CHAPTER

10

Mathematical Induction

Introduction

There are two basic processes of reasoning which are commonly used to draw mathematical or scientific conclusions. Reasoning or drawing conclusions can be classified in two categories:

(i) Inductive reasoning
(ii) Deductive reasoning

(i) **Inductive reasoning** This is the process of reasoning from particular to general.

The numbers 324, 576, 603, 732 are all divisible by 3. From these particular results, we can hope to have a general result that all numbers of 3-digits are divisible by 3. But this is not true, because 692 is not divisible by 3.

If at all this conjuctive were true, we would have to establish its validity either by verifying the conjuctive for all possible 3-digit numbers or by using some mathematical process. The process of reasoning a valid general result from particular results is called inductive reasoning.

(ii) **Deductive reasoning** This is the process of reasoning from general to particular.

The sum of first n natural numbers is $\frac{n(n+1)}{2}$. This is a general result. From this, we can deduce that the sum of first 100 natural numbers is $5050\left(=\frac{100(100+1)}{2}\right)$. This process of reasoning a valid particular result from general result is called deductive reasoning.

The principle of mathematical induction is a mathematical process which is used to establish the validity of a general result involving **natural numbers**.

Statement

A sentence or description which is either definitely true or definitely false is called a **statement**. *For example,*

1. Mumbai is the capital of Maharashtra is a true statement.
2. There are 30 days in February is a false statement.
3. Umang is a good boy is not a statement (as it is not a definite sentence. One day whose name is Umang may be a good boy while the other boy whose name is also Umang may not be a good boy. Also, the word 'good' is not well-defined).

Mathematical Statement

A statement involving mathematical relation or relations is called **mathematical statement**.

A statement concerning the natural number 'n' is generally denoted by $P(n)$.

For example, If $P(n)$ denotes the statement "$n(n+1)$ is divisible by 2",

then $P(3)$: "$3(3+1)=12$ is divisible by 2"

and $P(8)$: "$8(8+1)=72$ is divisible by 2", etc.

Here, $P(3)$ and $P(8)$ are both true.

First Principle of Mathematical Induction

To prove that $P(n)$ is true for all natural numbers $n \geq i$, we proceed as follows:

***Step* I** (*Verification Step*) : Verify $P(n)$ for $n = i$.

***Step* II** (*Assumption Step*) : Assume $P(n)$ is true for $n = k > i$.

***Step* III** (*Induction Step*) : Using results in Step I and Step II, prove that $P(k+1)$ is true.

Remark

If $P(n)$ is true for $n = 1$ (i.e., for i =).

Second Principle of Mathematical Induction

Sometimes the above procedure will not work. Then, we consider the alternative principle called the second principle of mathematical induction, which consists of the following steps:

***Step* I** (*Verification Step*) : Verify $P(n)$ for $n = i$.

***Step* II** (*Assumption Step*) : Assume $P(n)$ is true for $i < n \leq k$.

***Step* III** (*Induction Step*) : Prove $P(n)$ for $n = k + 1$.

Remark

The second principle of mathematical induction is useful to prove recurrence relations which involve three successive terms,

For example, statements of the type

$$pT_{n+1} = qT_n + rT_{n-1}$$

Different Types of Problems of Mathematical Induction

Type I These problems are of the **Identity Type.** Examples of this type are as follows:

Example 1. Prove by mathematical induction that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ for every natural number n.

Sol. Let $P(n): 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ ...(i)

***Step* I** For $n = 1$, LHS of Eq. (i) $= 1^3 = 1$

and RHS of Eq. (i) $= \left[\frac{1(1+1)}{2}\right]^2 = 1^2 = 1$

$\therefore$ LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Assume $P(k)$ is true, then

$$P(k): 1^3 + 2^3 + 3^3 + \ldots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

***Step* III** For $n = k + 1$,

$$P(k+1): 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

$$\text{LHS} = 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3$$

$$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3$$

[by assumption step]

$$= \frac{(k+1)^2}{4}[k^2 + 4(k+1)]$$

$$= \frac{(k+1)^2(k^2 + 4k + 4)}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2 = \text{RHS}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction, $P(n)$ is true for all $n \in N$.

Example 2. Prove by mathematical induction that $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$ for every natural number.

