Unit 8 (Application Of Integrals)

Short Answer Type Questions

1. Find the area of the region bounded by the curves $y^2 = 9x$, $y = 3x$.

Sol. We have, $y^2 = 9x$ and $y = 3x$

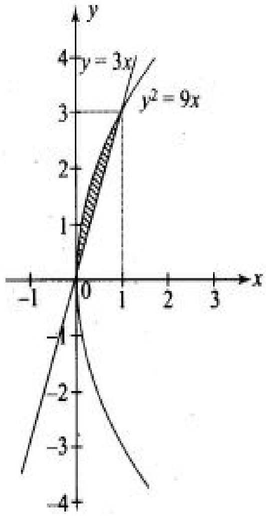
Solving $y^2 = 3(3x) = 3y$

$\Rightarrow$ $y = 0$ or 3

When $y = 0$, $x = 0$ and when $y = 3$, $x = 1$

So points of intersection are (0, 0) and (1, 3)

Graphs of parabola $y^2 = 9x$ and line $y = 3x$ are as shown in the adjacent figure.



From the figure, Area of shaded region

$$A = \int_0^1 (\sqrt{9x} - 3x)\,dx$$

$$= 3\int_0^1 x^{1/2}\,dx - 3\int_0^1 x\,dx$$

$$= 3\left[\frac{x^{3/2}}{3/2}\right]_0^1 - 3\left[\frac{x^2}{2}\right]_0^1$$

$$= 3\left(\frac{2}{3} - 0\right) - 3\left(\frac{1}{2} - 0\right)$$

$$= 2 - \frac{3}{2} = \frac{1}{2} \text{ sq. units.}$$

2. Find the area of the region bounded by the parabola $y^2 = 2px$, $x^2 = 2py$.

Sol. We have, $y^2 = 2px$ and $x^2 = 2py$

Solving curves, we get

$\therefore$ $x^4 = 4p^2y^2$

$\Rightarrow \quad x^4 = 4p^2 \cdot (2px)$

$\Rightarrow \quad x^4 = 8p^3x$

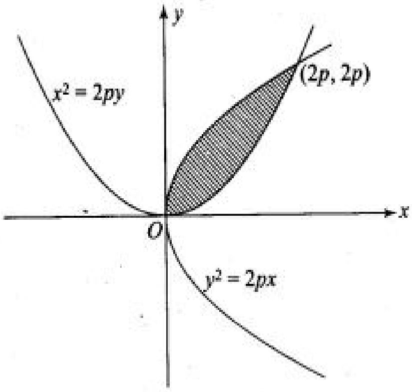
$\Rightarrow \quad x(x^3 - 8p^3) = 0$

$\Rightarrow \quad x = 0, 2p$

When $x = 0, y = 0$ and when $x = 2p, y = 2p$

So, points of intersection are $(0, 0)$ and $(2p, 2p)$

Graph of both the parabolas is as shown in the following figure.



From the figure, Area of shaded region,

$$A = \int_0^{2p} \left(\sqrt{2px} - \frac{x^2}{2p}\right) dx$$

$$= \sqrt{2p} \int_0^{2p} x^{1/2} dx - \frac{1}{2p} \int_0^{2p} x^2 dx$$

$$= \sqrt{2p}\left[\frac{2}{3}x^{3/2}\right]_0^{2p} - \frac{1}{2p}\left[\frac{x^3}{3}\right]_0^{2p}$$

$$= \sqrt{2p}\left(\frac{2}{3} \cdot 2\sqrt{2}p^{3/2}\right) - \frac{1}{2p}\left(\frac{1}{3}8p^3\right)$$

$$= \frac{8}{3}p^2 - \frac{4}{3}p^2 = \frac{4p^2}{3} \text{ sq. units}$$

3. Find the area of the region bounded by the curve $y = x^3$ and $y = x + 6$ and $x = 0$.

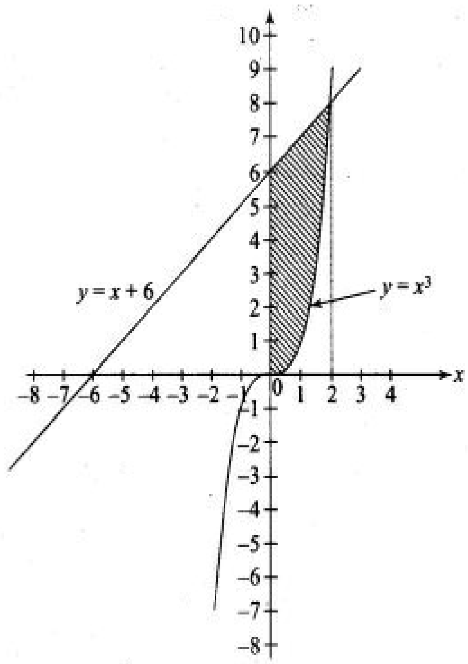
Sol. We have, $y = x^3$, $y = x + 6$ and $x = 0$

Graph of function are as shown in the following figure

Solving $y = x^3$ and $y = x + 6$, we get

$x^3 = x + 6$

$\Rightarrow \quad x^3 - x - 6 = 0$



Clearly $x = 2$ satisfies the above equation.

Also from the figure it is clear that there is only point of intersection.

$\therefore$ From the figure, area of shaded region,

$$A = \int_0^2 (x + 6 - x^3)\,dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^4}{4}\right]_0^2 = \frac{4}{2} + 12 - \frac{16}{4} = 10 \text{ sq. units}$$

4. Find the area of the region bounded by the curve $y^2 = 4x$, $x^2 = 4y$.

Sol. In q. no. 2, putting $p = 2$ we get the required area $\frac{4(2)^2}{3} = \frac{16}{3}$ sq. units

