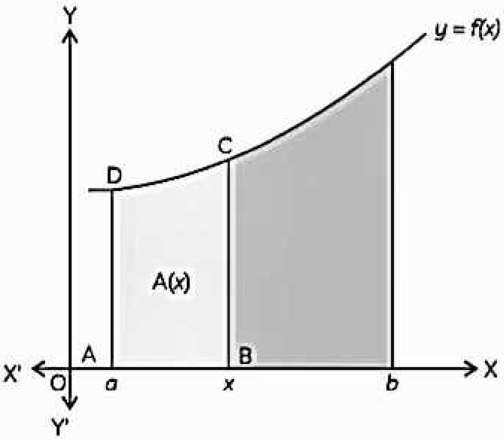
DEFINITE INTEGRALS AS AN AREA

The value of the definite integral $\int_a^b f(x)\,dx$ represents the area of the region bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and x-axis. Let us first understand the area function A(x) before we proceed further.

Area Function

Let x be any real number in $[a, b]$.

Then $\int_a^x f(x)\,dx$ represents the area of the shaded region ABCD shown in the figure.



[Here, it is assumed that $f(x) > 0$ for all $x \in [a, b]$].

Area of the shaded region depends upon the value of x. We denote this function of x by A(x) and call the function A(x) as Area function and is written as

$$A(x) = \int_a^x f(x)\,dx, x \geq a$$

Indefinite integral was a function but the definite integral $\int_a^b f(x)\,dx$ is a value of the area function A(x) at $x = b$.

On the basis of the above, we state the fundamental theorems of calculus.

First Fundamental Theroem

Theorem: Let f be a continuous function on the closed interval $[a, b]$ and let $A(x) = \int_a^b f(x)\,dx$ be the area function. Then,

$$A'(x) = f(x) \text{ for all } x \in [a, b] \text{ and } A(0) = 0$$

Second Fundamental Theroem

Theorem: Let f be a continuous function on the closed interval $[a, b]$ and let F be an anti-derivative, of f. Then,

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

This theorem enables us to evaluate definite integrals by making use of anti-derivative.

Important

- *In $\int_a^b f(x)\,dx$, the function f (the integrand) must be well defined and continuous in [a, b].*
- *While evaluating $\int_a^b f(x)\,dx$, there is no need to take constant of integration C, because if we take $\int f(x)dx = F(x) + C$, then $\int_a^b f(x)dx = [F(x) + C]_a^b$ = [F(b) + C] − [F(a) + C] − F(b) − F(a) = i.e., the constant of integration disappears while calculating definite integral.*

Illustration: Evaluate $\int_{-1}^{1} (x+1)\,dx$

Since $\int (x+1)dx = \frac{x^2}{2} + x$, by second fundamental theorem, we have

$$\int_{-1}^{1}(x+1)\,dx = \left[\frac{x^2}{2} + x\right]_{-1}^{1}$$

$$= \left[\frac{1^2}{2} + 1\right] - \left[\frac{(-1)^2}{2} + (-1)\right]$$

$$= \left[\frac{1}{2} + 1\right] - \left[\frac{1}{2} - 1\right], \text{ i.e., } 2$$

Example 4.1: Evaluate: (A) $\int_2^3 \frac{x}{x^2+1}\,dx$

(B) $\int_1^2 \frac{5x^2}{x^2+4x+3}\,dx$ **[NCERT]**

Ans. (A) Here, $\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx$

$$= \frac{1}{2}\log(x^2+1)$$

So, by second fundamental theorem, we have

$$\int_2^3 \frac{x}{x^2+1}\,dx = \left[\frac{1}{2}\log(x^2+1)\right]_2^3$$

$$= \frac{1}{2}[\log 10 - \log 5]$$

$$= \frac{1}{2}\log\frac{10}{5}$$

$$= \frac{1}{2}\log 2$$

(B) Consider $\int \frac{5x^2}{x^2+4x+3}dx$ ...(i)

Here, $\frac{5x^2}{x^2+4x+3} = 5 - \frac{20x+15}{x^2+4x+3}$

$$= 5 - \frac{20x+15}{(x+1)(x+3)} \quad \text{...(ii)}$$

Let us write $\frac{20x+15}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$ ...(iii)

$\Rightarrow$ $20x + 15 = A(x + 3) + B(x + 1)$

$\Rightarrow$ $20x + 15 = (A + B)x + (3A + B)$

$\Rightarrow$ $A + B = 20$ and $3A + B = 15$

$\Rightarrow$ $A = -\frac{5}{2}$ and $B = \frac{45}{2}$

Substituting these values of A and B in (iii), we have

$$\frac{20x+15}{(x+1)(x+3)} = \frac{-5/2}{x+1} + \frac{45/2}{x+3}$$

From (ii), we have

$$\frac{5x^2}{x^2+4x+3} = 5 - \frac{-5/2}{x+1} + \frac{45/2}{x+3}$$

From (i), we have

$$\int \frac{5x^2}{x^2+4x+3}dx$$

$$= \int\left[5 - \left(\frac{-\frac{5}{2}}{x+1} + \frac{\frac{45}{2}}{x+3}\right)\right]dx$$

$$= \int 5\,dx + \frac{5}{2}\int\frac{1}{x+1}dx - \frac{45}{2}\int\frac{1}{x+3}dx$$

$$= 5x + \frac{5}{2}\log|x+1| - \frac{45}{2}\log|x+3|$$

By second fundamental theorem, we have

$$\int_1^2 \frac{5x^2}{x^2+4x+3}dx$$

$$= \left[5x + \frac{5}{2}\log|x+1| - \frac{45}{2}\log|x+3|\right]_1^2$$

$$= \left[10 + \frac{5}{2}\log 3 - \frac{45}{2}\log 5\right] - \left[5 + \frac{5}{2}\log 2 - \frac{45}{2}\log 4\right]$$

$$= 5 + \frac{5}{2}(\log 3 - \log 2) - \frac{5}{2}(9\log 5 - 9\log 4)$$

$$= 5 + \frac{5}{2}\left(\log\frac{3}{2}\right) - \frac{5}{2}\left\{9\log\frac{5}{4}\right\}$$

$$= 5 - \frac{5}{2}\left(9\log\frac{5}{4} - \log\frac{3}{2}\right)$$

Example 4.2: Evaluate (A) $\int_0^{\pi/4} \sin 2x\,dx$ (B) $\int_0^{\pi/2} \cos^2 x\,dx$ **[NCERT]**

Ans. (A) Here, $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x$

So, by second fundamental theorem, we have

$$\int_0^{\pi/4} \sin 2x\,dx = \left[\frac{-1}{2}\cos 2x\right]_0^{\pi/4}$$

$$= \frac{-1}{2}\left[\cos\frac{\pi}{2} - \cos 0\right] = \frac{1}{2}$$

(B) Here, $\int \cos^2 x\,dx = \frac{1}{2}\int(1+\cos 2x)dx$

$$= \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)$$

So, by second fundamental theorem, we have

$$\int_0^{\pi/2} \cos^2 x\,dx = \left[\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right)\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - \left(0 + \frac{\sin 0}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right], \text{ i.e., } \frac{\pi}{4}$$

TOPIC 2

EVALUATION OF DEFINITE INTEGRALS BY SUBSTITUTION

To evaluate $\int_a^b f(x)\,dx$ by substitution, we take the following steps:

Step 1: Consider the integral without limits and substitute $y = f(x)$ or $x = g(y)$ and reduce the given integral to a known form.

Step 2: Change the upper and lower limits in the context of the new variable.

Step 3: Integrate the new integral with respect to the new variable without mentioning constant of integration.

Step 4: Find the difference of the values of the upper and lower limits.

Illustration: Evaluate $\int_0^2 x\sqrt{x+2}\,dx$

Let $I = \int_0^2 x\sqrt{x+2}\,dx$

Put $x + 2 = t$ so that $dx = dt$

For $x = 2, t = 4$ and for $x = 0, t = 2$.

Thus, $I = \int_2^4 (t-2)\sqrt{t}\,dt$

$$= \int_2^4 (t^{3/2} - 2t^{1/2})\,dt$$

$$= \left[\frac{t^{5/2}}{5/2} - 2\frac{t^{3/2}}{3/2}\right]_2^4$$

$$= \left[\frac{2}{5}t^{5/2} - \frac{4}{3}t^{3/2}\right]_2^4$$

$$= \left[\frac{2}{5}(4)^{5/2} - \frac{4}{3}(4)^{3/2}\right] - \left[\frac{2}{5}(2)^{5/2} - \frac{4}{3}(2)^{3/2}\right]$$

$$= \left[\frac{2}{5}(2)^5 - \frac{4}{3}(2)^3\right] - \left[\frac{2}{5}(4\sqrt{2}) - \frac{4}{3}(2\sqrt{2})\right]$$

$$= \left(\frac{64}{5} - \frac{32}{3}\right) + \left(\frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}\right)$$

$$= \frac{32}{15} + \frac{16\sqrt{2}}{15}, \text{ or } \frac{16\sqrt{2}}{15}(\sqrt{2}+1)$$

Example 4.3: (A) $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

(B) $\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$ [NCERT]

Ans. (A) Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Put $x = \tan t$, $t = \tan^{-1} x \Rightarrow dx = \sec^2 t\,dt$

For $x = 1$, $t = \frac{\pi}{4}$ and for $x = 0$, $t = 0$.

Thus,

$$I = \int_0^{\pi/4} \sin^{-1}\left(\frac{2\tan t}{1+\tan^2 t}\right)\sec^2 t\,dt$$

$$= \int_0^{\pi/4} \sin^{-1}(\sin 2t)\sec^2 t\,dt$$

$$= 2\int_0^{\pi/4} t.\sec^2 t\,dt \quad ...(i)$$

Using integration by parts, we get

$$\int t.\sec^2 t\,dt = t.\tan t - \int 1.\tan t\,dt$$

$$= t.\tan t - \log|\sec t|$$

From (i), we have

$$I = 2\int_0^{\pi/4} t.\sec^2 t\,dt$$

$$= 2[t.\tan t - \log|\sec t|]_0^{\pi/4}$$

$$= 2\left[\left(\frac{\pi}{4}\cdot\tan\frac{\pi}{4} - \log\left|\sec\frac{\pi}{4}\right|\right) - (0.\tan 0 - \log|\sec 0|)\right]$$

$$= 2\left[\left(\frac{\pi}{4} - \log\sqrt{2}\right) - (0-0)\right]$$

$$= \frac{\pi}{2} - 2\log\sqrt{2}, \text{ or } \frac{\pi}{2} - \log 2$$

(B) Let $I = \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx$

Put $\cos x = t$ so that $(-\sin x)dx = dt$

For $x = \frac{\pi}{2}$, $t = 0$ and for $x = 0$, $t = 1$

Thus, $I = -\int_1^0 \frac{1}{1+t^2}\,dt$

$$= -[\tan^{-1} t]_1^0$$

$$= -\tan^{-1} 0 + \tan^{-1} 1 = \frac{\pi}{4}$$

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$ equals: [NCERT]

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{12}$

Ans. (d) $\frac{\pi}{12}$

Explanation:

$$\int_1^{\sqrt{3}} \frac{dx}{1+x^2} = [\tan^{-1} x]_1^{\sqrt{3}}$$

$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

2. $\int_0^{2/3} \frac{dx}{4+9x^2}$ **equals:**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{12}$