Sol. Let $P(n): 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + n(n+1)(n+2)$

$$= \frac{n(n+1)(n+2)(n+3)}{4} \quad \text{...(i)}$$

***Step* I** For $n = 1$, LHS of Eq. (i) $= 1\cdot 2\cdot 3 = 6$

and RHS of Eq. (i) $= \frac{1\cdot(1+1)(1+2)(1+3)}{4} = 6$

$\therefore$ LHS = RHS

Therefore, $P(1)$ is true.

***Step* II** Assume that $P(k)$ is true, then

$$P(k): 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$

***Step* III** For $n = k + 1$

$$P(k+1): 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$\therefore$ LHS $= 1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3)$$

[by assumption step]

$$= \frac{(k+1)(k+2)(k+3)}{4}(k+4)$$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4} = \text{RHS}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example 3. Prove by mathematical induction that $\frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$, $\forall\, n \in N$.

Sol. Let $P(n): \frac{1}{1\cdot 2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \ldots + \frac{1}{n(n+1)(n+2)}$

$$= \frac{n(n+3)}{4(n+1)(n+2)} \quad \text{...(i)}$$

***Step* I** For $n = 1$,

LHS of Eq. (i) $= \frac{1}{1\cdot 2\cdot 3} = \frac{1}{6}$

and RHS of Eq. (i) $= \frac{1\cdot(1+3)}{4(1+1)(1+2)} = \frac{1}{6}$

Therefore, $P(1)$ is true.

***Step* II** Assume that $P(k)$ is true, then

$$P(k): \frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\ldots+\frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)}$$

***Step* III** For $n = k+1$,

$$P(k+1): \frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\ldots+\frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} = \frac{(k+1)(k+4)}{4(k+2)(k+3)}$$

$$\therefore \text{ LHS} = \frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\ldots+\frac{1}{k(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$ [by assumption step]

$$= \frac{k(k+3)^2+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{k^3+6k^2+9k+4}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$= \frac{(k+1)(k+4)}{4(k+2)(k+3)} = \text{RHS}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example 4. *Prove by mathematical induction that* $\sum_{r=0}^{n} r\,{}^nC_r = n\cdot 2^{n-1}, \forall n \in N$.

Sol. Let $P(n): \sum_{r=0}^{n} r\,{}^nC_r = n\cdot 2^{n-1}$

***Step* I** For $n = 1$,

LHS of Eq. (i) $= \sum_{r=0}^{1} r\cdot{}^1C_r = 0 + 1\cdot{}^1C_1 = 1$

and RHS of Eq. (i) $= 1\cdot 2^{1-1} = 2^0 = 1$

Therefore, $P(1)$ is true.

***Step* II** Assume that $P(k)$ is true, then $P(k)$

$$: \sum_{r=0}^{k} r\cdot{}^kC_r = k\cdot 2^{k-1}$$

***Step* III** For $n = k+1$

$$P(k+1): \sum_{r=0}^{k+1} r\cdot{}^{k+1}C_r = (k+1)\cdot 2^k$$

$$\therefore \text{ LHS} = \sum_{r=0}^{k+1} r\cdot{}^{k+1}C_r = 0 + \sum_{r=1}^{k+1} r\cdot{}^{k+1}C_r$$

$$= \sum_{r=1}^{k+1} r\cdot{}^{k+1}C_r = \sum_{r=1}^{k} r\cdot{}^{k+1}C_r + (k+1)\,{}^{k+1}C_{k+1}$$

$$= \sum_{r=1}^{k} r\,({}^kC_r + {}^kC_{r-1}) + (k+1)$$

$$= \sum_{r=1}^{k} r\cdot{}^kC_r + \sum_{r=1}^{k} r\cdot{}^kC_{r-1} + (k+1)$$

$$= \sum_{r=0}^{k} r\cdot{}^kC_r + \sum_{r=0}^{k} r\cdot{}^kC_{r-1} + (k+1)$$

$$= \sum_{r=0}^{k} r\cdot{}^kC_r + \sum_{r=0}^{k+1} r\cdot{}^kC_{r-1}$$

$$= \sum_{r=0}^{k} r\cdot{}^kC_r + \sum_{r=0}^{k} (r+1)\cdot{}^kC_r$$

$$= \sum_{r=0}^{k} r\cdot{}^kC_r + \sum_{r=0}^{k} r\cdot{}^kC_r + \sum_{r=0}^{k} {}^kC_r$$