5. Find the area of the region included between $y^2 = 9x$ and $y = x$

Sol. We have, $y^2 = 9x$ and $y = x$

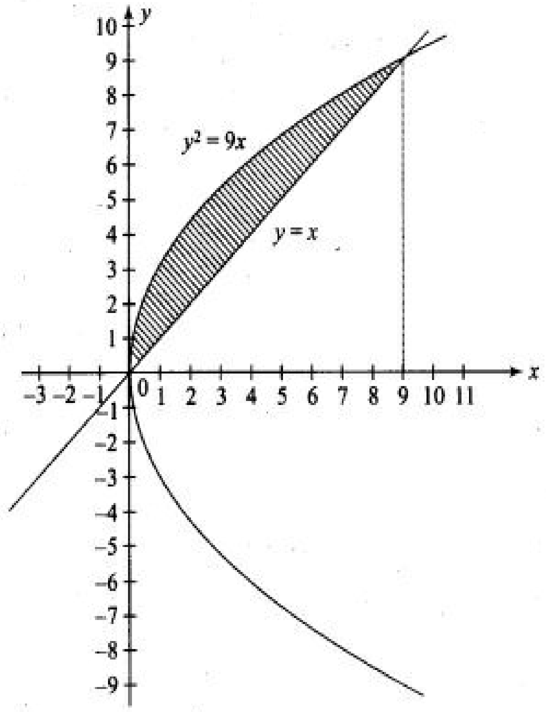
Solving $\quad y^2 = 9y$

$\Rightarrow \qquad y = 0$ or 9

When $y = 0$, $x = 0$ and when $y = 9$, $x = 9$

So points of intersection are $(0, 0)$ and $(9, 9)$

Graphs of parabola $y^2 = 9x$ and $y = x$ are as shown in the following figure.



From the figure, area of shaded region

$$A = \int_0^9 (\sqrt{9x} - x)\,dx$$

$$= 3\int_0^9 x^{1/2}\,dx - \int_0^9 x\,dx$$

$$= 3\left[\frac{x^{3/2}}{3/2}\right]_0^9 - \left[\frac{x^2}{2}\right]_0^9$$

$$= 3\left(\frac{2}{3}\cdot 27 - 0\right) - \left(\frac{81}{2} - 0\right) = 54 - \frac{81}{2} = \frac{27}{2} \text{ sq. units}$$

6. Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$

Sol. We have, $x^2 = y$ and $y = x + 2$

Solving curves, we get

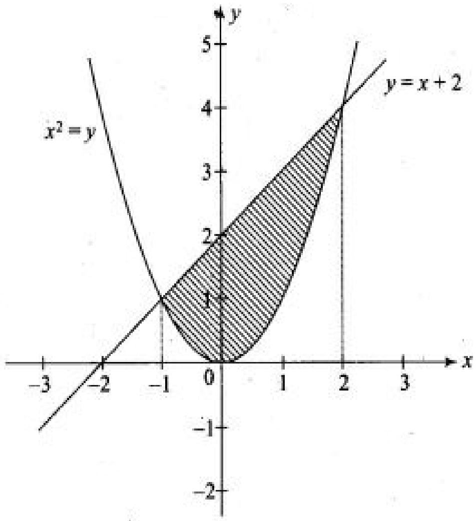
$\Rightarrow \quad x^2 = x + 2$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad (x - 2)(x + 1) = 0$

When $x = 2, y = 4$ and when $x = -1, y = 1$

Graph of parabola and straight line are as shown in the following figure.



∴ From the figure, area of shaded region

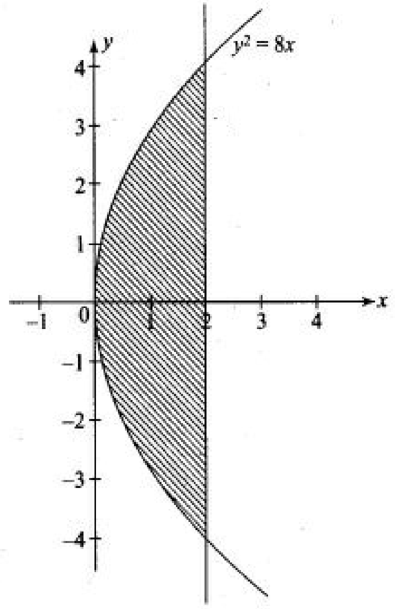
$$= \int_{-1}^{2} (x + 2 - x^2)dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{4}{2} + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}\right] = 8 - \frac{1}{2} - \frac{9}{3} = 5 - \frac{1}{2} = \frac{9}{2} \text{sq. units}$$

7. Find the area of region bounded by the line $x = 2$ and the parabola $y^2 = 8x$

Sol. We have, $y^2 = 8x$ and $x = 2$

Graphs of parabola and line are as shown in the following figure.



From the figure, area of shaded region

$$A = 2\int_0^2 \sqrt{8x}\,dx$$

$$= 4\sqrt{2}\int_0^2 x^{1/2}dx = 4\sqrt{2}\left[\frac{2}{3}x^{3/2}\right]_0^2$$

$$= 4\sqrt{2}\left[\frac{2}{3}\cdot 2\sqrt{2} - 0\right] = \frac{32}{3}\text{sq. units}$$

8. Sketch the region $\{(x, 0) : y = \sqrt{4 - x^2}\}$ and x-axis. Find the area of the region using integration.

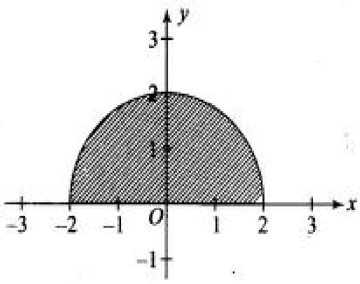
Sol. We have $y = \sqrt{4 - x^2}$ (i)

$\Rightarrow \quad y^2 = 4 - x^2, y \geq 0$

$\Rightarrow \quad x^2 + y^2 = 4, y \geq 0$

Graph of above function is semi-circle lying above x-axis.

The graph is as shown in the adjacent figure.



$\therefore$ From the figure, area of shaded region,

$$A = \int_{-2}^{2} \sqrt{4 - x^2}\,dx$$

$$= \int_{-2}^{2} \sqrt{2^2 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\cdot\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$= 0 + 2\sin^{-1}1 - 0 - 2\sin^{-1}(-1) = 2\frac{\pi}{2} - 2\left(-\frac{\pi}{2}\right) = 2\pi \text{ sq. units.}$$

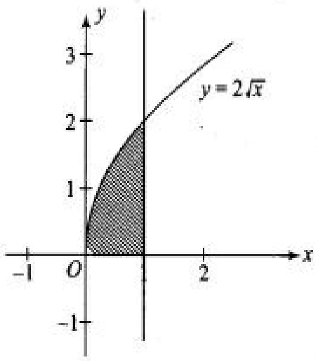
9. Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.

Sol. We have, $y = 2\sqrt{x}$

or $\quad y^2 = 4x, x \geq 0$

The graph of above function is part of parabola lying above x-axis.

