Session 2

Transpose of a Matrix, Symmetric Matrix, Orthogonal Matrix, Complex Conjugate (or Conjugate) of a Matrix, Hermitian Matrix, Unitary Matrix, Determinant of a Matrix, Singular and Non-Singular Matrices,

Transpose of a Matrix

Let $A=[a_{ij}]_{m\times n}$ be any given matrix, then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. Transpose of the matrix A is denoted by A' or A^T or A^t. In other words, if $A=[a_{ij}]_{m\times n}$, then $A'=[a_{ji}]_{n\times m}$.

For example,

If $$A=\begin{bmatrix} 2 & 3 & 4 & 5 \\ -2 & -1 & 4 & 8 \\ 7 & 5 & 3 & 1 \end{bmatrix}_{3\times 4},$$

then $$A'=\begin{bmatrix} 2 & -2 & 7 \\ 3 & -1 & 5 \\ 4 & 4 & 3 \\ 5 & 8 & 1 \end{bmatrix}_{4\times 3}$$

Properties of Transpose Matrices

If A' and B' denote the transpose of A and B respectively, then

(i) $(A')'=A$

(ii) $(A\pm B)'=A'\pm B'$; A and B are conformable for matrix addition.

(iii) $(kA)'=kA'$; k is a scalar.

(iv) $(AB)'=B'A'$; A and B are conformable for matrix product AB.

In general, $(A_1A_2A_3\dots A_{n-1}A_n)'=A'_n\ A'_{n-1}\dots A'_3\ A'_2\ A'_1$ (reversal law for transpose).

Remark

$I'=I$, where I is an identity matrix.

Example 19. *If* $A=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, ***find the values of*** θ ***satisfying the equation*** $A^T+A=I_2$.

Sol. We have, $A^T+A=I_2$

$$\Rightarrow \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}+\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2\cos\theta & 0 \\ 0 & 2\cos\theta \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \cos\theta=\frac{1}{2}=\cos\left(\frac{\pi}{3}\right) \Rightarrow \theta=2n\pi\pm\frac{\pi}{3}, n\in I$$

Symmetric Matrix

A square matrix $A=[a_{ij}]_{n\times n}$ is said to be symmetric, if $A'=A$ i.e., $a_{ij}=a_{ji}$, $\forall\ i, j$.

For example,

If $A=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, then $A'=\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

Here, A is symmetric matrix as $A'=A$.

Note

1. Maximum number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.
2. For any square matrix A with real number entries, then $A+A'$ is a symmetric matrix.

Proof $(A+A')'=A'+(A')'=A'+A=A+A'$

Skew-Symmetric Matrix

A square matrix $A=[a_{ij}]_{n\times n}$ is said to be skew-symmetric matrix, if $A'=-A$, i.e. $a_{ij}=-a_{ji}$, $\forall\ i, j$. (the pair of conjugate elements are additive inverse of each other)

Now, if we put $i=j$, we have $a_{ii}=-a_{ii}$.

Therefore, $2a_{ii}=0$ or $a_{ii}=0$, $\forall\ i$'s.

This means that all the diagonal elements of a skew-symmetric matrix are zero, but not the converse.

For example,

If $A=\begin{bmatrix}0 & h & g\\ -h & 0 & f\\ -g & -f & 0\end{bmatrix}$, then

$$A'=\begin{bmatrix}0 & -h & -g\\ h & 0 & -f\\ g & f & 0\end{bmatrix}=-\begin{bmatrix}0 & h & g\\ -h & 0 & f\\ -g & -f & 0\end{bmatrix}=-A$$

Here, A is skew-symmetric matrix as $A'=-A$.

Note

1. Trace of a skew-symmetric matrix is always 0.
2. For any square matrix A with real number entries, then $A-A'$ is a skew-symmetric matrix.

   Proof $(A-A')'=A'-(A')'=A'-A=-(A-A')$
3. Every square matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix.

   i.e. If A is a square matrix, then we can write

   $$A=\frac{1}{2}(A+A')+\frac{1}{2}(A-A')$$

Example 20. ***The square matrix $A=[a_{ij}]_{m\times m}$ given by $a_{ij}=(i-j)^n$, show that A is symmetric and skew-symmetric matrices according as n is even or odd, respectively.***

Sol. $\because a_{ij}=(i-j)^n=(-1)^n(j-i)^n$

$$=(-1)^n a_{ji}=\begin{cases}a_{ji}, & n \text{ is even integer}\\ -a_{ji}, & n \text{ is odd integer}\end{cases}$$

Hence, A is symmetric if n is even and skew-symmetric if n is odd integer.