$= P(k) + P(k) + 2^k$ [by assumption step]

$$= k\cdot 2^{k-1} + k\cdot 2^{k-1} + 2^k = 2\cdot k\cdot 2^{k-1} + 2^k$$

$$= k\cdot 2^k + 2^k = (k+1)\cdot 2^k = \text{RHS}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Type II These problems are of the **Divisibility Type**. Examples of this type are as follows:

Example 5. Use the principle of mathematical induction to show that $5^{2n+1} + 3^{n+2}\cdot 2^{n-1}$ divisible by 19 for all natural numbers n.

Sol. Let $P(n) = 5^{2n+1} + 3^{n+2}\cdot 2^{n-1}$

***Step* I** For $n = 1$, $P(1) = 5^{2+1} + 3^{1+2}\cdot 2^{1-1}$

$= 125 + 27 = 152$, which is divisible by 19.

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$, i.e.,

$P(k) = 5^{2k+1} + 3^{k+2}\cdot 2^{k-1}$ is divisible by 19.

$\Rightarrow$ $P(k) = 19r$, where r is an integer.

***Step* III** For $n = k+1$,

$$P(k+1) = 5^{2(k+1)+1} + 3^{k+1+2}\cdot 2^{k+1-1}$$

$$= 5^{2k+3} + 3^{k+3}\cdot 2^k$$

$$= 25\cdot 5^{2k+1} + 3\cdot 3^{k+2}\cdot 2\cdot 2^{k-1}$$

$$= 25\cdot 5^{2k+1} + 6\cdot 3^{k+2}\cdot 2^{k-1}$$

Now, $5^{2k+1} + 3^{k+2}\cdot 2^{k-1}$) $25\cdot 5^{2k+1} + 6\cdot 3^{k+2}\cdot 2^{k-1}$ (25

$25\cdot 5^{2k+1} + 25\cdot 3^{k+2}\cdot 2^{k-1}$

$-\quad -$

$-19\cdot 3^{k+2}\cdot 2^{k-1}$

$$\therefore \quad 25\cdot 5^{2k+1} + 6\cdot 3^{k+2}\cdot 2^{k-1}$$
$$= 25\cdot(5^{2k+1} + 3^{k-2}\cdot 2^{k-2}) - 19\cdot 3^{k+2}\cdot 2^{k-1}$$

i.e., $P(k+1) = 25\,P(k) - 19\cdot 3^{k+2}\cdot 2^{k-1}$

But we know that $P(k)$ is divisible by 19. Also, $19\cdot 3^{k+2}\cdot 2^{k-1}$ is clearly divisible by 19.

Therefore, $P(k+1)$ is divisible by 19. This shows that the result is true for $n = k+1$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

Example 6. Use the principle of mathematical induction to show that $a^n - b^n$ is divisible by $a-b$ for all natural numbers n.

Sol. Let $P(n) = a^n - b^n$

***Step* I** For $n = 1$,

$P(1) = a - b$, which is divisible by $a - b$.

Therefore, the result is true for $n = 1$.

***Step* II** Assume that the result is true for $n = k$,

i.e., $P(k) = a^k - b^k$ is divisible by $a - b$.

$\Rightarrow \quad P(k) = (a-b)\,r$, where r is an integer.

***Step* III** For $n = k+1$,

$$\therefore \quad P(k+1) = a^{k+1} - b^{k+1}$$

Now, $a^k - b^k \,)\, a^{k+1} - b^{k+1} \,(\, a$

$$\underline{-a^{k+1} \mp ab^k}$$

$$ab^k - b^{k+1} = b^k(a-b)$$

$$\therefore \quad a^{k+1} - b^{k+1} = a(a^k - b^k) + b^k(a-b)$$

i.e., $$P(k+1) = a\,P(k) + b^k\,(a-b)$$

But we know that $P(k)$ is divisible by $a-b$. Also, $b^k\,(a-b)$ is clearly divisible by $a-b$.

Therefore, $P(k+1)$ is divisible by $a-b$.

This shows that the result is true for $n = k+1$.

Hence, by the principle of mathematical induction, the result is true for all $n \in N$.