(c) $\frac{\pi}{24}$ (d) $\frac{\pi}{4}$

3. $\int_{\frac{1}{3}}^{1} \frac{(x-x^3)^{1/3}}{x^4} dx$ **equals:**

(a) 6 (b) 0

(c) 3 (d) 4

Ans. *(a) 6*

Explanation: Let $I = \int_{\frac{1}{3}}^{3} \frac{(x-x^3)^{1/3}}{x^4} dx$

$$= \int_{\frac{1}{3}}^{1} \frac{x\left(\frac{1}{x^2}-1\right)^{1/3}}{x^4} dx$$

$$= \int_{\frac{1}{3}}^{1} \frac{\left(\frac{1}{x^2}-1\right)^{1/3}}{x^3} dx$$

Put $\left(\frac{1}{x^2}-1\right) = t$ so that $-\frac{2}{x^3} dx = dt$,

or $\frac{1}{x^3} dx = -\frac{1}{2} dt$

When $x = 1, t = 0$ and when $x = \frac{1}{3}, t = 8$

Thus, $I = \int_8^0 t^{1/3}\left(-\frac{1}{2} dt\right)$

$$= \frac{1}{2}\int_0^8 t^{1/3}\, dt$$

$$= \frac{1}{2}\left[\frac{3}{4} t^{4/3}\right]_0^8$$

$$= \frac{1}{2} \cdot \frac{3}{4} \cdot (16) = 6$$

4. The value of $\int_{e^2}^{e^3} \frac{dx}{x \log x}$ **is:**

(a) $\log \frac{2}{3}$ (b) $\log \frac{3}{2}$

(c) $\log 3$ (d) $\log 2$

Ans. *(b)* $\log \frac{3}{2}$

Explanation: Let $I = \int_{e^2}^{e^3} \frac{dx}{x \log x}$

Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

When $x = e^3, t = \log e^3 = 3 \log e = 3$

When $x = e^2, t = \log e^2 = 2 \log e = 2$

$\therefore \quad I = \int_2^3 \frac{dt}{t}$

$$= [\log|t|]_2^3 = \log(3) - \log(2)$$

$$= \log\frac{3}{2}$$

5. $\int_2^4 \frac{x^3-1}{x^2} dx$ **is equal to:**

(a) $\frac{19}{4}$ (b) $\frac{21}{4}$

(c) $\frac{4}{23}$ (d) $\frac{23}{4}$

6. Evaluate: $\int_0^1 \log\left(\frac{1-x}{x}\right) dx$.

(a) $\log 3 - \log 2$ (b) $\log 2 - \log 3$

(c) $\log 4 - \log 52$ (d) 0 **[DIKSHA]**

Ans. *(d) 0*

Explanation:

Let $I = \int_0^1 \log\left(\frac{1-x}{x}\right) dx$

$$I = \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right) dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^1 \log\left(\frac{x}{1-x}\right) dx \quad \text{...(ii)}$$

On adding (i) and (ii), we get

$$2I = \int_0^1 \log\left(\frac{1-x}{x}\right) + \log\left(\frac{x}{1-x}\right) dx$$

$$= \int_0^1 \log 1\, dx = 0$$

$\therefore \quad I = 0$

7. $\int_{-/4}^{/4} \frac{dx}{1 \quad \cos 2}$ is equal to:

(a) 1 (b) 2

(c) 3 (d) 4 [NCERT Exemplar]

Ans. *(a) 1*

Explanation:

Let $I = \int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$

$= \int_{-\pi/4}^{\pi/4} \frac{dx}{2\cos^2 x}$

$= \frac{1}{2} \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx$

$= \int_{-\pi/4}^{\pi/4} \sec^2 x \, dx$

$= [\tan x]_0^{\pi/4} = \tan \frac{\pi}{4} - \tan 0 = 1$

8. $\int_2^3 \frac{x}{x^2+1} dx$ is:

(a) $\frac{1}{2}\log 2$ (b) $\log 2$

(c) $\log \frac{1}{2}$ (d) $\frac{1}{2}\log\frac{1}{2}$

9. $\int_0^{\pi/4} \tan x \, dx$ is equal to:

(a) $\frac{1}{3}\log 2$ (b) $\frac{1}{2}\log 2$

(c) $\log 3$ (d) $2 \log 2$

Ans. *(b)* $\frac{1}{2}\log 2$

Explanation: Let

$I = \int_0^{\pi/4} \tan x \, dx$

$= [\log|\sec x|]_0^{\pi/4}$

$= \log\left|\sec\frac{\pi}{4}\right| - \log|\sec 0|$

$= \log\sqrt{2} - \log(1)$

$= \log\sqrt{2} - 0$

$= \log 2^{1/2} = \frac{1}{2}\log 2$

10. $\int_0^1 \frac{1-x}{1+x} dx$ is:

(a) $2\log\frac{1}{2}$

(b) $2 \log 2 + 1$

(c) $2 \log 2 - 1$

(d) $\log 2$

Ans. *(c) $2\log 2 - 1$*

Explanation: Let

$I = \int_0^1 \frac{1-x}{1+x} dx$

$= \int_0^1 \frac{-x-1+2}{1+x} dx$

$= \int_0^1 \left[-1 + \frac{2}{1+x}\right] dx$

$= [-x + 2\log|1+x|]_0^1$

$= [-1 + 2\log|1+1| - (-0 + 2\log|1+0|]$

$= [-1 + 2\log 2 - 2\log 1]$

$= -1 + 2\log 2 - 0$

$= 2\log 2 - 1$

11. The value of $\int_2^3 \log x \, dx$ is:

(a) $\log\frac{27}{4} + 1$ (b) $\log\frac{27}{4} - 1$

(c) $\log\frac{9}{2} - 1$ (d) $\log\frac{3}{2} + 1$

12. $\int_0^{\pi/4} \frac{1}{\cos x} dx$ is:

(a) $\log(\sqrt{2}+2)$ (b) $\log(\sqrt{2}-1)$

(c) $\frac{1}{2}\log(\sqrt{2}+1)$ (d) $\log(\sqrt{2}+1)$

Ans. *(d) $\log(\sqrt{2}+1)$*

Explanation: Let

$I = \int_0^{\pi/4} \frac{1}{\cos x} dx$

$= \int_0^{\pi/4} \sec x \, dx$

$= [\log|\sec x + \tan|]_0^{\pi/4}$

$= \log\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \log|\sec 0 + \tan 0|$

$= \log(\sqrt{2}+1) - \log|1+0|$

$= \log(\sqrt{2}+1) - \log 1$

$= \log(\sqrt{2}+1) - 0$

$= \log(\sqrt{2}+1)$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

13. Evaluate $\int_2^3 3^x\,dx$. **[CBSE 2017]**

Ans. Let
$$I = \int_2^3 3^x\,dx$$
$$= \frac{1}{\log 3}[3^x]_2^3$$
$$= \frac{1}{\log 3}[3^3 - 3^2]$$
$$= \frac{1}{\log 3}(27-9)$$
$$= \frac{18}{\log 3} = 18\log_3 e$$

14. Simplify the integral $\int_1^3 (x^2+1)\,dx$.

15. Find $\int_0^{\pi/2} \cos^2 x\,dx$.

16. Evaluate $\int_0^1 xe^{x^2}\,dx$. **[CBSE 2014]**

Ans. Let
$$I = \int_0^1 xe^{x^2}\,dx$$

Put $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{dt}{2x}$

Lower limit, when $x = 0$, then $t = 0^2 = 0$

Upper limit when $x = 1$, then $t = 1^2 = 1$

$$\therefore \quad I = \int_0^1 xe^t \frac{dt}{2x}$$
$$= \frac{1}{2}\int_0^1 e^t\,dt$$
$$= \frac{1}{2}[e^t]_0^1$$
$$= \frac{1}{2}[e^1 - e^0]$$
$$= \frac{1}{2}[e-1]$$

17. Evaluate $\int_{\pi/6}^{\pi/4} \cot x\,dx$.

18. If $\int_0^b \frac{1}{9+x^2}\,dx = \frac{\pi}{9}$, then find the value of b.

Ans. We have,
$$\int_0^b \frac{1}{9+x^2}\,dx = \frac{\pi}{9}$$
$$\Rightarrow \frac{1}{3}\left[\tan^{-1}\frac{b}{3} - 0\right] = \frac{\pi}{9}$$
$$\Rightarrow \tan^{-1}\frac{b}{3} = \frac{\pi}{3}$$
$$\Rightarrow \frac{b}{3} = \tan\left(\frac{\pi}{3}\right)$$
$$\Rightarrow b = 3\times\sqrt{3}$$
$$\Rightarrow b = 3\sqrt{3}$$

Hence, the value of b is $3\sqrt{3}$.

19. Evaluate $\int_2^4 \frac{x}{x^2+1}\,dx$. **[CBSE 2014]**

20. Evaluate $\int_0^{\sqrt{3}} \frac{\tan^{-1}x}{1+x^2}\,dx$.

Ans. Let
$$I = \int_0^{\sqrt{3}} \frac{\tan^{-1}x}{1+x^2}\,dx$$

Put $\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

Upper limit, when $x = \sqrt{3}, \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Lower limit, when $x = 0$, $\tan^{-1}0 = 0$

$$\therefore \quad I = \int_0^{\pi/3} t\,dt$$
$$= \left[\frac{t^2}{2}\right]_0^{\pi/3}$$
$$= \frac{1}{2}\left[\left(\frac{\pi}{3}\right)^2 - 0\right]$$
$$= \frac{\pi^2}{18}$$

21. Find $\int_0^3 \sqrt{9-x^2}\,dx$.

22. If $\int_0^k \frac{1}{2+8x^2}\,dx = \frac{\pi}{16}$, find the value of k.

Ans. We have, $\int_0^k \frac{1}{2+8x^2}dx = \frac{1}{8}\int_0^k \frac{1}{\frac{1}{4}+x^2}dx$

$$= \frac{1}{8}\frac{1}{\left(\frac{1}{2}\right)}\left[\tan^{-1}\frac{x}{1/2}\right]_0^k$$

$$= \frac{1}{4}[\tan^{-1}2x]_0^k$$

$$= \frac{1}{4}[\tan^{-1}2k - \tan^{-1}0]$$

$$= \frac{1}{4}[\tan^{-1}2k]$$

$$\therefore \quad \frac{1}{4}\tan^{-1}2k = \frac{\pi}{16}$$

$$\Rightarrow \quad \tan^{-1}2k = \frac{\pi}{4}$$

$$\Rightarrow \quad 2k = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad 2k = 1$$

$$\Rightarrow \quad k = \frac{1}{2}$$

Hence, the value of k is $\frac{1}{2}$.

23. If $f(x) = \int_0^x t\sin t\,dt$, then write the value of $f'(x)$. **[CBSE 2014]**

Ans. Given $f(x) = \int_0^x \underset{\text{I}}{t}\,\underset{\text{II}}{\sin t}\,dt$

By using integration by parts, we get

$$f(x) = \left[t\int \sin t\,dt - \int\left[\frac{d}{dt}(t)\int \sin t\,dt\right]dt\right]_0^x$$

$$= [t(-\cos t)]_0^x + \int_0^x 1.\cos t\,dt$$

$$= [-t\cos t]_0^x + [\sin t]_0^x$$

$$= -x\cos x + 0 + \sin x - 0$$

$$= \sin x - x\cos x$$

Thus, $f(x) = \sin x - x\cos x$

On differentiating both sides with respect to x, we get

$$f'(x) = \cos x - \left[x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x)\right]$$

$$= \cos x - [x(-\sin x) + \cos x]$$

$$= \cos x + x\sin x - \cos x = x\sin x.$$

24. Sonam was observing a few boys playing with lattoo or top. She then noticed that one boy started rotating the top on his palm. She wanted to use her knowledge of integration and find the area of the surface formed if the function is given by cos 3x.