Example 21. Express A as the sum of a symmetric and a skew-symmetric matrix, where $A=\begin{bmatrix}3 & 5\\ -1 & 2\end{bmatrix}$.

Sol. We have,

$$A=\begin{bmatrix}3 & 5\\ -1 & 2\end{bmatrix}, \text{ then } A'=\begin{bmatrix}3 & -1\\ 5 & 2\end{bmatrix}$$

Let $P=\frac{1}{2}(A+A')=\frac{1}{2}\begin{bmatrix}6 & 4\\ 4 & 4\end{bmatrix}=\begin{bmatrix}3 & 2\\ 2 & 2\end{bmatrix}=P'$

Thus, $P=\frac{1}{2}(A+A')$ is a symmetric matrix.

Also, let $Q=\frac{1}{2}(A-A')=\frac{1}{2}\begin{bmatrix}0 & 6\\ -6 & 0\end{bmatrix}=\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}$

Then, $Q'=\begin{bmatrix}0 & -3\\ 3 & 0\end{bmatrix}=-\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}=-Q$

Thus, $Q=\frac{1}{2}(A-A')$ is a skew-symmetric matrix.

Now, $P+Q=\begin{bmatrix}3 & 2\\ 2 & 2\end{bmatrix}+\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}=\begin{bmatrix}3 & 5\\ -1 & 2\end{bmatrix}=A$

Hence, A is represented as the sum of a symmetric and a skew-symmetric matrix.

Properties of Symmetric and Skew-Symmetric Matrices

(i) If A be a square matrix, then AA' and $A'A$ are symmetric matrices.

(ii) All positive integral powers of a symmetric matrix are symmetric, because

$$(A^n)'=(A')^n$$

(iii) All positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric, because $(A^n)'=(A')^n$

(iv) If A be a symmetric matrix and B be a square matrix of order that of A, then $-A$, kA, A', A^{-1}, A^n and $B'AB$ are also symmetric matrices, where $n\in N$ and k is a scalar.

(v) If A be a skew-symmetric matrix, then

(a) A^{2n} is a symmetric matrix for $n\in N$.

(b) A^{2n+1} is a skew-symmetric matrix for $n\in N$.

(c) kA is a skew-symmetric matrix, where k is scalar.

(d) $B'AB$ is also skew-symmetric matrix, where B is a square matrix of order that of A.

(vi) If A and B are two symmetric matrices, then

(a) $A\pm B$, $AB+BA$ are symmetric matrices.

(b) $AB-BA$ is a skew-symmetric matrix

(c) AB is a symmetric matrix, iff $AB=BA$

(where A and B are square matrices of same order)

(vii) If A and B are two skew-symmetric matrices, then

(a) $A\pm B$, $AB-BA$ are skew-symmetric matrices.

(b) $AB+BA$ is a symmetric matrix.

(where A and B are square matrices of same order)

(viii) If A be a skew-symmetric matrix and C is a column matrix, then $C'AC$ is a zero matrix, where $C'AC$ is conformable.

Orthogonal Matrix

A square matrix A is said to be orthogonal matrix, iff $AA'=I$, where I is an identity matrix.

Note

1. If $AA'=I$, then $A^{-1}=A$.
2. If A and B are orthogonal, then AB is also orthogonal.
3. If A is orthogonal, then A^{-1} and A' are also orthogonal.

Example 22. If $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal, then find the value of $2\alpha^2 + 6\beta^2 + 3\gamma^2$.

Sol. Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$, then $A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$

Since, A is orthogonal.