The graph is as shown in the adjacent figure.



From the figure, area of shaded region,

$$A = \int_0^1 2\sqrt{x}\,dx$$

$$= 2\left[\frac{2}{3}x^{3/2}\right]_0^1$$

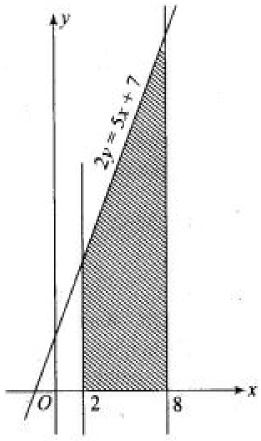
$$= 2\left(\frac{2}{3}\right) = \frac{4}{3}\text{ sq. units}$$

10. Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$.

Sol. We have, $2y = 5x + 7$

or $$y = \frac{5x}{2} + \frac{7}{2}$$

The graph is as shown in the adjacent figure.



From the figure, area of shaded region

$$= \int_2^8 \frac{5x+7}{2}\,dx$$

$$= \frac{1}{2}\left[5\cdot\frac{x^2}{2} + 7x\right]_2^8$$

$$= \frac{1}{2}\left[5 \times 32 + 7 \times 8 - 10 - 14\right]$$

$$= \frac{1}{2}\left[160 + 56 - 24\right] = 96\text{ sq. units}$$

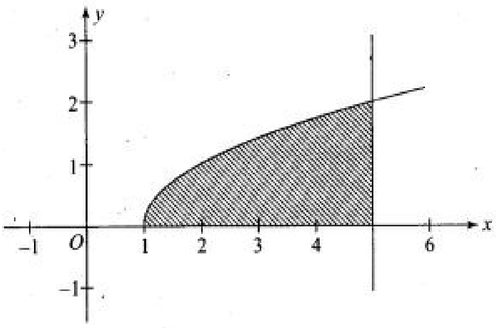
11. Draw a rough sketch of the curve $y = \sqrt{x-1}$ in the interval [1, 5]. Find the area under the curve and between the lines $x = 1$ and $x = 5$.

Sol. We have $y = \sqrt{x-1}$

$\Rightarrow \quad y^2 = x - 1$

The graph of above function is parabola with vertex at (1, 0) and lying above x-axis

For $x \in [1, 5]$, graph is as shown in the following figure.



From the figure, area of shaded region,

$$A = \int_1^5 (x-1)^{1/2}\,dx$$

$$= \left[\frac{2}{3}(x-1)^{3/2}\right]_1^5$$

$$= \left[\frac{2}{3}(5-1)^{3/2} - 0\right] = \frac{16}{3}\text{sq. units}$$

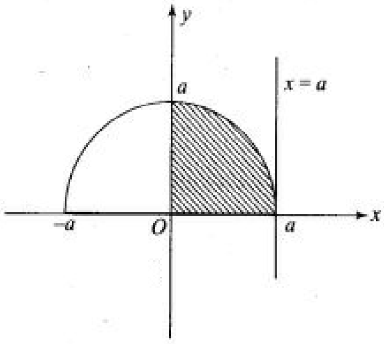
12. Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included between the lines $x = 0$ and $x = a$.

Sol. We have $y = \sqrt{a^2 - x^2}$

$\Rightarrow \quad y^2 = a^2 - x^2$

$\Rightarrow \quad x^2 + y^2 = a^2$

Graph of above function is semi-circle lying above x-axis.
The graph is as shown in the following figure.



From the figure, area of shaded region,

$$A = \int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \left[0 + \frac{a^2}{2}\sin^{-1}1 - 0 - \frac{a^2}{2}\sin^{-1}0\right] = \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}\text{sq. units}$$

13. Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.

Sol. We have $y = \sqrt{x}$ and $y = x$

Solving, we get

$$x = \sqrt{x}$$

$\Rightarrow \quad x^2 = x$

$\Rightarrow \quad x^2 - x = 0$

$\Rightarrow \quad x = 0, 1$

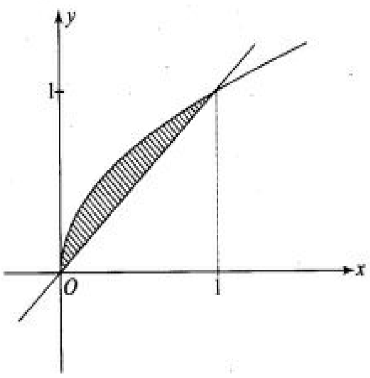
At $x = 0, y = 0$

and at $x = 1, y = 1$

Thus curves intersect at (0, 0) and (1, 1)

Graph of $y = \sqrt{x}$ is part of parabola lying above x-axis.

The graph is as shown in the adjacent figure.



From the figure, area of shaded region,

$$A = \int_0^1 (\sqrt{x} - x)\,dx = \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \text{sq. units}$$

14. Find the area enclosed by the curve $y = -x^2$ and the straight line $x + y + 2 = 0$.

Sol. We have, $y = -x^2$ and $x + y + 2 = 0$

Solving we get,

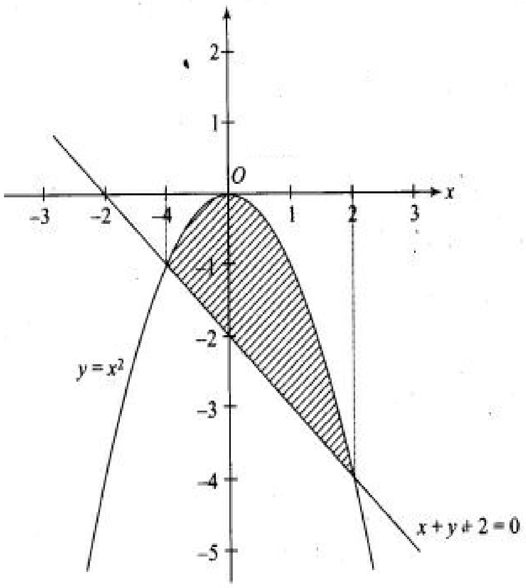
$x^2 = x + 2$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad (x - 2)(x + 1) = 0$

$\Rightarrow \quad x = 2, -1$

The graph of above function is downward parabola.