Evaluate $\int_0^{\pi/3} \cos 3x\,dx$

Ans. Let $\quad I = \int_0^{\pi/3} \cos 3x\,dx$

$$= \left[\frac{\sin 3x}{3}\right]_0^{\pi/3}$$

$$= \frac{1}{3}[\sin\pi - \sin 0]$$

$$= \frac{1}{3}[0 - 0] = 0$$

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

25. Evaluate $\int_0^{\pi/4}(2\sec^2 x + x^3 + 2)\,dx$. **[NCERT]**

26. Evaluate $\int_1^2\left[\frac{1}{x} - \frac{1}{2x^2}\right]e^{2x}\,dx$. **[CBSE 2020]**

Ans. Let $\quad I = \int_1^2\left[\frac{1}{x} - \frac{1}{2x^2}\right]e^{2x}\,dx$

$$= \int_1^2 \underset{\text{I}}{\frac{1}{x}}\,\underset{\text{II}}{e^{2x}}\,dx - \int_1^2 \frac{1}{2x^2}e^{2x}\,dx$$

By using integration by parts in first integral

$$I = \left[\frac{1}{x}\frac{e^{2x}}{2}\right]_1^2 - \left[\int\left\{\frac{d}{dx}\left(\frac{1}{x}\right)\int e^{2x}dx\right\}dx\right]_1^2 - \int_1^2 \frac{1}{2x^2}e^{2x}\,dx$$

$$= \left[\frac{e^{2x}}{2x}\right]_1^2 - \left[\int_1^2 -\frac{1}{x^2}\times\frac{e^{2x}}{2}\,dx\right] - \int_1^2 \frac{e^{2x}}{2x^2}\,dx$$

$$= \left[\frac{e^{2x}}{2x}\right]_1^2$$

$$= \frac{e^4}{4} - \frac{e^2}{2}$$

$$= \frac{e^2(e^2-2)}{4}$$

27. Prove that $\int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2} - 1$

28. Evaluate $\int_0^1 \frac{dx}{e^x + e^{-x}}$. **[NCERT Exemplar]**

Ans. Let, $I = \int_0^1 \frac{dx}{e^x + e^{-x}}$

$$= \int_0^1 \frac{e^x}{e^{2x}+1} dx$$

Consider,

$$e^x = t$$

$\Rightarrow$ $e^x dx = dt$

When $x = 0$, $t = e^0 = 1$

When $x = 1$, $t = e^1 = e$

$\therefore$ $I = \int_1^e \frac{dt}{t^2+1} = \left[\tan^{-1} t\right]_1^e$

$$= \tan^{-1} e - \tan^{-1} 1$$

$$= \tan^{-1} e - \frac{\pi}{4}$$

29. Find $\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x} dx$.

Ans. Let $I = \int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

Upper limit, when $x = \frac{\pi}{2}, t = \sin\frac{\pi}{2} = 1$

Lower limit, when $x = 0$, $t = \sin 0 = 0$

$\therefore$ $I = \int_0^1 \frac{dt}{1+t^2}$

$$= \left[\tan^{-1} t\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

30. Evalaute $\int_0^3 x\sqrt{x+6}\, dx$.

Ans. Let

$$I = \int_0^3 x\sqrt{x+6}\, dx$$

$$= \int_0^3 (x+6-6)\sqrt{x+6}\, dx$$

$$= \int_0^3 [(x+6)^{3/2} dx - 6\sqrt{x+6}\, dx]$$

$$= \left[\frac{(x+6)^{\frac{3}{2}+1}}{\frac{3}{2}+1} - \frac{6(x+1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_0^3$$

$$= \left[\frac{(x+6)^{5/2}}{\frac{5}{2}} - 4(x+1)^{3/2}\right]_0^3$$

$$= \left\{\frac{(3+6)^{5/2}}{5/2} - 4(3+6)^{3/2}\right\}$$

$$-\left\{\frac{(0+6)^{5/2}}{5/2} - 4(0+6)^{3/2}\right\}$$

$$= \left\{\frac{2}{5}(9)^{5/2} - 4(9)^{3/2}\right\} - \left\{\frac{2}{5}(6)^{5/2} - 4(6)^{3/2}\right\}$$

$$= \left\{\frac{2}{5}(3^2)^{5/2} - 4(3^2)^{3/2}\right\} - \left\{\frac{2}{5}(6)^{2+\frac{1}{2}} - 4(6)^{1+\frac{1}{2}}\right\}$$

$$= \left\{\frac{2}{5}\times 3^5 - 4\times 3^3\right\} - \left\{\frac{2}{5}\times 36\sqrt{6} - 4\times 6\times\sqrt{6}\right\}$$

$$= \left\{\frac{486-540}{5}\right\} - \left\{\frac{72\sqrt{6} - 120\sqrt{6}}{5}\right\}$$

$$= -\frac{54}{5} + \frac{48\sqrt{6}}{5} = \frac{48\sqrt{6} - 54}{5}$$

31. Evaluate $\int_{-3}^{3} \frac{1}{x^2 + 6x + 45} dx$.

32. Find $\int_0^{\pi/12} \frac{1+\tan x}{1-\tan x} dx$.

Ans. Let $I = \int_0^{\pi/12} \frac{1+\tan x}{1-\tan x} dx$

$$= \int_0^{\pi/12} \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x} dx$$

$$= \int_0^{\pi/12} \tan\left(\frac{\pi}{4} + x\right) dx$$

$$= \left[\log\left\{\sec\left(\frac{\pi}{4}+x\right)\right\}\right]_0^{\pi/12}$$

$$= \log\left\{\sec\left(\frac{\pi}{4}+\frac{\pi}{12}\right)\right\} - \log\left\{\sec\left(\frac{\pi}{4}+0\right)\right\}$$

$$= \log\left(\sec\frac{4\pi}{12}\right) - \log\left(\sec\frac{\pi}{4}\right)$$

$$= \log\left(\sec\frac{\pi}{3}\right) - \log\left(\sec\frac{\pi}{4}\right)$$

$$= \log 2 - \log\sqrt{2}$$

$$= \log\frac{2}{\sqrt{2}} = \log\sqrt{2}$$

33. Evaluate $\int_0^{\pi/4} \sin 3x \sin 2x \, dx$

SHORT ANSWER Type-II Questions (SA-II)

[**3** marks]

34. Prove that $\int_0^{\sqrt{3}} \frac{x^4+1}{x^2+1} dx = \frac{2\pi}{3}$

Ans. Let

$$I = \int_0^{\sqrt{3}} \frac{x^4+1}{x^2+1} dx$$

$$= \int_0^{\sqrt{3}} \frac{(x^4-1)+2}{x^2+1} dx$$

$$= \int_0^{\sqrt{3}} \left[\frac{(x^4-1)}{x^2+1} + \frac{2}{x^2+1}\right] dx$$

$$= \int_0^{\sqrt{3}} \frac{(x^2-1)(x^2+1)}{x^2+1} dx + \int_0^{\sqrt{3}} \frac{2}{x^2+1} dx$$

$$= \int_0^{\sqrt{3}} (x^2-1) dx + \int_0^{\sqrt{3}} \frac{2}{x^2+1} dx$$

$$= \left[\frac{x^3}{3} - x\right]_0^{\sqrt{3}} + \left[2 \times \frac{1}{1}\tan^{-1}\frac{x}{1}\right]_0^{\sqrt{3}}$$

$$= \left[\frac{(\sqrt{3})^3}{3} - \sqrt{3} - 0 + 0\right] + \left[2\tan^{-1}\sqrt{3} - 0\right]$$

$$= \frac{3\sqrt{3}}{3} - \sqrt{3} + 2 \times \frac{\pi}{3}$$

$$= 0 + \frac{2\pi}{3} = \frac{2\pi}{3}$$

Hence, proved.

35. Evaluate: $\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}} dx$

[NCERT Exemplar]

36. Evaluate $\int_0^{\sqrt{3}/2} \sin^{-1} x \, dx$

Ans. Let $\quad I = \int_0^{\sqrt{3}/2} \sin^{-1} x \, dx$

Put $\sin^{-1} x = \theta \Rightarrow \sin\theta = x$

$\Rightarrow \quad \cos\theta \, d\theta = dx$

Upper limit, when $x = \frac{\sqrt{3}}{2}, \theta = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

Lower limit, when $x = 0, \theta = \sin^{-1} 0 = 0$

$\therefore \quad I = \int_0^{\pi/3} \underset{\text{I}}{\theta} . \underset{\text{II}}{\cos\theta} \, d\theta$

By using integration by parts, we have

$$I = \left[\theta \int \cos\theta d\theta\right]_0^{\pi/3} - \left[\int\left\{\frac{d}{d\theta}(\theta)\int\cos\theta d\theta\right\}d\theta\right]$$

$$= [\theta \sin\theta]_0^{\pi/3} - \left[\int 1.\sin\theta \, d\theta\right]_0^{\pi/3}$$

$$= \left[\frac{\pi}{3}\sin\frac{\pi}{3} - 0\right] - [-\cos\theta]_0^{\pi/3}$$

$$= \frac{\pi}{3} \times \frac{\sqrt{3}}{2} + \left[\cos\frac{\pi}{3} - \cos 0\right]$$

$$= \frac{\pi}{2\sqrt{3}} + \left[\frac{1}{2} - 1\right]$$

$$= \frac{\pi}{2\sqrt{3}} - \frac{1}{2}$$

37. Evaluate $\int_0^{\pi/2} \frac{\tan x}{1+m^2\tan^2 x} dx$

[NCERT Exemplar]

Ans. Let, $\quad I = \int_0^{\pi/2} \frac{\tan x}{1+m^2\tan^2 x} dx$

$$= \int_0^{\pi/2} \frac{\frac{\sin x}{\cos x}}{1+m^2\frac{\sin^2 x}{\cos^2 x}} dx$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + m^2 \sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x}{1-\sin^2 x + m^2 \sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\sin x \cos x}{1+(m^2-1)\sin^2 x} dx$$

Put $\sin^2 x = t$

$\Rightarrow \quad 2\sin x \cos x\, dx = dt$

When $x = 0$, $t = \sin^2 0 = 0$

When $x = \frac{\pi}{2}$, $t = \sin^2 \frac{\pi}{2} = 1$

$$\therefore \quad I = \int_0^1 \frac{dt}{2(1+(m^2-1)t)}$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{1+(m^2-1)t}$$

$$= \frac{1}{2}\left[\frac{1}{m^2-1}\log\left[(1+(m^2-1)t\right]\right]_0^1$$

$$= \frac{1}{2(m^2-1)}[\log(1+(m^2-1)) - \log 1]$$

$$= \frac{\log m^2}{2(m^2-1)} = \frac{2\log m}{2(m^2-1)} = \frac{\log m}{(m^2-1)}$$

38. Evaluate $\int_1^3 \frac{1}{x^2(x+1)} dx$

39. Evaluate $\int_{\pi/4}^{\pi/2} \sqrt{1-\sin 2x}\, dx$

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

40. Evaluate $\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx.$

[CBSE 2018]

Ans. Let

$$I = \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\sin 2x} dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{9(2\sin x \cos x) + 16} dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{-9(-2\sin x \cos x) + 16} dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{-9(\sin^2 x + \cos^2 x - 2\sin x \cos x - 1) + 16} dx$$

$$= \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 9(\sin x - \cos x)^2} dx$$

Put $\sin x - \cos x = t \Rightarrow (\cos x + \sin x)dx = dt$

Lower limit, when $x = 0$, then

$t = \sin 0 - \cos 0 = -1$

Upper limit, when $x = \frac{\pi}{4}$, then

$$t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$$

$$\therefore \quad I = \int_{-1}^{0} \frac{dt}{25 - 9t^2}$$

$$= \frac{1}{9}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{3}\right)^2 - (t)^2}$$

$$= \frac{1}{9} \times \frac{1}{2\times\frac{5}{3}}\left[\log\frac{\frac{5}{3}+t}{\frac{5}{3}-t}\right]_{-1}^{0}$$

$$= \frac{1}{30}\left[\log\left|\frac{5+3t}{5-3t}\right|\right]_{-1}^{0}$$

$$= \frac{1}{30}\left[\log 1 - \log\left(\frac{2}{8}\right)\right]$$

$$= \frac{1}{30}\left[0 - \log\frac{1}{4}\right] = \frac{1}{30}\log 4$$

41. Evaluate $\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2\sin^2 x)^2}$.

[NCERT Exemplar]

42. Evaluate $\int_0^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)} dx.$

Ans. Consider $x^2 = y$. Then

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{1}{(y+a^2)(y+b^2)} \quad ...(i)$$

Let $\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{A}{y+a^2}+\frac{B}{y+b^2}$

$$1 = A(y+b^2)+B(y+a^2) \qquad \text{...(ii)}$$

Putting $y = -a^2$ and $y = -b^2$ successively in (ii),

we get $A = \frac{1}{b^2-a^2}$ and $B = \frac{1}{a^2-b^2}$

Substituting the values of A and B in (i), we get

$$\frac{1}{(y+a^2)(y+b^2)}$$

$$= \frac{1}{a^2-b^2}\left\{\frac{1}{y+b^2}-\frac{1}{y+a^2}\right\}$$

$$= \frac{1}{a^2-b^2}\left\{\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right\}$$

$$\Rightarrow \quad I = \int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}dx$$

$$= \frac{1}{a^2-b^2}\int_0^\infty \left(\frac{1}{x^2+b^2}-\frac{1}{x^2+a^2}\right)dx$$

$$I = \frac{1}{a^2-b^2}\left[\frac{1}{b}\tan^{-1}\frac{x}{b}-\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^\infty$$

$$= \frac{1}{a^2-b^2}\left[\left(\frac{1}{b}\tan^{-1}\infty-\frac{1}{a}\tan^{-1}\infty\right)\right.$$

$$\left.-\left(\frac{1}{b}\tan^{-1}0-\frac{1}{a}\tan^{-1}0\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[\left(\frac{\pi}{2b}-\frac{\pi}{2a}\right)-(0-0)\right]$$

$$= \frac{\pi}{2ab(a+b)}$$

PROPERTIES OF DEFINITE INTEGRALS 5

TOPIC 1

EVALUATION OF DEFINITE INTEGRALS USING PROPERTIES

Here are some important properties of definite integrals which are very useful in evaluating them.