$\therefore \quad AA' = I$

$$\Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we get

$$4\beta^2 + \gamma^2 = 1 \quad \text{...(i)}$$

$$2\beta^2 - \gamma^2 = 0 \quad \text{...(ii)}$$

and $\quad \alpha^2 + \beta^2 + \gamma^2 = 1 \quad$ ...(iii)

From Eqs. (i) and (ii), we get

$$\beta^2 = \frac{1}{6} \text{ and } \gamma^2 = \frac{1}{3}$$

From Eq. (iii),

$$\alpha^2 = 1 - \beta^2 - \gamma^2 = 1 - \frac{1}{6} - \frac{1}{3} = \frac{1}{2}$$

Hence, $2\alpha^2 + 6\beta^2 + 3\gamma^2 = 2 \times \frac{1}{2} + 6 \times \frac{1}{6} + 3 \times \frac{1}{3} = 3$

Aliter

The rows of matrix A are unit orthogonal vectors

$$\vec{R}_1 \cdot \vec{R}_2 = 0 \Rightarrow 2\beta^2 - \gamma^2 = 0 \Rightarrow 2\beta^2 = \gamma^2 \quad \text{...(i)}$$

$$\vec{R}_2 \cdot \vec{R}_3 = 0 \Rightarrow \alpha^2 - \beta^2 - \gamma^2 = 0 \Rightarrow \beta^2 + \gamma^2 = \alpha^2 \quad \text{...(ii)}$$

and $\vec{R}_3 \cdot \vec{R}_3 = 1 \Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 1 \quad$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$\alpha^2 = \frac{1}{2}, \beta^2 = \frac{1}{6} \text{ and } \gamma^2 = \frac{1}{3}$$

$\therefore \quad 2\alpha^2 + 6\beta^2 + 3\gamma^2 = 3$

Example 23. ***If*** $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ ***is a matrix satisfying*** $AA' = 9I_3$***, find the value of*** $|a| + |b|$.

Sol. Since, $AA' = 9I_3$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & 0 & a + 2b + 4 \\ 0 & 9 & 2a - 2b + 2 \\ a + 2b + 4 & 2a - 2b + 2 & a^2 + b^2 + 4 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

Equating the corresponding elements, we get

$$a + 2b + 4 = 0 \quad \text{...(i)}$$

$$2a - 2b + 2 = 0 \quad \text{...(ii)}$$

and $\quad a^2 + b^2 + 4 = 9 \quad$ ...(iii)

From Eqs. (i) and (ii), we get

$$a = -2 \text{ and } b = -1$$

Hence, $|a| + |b| = |-2| + |-1| = 2 + 1 = 3$

Complex Conjugate (or Conjugate) of a Matrix

If a matrix A is having complex numbers as its elements, the matrix obtained from A by replacing each element of A by its conjugate ($\overline{a \pm ib} = a \mp ib$, where $i = \sqrt{-1}$) is called the conjugate of matrix A and is denoted by $\overline{A}$.

For example, If $A = \begin{bmatrix} 2 + 5i & 3 - i & 7 \\ -2i & 6 + i & 7 - 5i \\ 1 - i & 3 & 6i \end{bmatrix}$, where $i = \sqrt{-1}$,

then $\quad \overline{A} = \begin{bmatrix} 2 - 5i & 3 + i & 7 \\ 2i & 6 - i & 7 + 5i \\ 1 + i & 3 & -6i \end{bmatrix}$

Note

If all elements of A are real, then $\overline{A} = A$.

Properties of Complex Conjugate of a Matrix

If A and B are two matrices of same order, then

(i) $(\overline{\overline{A}}) = A$

(ii) $(\overline{A + B}) = \overline{A} + \overline{B}$, where A and B being conformable for addition.

(iii) $(\overline{kA}) = k\,\overline{A}$, where k is real.

(iv) $(\overline{AB}) = \overline{A}\,\overline{B}$, where A and B being conformable for multiplication.

Conjugate Transpose of a Matrix

The conjugate of the transpose of a matrix A is called the conjugate transpose of A and is denoted by A^θ i.e.
$A^\theta =$ Conjugate of $A' = (\overline{A'})$

For example,

If $A = \begin{bmatrix} 2+4i & 3 & 5-9i \\ 4 & 5+2i & 3i \\ 2 & -5 & 4-i \end{bmatrix}$,

where $i = \sqrt{-1}$,

then $A^\theta = (\overline{A'}) = \begin{bmatrix} 2-4i & 4 & 2 \\ 3 & 5-2i & -5 \\ 5+9i & -3i & 4+i \end{bmatrix}$

Properties of Transpose Conjugate Matrix

If A and B are two matrices of same order, then

(i) $(\overline{A})' = (\overline{A'})$ (ii) $(A^\theta)^\theta = A$

(iii) $(A+B)^\theta = A^\theta + B^\theta$, where A and B being conformable for addition.