From the figure, area of shaded region,

$$A = \int_{-1}^{2} (-x^2 - (-x-2))\,dx$$

$$= \int_{-1}^{2} (x + 2 - x^2)\,dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{2^2}{2} + 2(2) - \frac{2^3}{3}\right] - \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1)^3}{3}\right]$$

$$= \left[6 - \frac{8}{3}\right] - \left[\frac{1}{2} - 2 + \frac{1}{3}\right] = \frac{9}{2} \text{ sq. units}$$

15. Find the area bounded by the curve $y = \sqrt{x}$, $x = 2y + 3$ in the first quadrant and x-axis.

Sol. We have $y = \sqrt{x}$ and $x = 2y + 3$

Solving we get

$$y = \sqrt{2y+3},\ y \geq 0$$

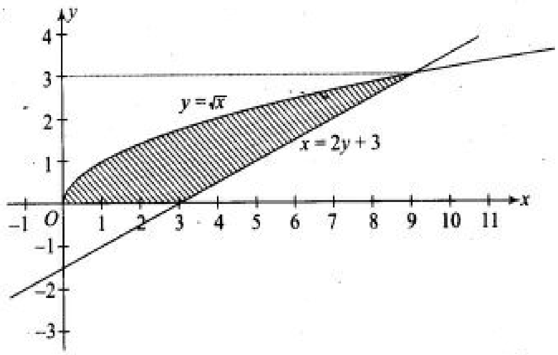
$\Rightarrow \quad y^2 = 2y + 3, y \geq 0$

$\Rightarrow \quad y^2 - 2y - 3 = 0, y \geq 0$

$\Rightarrow \quad (y-3)(y+1) = 0, y \geq 0$

$\Rightarrow \quad y = 3$

The graph of function $y = \sqrt{x}$ is part of parabola $y^2 = x$ lying above x-axis. The graph is as shown in the following figure.



From the figure, area of shaded region,

$$A = \int_{0}^{3} (2y + 3 - y^2)\,dy$$

$$= \left[\frac{2y^2}{2} + 3y - \frac{y^3}{3}\right]_0^3 = \left[\frac{18}{2} + 9 - 9 - 0\right] = 9 \text{ sq. units}$$

Long Answer Type Questions

16. Find the area of the region bounded by the curve $y^2 = 2x$ and $x^2 + y^2 = 4x$.

Sol. We have, $y^2 = 2x$ and $x^2 + y^2 = 4x$

$y^2 = 2x$ is parabola opening to the right of positive direction x-axis

$$x^2 + y^2 = 4x$$

$\Rightarrow (x-2)^2 + y^2 = 4$, which is circle having centre at (2, 0) and radius '2'

Solving the curves we get,

$$x^2 + 2x = 4x$$

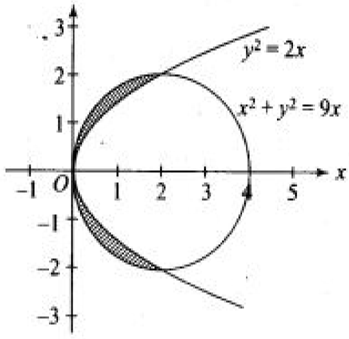
$$\Rightarrow \quad x^2 - 2x = 0$$

$$\Rightarrow \quad x = 0, 2$$

When $x = 0, y = 0$ and when $x = 2, y = \pm 2$

Thus points of intersection are (0, 0), (2, 2) and (2, –2)

The graph is as shown in the following figure.



From the figure, area of shaded region,

$$= 2 \cdot \int_0^2 [\sqrt{2^2 - (x-2)^2} - \sqrt{2x}]dx$$

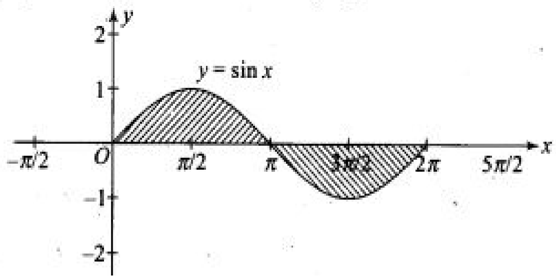
$$= 2\left[\left[\frac{x-2}{2} \cdot \sqrt{2^2 - (x-2)^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^2 - \left[\sqrt{2} \cdot \frac{x^{3/2}}{3/2}\right]_0^2\right]$$

$$= 2\left[\left(0 + 0 - 0 + 2 \cdot \frac{\pi}{2}\right) - \frac{2\sqrt{2}}{3}(2^{3/2} - 0)\right] = 2\pi - \frac{16}{3} \text{sq. units}$$

17. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.

Sol. We have $y = \sin x$, $0 \le x \le 2\pi$

The graph is as shown in the following figure.



From the figure, area of shaded region,

$$= 2\int_0^{\pi} \sin x dx = 2[-\cos x]_0^{\pi} = 2[-\cos \pi + \cos 0] = 4$$

18. Find the area of region bounded by the triangle whose vertices are (–1, 1), (0, 5) and (3, 2), using integration.

Sol. We have vertices of a ΔABC as $A(-1, 1)$, $B(0, 5)$ and $C(3, 2)$

$\therefore$ Equation of AB is $y - 1 = \left(\frac{5-1}{0+1}\right)(x+1)$

$\Rightarrow \quad y - 1 = 4x + 4$

$\Rightarrow \quad y = 4x + 5$

Equation of BC is $y - 5 = \left(\frac{2-5}{3-0}\right)(x-0)$

$\Rightarrow \quad y - 5 = -x$

$\Rightarrow \quad y = 5 - x$

Equation of line AC is $y - 1 = \left(\frac{2-1}{3+1}\right)(x+1)$

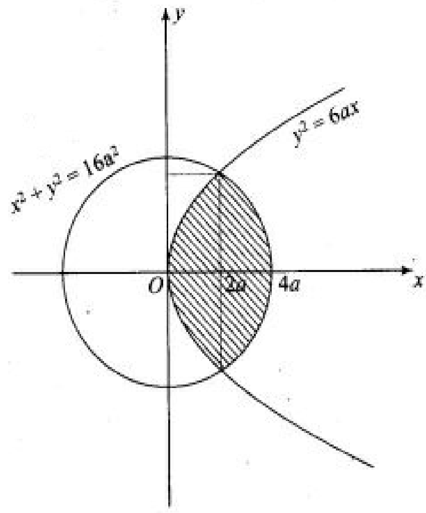
$\Rightarrow \quad y - 1 = \frac{1}{4}(x+1)$

$\Rightarrow \quad 4y = x + 5$

Now see the solution of q. no. 21

19. Draw a rough sketch of the region $\{(x, y) : y^2 \le 6ax \text{ and } x^2 + y^2 \le 16a^2\}$. Also find the area of the region sketched using method of integration.