Type III These problems are of the **Inequality Type**. Examples of this type are as follows:

Example 7. ***Using mathematical induction, show that*** $\tan n\alpha > n\tan\alpha$***, where*** $0 < \alpha < \dfrac{\pi}{4(n-1)}, \forall\, n \in N$ ***and*** $n > 1$**.**

Sol. Let $P(n): \tan n\alpha > n\tan\alpha$

***Step* I** For $n = 2$, $\quad \tan 2\alpha > 2\tan\alpha$

$$\Rightarrow \quad \frac{2\tan\alpha}{1-\tan^2\alpha} - 2\tan\alpha > 0$$

$$\Rightarrow \quad 2\tan\alpha\left(\frac{1-(1-\tan^2\alpha)}{1-\tan^2\alpha}\right) > 0$$

$$\Rightarrow \quad \tan^2\alpha\cdot\tan 2\alpha > 0 \qquad \left[\because 0 < \alpha < \frac{\pi}{4} \text{ for } n = 2\right]$$

$$\Rightarrow \quad \tan 2\alpha > 0 \qquad \left[\because 0 < 2\alpha < \frac{\pi}{2}\right]$$

which is true ($\because$ in first quadrant, $\tan 2\alpha$ is positive)

Therefore, $P(2)$ is true.

***Step* II** Assume that $P(k)$ is true, then $P(k)$: $\tan k\alpha > k\tan\alpha$

***Step* III** For $n = k+1$, we shall prove that $\tan(k+1)\alpha > (k+1)\tan\alpha$

$$\because \quad \tan(k+1)\alpha = \frac{\tan k\alpha + \tan\alpha}{1 - \tan k\alpha\tan\alpha} \qquad \text{...(i)}$$

when $0 < \alpha < \dfrac{\pi}{4k}$ or $0 < k\alpha < \dfrac{\pi}{4}$

i.e. $\quad 0 < \tan k\alpha < 1$, also $0 < \tan\alpha < 1$

$\therefore \quad \tan k\alpha\tan\alpha < 1$

$1 - \tan k\alpha\tan\alpha > 0$ and $1 - \tan k\alpha\tan\alpha < 1$ ...(ii)

From Eqs. (i) and (ii), we get

$$\tan(k+1)\alpha > \frac{\tan k\alpha + \tan\alpha}{1}$$
$$> \tan k\alpha + \tan\alpha > k\tan\alpha + \tan\alpha$$

[by assumption step]

$$\therefore \quad \tan(k+1)\alpha > (k+1)\tan\alpha$$

Therefore, $P(k+1)$ is true. Hence by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example 8. Show using mathematical induction that $n! < \left(\dfrac{n+1}{2}\right)^n$, where $n \in N$ and $n > 1$.

Sol. Let $P(n): n! < \left(\dfrac{n+1}{2}\right)^n$

***Step* I** For $n = 2$, $2! < \left(\dfrac{2+1}{2}\right)^2 \Rightarrow 2 < \dfrac{9}{4}$

$\Rightarrow \quad 2 < 2\cdot 25$, which is true.

Therefore, $P(2)$ is true.

***Step* II** Assume that $P(k)$ is true, then

$$P(k): k! < \left(\frac{k+1}{2}\right)^k$$

***Step* III** For $n = k+1$, we shall prove that

$$P(k+1): (k+1)! < \left(\frac{k+2}{2}\right)^{k+1}$$

From assumption step $k! < \dfrac{(k+1)^k}{2^k}$

$$\Rightarrow \quad (k+1)k! < \frac{(k+1)^{k+1}}{2^k}$$

$$\Rightarrow \quad (k+1)! < \frac{(k+1)^{k+1}}{2^k} \quad \text{...(i)}$$

Let us assume, $\frac{(k+1)^{k+1}}{2^k} < \left(\frac{k+2}{2}\right)^{k+1}$ ...(ii)

$$\Rightarrow \quad \left(\frac{k+2}{k+1}\right)^{k+1} > 2 \Rightarrow \left(1+\frac{1}{k+1}\right)^{k+1} > 2$$

$$\Rightarrow \quad 1+(k+1)\cdot\frac{1}{(k+1)} + {}^{k+1}C_2\left(\frac{1}{k+1}\right)^2 + \ldots > 2$$

$$\Rightarrow \quad 1+1+{}^{k+1}C_2\left(\frac{1}{k+1}\right)^2 + \ldots > 2$$

which is true, hence Eq. (ii) is true.