Property 1 (Change of Variable):

$$\int_a^b f(x)dx = \int_a^b f(t)\,dt$$

Property 2 (Interchange of limits):

$$\int_a^b f(x)dx = -\int_b^a f(x)\,dx.$$

In particular, $\int_a^a f(x)\,dx = 0$

Property 3: $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

Important

➥ *This property is generally used for evaluating the modulus function etc.*

Property 4: $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$

In particular, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Property 5: $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

Property 6:

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

Property 7: $\int_{-a}^a f(x)\,dx$

$$= \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(-x) = f(x), \text{ i.e., if } f \text{ is an even function.} \\ 0, & \text{if } f(-x) = -f(x), \text{ i.e., if } f \text{ is an odd function.} \end{cases}$$

Proofs of the Properties

Property 1: Let $\int f(x)\,dx = F(x)$. Then, $\int f(t)\,dt = F(t)$.

$\therefore$ LHS $= \int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ ...(i)

RHS $= \int_a^b f(t)\,dt = [F(t)]_a^b = F(b) - F(a)$ ...(ii)

From (i) and (ii), we conclude that

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

Therefore, in a definite integral, we can change the variable throughout.

Property 2: Let $\int f(x)\,dx = F(x)$. Then,

LHS $= \int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ ...(i)

RHS $= -\int_b^a f(x)\,dx = -[F(x)]_b^a = -[F(a) - F(b)]$

$= F(b) - F(a)$ ...(ii)

From (i) and (ii), we conclude that

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Further, we observe that if $a = b$, then $\int_a^a f(x)dx = 0$

Property 3: Let $\int f(x)\,dx = F(x)$. Then,

LHS $= \int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ ...(i)

RHS $= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$= [F(x)]_a^c + [F(x)]_c^b$

$= [F(c) - F(a)] + [F(b) - F(c)]$

$= F(b) - F(a)$ ...(ii)

From (i) and (ii), we conclude that

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

where, $a < c < b$.

This property can be generalised as

$$\int_a^b f(x)\,dx = \int_a^c f(x)dx + \int_c^d f(x)\,dx + \int_d^e f(x)\,dx + \dots + \int_z^b f(x)\,dx$$

where $a < c < d < e < \dots < z < b$

Property 4: Let $t = a + b - x$. Then, $dt = -dx$

When $x = a$, $t = b$ and when $x = b$, $t = a$

So, $\int_a^b f(x)\,dx = -\int_b^a f(a+b-t)\,dt$

$= \int_a^b f(a+b-t)\,dt$ {Using Property 2}

$= \int_a^b f(a+b-x)\,dx$ {Using Property 1}

Taking $a = 0$ and $b = a$, we have

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Property 5: Using Property 3, we have

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx \quad ...(i)$$

Consider $\int_a^{2a} f(x)\,dx$.

Put $x = 2a - t$ so that $dx = -dt$
When $x = a, t = a$ and when $x = 2a, t = 0$

So, $\int_a^{2a} f(x)\,dx = -\int_a^0 f(2a-t)\,dt$

$$= \int_0^a f(2a-t)\,dt \quad \text{\{Using Property 2\}}$$

$$= \int_0^a f(2a-x)\,dx \quad \text{\{Using Property 1\}}$$

Thus, from (i), we have

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Property 6: Using property 5, we have

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$= \int_0^a f(x)\,dx + \int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x)$$

$$= 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x)$$

Further, $\int_0^{2a} f(x)\,dx$

$$= \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

$$= \int_0^a f(x)\,dx - \int_0^a f(x)\,dx,$$

if $f(2a-x) = -f(x)$

$= 0,$

if $f(2a-x) = -f(x)$

Property 7: Using property 3, we have

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx \quad ...(i)$$

Consider $\int_{-a}^0 f(x)\,dx$.

Put $x = -t$ so that $dx = -dt$
When $x = -a, t = a$ and when $x = 0, t = 0$

So, $\int_{-a}^0 f(x)dx = \int_a^0 f(-t)(-dt)$

$$= \int_0^a f(-t)\,dt \quad \text{\{Using Property 2\}}$$

$$= \int_0^a f(-x)\,dx \quad \text{\{Using Property 1\}}$$

Thus, from (i), we have

$$\int_{-a}^a f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx \quad ...(ii)$$

(a) If f is an even function, *i.e.*, when $f(-x) = f(x)$, equation (ii) becomes

$$\int_{-a}^a f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(x)\,dx,$$

$$\text{or } 2\int_0^a f(x)\,dx$$

(b) If f is an odd function, *i.e.*, when $f(-x) = -f(x)$, equation (ii) becomes

$$\int_{-a}^a f(x)\,dx = -\int_0^a f(x)\,dx + \int_0^a f(x)\,dx, \text{ or } 0$$

Example 5.1: Evaluate:

(A) $\int_{-5}^5 |x+2|\,dx$

(B) $\int_2^8 |x-5|\,dx$ **[NCERT]**

Ans. (A) We know that

$$|x+2| = \begin{cases} x+2, & \text{when } x+2 \geq 0, \text{ or } x \geq -2 \\ -(x+2), & \text{when } x+2 < 0, \text{ or } x < -2 \end{cases}$$

So, $\int_{-5}^5 |x+2|\,dx$

$$= \int_{-5}^{-2} |x+2|\,dx + \int_{-2}^5 |x+2|\,dx$$

{Using Property 3}

$$= -\int_{-5}^{-2} (x+2)dx + \int_{-2}^5 (x+2)\,dx$$

$$= \left[\frac{x^2}{2} + 2x\right]_{-2}^5 - \left[\frac{x^2}{2} + 2x\right]_{-5}^{-2}$$

$$= \left[\frac{25}{2} + 10 - 2 + 4\right] - \left[2 - 4 - \frac{25}{2} + 10\right]$$

$= 29$

(B) We know that

$$|x-5| = \begin{cases} x-5, & \text{when } x-5 \geq 0, \text{ or } x \geq 5 \\ -(x-5), & \text{when } x-5 < 0, \text{ or } x < 5 \end{cases}$$

So, $\int_2^8 |x-5|\,dx$

$$= \int_2^5 |x-5|\,dx + \int_5^8 |x-5|\,dx$$

{Using Property 3}

$$= -\int_2^5 (x-5)\,dx + \int_5^8 (x-5)\,dx$$

$$= -\left[\frac{x^2}{2} - 5x\right]_2^5 + \left[\frac{x^2}{2} - 5x\right]_5^8$$

$$= -\left[\frac{25}{2} - 25 - 2 + 10\right] + \left[32 - 40 - \frac{25}{2} + 25\right]$$

$= 9$

Example 5.2: Evaluate: $\int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$.

Ans. Let

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad \text{...(i)}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}dx$$

{Using property 4}

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$\Rightarrow \quad I + I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$$

$$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} 1.dx$$

$$= [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$$

$$\Rightarrow \quad I = \frac{\pi}{12}$$

Example 5.3: Evaluate:

(A) $\int_0^2 x\sqrt{2-x}\,dx$

(B) $\int_0^{\pi/4} \log(1+\tan x)\,dx$ **[NCERT]**

Ans. (A) Let

$$I = \int_0^2 x\sqrt{2-x}\,dx$$

$$= \int_0^2 (2-x)\sqrt{x}\,dx \text{ {Using property 4}}$$

$$= \int_0^2 (2\sqrt{x}-x\sqrt{x})\,dx$$

$$= \left[\frac{4}{3}x^{3/2}-\frac{2}{5}x^{5/2}\right]_0^2$$

$$= \frac{4}{3}(2)^{3/2}-\frac{2}{5}(2)^{5/2}$$

$$= \frac{8}{3}\sqrt{2}-\frac{8}{5}\sqrt{2}, \text{ or } \frac{16\sqrt{2}}{15}$$

(B) Let

$$I = \int_0^{\pi/4} \log[1+\tan x]dx$$

$$= \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]dx$$

{Using property 4}

$$= \int_0^{\pi/4} \log\left[1+\left(\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)\right]dx$$

$$= \int_0^{\pi/4} \log\left[1+\left(\frac{1-\tan x}{1+1\times\tan x}\right)\right]dx$$

$$= \int_0^{\pi/4} \log\left[\frac{2}{1+\tan x}\right]dx$$

$$= \int_0^{\pi/4} \log 2\,dx - \int_0^{\pi/4} \log[1+\tan x]dx$$

$$= \int_0^{\pi/4} \log 2\,dx - I$$

$$\Rightarrow 2I = \int_0^{\pi/4} \log 2\,dx = \log 2\,[x]_0^{\pi/4} = \frac{\pi}{4}\log 2$$

$$\Rightarrow \quad I = \frac{\pi}{8}\log 2$$

Example 5.4: Evaluate:

(A) $\int_{-\pi/2}^{\pi/2} \sin^7 x\,dx$.

(B) $\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$. **[NCERT]**

Ans. (A) Let $f(x) = \sin^7 x$.

Then, $f(-x) = \sin^7(-x) = -\sin^7 x = -f(x)$

$\therefore$ Using Property 7, we have

$$\int_{-\pi/2}^{\pi/2} \sin^7 x\,dx = 0$$

(B) Let $f(x) = \sin^2 x$.

Then, $f(-x) = \sin^2(-x) = \sin^2 x = f(x)$

$\therefore$ Using Property 7, we have

$$\int_{-\pi/2}^{\pi/2} \sin^2 x\,dx = 2\int_0^{\pi/2} \sin^2 x\,dx$$

$$= \int_0^{\pi/2} (1-\cos 2x)\,dx$$

$[\because \cos 2x = 1-2\sin^2 x]$

$$= \left[x-\frac{1}{2}\sin 2x\right]_0^{\pi/2}$$

$$= \frac{\pi}{2}$$

Example 5.5: Show that

$$\int_0^a f(x)\,g(x)dx = 2\int_0^a f(x)\,dx,$$

if f and g are defined as $f(x) = f(a-x)$ and $g(x) + g(a-x) = 4$ **[NCERT]**

Ans. Let $$I = \int_0^a f(x)\,g(x)\,dx$$

$$= \int_0^a f(a-x)g(a-x)\,dx$$

{Using property 4}

$$= \int_0^a f(x)\{4-g(x)\}\,dx$$

$$= \int_0^a 4f(x)\,dx - \int_0^a f(x)g(x)\,dx$$

$$= 4\int_0^a f(x)\,dx - I$$

$$\Rightarrow \quad 2I = 4\int_0^a f(x)\,dx$$

$$\Rightarrow \quad I = 2\int_0^a f(x)\,dx$$

Hence, proved.