Sol. We have, $y^2 \le 6ax$, which represents the region interior to parabola $y^2 = 6ax$ towards focus.



And $x^2 + y^2 \le 16a^2$, which represents the region interior to circle $x^2 + y^2 = 16a^2$.

Solving circle and parabola, we get

$x^2 + 6ax = 16a^2$

$\Rightarrow \quad x^2 + 6ax - 16a^2 = 0$

$\Rightarrow \quad (x - 2a)(x + 8a) = 0$

$\Rightarrow \quad x = 2a$

(as $x = -8a$ is not possible)

Putting $x = 2a$ in parabola, we get

The graph of functions are as shown in the adjacent figure.

From the figure, area of the shaded region

$$A = 2\left[\int_0^{2a}\sqrt{6ax}\,dx + \int_{2a}^{4a}\sqrt{(4a)^2 - x^2}\,dx\right]$$

$$= 2\left[\sqrt{6a}\left(\frac{2}{3}x^{3/2}\right)_0^{2a} + \left(\frac{x}{2}\sqrt{(4a)^2 - x^2} + \frac{(4a)^2}{2}\sin^{-1}\frac{x}{4a}\right)_{2a}^{4a}\right]$$

$$= 2\left[\sqrt{6a}\frac{2}{3}(2a)^{3/2} + 8a^2\cdot\frac{\pi}{2} - \frac{2a}{2}\sqrt{16a^2 - 4a^2} - 8a^2\cdot\frac{\pi}{6}\right]$$

$$= 2\left[\sqrt{6a}\frac{2}{3}\cdot 2\sqrt{2}a^{3/2} + 4\pi a^2 - a\cdot 2\sqrt{3}a - \frac{4a^2}{3}\pi\right]$$

$$= 2\left[\frac{8}{3}\sqrt{3}a^2 + 4\pi a^2 - 2\sqrt{3}a^2 - \frac{4a^2\pi}{3}\right]$$

$$= 2\left[\frac{2}{3}\sqrt{3}\,a^2 + \frac{8a^2\pi}{3}\right] = \frac{4}{3}a^2[\sqrt{3} + 4\pi]$$

20. Compute the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.

Sol. We have lines

$x + 2y = 2$ (i)

$y - x = 1$ (ii)

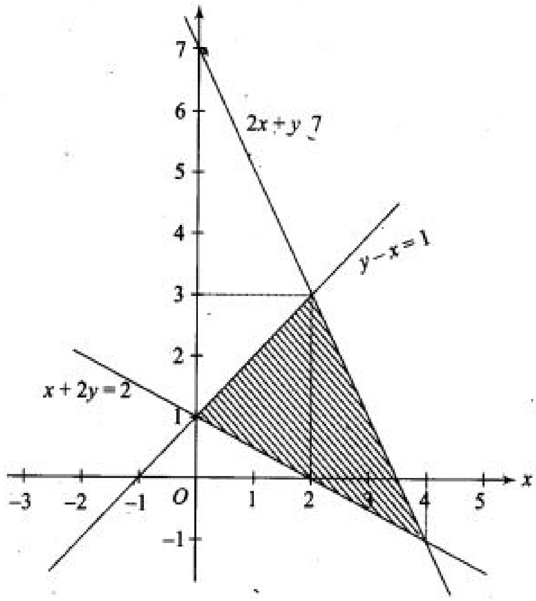
and $2x + y = 7$ (iii)

Solving (i) and (ii), we get point of intersection (0, 1)

Solving (ii) and (iii), we get point of intersection (2, 3)

Solving (i) and (iii), we get point of intersection (4, –1)

These lines are plotted on coordinate plane as shown in the following figure.,



∴ From the figure, area of the shaded region

$$A=\int_0^2\left(x+1-\frac{2-x}{2}\right)dx+\int_2^4\left(7-2x-\frac{2-x}{2}\right)dx$$

$$=\int_0^2\frac{3x}{2}dx+\int_2^4\left(6-\frac{3}{2}x\right)dx$$

$$=\left[\frac{3x^2}{4}\right]_0^2+\left[6x-\frac{3x^2}{4}\right]_2^4$$

$$=3+(24-12)-(12-3)=6 \text{ sq. units.}$$

21. Find the area bounded by the lines $y = 4x + 5$, $y = 5 - x$ and $4y = x + 5$.

Sol. We have lines

$y = 4x + 5$ (i)

$y = 5 - x$ (ii)

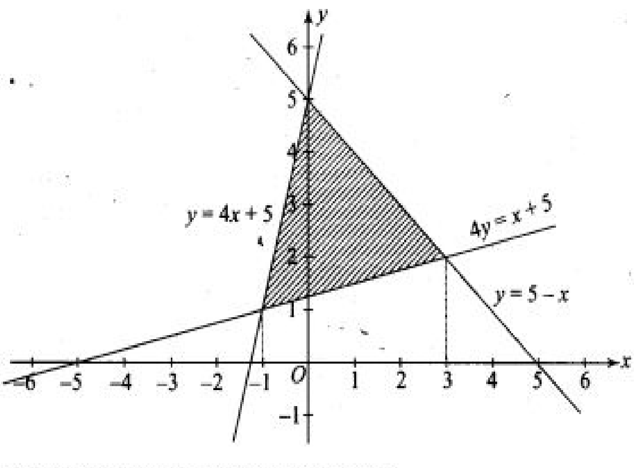
and $4y = x + 5$ (iii)

Solving (i) and (ii), we get point of intersection (0, 5)

Solving (ii) and (iii), we get point of intersection (3, 2)

Solving (i) and (iii), we get point of intersection (–1, 1)

These lines are plotted on coordinate plane as shown in the following figure.,



From the figure, area of the shaded region

$$A=\int_{-1}^{0}(4x+5)dx+\int_0^3(5-x)dx-\int_{-1}^{3}\frac{x+5}{4}dx$$

$$=\left[\frac{4x^2}{2}+5x\right]_{-1}^{0}+\left[5x-\frac{x^2}{2}\right]_0^3-\frac{1}{4}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$$