(iv) $(kA)^\theta = k\,A^\theta$, where k is real.

(v) $(AB)^\theta = B^\theta A^\theta$, where A and B being conformable for multiplication

Hermitian Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be hermitian, if $A^\theta = A$ i.e., $a_{ij} = \overline{a}_{ji}$, $\forall\, i, j$. If we put $j = i$, we have $a_{ii} = \overline{a}_{ii}$
$\Rightarrow a_{ii}$ is purely real for all i's.

This means that all the diagonal elements of a hermitian matrix must be purely real.

For example,

If $A = \begin{bmatrix} \alpha & \lambda + i\mu & \theta + i\phi \\ \lambda - i\mu & \beta & x+iy \\ \theta - i\phi & x-iy & \gamma \end{bmatrix}$

where $\alpha, \beta, \gamma, \lambda, \mu, \theta, \phi, x, y \in R$ and $i = \sqrt{-1}$, then

$$A' = \begin{bmatrix} \alpha & \lambda - i\mu & \theta - i\phi \\ \lambda + i\mu & \beta & x-iy \\ \theta + i\phi & x+iy & \gamma \end{bmatrix}$$

$$\therefore \quad A^\theta = (\overline{A'}) = \begin{bmatrix} \alpha & \lambda + i\mu & \theta + i\phi \\ \lambda - i\mu & \beta & x+iy \\ \theta - i\phi & x-iy & \gamma \end{bmatrix} = A$$

Here, A is hermitian matrix as $A^\theta = A$.

Note

For any square matrix A with complex number entries, then $A + A^\theta$ is a Hermitian matrix.

Proof $(A + A^\theta)^\theta = A^\theta + (A^\theta)^\theta = A^\theta + A = A + A^\theta$

Skew-Hermitian Matrix

A square matrix $A = [a_{ij}]_{n \times n}$ is said to be skew-hermitian matrix. If $A^\theta = -A$, i.e. $a_{ij} = -\overline{a}_{ij}$, $\forall\, i, j$. If we put $j = i$, we have $a_{ii} = -\overline{a}_{ii} \Rightarrow a_{ii} + \overline{a}_{ii} = 0 \Rightarrow a_{ii}$ is purely imaginary for all i's. This means that all the diagonal elements of a skew-hermitian matrix must be purely imaginary or zero.

For example,

If $A = \begin{bmatrix} 2i & -2-3i & -2+i \\ 2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix}$, where $i = \sqrt{-1}$,

then $A' = \begin{bmatrix} 2i & 2-3i & 2+i \\ -2-3i & -i & 3i \\ -2+i & 3i & 0 \end{bmatrix}$

$$\therefore \quad A^\theta = (\overline{A'}) = \begin{bmatrix} -2i & 2+3i & 2-i \\ -2+3i & i & -3i \\ -2-i & -3i & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 2i & -2-3i & -2+i \\ 2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix} = -A$$

Hence, A is skew-hermitian matrix.

Note

1. For any square matrix A with complex number entries, then $A - A^\theta$ is a skew-hermitian matrix.
   Proof $(A - A^\theta)^\theta = (A^\theta) - (A^\theta)^\theta = A^\theta - A = -(A - A^\theta)$
2. Every square matrix (with complex elements) can be uniquely expressed as the sum of a hermitian and a skew-hermitian matrix i.e.
   If A is a square matrix, then we can write
   $$A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$$

Example 24. ***Express*** A ***as the sum of a hermitian and a skew-hermitian matrix, where***
$A = \begin{bmatrix} 2+3i & 7 \\ 1-i & 2i \end{bmatrix}$, $i = \sqrt{-1}$.

Sol. We have, $A = \begin{bmatrix} 2+3i & 7 \\ 1-i & 2i \end{bmatrix}$, then $A^\theta = (\overline{A'}) = \begin{bmatrix} 2-3i & 1+i \\ 7 & -2i \end{bmatrix}$

Let $P = \frac{1}{2}(A + A^\theta) = \frac{1}{2}\begin{bmatrix} 4 & 8+i \\ 8-i & 0 \end{bmatrix} = \begin{bmatrix} 2 & 4 + \frac{i}{2} \\ 4 - \frac{i}{2} & 0 \end{bmatrix} = P^\theta$

Thus, $P = \frac{1}{2}(A + A^\theta)$ is a hermitian matrix.