From Eqs. (i) and (ii), we get

$$(k+1)! < \frac{(k+1)^{k+1}}{2^k} < \left(\frac{k+2}{2}\right)^{k+1}$$

$$\Rightarrow \quad (k+1)! < \left(\frac{k+2}{2}\right)^{k+1}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Type IV These problems are of the **Second principle of induction**. Examples of this type are as follows:

Example 9. If $a+b=c+d$ and $a^2+b^2=c^2+d^2$, then show by mathematical induction

$$a^n+b^n=c^n+d^n$$

Sol. $P(n): a^n + b^n = c^n + d^n$

***Step* I** For $n=1$ and $n=2$,

$P(1): a+b=c+d$ and $P(2): a^2+b^2=c^2+d^2$

which are true (from given conditions).

Therefore, $P(1)$ and $P(2)$ are true.

***Step* II** Assume $P(k-1)$ and $P(k)$ to be true

$$\therefore \quad a^{k-1}+b^{k-1}=c^{k-1}+d^{k-1} \quad \text{...(i)}$$

$$\text{and} \quad a^k+b^k=c^k+d^k \quad \text{...(ii)}$$

***Step* III** For $n=k+1$,

$P(k+1): a^{k+1}+b^{k+1}=c^{k+1}+d^{k+1}$

$$\therefore \quad \text{LHS} = a^{k+1}+b^{k+1}$$

$$= (a+b)(a^k+b^k) - ab^k - ba^k$$

$$= (a+b)(a^k+b^k) - ab(a^{k-1}+b^{k-1})$$

[given $a+b=c+d$ and $a^2+b^2=c^2+d^2$, then $ab=cd$]

$$= (c+d)(c^k+d^k) - cd(c^{k-1}+d^{k-1})$$

[from Eqs. (i) and (ii)]

$$= c^{k+1}+d^{k+1} = \text{RHS}$$

Therefore, $P(k+1)$ is true. Hence, by the principle of mathematical induction $P(n)$ is true for all $n \in N$.

Example 10. *Let* $I_m = \int_0^{\pi}\left(\frac{1-\cos mx}{1-\cos x}\right)dx$ *use mathematical induction to prove that* $I_m = m\pi$, $m = 0, 1, 2, \ldots$

Sol. $\because \; I_m = \int_0^{\pi}\left(\frac{1-\cos mx}{1-\cos x}\right)dx$

***Step* I** For $m=1$, $I_1 = \int_0^{\pi}\left(\frac{1-\cos x}{1-\cos x}\right)dx$

$\therefore \; I_1 = \pi$ and for $m=2$,

$$I_2 = \int_0^{\pi}\left(\frac{1-\cos 2x}{1-\cos x}\right)dx$$

$$= \int_0^{\pi}\frac{2\sin^2 x(1+\cos x)}{(1-\cos x)(1+\cos x)}dx$$

$$= \int_0^{\pi}\frac{2\sin^2 x(1+\cos x)}{\sin^2 x}dx = 2\int_0^{\pi}(1+\cos x)dx$$

$$= 2[x+\sin x]_0^{\pi} = 2[(\pi+0)-(0+0)] = 2\pi$$

which are true, therefore I_1 and I_2 are true.

***Step* II** Assume I_{k-1} and I_k to be true

$$\therefore \quad I_{k-1} = (k-1)\pi \quad \text{...(i)}$$

$$\text{and} \quad I_k = k\pi \quad \text{...(ii)}$$

***Step* III** For $m=k+1$,

$$I_{k+1} = \int_0^{\pi}\frac{1-\cos(k+1)x}{1-\cos x}dx$$

$$\therefore \; I_{k+1} - I_k = \int_0^{\pi}\frac{\cos kx - \cos(k+1)x}{1-\cos x}dx$$

$$= \int_0^{\pi}\frac{2\sin\left(\frac{2k+1}{2}\right)x\cdot\sin\left(\frac{x}{2}\right)}{2\sin^2\left(\frac{x}{2}\right)}dx$$

$$= \int_0^{\pi}\frac{\sin\left(\frac{2k+1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}dx \quad \text{...(iii)}$$