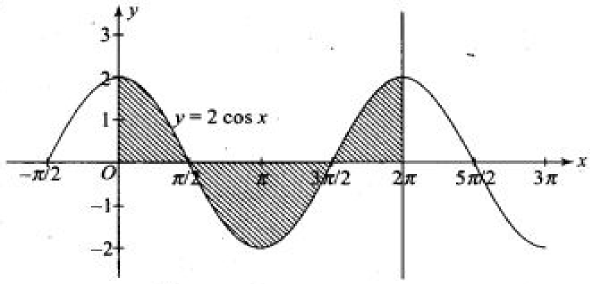
$$=[0-2+5]+\left[15-\frac{9}{2}-0\right]-\frac{1}{4}\left[\frac{9}{2}+15-\frac{1}{2}+5\right]$$

$$=3+\frac{21}{2}-\frac{1}{4}\cdot 24=\frac{15}{2}\text{sq. units}$$

22. Find the area bounded by the curve $y = 2\cos x$ and the x-axis from $x = 0$ to $x = 2\pi$.

Sol. We have $y = 2\cos x,\ 0 \le x \le 2\pi$

Graph of the functions is as shown in the following figure.



From the figure, area of the shaded region

$$A = \int_0^{2\pi} |2\cos x|\,dx = 4\int_0^{\pi/2} (2\cos x)\,dx = 8[\sin x]_0^{\pi/2} = 8 \text{ sq. units}$$

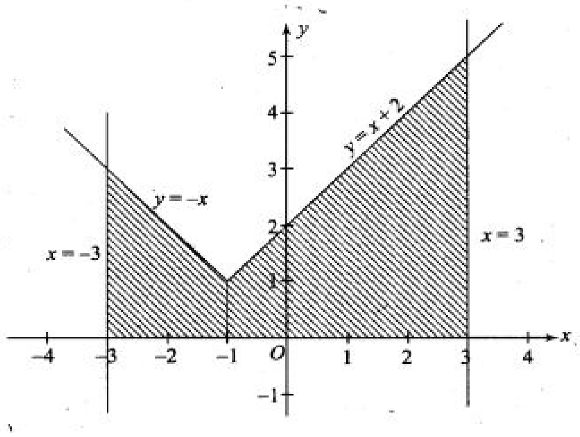
23. Draw a rough sketch of the given curve $y = 1 + |x + 1|$, $x = -3$, $x = 3$, $y = 0$ and find the area of the region bounded by them, using integration.

Sol. We have, $y = 1 + |x + 1|$, $x = -3$, $x = 3$ and $y = 0$

Now $|x+1| = \begin{cases} -x-1, & x < -1 \\ x+1, & x \ge -1 \end{cases}$

$\therefore \quad y = 1 + |x+1| = \begin{cases} -x, & x < -1 \\ x+2, & x \ge -1 \end{cases}$

Graph of the above function with $x = -3$, $x = 3$ as shown in the following figure



From the figure, Area of shaded region,

$$A = \int_{-3}^{-1} -x\,dx + \int_{-1}^{3} (x+2)\,dx$$

$$= -\left[\frac{x^2}{2}\right]_{-3}^{-1} + \left[\frac{x^2}{2} + 2x\right]_{-1}^{3}$$

$$= -\left[\frac{1}{2} - \frac{9}{2}\right] + \left[\frac{9}{2} + 6 - \frac{1}{2} + 2\right] = 4 + 12 = 16 \text{ sq. units.}$$

Objective Type Questions

24. The area of the region bounded by the y-axis, $y = \cos x$ and $y = \sin x$, $0 \le x \le \frac{\pi}{2}$ is

(a) $\sqrt{2}$ sq. units (b) $(\sqrt{2}+1)$ sq. units

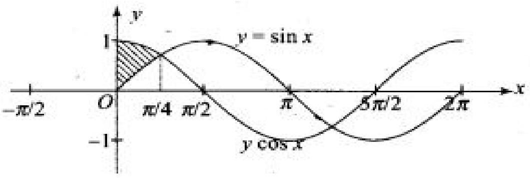
(c) $(\sqrt{2}-1)$ sq. units (d) $(2\sqrt{2}-1)$ sq. units

Sol. **(c)** We have $y = \cos x$ and $y = \sin x$, where $0 \le x \le \frac{\pi}{2}$

Solving curve, we get $\sin x = \cos x$

$\therefore \quad x = \frac{\pi}{4}$

Graphs are as shown in the following figure.



From the figure, area of the shaded region,

$$A = \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$= [\sin x + \cos x]_0^{\pi/4}$$

$$= \left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - \sin 0 - \cos 0\right)$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 = (\sqrt{2}-1) \text{ sq. units}$$

25. The area of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is

(a) $\frac{3}{8}$ sq. units (b) $\frac{5}{8}$ sq. units

(c) $\frac{7}{8}$ sq. units (d) $\frac{9}{8}$ sq. units

Sol. **(d)** We have parabola $x^2 = 4y$ and the straight line $x = 4y - 2$

Solving we get

$x^2 = x + 2$

$\Rightarrow \quad x^2 - x - 2 = 0$

$\Rightarrow \quad (x-2)(x+1) = 0$

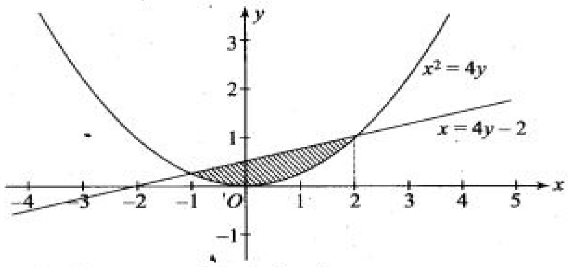
$\Rightarrow \quad x = -1, 2$

For $x = -1$, $y = 1/4$

and for $x = 2$, $y = 1$.

Thus point of intersection are $(-1, 1/4)$ and $(2, 1)$

Graphs of parabola $x^2 = 4y$ and $x = 4y - 2$ are as shown in the following figure.