OBJECTIVE Type Questions

[**1** mark]

Multiple Choice Questions

1. $\int_0^{\pi/2} \sin^2 x\,dx$ **is equal to:**

(a) $\frac{\pi}{3}$ (b) 0

(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

Ans. *(c)* $\frac{\pi}{4}$

Explanation: Let $I = \int_0^{\pi/2} \sin^2 x\,dx$ ...(i)

$$I = \int_0^{\pi/2} \sin^2\left(\frac{\pi}{2}-x\right)dx$$

$$I = \int_0^{\pi/2} \cos^2 x\,dx \quad \text{...(ii)}$$

On adding eq. (i) and (ii), we get

$$2I = \int_0^{\pi/2}(\sin^2 x + \cos^2 x)\,dx$$

$$= \int_0^{\pi/2} 1\,dx$$

$$I = \frac{1}{2}[x]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{\pi}{2}-0\right]$$

$$= \frac{\pi}{4}$$

2. $\int_{-\pi/4}^{\pi/4}(\sec^2 x)\,dx$ **is equal to:**

(a) – 1 (b) 0

(c) 1 (d) 2 **[CBSE 2020]**

Ans. *(d)* 2

Explanation: Let $I = \int_{-\pi/4}^{\pi/4}(\sec^2 x)\,dx$

Consider $f(x) = \sec^2 x$

$$\Rightarrow \quad f(-x) = \sec^2(-x)$$

$$= \sec^2 x = f(x)$$

So, $f(x)$ is an even function.

$$\therefore \quad I = 2\int_0^{\pi/4} \sec^2 x\,dx$$

$$= 2[\tan x]_0^{\pi/4}$$

$$= 2\left[\tan\frac{\pi}{4} - \tan 0\right]$$

$$= 2\,[1-0] = 2$$

3. $\int_{-\pi/2}^{\pi/2} \sin^5 x\,dx$ **is:**

(a) 2 (b) – 1

(c) 0 (d) 1

4. If $f(x) = \begin{cases} 2x+8, & 1 \le x \le 2 \\ 6x, & 2 \le x \le 4 \end{cases}$, then $\int_1^4 f(x)\,dx$ **is:**

(a) 43 (b) 45

(c) 47 (d) 46

Ans. *(c)* 47

Explanation: We have

$$f(x) = \begin{cases} 2x+8, & 1 \le x \le 2 \\ 6x, & 2 \le x \le 4 \end{cases}$$

$$\therefore \quad \int_1^4 f(x)\,dx = \int_1^2 (2x+8)\,dx + \int_2^4 6x\,dx$$

$$= \left[\frac{2x^2}{2}+8x\right]_1^2 + \left[\frac{6x^2}{2}\right]_2^4$$

$$= [4+16-(1+8)] + 3[16-4]$$

$$= (20-9) + 3 \times 12$$

$$= 11 + 36 = 47$$

5. $\int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ equals:

(a) 2 (b) $\frac{3}{4}$

(c) 0 (d) -2

Ans. *(c) 0*

Explanation: Let

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx \quad ...(i)$$

$$= \int_0^{\pi/2} \log\left(\frac{4+3\sin\left(\frac{\pi}{2}-x\right)}{4+3\cos x\left(\frac{\pi}{2}-x\right)}\right)dx$$

$$I = \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$I + I = \int_0^{\pi/2} \log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx + \int_0^{\pi/2} \log\left(\frac{4+3\cos x}{4+3\sin x}\right)dx$$

$$= \int_0^{\pi/2} \log\left[\left(\frac{4+3\sin x}{4+3\cos x}\right)\times\left(\frac{4+3\cos x}{4+3\sin x}\right)\right]dx$$

$$= \int_0^{\pi/2} \log 1\,dx$$

$$= \int_0^{\pi/2} (0)\,dx$$

$\Rightarrow 2I = 0$

$\Rightarrow I = 0$

6. If $f(a+b-x) = f(x)$, then $\int_a^b xf(x)\,dx$ equals:

(a) $\frac{a+b}{20}\int_a^b f(b-x)\,dx$

(b) $\frac{a+b}{2}\int_a^b f(b+x)\,dx$

(c) $\frac{b-a}{2}\int_a^b f(x)\,dx$

(d) $\frac{a+b}{2}\int_a^b f(x)\,dx$

7. $\int_{-\pi/2}^{\pi/2} x^3 e^{\cos x}dx$ equals:

(a) 1 (b) 0

(c) 3 (d) 4

Ans. *(b) 0*

Explanation: Let $f(x) = x^3 e^{\cos x}$.

Then, $f(-x) = (-x)^3 e^{\cos(-x)} = -x^3 e^{\cos x} = -f(x)$

i.e., $f(x)$ is an odd functions

$\therefore$ By property, we have

$$\int_{-\pi/2}^{\pi/2} x^3 e^{\cos x}dx = 0$$

8. $\int_0^{\pi/2} \sqrt{1-\sin 2x}\,dx$ equals:

(a) $2\sqrt{2}$ (b) $2(\sqrt{2}+1)$

(c) 2 (d) $2(\sqrt{2}-1)$

Ans. *(d) $2(\sqrt{2}-1)$*

Explanation: Let $I = \int_0^{\pi/2} \sqrt{1-\sin 2x}\,dx$

$$= \int_0^{\pi/2} \sqrt{(\cos x - \sin x)^2}\,dx$$

$$= \int_0^{\pi/2} |\cos x - \sin x|\,dx$$

When $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$.

So, $|\cos x - \sin x| = \cos x - \sin x$

When $\frac{\pi}{4} < x < \frac{\pi}{2}$, $\cos x < \sin x$.

So, $|\cos x - \sin x| = -(\cos x - \sin x)$

$= \sin x - \cos x$

$$\therefore \quad I = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx$$

$$\Rightarrow \quad I = [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0\right) + \left(-\cos\frac{\pi}{2} - \sin\frac{\pi}{2} + \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1\right) - 0 - 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}} - 2, \text{ or } 2(\sqrt{2}-1)$$

9. Evaluate: $\int_0^1 \log\left(\frac{1-x}{x}\right)dx$.

(a) log 3 – log 2 (b) log 2 – log 3

(c) log 4 – log 52 (d) 0 [DIKSHA]

10. The value of $\int_0^{\pi} |\cos x|\,dx$ is:

(a) 1 (b) – 3

(c) 2 (d) – 2

11. The value of $\int_0^2 [x]\,dx$, where $[x]$ is a greatest integer which is less than equal to x, is:

(a) –1 (b) 2
(c) 1 (d) 3

Ans. *(c)* 1

Explanation: Let
$$I = \int_0^2 [x]\,dx$$
$$= \int_0^1 [x]\,dx + \int_1^2 [x]\,dx$$
$$= \int_0^1 0\,dx + \int_1^2 1\,dx$$
$$= 0 + [x]_1^2$$
$$= 0 + [2-1] = 1$$

12. ⓐ $\int_{-3}^{3} (x^3 \cos x)\,dx$ is:

(a) 0 (b) 2
(c) –1 (d) –2

13. $\int_0^{\pi/2} \cos^2 x\,dx$ is:

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

Ans. *(c)* $\frac{\pi}{4}$

Explanation: Let
$$I = \int_0^{\pi/2} \cos^2 x\,dx \quad ...(i)$$
$$I = \int_0^{\pi/2} \cos^2\left(\frac{\pi}{2} - x\right)dx$$
$$I = \int_0^{\pi/2} \sin^2 x\,dx \quad ...(ii)$$

On adding Eqs. (i) and (ii), we get
$$2I = \int_0^{\pi/2} (\cos^2 x + \sin^2 x)\,dx$$
$$= \int_0^{\pi/2} 1\,dx$$
$$= [x]_0^{\pi/2}$$
$$\Rightarrow \quad 2I = \frac{\pi}{2} - 0$$
$$\Rightarrow \quad I = \frac{\pi}{4}$$

14. ⓐ $\int_0^1 \tan^{-1}\left(\frac{2x-1}{1+x-x^2}\right)dx$ equals:

(a) 1 (b) 0
(c) –1 (d) $\frac{\pi}{4}$

15. Samrat, Kanchan and few friends were playing a game with die. They discussed ways to calculate the probability of rolling a particular digit. Samrat came up with the idea of calculating the probability from the probability distribution function, which he assumed to be $e^{|x|dx}$.

$\int_{-1}^{1} e^{|x|dx}\,dx$ is:

(a) $2e-3$ (b) $2e+2$
(c) $2e-4$ (d) $2e-2$

Ans. *(d)* $2e-2$

Explanation: Let $I = \int_{-1}^{1} e^{|x|}\,dx$

Here
$$|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$$

$$\therefore \quad I = \int_{-1}^{0} e^{-x}\,dx + \int_0^1 e^x\,dx$$
$$= \left[\frac{e^{-x}}{-1}\right]_{-1}^{0} + [e^x]_0^1$$
$$= -[e^0 - e^1] + [e - e^0]$$
$$= -[1-e] + [e-1]$$
$$= 2e - 2$$

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

16. *Let [x] denote the greatest integer, less than or equal to x and n is a positive integer.*

(A) ⓐ $[5.79]$ is equal to:

(a) 5 (b) 6
(c) 5.7 (d) 5.8

(B) The value $\int_{-n}^{n} [x]\,dx$, for $n = 1$ is:

(a) 2 (b) 1
(c) −1 (d) 0

(C) The value of $\int_{-n}^{n} [x]\,dx$, for $n = 2$ is:

(a) 4 (b) 2
(c) −2 (d) 0

(D) The value of $\int_{0}^{n} [x]\,dx$, for $n = 1$ is:

(a) 1 (b) −1
(c) −2 (d) 0

(E) The value of $\int_{0}^{n} [x]\,dx$, for $n = 2$ is:

(a) 0 (b) 1
(c) 2 (d) 4

Ans. (B) *(c) −1*

Explanation: For $n = 1$,

$$\int_{-n}^{n} [x]\,dx = \int_{-1}^{1} [x]\,dx$$

$$= \int_{-1}^{0} (-1)\,dx + \int_{0}^{1} (0)\,dx$$

$$= [-x]_{-1}^{0}$$

$$= -[0-(-1)] = -1$$

(D) *(d) 0*

Explanation: For $n = 1$,

$$\int_{0}^{n} [x]\,dx = \int_{0}^{1} [x]\,dx = \int_{0}^{1} 0\,dx = 0$$

17. *We know that a given function f(x) is even $f(-x) = f(x)$; and it is odd, if $f(-x) = -f(x)$. Also,*

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x) & , \text{if } f(x) \text{ is even} \\ 0 & , \text{if } f(x) \text{ is odd} \end{cases}$$

(A) $f(x) = x^3 \sin^{40} x$ is a/an:

(a) odd function
(b) even function
(c) neither odd nor even
(d) odd as well as even

(B) $f(x) = \{\sin|x| + \cos|x|\}$ is/an:

(a) odd function
(b) even function
(c) neither odd nor even
(d) odd as well as even

(C) $\int_{-\pi}^{\pi} x^{17} \cos^4 x\,dx$ is equal to:

(a) 0 (b) π
(c) $\frac{\pi}{3}$ (d) π

(D) $\int_{-\pi/2}^{\pi/2} |\sin x|\,dx$ is equal to:

(a) 0 (b) 1
(c) 2 (d) 3

(E) $\int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1)\,dx$ is equal to:

(a) 0 (b) 1
(c) 2 (d) π

Ans. (A) *(a) odd function*

Explanation: Let $f(x) = x^3 \sin^{40} x$. Then,

$$\begin{aligned} f(-x) &= (-x)^3 \sin^{40}(-x) \\ &= (-x^3)(-\sin x)^{40} \\ &= (-x^3)(\sin^{40} x) \\ &= -(x^3 \sin^{40} x) \\ &= -f(x) \end{aligned}$$

$\Rightarrow f(x)$ is an odd function.

(E) *(d) π*

Explanation: Let $f(x) = x^3 + x\cos x + \tan^5 x$. Then,

$$\begin{aligned} f(-x) &= (-x)^3 + (-x)\cos(-x) + \tan^5(-x) \\ &= -x^3 - x\cos x - \tan^5 x \\ &= -f(x) \end{aligned}$$

i.e., $f(x)$ is an odd function.