Also, let $Q = \frac{1}{2}(A - A^\theta) = \frac{1}{2}\begin{bmatrix} 6i & 6-i \\ -6-i & 4i \end{bmatrix}$

$$= \begin{bmatrix} 3i & 3-\frac{i}{2} \\ -3-\frac{i}{2} & 2i \end{bmatrix} = -\begin{bmatrix} -3i & -3+\frac{i}{2} \\ 3+\frac{i}{2} & -2i \end{bmatrix} = -Q^\theta$$

Thus, $Q = \frac{1}{2}(A - A^\theta)$ is a skew-hermitian matrix.

Now, $P + Q = \begin{bmatrix} 2 & 4+\frac{i}{2} \\ 4-\frac{i}{2} & 0 \end{bmatrix} + \begin{bmatrix} 3i & 3-\frac{i}{2} \\ -3-\frac{i}{2} & 2i \end{bmatrix}$

$$= \begin{bmatrix} 2+3i & 7 \\ 1-i & 2i \end{bmatrix} = A$$

Hence, A is represented as the sum of a hermitian and a skew-hermitian matrix.

Properties of Hermitian and Skew-Hermitian Matrices

(i) If A be a square matrix, then AA^θ and $A^\theta A$ are hermitian matrices.

(ii) If A is a hermitian matrix, then
 (a) iA is skew-hermitian matrix, where $i = \sqrt{-1}$.
 (b) iff $\overline{A}$ is hermitian matrix.
 (c) kA is hermitian matrix, where $k \in R$.

(iii) If A is a skew-hermitian matrix, then
 (a) iA is hermitian matrix, where $i = \sqrt{-1}$.
 (b) iff $\overline{A}$ is skew-hermitian matrix.
 (c) kA is skew-hermitian matrix, where $k \in R$.

(iv) If A and B are hermitian matrices of same order, then
 (a) $k_1 A + k_2 B$ is also hermitian, where $k_1, k_2 \in R$.
 (b) AB is also hermitian, if $AB = BA$.
 (c) $AB + BA$ is a hermitian matrix.
 (d) $AB - BA$ is a skew-hermitian matrix.

(v) If A and B are skew-hermitian matrices of same order, then $k_1 A + k_2 B$ is also skew-hermitian matrix.

Unitary Matrix

A square matrix A is said to be unitary matrix iff $AA^\theta = I$, where I is an identity matrix.

Note

1. If $AA^\theta = I$, then $A^{-1} = A^\theta$
2. If A and B are unitary, then AB is also unitary.
3. If A is unitary, then A^{-1} and A' are also unitary.

Example 25. Verify that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary, where $i = \sqrt{-1}$.

Sol. Let $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$, then

$$A^\theta = (\overline{A'}) = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\therefore \quad AA^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, A is unitary matrix.

Determinant of a Matrix

Let A be a square matrix, then the determinant formed by the elements of A without changing their respective positions is called the determinant of A and is denoted by det A or $|A|$.

i.e., If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, then $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Properties of the Determinant of a Matrix

If A and B are square matrices of same order, then

(i) $|A|$ exists $\Leftrightarrow A$ is a square matrix.

(ii) $|A'| = |A|$

(iii) $|AB| = |A||B|$ and $|AB| = |BA|$

(iv) If A is orthogonal matrix, then $|A| = \pm 1$

(v) If A is skew-symmetric matrix of odd order, then $|A| = 0$

(vi) If A is skew-symmetric matrix of even order, then $|A|$ is a perfect square.

(vii) $|kA| = k^n |A|$, where n is order of A and k is scalar.

(viii) $|A^n| = |A|^n$, where $n \in N$

(ix) If $A = \text{diag}(a_1, a_2, a_3, \ldots, a_n)$, then $|A| = a_1 \cdot a_2 \cdot a_3 \ldots a_n$

Example 26. If A, B and C are square matrices of order n and $\det(A) = 2$, $\det(B) = 3$ and $\det(C) = 5$, then find the value of $10\det(A^3 B^2 C^{-1})$.

Sol. Given, $|A| = 2$, $|B| = 3$ and $|C| = 5$.