$\therefore$ From the figure, area of shaded region

$$A = \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \frac{1}{4}\left[\left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right] = \frac{1}{4}\left[8 - \frac{1}{2} - 3\right] = \frac{9}{8} \text{sq. units}$$

26. The area of the region bounded by the curve $y = \sqrt{16 - x^2}$ and x-axis is

(a) 8π sq. units (b) 20π sq. units (c) 16π sq. units (d) 256π sq. units

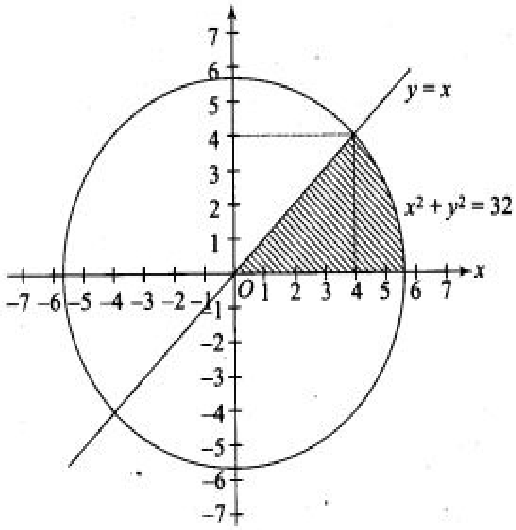
Sol. **(a)** It is similar to q. no. 8.

Solve yourself

27. Area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$ is

(a) 16π sq. units (b) 4π sq. units (c) 32π sq. units (d) 24 sq. units

Sol. **(b)** We have, $y = 0$, $y = x$ and the circle $x^2 + y^2 = 3$ in the first quadrant



Solving $y = x$ with the circle

$$x^2 + x^2 = 32$$

$\Rightarrow \quad x^2 = 16$

$\Rightarrow \quad x = 4$ (In first quadrant)

When $x = 4$, $y = 4$

For point of intersection of circle with the x-axis,

Put $y = 0$

$\therefore \quad x^2 + 0 = 32$

$\Rightarrow \quad x = \pm 4\sqrt{2}$

So, the circle intersects the x-axis at $(\pm 4\sqrt{2}, 0)$

From the figure, area of shaded region

$$A = \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{(4\sqrt{2})^2 - x^2}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x}{2}\sqrt{(4\sqrt{2})^2 - x^2} + \frac{(4\sqrt{2})^2}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_0^{4\sqrt{2}}$$

$$= \frac{16}{2} + \left[0 + 16\sin^{-1}1 - \frac{4}{2}\sqrt{(4\sqrt{2})^2 - 16} - 16\sin^{-1}\frac{4}{4\sqrt{2}}\right]$$

$$= 8 + \left[16 \cdot \frac{\pi}{2} - 2 \cdot \sqrt{16} - 16 \cdot \frac{\pi}{4}\right]$$

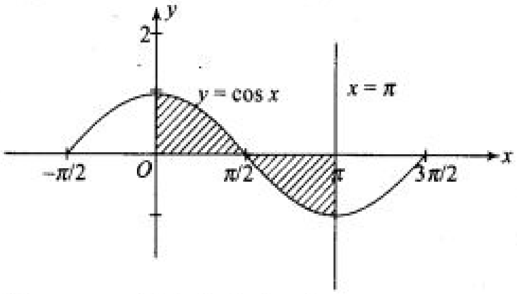
$$= 8 + [8\pi - 8 - 4\pi] = 4\pi \text{ sq. units}$$

28. Area of the region bounded by the curve $y = \cos x$ between $x = 0$ and $x = \pi$ is

(a) 2 sq. units (b) 4 sq. units

(c) 3 sq. units (d) 1 sq. units

Sol. **(a)** We have $y = \cos x$, $x = 0$ and $x = \pi$



From the figure, area of the shaded region,

$$A = \int_0^{\pi} |\cos x| dx = 2\int_0^{\pi/2} \cos x \, dx = 2[\sin x]_0^{\pi/2} = 2 \text{ sq. units}$$

29. The area of the region bounded by parabola $y^2 = x$ and the straight line $2y = x$ is

(a) $\frac{4}{3}$ sq. units (b) 1 sq. units (c) $\frac{2}{3}$ sq. units (d) $\frac{1}{3}$ sq. units

Sol. **(a)** We have $y^2 = x$ and $2y = x$

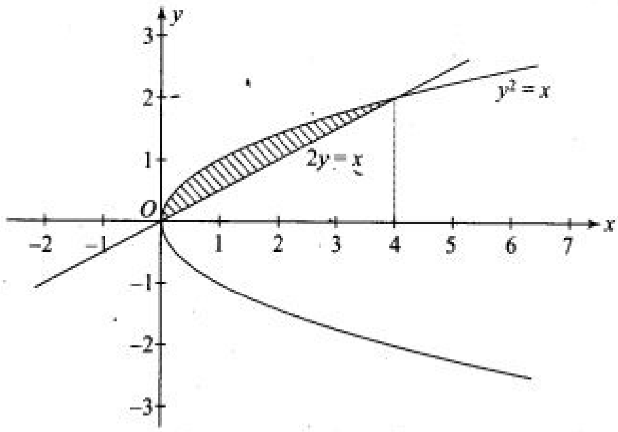
Solving, we get $y^2 = 2y$

$\Rightarrow \quad y = 0, 2$

When $y = 0$, $x = 0$ and when $y = 2$, $x = 4$

So, points of intersection are (0, 0) and (4, 2)

Graphs of parabola $y^2 = x$ and $2y = x$ are as shown in the following figure.



From the figure, area of the shaded region,

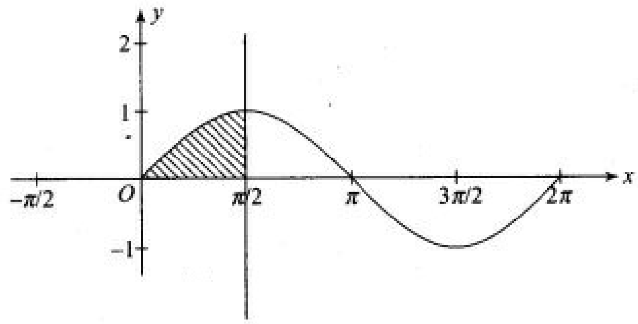
$$A = \int_0^4 \left[\sqrt{x} - \frac{x}{2}\right] dx$$

$$= \left[\frac{2}{3}x^{3/2} - \frac{1}{2}\cdot\frac{x^2}{2}\right]_0^4 = \frac{2}{3}\cdot 4^{3/2} - \frac{16}{4} - 0 = \frac{16}{3} - 4 = \frac{4}{3} \text{ sq. units}$$

30. The area of the region bounded by the curve $y = \sin x$ between the ordinates $x = 0, x = \frac{\pi}{2}$ and the x-axis is