Now, $\int_{-\pi/2}^{\pi/2} (x^3 + x\cos x + \tan^5 x + 1)\,dx$

$$= \int_{-\pi/2}^{\pi/2} \{x^3 + x\cos x + \tan^5 x\}\,dx + \int_{-\pi/2}^{\pi/2} 1\,dx$$

$$= \int_{-\pi/2}^{\pi/2} f(x)\,dx + [x]_{-\pi/2}^{\pi/2}$$

$$= 0 + \left(\frac{\pi}{2} + \frac{\pi}{2}\right), \text{ or } \pi$$

18. *We know that a given function f(x) is a continuous function defined on [0, a], then*

$$\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$$

(A) If $f(x) = \dfrac{\sin x - \cos x}{1 + \sin x \cos x}$, then what is the value of $f\left(\dfrac{\pi}{2} - x\right)$? Hence, evaluate

$$\int_{0}^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x}\,dx$$

(B) Evaluate $\int_{0}^{\pi/2} \dfrac{\sin x}{\sin x + \cos x}\,dx$

Ans. (A) If $f(x) = \dfrac{\sin x - \cos x}{1 + \sin x \cos x}$, then

$$f\left(\frac{\pi}{2} - x\right) = \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}$$

$$= \frac{\cos x - \sin x}{1 + \cos x \sin x}$$

$$= -\frac{\sin x - \cos x}{1 + \sin x \cos x} = -f(x)$$

Now,Let $I = \int_0^{\pi/2} \dfrac{\sin x - \cos x}{1 + \sin x \cos x} dx$ ...(i)

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{1 + \sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \left[\frac{\sin x - \cos x}{1 + \sin x \cos x} + \frac{\cos x - \sin x}{1 + \cos x \sin x}\right] dx$$

$$= \int_0^{\pi/2} 0 dx = 0$$

$\Rightarrow \quad I = 0$

(B) Let $I = \int_0^{\pi/2} \dfrac{\sin x}{\sin x + \cos x} dx$ ...(i)

$$= \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

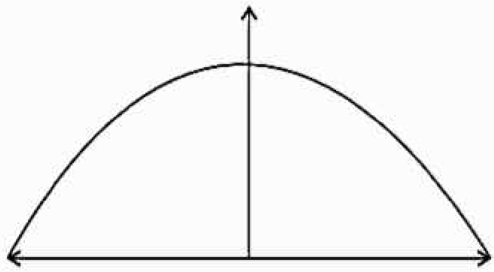
$$2I = \int_0^{\pi/2} \left[\frac{\sin x}{\sin x + \cos x} + \frac{\cos x}{\cos x + \sin x}\right] dx$$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} 1.dx$$

$$= [x]_0^{\pi/2} = \left[\frac{\pi}{2} - 0\right]$$

$\Rightarrow I = \dfrac{\pi}{4}$

19. *The bridge connects two hills 100 feet apart. The arch on the bridge is in a parabolic form. The highest point on the bridge is 10 feet above the road at the middle of the bridge as shown in the figure.*



(A) Find the value of the integral $\int_{-50}^{50} \dfrac{x^2}{250} dx$

(B) Find the area formed by the curve $x^2 = 250\,y$, $y = 0$ and $y = 10$.

Ans. (A) $\int_{-50}^{50} \dfrac{x^2}{250} dx$

$$= 2\int_0^{50} \frac{x^2}{250} dx$$

$$\left[\because \frac{x^2}{250} \text{ is an even function}\right]$$

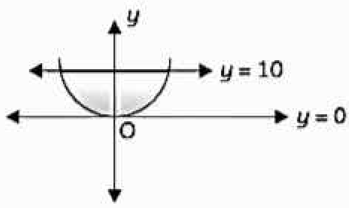
$$= \frac{2}{250}\left(\frac{x^3}{3}\right)_0^{50}$$

$$= \frac{2}{750}[(50)^3 - 0]$$

$$= \frac{1000}{3}$$

(B) Given, $x^2 = 250y$

$\Rightarrow \quad x = \sqrt{250y}$



$\therefore$ Required area $= 2\int_0^{10} \sqrt{250y}\, dy$

$$= 2\sqrt{250}\frac{[y^{3/2}]_0^{10}}{3/2}$$

$$= 2\sqrt{250}\times\frac{2}{3}[10^{3/2}-0]$$

$$= 2\times 5\sqrt{10}\times\frac{2}{3}\times 10\sqrt{10}$$

$$= 2\times\frac{2000}{3} \text{ sq. units}$$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

20. Evaluate $\int_0^1 x(1-x)^n\, dx$

Ans. Let $I = \int_0^1 x(1-x)^n dx$ ...(i)

$$I = \int_0^1 (1-x)[1-(1-x)]^n dx$$

$$= \int (1-x)x^n dx$$

$$= \int_0^1 (x^n - x^{n+1})dx$$

$$= \left[\frac{x^{n+1}}{n+1}\quad\frac{x^{n+2}}{n+2}\right]$$

$$= \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{n^2+3n+2}$$

21. Evaluate $\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$**.**

Ans. Let $I = \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$

Consider $f(x) = \sec^2 x$

So, $f(-x) = \sec^2(-x) = \sec^2 x$, which is an even function

$\therefore$ $I = 2\int_0^{\pi/4} \sec^2 x\, dx$

$$= 2[\tan x]_0^{\pi/4}$$

$$= 2\left[\tan\frac{\pi}{4} - \tan 0\right]$$

$$= 2\times 1 = 2$$

22. Find $\int_{-\pi/3}^{\pi/3} \tan^3 x\, dx$**.**

23. If $f(x) = \begin{cases} \sin x, & 0 \le x < \frac{\pi}{2} \\ \cos x, & \frac{\pi}{2} \le x < \pi \end{cases}$**, then find** $\int_0^{\pi} f(x)\, dx$**.**

24. Evaluate $\int_{-2}^{2} |x|\, dx$**.** **[CBSE 2020]**

Ans. Let $I = \int_{-2}^{2} |x|\, dx$

Here $|x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0 \end{cases}$

$\therefore$ $I = \int_{-2}^{0} (-x)\, dx + \int_0^2 x\, dx$

$$= -\left[\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_0^2$$

$$= -\left[0 - \frac{(-2)^2}{2}\right] + \frac{1}{2}[2^2 - 0]$$

$$= \frac{4}{2} + \frac{4}{2}$$

$$= 2 + 2 = 4$$

25. Evaluate $\int_{-\pi/4}^{\pi/4} |\sin x|\, dx$**.**

Ans. Let $I = \int_{-\pi/4}^{\pi/4} |\sin x|\, dx$

Hee $|\sin x|$ is an absolute function, so it is an even function.

$\therefore$ $I = 2\int_0^{\pi/4} \sin x\, dx$

$$= 2[-\cos x]_0^{\pi/4}$$

$$= -2\left[\cos\frac{\pi}{4} - \cos 0\right]$$

$$= -2\left[\frac{1}{\sqrt{2}}-1\right]$$

$$= 2-\sqrt{2} \text{ or } \sqrt{2}\left(\sqrt{2}-1\right)$$

26. If $g(x)=\begin{cases}7x+3, & 1\le x\le 3\\ 8x, & 3\le x<4\end{cases}$ then find $\int_1^4 g(x)\,dx$.

27. Evaluate $\int_{-8}^{-3}\log x\,dx$.

Ans. Let $\quad I=\int_{-8}^{-3}\log x\,dx$

Consider $\quad f(x)=\log x$

Since $f(x)$ is not defined for $-8\le x\le -3$, so given integrand can not be determined.

28. Find $\int_{-1}^{1}[x]dx$, where [ ] is a greatest integer which is less than equal to x.

Ans. Let $\quad I=\int_{-1}^{1}[x]dx$

$$=\int_{-1}^{0}[x]dx+\int_{0}^{1}(x)dx$$

$$=\int_{-1}^{0}-1\,dx+\int_{0}^{1}0\,dx$$

$$=[-x]_{-1}^{0}+0$$

$$=0+(-1)=-1$$

29. Find $\int_{-\pi/8}^{\pi/8}(\sin^5 2x)\,dx$.

30. Evaluate $\int_1^3|\,2x-1\,|dx$. **[CBSE 2020]**

Ans. Let $\quad I=\int_1^3|\,2x-1\,|dx$

Here

$$|2x-1|=\begin{cases}(2x-1)\ ;\ \text{when } 2x-1\ge 0 \text{ or } x\ge \frac{1}{2}\\ -(2x-1)\ ;\ \text{when } 2x-1<0 \text{ or } x<\frac{1}{2}\end{cases}$$

$$\therefore\quad I=\int_1^3(2x-1)dx=\left[\frac{2x^2}{2}-x\right]_1^3$$

$$\therefore\quad I=[(3)^2-3]-[(1)^2-1]$$

$$=9-3=6$$

31. Find $\int_0^5\frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}dx$

Ans. Let $I=\int_0^5\frac{\sqrt{x}}{\sqrt{x}+\sqrt{5-x}}dx \quad$ ...(i)

Using property $\int_0^a f(x)=\int_0^a f(a-x)\,dx$, we have

$$I=\int_0^5\frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{5-(5-x)}}dx$$

$$I=\int_0^5\frac{\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I=\int_0^5\frac{\sqrt{x}+\sqrt{5-x}}{\sqrt{5-x}+\sqrt{x}}dx$$

$$=\int_0^5 1\,dx=[x]_0^5$$

$$=5-0=5$$

$$\therefore\quad I=\frac{5}{2}$$

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

32. By using of properties of definite integrals, evaluate $\int_0^2(2-x)^m\,x\,dx$.

Ans. Let $\quad I=\int_0^2(2-x)^m\,x\,dx$

$$=\int_0^2[2-(2-x)]^m(2-x)\,dx$$

$$=\int_0^2 x^m(2-x)\,dx$$

$$=\int_0^2(2x^m-x^{m+1})\,dx$$

$$=\left[\frac{2x^{m+1}}{m+1}-\frac{x^{m+2}}{m+2}\right]_0^2$$

$$= \frac{2.2^{m+1}}{m+1} - \frac{2^{m+2}}{m+2} - (0-0)$$

$$= \frac{2^{m+2}}{m+1} - \frac{2^{m+2}}{m+2}$$

$$= 2^{m+2}\left[\frac{1}{(m+1)} - \frac{1}{(m+2)}\right]$$

$$= 2^{m+2}\left[\frac{m+2-(m+1)}{(m+1)(m+2)}\right]$$

$$= \frac{2^{m+2}}{(m+1)(m+2)}$$

33. **Evaluate $\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\,dx$.**

34. Evaluate $\int_{-\pi}^{\pi} (1-x^2)\sin x.\cos^2 x\,dx$.

[CBSE 2019]

Ans. Let $I = \int_{-\pi}^{\pi} (1-x^2)\sin x.\cos^2 x\,dx$

If $f(x) = (1-x^2)\sin x\cos^2 x$, then

$f(-x) = [1-(-x)^2]\sin(-x)\cos^2(-x)$

$= (1-x^2)(-\sin x)\cos^2 x$

$= -(1-x^2)\sin x\cos^2 x$

$= -f(x)$, which is an odd function

$\therefore \quad I = 0$

$\left[\because \int_{-a}^{a} f(x)\,dx = 0, \text{ if } f(x) \text{ is odd function}\right]$

35. **Evaluate $\int_{-3}^{3} \frac{x^2}{1+7^x}\,dx$.**

36. Evaluate $\int_{-1}^{2} \frac{|x|}{x}dx$. **[CBSE 2019]**

Ans. Let $\quad I = \int_{-1}^{2} \frac{|x|}{x}dx$

$\because \quad |x| = \begin{cases} x \geq 0 \\ -x < 0 \end{cases}$

$\therefore \quad I = \int_{-1}^{0} \frac{-x}{x}dx + \int_0^2 \frac{x}{x}dx$

$= \int_{-1}^{0} -1\,dx + \int_0^2 1\,dx$

$= [-x]_{-1}^{0} + [x]_0^2$

$= -[0+1] + [2-0]$

$= -1 + 2 = 1$

37. **Find $\int_{-6}^{6} |x+3|\,dx$.**

38. Evaluate $\int_0^{\pi/2} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}dx$.

Ans. Let $\quad I = \int_0^{\pi/2} \frac{\cos^4 x}{\cos^4 x + \sin^4 x}dx \quad$ ...(i)

$$= \int_0^{\pi/2} \frac{\cos^4\left(\frac{\pi}{2}-x\right)dx}{\cos^4\left(\frac{\pi}{2}-x\right)+\sin^4\left(\frac{\pi}{2}-x\right)}$$

$I = \int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x}dx \quad$ ...(ii)

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\cos^4 x + \sin^4 x}{\cos^4 x + \sin^4 x}dx$$

$$2I = \int_0^{\pi/2} 1\,dx$$

$$2I = [x]_0^{\pi/2}$$

$$I = \frac{1}{2}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{4}$$

39. Evaluate $\int_0^{2\pi} \cos^5 x\,dx$.

Ans. Let $\quad I = \int_0^{2\pi} \cos^5 x\,dx$

Consider $\quad f(x) = \cos^5 x$

Now $\quad f(2\pi - x) = \cos^5(2\pi - x)$

$= \cos^5 x$

We know,

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx\ , \text{ if } f(2a-x) = f(x) \\ 0 \quad , f(2a-x) = -f(x) \end{cases}$$

$\therefore \quad I = 2\int_0^{\pi} \cos^5 x\,dx$

Again, $\quad f(\pi - x) = \cos^5(\pi - x)$

$= -\cos^5 x$

$= -f(x)$

$\therefore \quad I = 2\int_0^{\pi} \cos^5 x\,dx$

$= 0$

[by using property of definite integral]

40. Show that $\int_0^a f(x)\,h(x)\,dx = 3\int_0^a f(x)\,dx$, if f and h are defined as $f(x) = f(a-x)$ and $h(x) + h(a-x) = 6$.