Similarly, $I_k - I_{k-1} = \int_0^{\pi}\frac{\sin\left(\frac{2k-1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}dx$ ...(iv)

On subtracting Eq. (iv) from Eq. (iii), we get

$$I_{k+1} - 2I_k + I_{k-1} = \int_0^\pi \frac{\sin\left(\frac{2k+1}{2}\right)x - \sin\left(\frac{2k-1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} dx$$

$$= \int_0^\pi \frac{2\cos(kx)\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} dx = 2\int_0^\pi \cos kx \, dx = 2\left[\frac{\sin kx}{k}\right]_0^\pi = 0$$

$$\Rightarrow \quad I_{k+1} = 2I_k - I_{k-1} = 2k\pi - (k-1)\pi$$

[by assumption step]

$$= k\pi + \pi = (k+1)\pi$$

This shows that the result is true for $m = k + 1$. Hence, by the principle of mathematical induction the result is true for all $m \in N$.

Type V These problems are of the **Recursion Type**. Examples of this type are as follows:

Example 11. *Given* $u_{n+1} = 3u_n - 2u_{n-1}$ *and* $u_0 = 2$, $u_1 = 3$. *Prove that* $u_n = 2^n + 1, \forall n \in N$.

Sol. $\because u_{n+1} = 3u_n - 2u_{n-1}$ ...(i)

***Step* I** Given, $u_1 = 3 = 2 + 1 = 2^1 + 1$ which is true for $n = 1$.

Putting $n = 1$ in Eq. (i), we get

$$u_{1+1} = 3u_1 - 2u_{1-1}$$

$$\Rightarrow \quad u_2 = 3u_1 - 2u_0 = 3\cdot3 - 2\cdot2 = 5 = 2^2 + 1$$

which is true for $n = 2$.

Therefore, the result is true for $n = 1$ and $n = 2$.

***Step* II** Assume it is true for $n = k$, then it is also true for $n = k - 1$.

Then, $u_k = 2^k + 1$ ...(ii)

and $u_{k-1} = 2^{k-1} + 1$ ...(iii)

***Step* III** Putting $n = k$ in Eq. (i), we get

$$u_{k+1} = 3u_k - 2u_{k-1}$$

$$= 3(2^k + 1) - 2(2^{k-1} + 1)$$

[from Eqs. (ii) and (iii)]

$$= 3\cdot2^k + 3 - 2\cdot2^{k-1} - 2 = 3\cdot2^k + 3 - 2^k - 2$$

$$= (3-1)2^k + 1 = 2\cdot2^k + 1 = 2^{k+1} + 1$$

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction the result is true for all $n \in N$.

Example 12. Let $u_1 = 1$, $u_2 = 2$, $u_3 = \frac{7}{2}$ and $u_{n+3} = 3u_{n+2} - \left(\frac{3}{2}\right)u_{n+1} - u_n$. Use the principle of mathematical induction to show that $u_n = \frac{1}{3}\left[2^n + \left(\frac{1+\sqrt{3}}{2}\right)^n + \left(\frac{1-\sqrt{3}}{2}\right)^n\right] \forall n \geq 1$.

Sol. $\because \quad u_n = \frac{1}{3}\left[2^n + \left(\frac{1+\sqrt{3}}{2}\right)^n + \left(\frac{1-\sqrt{3}}{2}\right)^n\right]$ ...(i)

***Step* I** For $n = 1$, $u_1 = \frac{1}{3}\left[2^1 + \left(\frac{1+\sqrt{3}}{2}\right)^1 + \left(\frac{1-\sqrt{3}}{2}\right)^1\right]$

$$= \frac{1}{3}[2 + 1] = 1$$

which is true for $n = 1$ and for $n = 2$,

$$u_2 = \frac{1}{3}\left[2^2 + \left(\frac{1+\sqrt{3}}{2}\right)^2 + \left(\frac{1-\sqrt{3}}{2}\right)^2\right]$$

$$= \frac{1}{3}\left[4 + \left(\frac{4+2\sqrt{3}}{4}\right) + \left(\frac{4-2\sqrt{3}}{4}\right)\right] = \frac{1}{3}[6] = 2$$

which is true for $n = 2$.

Therefore, the result is true for $n = 1$ and $n = 2$.