(a) 2 sq. units (b) 4 sq. units (c) 3 sq. units (d) 1 sq. units

Sol. (d) We have $y = \sin x, 0 \le x \le \frac{\pi}{2}$



From the figure, area of the shaded region

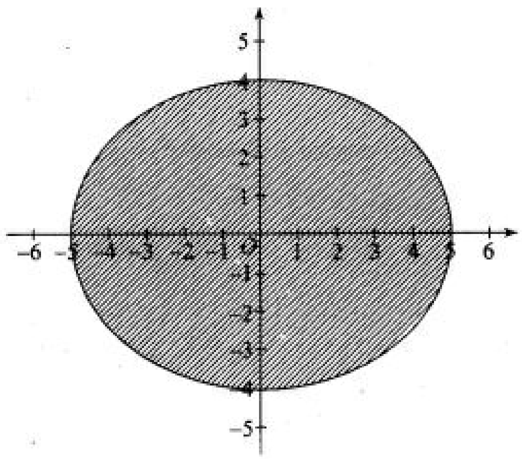
$$A = \int_0^{\pi/2} \sin x dx$$

$$= [-\cos x]_0^{\pi/2} = -\cos\frac{\pi}{2} + \cos 0 = 0 + 1 = 1 \text{ sq. units}$$

31. The area of the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is

(a) 20π sq. units (b) $20\pi^2$ sq. units
(c) $16\pi^2$ sq. units (d) 25π sq. units

Sol. (a) We have, $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$, which is ellipse with its axis as coordinate axis.



$$\frac{y^2}{4^2} = 1 - \frac{x^2}{5^2}$$

$$\Rightarrow \quad y^2 = 16\left(1 - \frac{x^2}{25}\right)$$

$$\Rightarrow \quad y = \frac{4}{5}\sqrt{5^2 - x^2}$$

From the figure, area of the shaded region

$$A = 4\int_0^5 \frac{4}{5}\sqrt{5^2 - x^2}\,dx$$

$$= \frac{16}{5}\left[\frac{x}{2}\sqrt{5^2 - x^2} + \frac{5^2}{2}\sin^{-1}\frac{x}{5}\right]_0^5$$

$$= \frac{16}{5}\left[0 + \frac{5^2}{2}\sin^{-1}1 - 0 - 0\right] = \frac{16}{5}\cdot\frac{25}{2}\cdot\frac{\pi}{2} = 20\pi \text{ sq. units}$$

32. The area of the region bounded by the circle $x^2 + y^2 = 1$ is

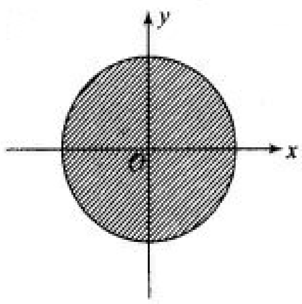
(a) 2π sq. units (b) π sq. units (c) 3π sq. units (d) 4π sq. units

Sol. **(b)** We have, $x^2 + y^2 = 1$, which is circle having centre at (0, 0) and radius '1'

$$\Rightarrow \quad y^2 = 1 - x^2$$

$$\Rightarrow \quad y = \sqrt{1 - x^2}$$

From the figure, area of the shaded region



$$= 4\int_0^1 \sqrt{1^2 - x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{1^2 - x^2} + \frac{1^2}{2}\sin^{-1}\frac{x}{1}\right]_0^1$$

$$= 4\left[0 + \frac{1}{2}\cdot\frac{\pi}{2} - 0 - 0\right]$$

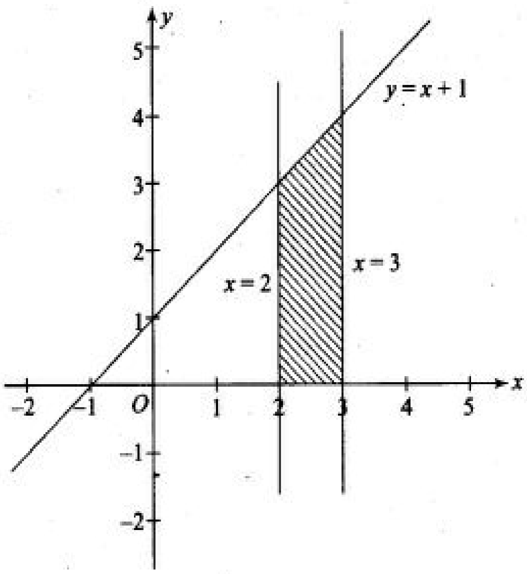
$$= \pi \text{ sq. units}$$

33. The area of the region bounded by the curve $y = x + 1$ and the lines $x = 2$ and $x = 3$ is

(a) $\frac{7}{2}$ sq. units (b) $\frac{9}{2}$ sq. units

(c) $\frac{11}{2}$ sq. units (d) $\frac{13}{2}$ sq. units

Sol. (a)



From the figure, are of the shaded region,

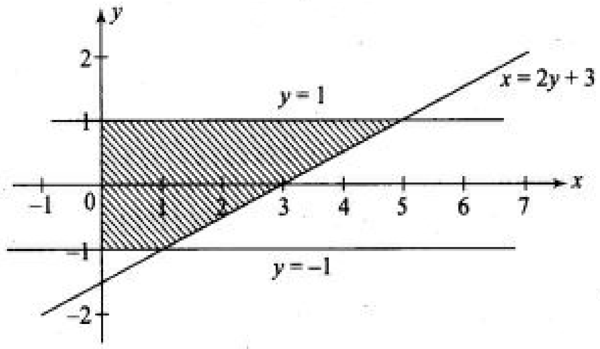
$$A=\int_2^3 (x+1)\,dx=\left[\frac{x^2}{2}+x\right]_2^3=\left[\frac{9}{2}+3-\frac{4}{2}-2\right]=\frac{7}{2}\text{ sq. units}$$

34. The area of the region bounded by the curve $x = 2y + 3$ and the y lines $y = 1$ and $y = -1$ is

(a) 4 sq. units (b) $\frac{3}{2}$ sq. units

(c) 6 sq. units (d) 8 sq. units

Sol. (c)



From the figure, are of the shaded region,

$$A=\int_{-1}^{1}(2y+3)\,dy=[y^2+3y]_{-1}^{1}=[1+3-1+3]=6\text{ sq. units.}$$