Ans. Let $$I = \int_0^a f(x)\,h(x)\,dx \quad \text{...(i)}$$

$$= \int_0^a f(a-x).\,h(a-x)\,dx$$

[Using property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$]

$$I = \int_0^a f(x)\{6 - h(x)\}\,dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = 6\int_0^a f(x)\,dx$$

$$\Rightarrow \quad I = 3\int_0^a f(x)\,dx$$

Hence, proved.

41. Evaluate $\int_{-\pi/4}^{\pi/4} \tan^2 x\,dx$

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

42. Evaluate $\int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}\,dx$

Ans. Let $$I = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x}\,dx \quad \text{...(i)}$$

$$I = \int_0^{\pi/2} \frac{\cos^2\left(\frac{\pi}{2} - x\right)}{\cos\left(\frac{\pi}{2} - x\right) + \sin\left(\frac{\pi}{2} - x\right)}\,dx$$

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\,dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\cos^2 x + \sin^2 x}{\cos x + \sin x}\,dx$$

$$= \int_0^{\pi/2} \frac{1}{\cos x + \sin x}\,dx$$

$$= \int_0^{\pi/2} \frac{1}{\frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} + \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}}\,dx$$

$$= \int_0^{\pi/2} \frac{\left(1 + \tan^2\frac{x}{2}\right)}{1 - \tan^2\frac{x}{2} + 2\tan\frac{x}{2}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{1 - \tan^2\frac{x}{2} + 2\tan\frac{x}{2}}\,dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

Upper limit, when $x = \frac{\pi}{2}, t = \tan\frac{\pi}{4} = 1$

Lower limit, when $x = 0$, $t = \tan 0 = 0$

$$\therefore \quad 2I = \int_0^1 \frac{2}{1 - t^2 + 2t}\,dt$$

$$= 2\int_0^1 \frac{dt}{-(t^2 - 2t - 1)}$$

$$= 2\int_0^1 \frac{dt}{-[(t-1)^2 - 2]}$$

$$= 2\int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2}$$

$$= 2 \times \left[\frac{1}{2\sqrt{2}} \log\left|\frac{\sqrt{2} + (t-1)}{\sqrt{2} - (t-1)}\right|\right]_0^1$$

$$= \frac{1}{\sqrt{2}}\left[\log\left|\frac{\sqrt{2} + 1 - 1}{\sqrt{2} - 1 + 1}\right| - \log\left|\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right|\right]$$

$$= \frac{1}{\sqrt{2}}\left[\log\left|\frac{\sqrt{2}}{\sqrt{2}}\right| - \log\left|\frac{\sqrt{2} - 1}{\sqrt{2} + 1}\right|\right]$$

$$= \frac{1}{\sqrt{2}}\left[0 + \log\left|\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right|\right]$$

$$= \frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2} + 1}{\sqrt{2} - 1}\right|$$

43. Evaluate $\int_0^{\pi} x \sin x \cos^2 x\,dx$

[NCERT Exemplar]

44. Evaluate $\int_{-1}^{2} | x^3 - x |\,dx$ **[CBSE 2020]**

Ans. Let $$I = \int_{-1}^{2} | x^3 - x |\,dx$$

Here absolute function is defined as

$$|x^3 - x| = |x\,(x-1)(x+1)|$$

$$= \begin{cases} +(x^3 - x), -1 \le x \le 0 \\ -(x^3 - x), 0 \le x \le 1 \\ +(x^3 - x), 1 \le x \le 2 \end{cases}$$

$$\therefore \ I = \int_{-1}^{0} +(x^3 - x)\,dx + \int_0^1 [-(x^3 - x)]dx + \int_1^2 (x^3 - x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_0^1 + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_1^2$$

$$= \left[0 - 0 - \left(\frac{1}{4} - \frac{1}{2}\right)\right] - \left[\frac{1}{4} - \frac{1}{2} - 0\right] + \left[\frac{16}{4} - \frac{4}{2} - \left(\frac{1}{4} - \frac{1}{2}\right)\right]$$

$$= \frac{1}{4} - \left(-\frac{1}{4}\right) + \left[\frac{15}{4} - \frac{3}{2}\right]$$

$$= \frac{1}{2} + \left[\frac{9}{4}\right]$$

$$= \frac{11}{4}$$

45. **Solve** $\int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$

Ans. Let $I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}}$

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\frac{\sin x}{\cos x}}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx \quad \text{...(i)}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)} + \sqrt{\sin\left(\frac{\pi}{3} + \frac{\pi}{6} - x\right)}}\,dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}}\,dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

$$2I = \int_{\pi/6}^{\pi/3} 1\,dx$$

$$I = \frac{1}{2}[x]_{\pi/6}^{\pi/3}$$

$$= \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{6}\right] = \frac{\pi}{12}$$

46. **Prove that** $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$, **hence evaluate** $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x}\,dx$. **[CBSE 2019]**

Ans. To prove: $\int_0^a f(x)\,dx = \int_0^a f(a - x)dx$

Consider RHS = $\int_0^a f(a - x)\,dx$

Putting $t = a - x$, then $dt = -dx$

Lower limit, when $x = 0$; then $t = a - 0 = a$

Upper limit, when $x = a$, then $t = a - a = 0$

Now, RHS = $-\int_a^0 f(t)\,dt$

$$= \int_0^a f(t)\,dt$$

$$= \int_0^a f(x)\,dx = \text{LHS}$$

Hence, proved.

Now, Let $I = \int_0^\pi \frac{x \sin x}{1 + \cos^2 x}dx \quad \text{...(i)}$

We know,

$$\int_0^a f(x) = \int_0^a f(a - x)\,dx$$

$$\therefore \quad I = \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}dx$$

$$= \int_0^\pi \frac{(\pi - x)\sin x}{1 + \cos^2 x}dx$$

$$= \pi\int_0^\pi \frac{\sin x}{1 + \cos^2 x}dx - \int_0^\pi \frac{x \sin x}{1 + \cos^2 x}dx$$

$$I = \pi\int_0^\pi \frac{\sin x}{1 + \cos^2 x}dx - I \quad \text{[From Eq. (i)]}$$

$$\Rightarrow 2I = \pi\int_0^\pi \frac{\sin x}{1 + \cos^2 x}dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1 + \cos^2 x}dx$$

Put $\cos x = t$

$\Rightarrow \quad -\sin x\,dx = dt$

Lower limit, when $x = 0$, then $t = 1$

Upper limit, when $x = \pi$, then $t = -1$

Now,
$$I = \frac{-\pi}{2}\int_1^{-1}\frac{dt}{1+t^2}$$
$$= \frac{\pi}{2}\int_{-1}^{1}\frac{dt}{1+t^2}$$
$$= \frac{\pi}{2}[\tan^{-1}t]_{-1}^{1}$$
$$= \frac{\pi}{2}[\tan^{-1}(1) - \tan^{-1}(-1)]$$
$$= \frac{\pi}{2}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$
$$= \frac{\pi}{2}\left[\frac{\pi}{2}\right] = \frac{\pi^2}{4}$$

47. Evaluate $\int_0^1 |x\cos\pi x|\, dx$

Ans. Let, $I = \int_0^1 |x\cos\pi x|\, dx$

Consider $x\cos\pi x = 0 \Rightarrow x = 0$ or $\cos\pi x = 0$

$\Rightarrow x = 0$ or $\pi x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$\Rightarrow x = 0$ or $x = \frac{1}{2}$

$$\therefore \quad |x\cos\pi x| = \begin{cases} x\cos\pi x & , 0 \le x \le \frac{1}{2} \\ -x\cos\pi x & , \frac{1}{2} \le x \le \frac{3}{2} \end{cases}$$

$$\therefore I = \int_0^{1/2}(x\cos\pi x)\,dx + \int_{1/2}^{1} -(x\cos\pi x)\,dx$$

Now,
$$I_1 = \int \underset{\text{I}}{x}\,\underset{\text{II}}{\cos\pi x}\,dx$$
$$= x\frac{\sin\pi x}{\pi} - \int\frac{\sin\pi x}{\pi}dx$$
$$= x\frac{\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2}$$

$$\therefore I = \left[\frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2}\right]_0^{1/2} - \left[\frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2}\right]_{1/2}^{1}$$

$$= \left[\left(\frac{1}{2\pi}\sin\frac{\pi}{2} + \frac{1}{\pi^2}\cos\frac{\pi}{2}\right) - \left(0 + \frac{\cos 0}{\pi^2}\right)\right] - \left[\left(\frac{\sin\pi}{\pi} + \frac{\cos\pi}{\pi^2}\right) - \left(\frac{1}{2\pi}\sin\frac{\pi}{2} + \frac{1}{\pi^2}\cos\frac{\pi}{2}\right)\right]$$

$$= \left[\left(\frac{1}{2\pi}\times 1 + \frac{1}{\pi^2}\times 0\right) - \left(\frac{1}{\pi^2}\right)\right] - \left[\left(0 - \frac{1}{\pi^2}\right) - \left(\frac{1}{2\pi}\times 1 + \frac{1}{\pi^2}\times 0\right)\right]$$

$$= \left(\frac{1}{2\pi} - \frac{1}{\pi^2}\right) - \left[\left(-\frac{1}{\pi^2} - \frac{1}{2\pi}\right)\right]$$

$$= \frac{1}{2\pi} - \frac{1}{\pi^2} + \frac{1}{\pi^2} + \frac{1}{2\pi}$$

$$= \frac{1}{\pi}$$

48. Evaluate $\int_0^{\pi}\log(1+\cos x)\,dx$**, when** $\int_0^{\pi/2}\log\sin x\,dx = -\frac{\pi}{2}\log 2$

Ans. Let $I = \int_0^{\pi}\log(1+\cos x)\,dx$

By using property of definite integral

$$= \int_0^{\pi}\log[1+\cos(\pi - x)]\,dx$$
$$= \int_0^{\pi}\log(1-\cos x)\,dx$$
$$= \int_0^{\pi}\log\left(2\sin^2\frac{x}{2}\right)dx$$
$$= \int_0^{\pi}\left(\log 2 + 2\log\sin\frac{x}{2}\right)dx$$
$$= \int_0^{\pi}\log 2\,dx + 2\int_0^{\pi}\log\sin\frac{x}{2}\,dx$$