***Step* II** Assume it is true for $n = k$, then it is also true for $n = k - 1, k - 2$

$$\therefore \quad u_k = \frac{1}{3}\left[2^k + \left(\frac{1+\sqrt{3}}{2}\right)^k + \left(\frac{1-\sqrt{3}}{2}\right)^k\right] \quad \text{...(ii)}$$

$$u_{k-1} = \frac{1}{3}\left[2^{k-1} + \left(\frac{1+\sqrt{3}}{2}\right)^{k-1} + \left(\frac{1-\sqrt{3}}{2}\right)^{k-1}\right] \quad \text{...(iii)}$$

$$u_{k-2} = \frac{1}{3}\left[2^{k-2} + \left(\frac{1+\sqrt{3}}{2}\right)^{k-2} + \left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\right] \quad \text{...(iv)}$$

***Step* III** Given that, $u_{n+3} = 3u_{n+2} - \left(\frac{3}{2}\right)u_{n+1} - u_n$

Replace n by $k - 2$

Then, $u_{k+1} = 3u_k - \frac{3}{2}u_{k-1} - u_{k-2}$

$$= \frac{1}{3}\left[3\cdot2^k + 3\left(\frac{1+\sqrt{3}}{2}\right)^k + 3\left(\frac{1-\sqrt{3}}{2}\right)^k\right]$$

$$+ \frac{1}{3}\left[-\frac{3}{2}\cdot2^{k-1} - \frac{3}{2}\left(\frac{1+\sqrt{3}}{2}\right)^{k-1} - \frac{3}{2}\left(\frac{1-\sqrt{3}}{2}\right)^{k-1}\right]$$

$$+ \frac{1}{3}\left[-2^{k-2} - \left(\frac{1+\sqrt{3}}{2}\right)^{k-2} - \left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\right]$$

$$= \frac{1}{3}\left[3\cdot2^k - 3\cdot2^{k-2} - 2^{k-2} + \left(\frac{1+\sqrt{3}}{2}\right)^k - \frac{3}{2}\left(\frac{1+\sqrt{3}}{2}\right)^{k-1} - \left(\frac{1+\sqrt{3}}{2}\right)^{k-2} + 3\left(\frac{1-\sqrt{3}}{2}\right)^k - \frac{3}{2}\left(\frac{1-\sqrt{3}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\right]$$

$$= \frac{1}{3}\left[2^{k-2}(3 \cdot 4 - 3 - 1) + \left(\frac{1+\sqrt{3}}{2}\right)^{k-2} \left[3\left(\frac{1+\sqrt{3}}{2}\right)^2 - \frac{3}{2}\left(\frac{1+\sqrt{3}}{2}\right) - 1\right] + \left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\left[3\left(\frac{1-\sqrt{3}}{2}\right)^2 - \frac{3}{2}\left(\frac{1-\sqrt{3}}{2}\right) - 1\right]\right]$$

$$= \frac{1}{3}\left[2^{k-2} \cdot 8 + \left(\frac{1+\sqrt{3}}{2}\right)^{k-2}\left[\frac{3(1+\sqrt{3})^2 - 3(1+\sqrt{3}) - 4}{4}\right] + \left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\left[\frac{3(1-\sqrt{3})^2 - 3(1-\sqrt{3}) - 4}{4}\right]\right]$$

$$= \frac{1}{3}\left[2^{k+1} + \left(\frac{1+\sqrt{3}}{2}\right)^{k-2}\left[\frac{10+6\sqrt{3}}{8}\right] + \left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\left[\frac{10-6\sqrt{3}}{8}\right]\right]$$

$$= \frac{1}{3}\left[2^{k+1} + \left(\frac{1+\sqrt{3}}{2}\right)^{k-2}\left(\frac{1+\sqrt{3}}{2}\right)^3 + \left(\frac{1-\sqrt{3}}{2}\right)^{k-2}\left(\frac{1-\sqrt{3}}{2}\right)^3\right]$$

$$= \frac{1}{3}\left[2^{k+1} + \left(\frac{1+\sqrt{3}}{2}\right)^{k+1} + \left(\frac{1-\sqrt{3}}{2}\right)^{k+1}\right]$$

This shows that the result is true for $n = k + 1$. Hence, by the principle of mathematical induction the result is true for all $n \in N$.