Now, $10\det(A^3 B^2 C^{-1}) = 10 \times |A^3 B^2 C^{-1}|$

$$= 10 \times |A^3| \times |B^2| \times |C^{-1}| = 10 \times |A|^3 \times |B|^2 \times |C|^{-1}$$

$$= \frac{10 \times |A|^3 \times |B^2|}{|C|} = \frac{10 \times 2^3 \times 3^2}{5} = 144$$

Example 27. ***If*** $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, $abc = 1$, $A^T A = I$, ***then find the value of*** $a^3 + b^3 + c^3$.

Sol. $\because \quad A^T A = I$

$\Rightarrow \quad |A^T A| = |I| \quad \Rightarrow \quad |A^T||A| = 1$

$\Rightarrow \quad |A||A| = 1 \qquad [\because |A^T| = |A|]$

$\Rightarrow \quad |A| = \pm 1$

$$\Rightarrow \quad \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \pm 1$$

$\Rightarrow \quad 3abc - (a^3 + b^3 + c^3) = \pm 1$

or $\quad 3 - (a^3 + b^3 + c^3) = \pm 1 \qquad [\because abc = 1]$

or $\quad a^3 + b^3 + c^3 = 3 \pm 1 = 2$ or 4

Singular and Non-Singular Matrices

A square matrix A is said to be a singular, if $|A| = 0$ and a square matrix A is said to be non-singular, if $|A| \neq 0$.

For example,

(i) $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 4 & 6 \end{bmatrix}$ is a singular matrix.

Since, $|A| = 0$.

(ii) $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ is a non-singular matrix.

Since, $|A| = 10 - 12 = -2 \neq 0$

Example 28. If $\omega \neq 1$ is a complex cube root of unity, then prove that

$$\begin{bmatrix} 1 + 2\omega^{2017} + \omega^{2018} & \omega^{2018} & 1 \\ 1 & 1 + \omega^{2018} + 2\omega^{2017} & \omega^{2017} \\ \omega^{2017} & \omega^{2018} & 2 + \omega^{2017} + 2\omega^{2018} \end{bmatrix}$$ ***is singular.***

Sol. Let $A = \begin{bmatrix} 1 + 2\omega^{2017} + \omega^{2018} & \omega^{2018} & 1 \\ 1 & 1 + \omega^{2018} + 2\omega^{2017} & \omega^{2017} \\ \omega^{2017} & \omega^{2018} & 2 + \omega^{2017} + 2\omega^{2018} \end{bmatrix}$

$\because \quad \omega^3 = 1 \Rightarrow \omega^{2017} = \omega$

and $\quad \omega^{2018} = \omega^2$, then

$$A = \begin{bmatrix} 1 + 2\omega + \omega^2 & \omega^2 & 1 \\ 1 & 1 + \omega^2 + 2\omega & \omega \\ \omega & \omega^2 & 2 + \omega + 2\omega^2 \end{bmatrix}$$

$$= \begin{bmatrix} \omega & \omega^2 & 1 \\ 1 & \omega & \omega \\ \omega & \omega^2 & -\omega \end{bmatrix} \qquad [\because 1 + \omega + \omega^2 = 0]$$

Now, $|A| = \begin{vmatrix} \omega & \omega^2 & 1 \\ 1 & \omega & \omega \\ \omega & \omega^2 & -\omega \end{vmatrix} = \omega \begin{vmatrix} \omega & \omega & 1 \\ 1 & 1 & \omega \\ \omega & \omega & -\omega \end{vmatrix} = 0$

$[\because C_1 = C_2]$

Thus, $|A| = 0$.

Hence, A is singular matrix.

Example 29. Find the real values of x for which the matrix $\begin{bmatrix} x+1 & 3 & 5 \\ 1 & x+3 & 5 \\ 1 & 3 & x+5 \end{bmatrix}$ is non-singular.

Sol. Let $A = \begin{bmatrix} x+1 & 3 & 5 \\ 1 & x+3 & 5 \\ 1 & 3 & x+5 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} x+1 & 3 & 5 \\ 1 & x+3 & 5 \\ 1 & 3 & x+5 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, then

$$|A| = \begin{vmatrix} x+9 & 3 & 5 \\ x+9 & x+3 & 5 \\ x+9 & 3 & x+5 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, then

$$|A| = \begin{vmatrix} x+9 & 3 & 5 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix} = x^2(x+9)$$

$\because \quad A$ is non-singular.

$\therefore \quad |A| \neq 0 \quad \Rightarrow \quad x^2(x+9) \neq 0$

$\therefore \quad x \neq 0, -9$

Hence, $x \in R - \{0, -9\}$.