Put $t = \frac{x}{2}$ in second integrand, so $dx = 2\,dt$

Upper limit, when $x = \pi$, $t = \frac{\pi}{2}$

Lower limit, when $x = 0$, $t = 0$

$$\therefore \quad I = \log 2[x]_0^{\pi} + 2\int_0^{\pi/2}\log(\sin t)\,(2dt)$$
$$= \log 2[\pi - 0] + 4\int_0^{\pi/2}\log\sin t\,dt$$
$$= \pi\log 2 + 4\left(-\frac{\pi}{2}\log 2\right)$$
$$= \pi\log 2 - 2\pi\log 2$$
$$= -\pi\log 2$$

49. @**Evaluate** $\int_{\pi/4}^{\pi/2}\frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

50. Evaluate $\int_0^{\pi} \frac{x\tan x}{\sec x + \tan x} dx$ **[CBSE 2017]**

Ans. Let, $I = \int_0^{\pi} \frac{x\tan x}{\sec x + \tan x} dx$ ...(i)

We know,

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\therefore \quad I = \int_0^{\pi} \frac{(\pi - x)\tan(\pi - x)}{\sec(\pi - x) + \tan(\pi - x)}$$

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)\tan x}{\sec x + \tan x} dx \quad \text{...(ii)}$$

On adding eq. (i) and (ii) we get

$$\Rightarrow \quad 2I = \int_0^{\pi} \frac{\pi\tan x}{\sec x + \tan x} dx$$

Multiply numerator and denominator by $(\sec x - \tan x)$

$$I = \frac{\pi}{2}\int_0^{\pi} \frac{\tan x(\sec x - \tan x)}{(\sec x + \tan x)(\sec x - \tan x)} dx$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{(\tan x \sec x - \tan^2 x)dx}{(\sec^2 x - \tan^2 x)}$$

$$= \frac{\pi}{2}\int_0^{\pi} \frac{\tan x \sec x - \sec^2 x + 1}{1} dx$$

$$= \frac{\pi}{2}[\sec x - \tan x + x]_0^{\pi}$$

$$= \frac{\pi}{2}[(\sec \pi - \tan \pi + \pi) - (\sec 0 - \tan 0 + 0)]$$

$$= \frac{\pi}{2}\ [(-1 - 0 + \pi) - (1 - 0)]$$

$$= \frac{\pi}{2}\ [\pi - 2]$$

51. Evaluate: $\int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$

[NCERT Exemplar]

Ans. Let, $I = \int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\, dx$ ...(i)

$$= \int_{-\pi/4}^{\pi/4} \left[\log\left\{\sin\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right) + \cos\left(\frac{\pi}{4} - \frac{\pi}{4} - x\right)\right\}\right] dx$$

$$= \int_{-\pi/4}^{\pi/4} \log\{\sin(-x) + \cos(-x)\}\, dx$$

$$= \int_{-\pi/4}^{\pi/4} \log(\cos x - \sin x)\, dx \quad \text{...(ii)}$$

Adding equations (i) and (ii), we get

$$\Rightarrow \quad 2I = \int_{-\pi/4}^{\pi/4} [\log(\cos x + \sin x) + \log(\cos x - \sin x)]dx$$

$$\Rightarrow \quad 2I = \int_{-\pi/4}^{\pi/4} \log(\cos^2 x - \sin^2 x)\, dx$$

$$\Rightarrow \quad 2I = \int_{-\pi/4}^{\pi/4} \log(\cos 2x)\, dx$$

$$\Rightarrow \quad 2I = 2\int_0^{\pi/4} \log(\cos 2x)dx \quad \text{...(iii)}$$

$$\left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x), \text{if } f(-x) = f(x)\right]$$

Put $2x = t$

$$\Rightarrow \quad dx = \frac{dt}{2}$$

When $x = 0$, then $t = 0$

and when $x = \frac{\pi}{4}$ then, $t = \frac{\pi}{2}$

$$\therefore \quad 2I = \int_0^{\pi/2} \log(\cos t)\, dt \quad \text{...(iv)}$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log\left[\cos\left(\frac{\pi}{2} - t\right)\right] dt$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \log(\sin t)\, dt \quad \text{...(v)}$$

On adding equations (iv) and (v), we get

$$4I = \int_0^{\pi/2} (\log\sin t + \log\cos t)\, dt$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin t\cos t}{2}\right) dt$$

$$= \int_0^{\pi/2} (\log \sin 2t - \log 2)dt$$

Put $2t = u$

$$\Rightarrow 2dt = du \text{ or } dt = \frac{1}{2}\ du$$

When $t = 0, u = 0$

When $t = \frac{\pi}{2}, u = \pi$

$\therefore \quad 4I = \frac{1}{2}\int_0^{\pi}\log(\sin u)du - \frac{\pi}{2}\log 2$

$\Rightarrow \quad 4I = \frac{1}{2}\times 2\int_0^{\pi/2}\log(\sin u)\,du - \frac{\pi}{2}\log 2$

$\Rightarrow \quad 4I = \int_0^{\pi/2}\log(\sin t)dt - \frac{\pi}{2}\log 2$

$\left[\because \int f(x)\,dx = \int f(t)\,dt\right]$

$\Rightarrow \quad 4I = 2I - \frac{\pi}{2}\log 2 \qquad$ [Using (v)]

$\therefore \quad 2I = -\frac{\pi}{2}\log 2$

$\Rightarrow \quad I = -\frac{\pi}{4}\log 2$

⚠ Caution

➥ *We need to apply the property of definite integrals multiple number of times, to solve it.*

52. Evaluate : $\int_{-1}^{2}|x^3 - 3x^2 + 2x|dx$

[CBSE Term-2 SQP 2022]

Ans. The given definite integral

$\int_{-1}^{2}|x^3 - 3x^2 + 2x|dx$

$= \int_{-1}^{0}|x(x-1)(x-2)|dx + \int_{0}^{1}|x(x-1)(x-2)|dx + \int_{1}^{2}|x(x-1)(x-2)|dx$

$= -\int_{-1}^{0}(x^3 - 3x^2 + 2x)dx + \int_{0}^{1}(x^3 - 3x^2 + 2x)dx - \int_{1}^{2}(x^3 - 3x^2 + 2x)dx$

$= -\left[\frac{x^4}{4} - x^3 + x^2\right]_{-1}^{0} + \left[\frac{x^4}{4} - x^3 + x^2\right]_{0}^{1} - \left[\frac{x^4}{4} - x^3 + x^2\right]_{1}^{2}$

$= \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4}$

[CBSE Marking Scheme Term-2 SQP 2022]

Explanation: Consider,

$$I = \int_{-1}^{2}|x^3 - 3x^2 + 2x|dx$$

$$\Rightarrow \quad I = \int_{-1}^{0}|x^3 - 3x^2 + 2x| + \int_{0}^{1}|x^3 - 3x^2 + 2x|dx + \int_{1}^{2}|x^3 - 3x^2 + 2x|dx$$

$$= -\int_{-1}^{0}(x^3 - 3x^2 + 2x) + \int_{0}^{1}(x^3 - 3x^2 + 2x)dx - \int_{1}^{2}(x^3 - 3x^2 + 2x)\,dx$$

$$= -\left[\frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2}\right]_{-1}^{0} + \left[\frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2}\right]_{0}^{1} - \left[\frac{x^4}{4} - \frac{3x^3}{3} + \frac{2x^2}{2}\right]_{1}^{2}$$

$$= -\left[0 - \left(\frac{1}{4} + 1 + 1\right)\right] + \left[\left(\frac{1}{4} - 1 + 1\right) - 0\right] - \left[(4 - 8 + 4) - \left(\frac{1}{4} - 1 + 1\right)\right]$$

$$= \frac{9}{4} + \frac{1}{4} + \frac{1}{4} = \frac{11}{4}$$

53. $\int_1^4 \{|x-1| + |x-2| + |x-3|\}\,dx$

Ans. $\int_1^4 \{|x-1| + |x-2| + |x-3|\}\,dx$

$= \int_1^2 (|x-1| + |x-2| + |x-3|)\,dx + \int_2^3 \{|x-1| + |x-2| + |x-3|\}\,dx + \int_3^4 \{|x-1| + |x-2| + |x-3|\}\,dx$

$= \int_1^2 \{x - 1 - (x-2) - (x-3)\}\,dx + \int_2^3 \{x - 1 + x - 2 - (x-3)\}\,dx + \int_3^4 (x - 1 + x - 2 + x - 3)\,dx$

$= \int_1^2 (-x+4)dx + \int_2^3 x\,dx + \int_3^4 (3x-6)\,dx$

$= \left[\frac{-x^2}{2} + 4x\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3 + \left[\frac{3x^2}{2} - 6x\right]_3^4$

$= \left[\left(\frac{-4}{2} + 8\right) - \left(\frac{-1}{2} + 4\right)\right] + \frac{1}{2}(3^2 - 2^2)$

$$+\left[\left(\frac{3}{2}\times 4^2-6\times 4\right)-\left(\frac{3}{2}\times 3^2-6\times 3\right)\right]$$

$$=\left[6-\frac{7}{2}\right]+\frac{5}{2}+\left[(24-24)-\left(\frac{-9}{2}\right)\right]$$

$$=\frac{12-7+5}{2}+0+\frac{9}{2}=\frac{19}{2}$$

54. Ⓐ**Evaluate** $\int_0^{\pi} x\log\sin x\,dx$. **[NCERT Exemplar]**

55. Ⓐ**Prove that** $\int_0^{\pi/2}\log\sin x\,dx=-\frac{\pi}{2}\log 2$.

56. **Evaluate** $\int_0^{\pi/2}\frac{x}{\cos^2 x+\sin^2 x}dx$.

Ans. Let $I=\int_0^{\pi/2}\frac{x}{\cos^2 x+\sin^2 x}dx$ ...(i)

We know,

$$\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$$

$$\therefore \quad I=\int_0^{\pi/2}\frac{(\pi-x)}{\cos^2(\pi-x)+\sin^2(\pi-x)}$$

$$\Rightarrow \quad I=\int_0^{\pi/2}\frac{(\pi-x)}{\cos^2 x+\sin^2 x}dx \quad ...(ii)$$

On adding eq. (i) and (ii), we get

$$2I=\int_0^{\pi/2}\frac{\pi}{\cos^2 x+\sin^2 x}dx$$

Dividing numerator and denominator by $\cos^2 x$, we get

$$2I=\pi\int_0^{\pi/2}\frac{\frac{1}{\cos^2 x}}{1+\frac{\sin^2 x}{\cos^2 x}}dx$$

$$2I=\pi\int_0^{\pi/2}\frac{\sec^2 x}{1+\tan^2 x}dx$$

Put $\tan x=t\Rightarrow\sec^2 x\,dx=dt$

Lower limit, when $x=0$, $t=\tan 0=0$

Upper limit, when $x=\frac{\pi}{2}, t=\tan\frac{\pi}{2}=\infty$

$$\therefore \quad 2I=\pi\int_0^{\infty}\frac{dt}{1+t^2}$$

$$=\pi\left[\frac{1}{1}\tan^{-1}\frac{x}{1}\right]_0^{\infty}$$

$$=\pi[\tan^{-1}\infty-\tan^{-1}0]$$

$$=\pi\left[\frac{\pi}{2}-0\right]$$

$$=\frac{\pi^2}{2}$$

$$\Rightarrow \quad I=\frac{\pi^2}{4}$$

TOPPER'S CORNER

VERY SHORT ANSWER Type Questions (VSA)

[**1** mark]

1. If $f(x) = \int_0^x t \sin t \, dt$, then write the value of $f'(x)$.

Ans.

$$f(x) = \int_0^x t \sin t \, dt$$

$$f'(x) = x \sin x$$

[CBSE Topper 2014]

2. Find : $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$.

Ans.

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$$

$$= -2 \int \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} dx$$

$$= (-2) \int \frac{\cos 2x}{\sin 2x} dx$$

$$= (-2) \int \cot 2x \, dx$$

$$= (-2) \log |\sin 2x| \times \frac{1}{2} + C \qquad \left[\int \cot x \, dx = \log |\sin x| \right]$$

$$= -\log |\sin 2x| + c$$

$$= \log |\operatorname{cosec} 2x| + c$$

[CBSE Topper 2017]