Exercise for Session 2

1 If $A = \begin{bmatrix} 4 & x+2 \\ 2x-3 & x+1 \end{bmatrix}$ is symmetric, then x is equal to

(a) 2 (b) 3
(c) 4 (d) 5

2 If A and B are symmetric matrices, then ABA is

(a) symmetric matrix (b) skew-symmetric matrix
(c) diagonal matrix (d) scalar matrix

3 If A and B are symmetric matrices of the same order and $P = AB + BA$ and $Q = AB - BA$, then $(PQ)'$ is equal to

(a) PQ (b) QP
(c) $-QP$ (d) None of these

4 If A is a skew-symmetric matrix and n is odd positive integer, then A^n is

(a) a skew-symmetric matrix (b) a symmetric matrix
(c) a diagonal matrix (d) None of these

5 If A is symmetric as well as skew-symmetric matrix, then A is

(a) diagonal matrix (b) null matrix
(c) triangular matrix (d) None of these

6 If A is square matrix order 3, then $|(A - A')^{2015}|$ is

(a) $|A|$ (b) $|A'|$
(c) 0 (d) None of these

7 The maximum number of different elements required to form a symmetric matrix of order 6 is

(a) 15 (b) 17
(c) 19 (d) 21

8 If A and B are square matrices of order 3×3 such that A is an orthogonal matrix and B is a skew-symmetric matrix, then which of the following statement is true?

(a) $|AB| = 1$ (b) $|AB| = 0$
(c) $|AB| = -1$ (d) None of these

9 The matrix $A = \begin{bmatrix} i & 1-2i \\ -1-2i & 0 \end{bmatrix}$, where $i = \sqrt{-1}$, is

(a) symmetric (b) skew-symmetric
(c) hermitian (d) skew-hermitian

10 If A and B are square matrices of same order such that $A^* = A$ and $B^* = B$, where A^* denotes the conjugate transpose of A, then $(AB - BA)^*$ is equal to

(a) null matrix (b) $AB - BA$
(c) $BA - AB$ (d) None of these

11 If matrix $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & a \end{bmatrix}$, $i = \sqrt{-1}$ is unitary matrix, a is equal to

(a) 2 (b) -1
(c) 0 (d) 1

12 If A is a 3×3 matrix and $\det(3A) = k\{\det(A)\}$, k is equal to

(a) 9 (b) 6
(c) 1 (d) 27

13 If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$, then $|3AB|$ is equal to

(a) -9 (b) -81
(c) -27 (d) 81

14 If A is a square matrix such that $A^2 = A$, then $\det(A)$ is equal to

(a) 0 or 1
(b) -2 or 2
(c) -3 or 3
(d) None of these

15 If I is a unit matrix of order 10, the determinant of I is equal to

(a) 10
(b) 1
(c) $\frac{1}{10}$
(d) 9

16 If $A_i = \begin{bmatrix} 2^{-i} & 3^{-i} \\ 3^{-i} & 2^{-i} \end{bmatrix}$, then $\sum_{i=1}^{\infty} \det(A_i)$ is equal to

(a) $\frac{3}{4}$
(b) $\frac{5}{24}$
(c) $\frac{5}{4}$
(d) $\frac{7}{144}$

17 The number of values of x for which the matrix $A = \begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix}$ is singular, is

(a) 0
(b) 1
(c) 2
(d) 3

18 The number of values of x in the closed interval $[-4, -1]$, the matrix $\begin{bmatrix} 3 & -1+x & 2 \\ 3 & -1 & x+2 \\ x+3 & -1 & 2 \end{bmatrix}$ is singular, is

(a) 0
(b) 1
(c) 2
(d) 3

19 The values of x for which the given matrix $\begin{bmatrix} -x & x & 2 \\ 2 & x & -x \\ x & -2 & -x \end{bmatrix}$ will be non-singular are

(a) $-2 \le x \le 2$
(b) for all x other than 2 and -2
(c) $x \ge 2$
(d) $x \le -2$

Answers

Exercise for Session 2

1. (d)	2. (a)	3. (c)	4. (a)	5. (b)	6. (c)
7. (d)	8. (b)	9. (d)	10. (c)	11. (b)	12. (d)
13. (b)	14. (a)	15. (b)	16. (b)	17. (c)	18. (b)
19. (b)					