Session 4

Equations of Higher Degree, Rational Algebraic Inequalities, Roots of Equation with the Help of Graphs,

Equations of Higher Degree

The equation $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1} x + a_n = 0$, where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are constants but $a_0 \neq 0$, is a polynomial equation of degree n. It has n and only n roots.

Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_n$ be n roots, then

- $\Sigma \alpha_1 = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_{n-1} + \alpha_n = (-1)^1 \dfrac{a_1}{a_0}$ **[sum of all roots]**
- $\Sigma \alpha_1 \alpha_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \ldots + \alpha_1\alpha_n + \alpha_2\alpha_3 + \ldots + \alpha_2\alpha_n + \ldots + \alpha_{n-1}\alpha_n$ $= (-1)^2 \dfrac{a_2}{a_0}$ [sum of products taken two at a time]
- $\Sigma \alpha_1 \alpha_2 \alpha_3 = (-1)^3 \dfrac{a_3}{a_0}$ **[sum of products taken three at a time]**
- $\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n = (-1)^n \dfrac{a_n}{a_0}$ **[product of all roots]**

In general, $\Sigma \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_p = (-1)^p \dfrac{a_p}{a_0}$

Remark

1. A polynomial equation of degree n has n roots (real or imaginary).
2. If all the coefficients, i.e., $a_0, a_1, a_2, \ldots, a_n$ are real, then the imaginary roots occur in pairs, i.e. number of imaginary roots is always even.
3. If the degree of a polynomial equation is odd, then atleast one of the roots will be real.
4. $(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$ $= x^n + (-1)^1 \Sigma\alpha_1 \cdot x^{n-1} + (-1)^2 \Sigma \alpha_1 \alpha_2 \cdot x^{n-2} + \ldots + (-1)^n \alpha_1 \alpha_2 \alpha_3 \ldots \alpha_n$

In Particular

(i) For $n = 3$, if α, β, γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$, where a, b, c, d are constants and $a \neq 0$, then $\Sigma\alpha = \alpha + \beta + \gamma = (-1)^1 \dfrac{b}{a} = -\dfrac{b}{a}$,

$$\Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = (-1)^2 \frac{c}{a} = \frac{c}{a}$$

and $\alpha\beta\gamma = (-1)^3 \dfrac{d}{a} = -\dfrac{d}{a}$

or $ax^3 + bx^2 + cx + d = a(x-\alpha)(x-\beta)(x-\gamma)$

$= a(x^3 - \Sigma\alpha \cdot x^2 + \Sigma\alpha\beta \cdot x - \alpha\beta\gamma)]$

(ii) For $n = 4$, if $\alpha, \beta, \gamma, \delta$ are the roots of the equation $ax^4 + bx^3 + cx^2 + dx + e = 0$, where a, b, c, d, e are constants and $a \neq 0$, then

$$\Sigma\alpha = \alpha + \beta + \gamma + \delta = (-1)^1 \frac{b}{a} = -\frac{b}{a},$$

$$\Sigma\alpha\beta = (\alpha+\beta)(\gamma+\delta) + \alpha\beta + \gamma\delta = (-1)^2 \frac{c}{a} = \frac{c}{a},$$

$$\Sigma\alpha\beta\gamma = \alpha\beta(\gamma+\delta) + \gamma\delta(\alpha+\beta) = (-1)^3 \frac{d}{a} = -\frac{d}{a}$$

and $\alpha\beta\gamma\delta = (-1)^4 \dfrac{e}{a} = \dfrac{e}{a}$

or $ax^4 + bx^3 + cx^2 + dx + e = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$= a(x^4 - \Sigma\alpha \cdot x^3 + \Sigma\alpha\beta \cdot x^2 - \Sigma\alpha\beta\gamma \cdot x + \alpha\beta\gamma\delta)$

Example 42. Find the conditions, if roots of the equation $x^3 - px^2 + qx - r = 0$ are in

(i) AP (ii) GP (iii) HP

Sol. (i) Let roots of the given equation are $A - D, A, A + D$, then

$$A - D + A + A + D = p \Rightarrow A = \frac{p}{3}$$

Now, A is the roots of the given equation, then it must be satisfy

$$A^3 - pA^2 + qA - r = 0$$

$$\Rightarrow \left(\frac{p}{3}\right)^3 - p\left(\frac{p}{3}\right)^2 + q\left(\frac{p}{3}\right) - r = 0$$

$$\Rightarrow p^3 - 3p^3 + 9qp - 27r = 0$$

or $2p^3 - 9pq + 27r = 0$,

which is the required condition.

(ii) Let roots of the given equation are $\frac{A}{R}$, A, AR, then

$$\frac{A}{R}\cdot A\cdot AR=(-1)^3\cdot\left(-\frac{r}{1}\right)=r$$

$$\Rightarrow \quad A^3=r$$

$$\Rightarrow \quad A=r^{\frac{1}{3}}$$

Now, A is the roots of the given equation, then

$$A^3-pA^2+qA-r=0$$

$$\Rightarrow \quad r-p(r)^{2/3}=q(r)^{1/3}-r=0$$

or $\quad p(r)^{2/3}=q(r)^{1/3}$

or $\quad p^3r^2=q^3r$

or $\quad p^3r=q^3$

which is the required condition.

(iii) Given equation is

$$x^3-px^2+qx-r=0 \quad \text{...(i)}$$

On replacing x by $\frac{1}{x}$ in Eq. (i), then

$$\left(\frac{1}{x}\right)^3-p\left(\frac{1}{x}\right)^2+q\left(\frac{1}{x}\right)-r=0$$

$$\Rightarrow \quad rx^3-qx^2+px-1=0 \quad \text{...(ii)}$$

Now, roots of Eq. (ii) are in AP.

Let roots of Eq. (ii) are $A-P$, A, $A+P$, then

$$A-P+A+A+P=\frac{q}{r} \quad \text{or} \quad A=\frac{q}{3r}$$

$\because$ A is a root of Eq. (ii), then

$$rA^3-qA^2+pA-1=0$$

$$\Rightarrow \quad r\left(\frac{q}{3r}\right)^3-q\left(\frac{q}{3r}\right)^2+p\left(\frac{q}{3r}\right)-1=0$$

$$\Rightarrow \quad q^3-3q^3+9pqr-27r^2=0$$

$$\Rightarrow \quad 2q^3-9pqr+27r^2=0,$$

which is the required condition.

Example 43. Solve $6x^3-11x^2+6x-1=0$, if roots of the equation are in HP.

Sol. Put $x=\frac{1}{y}$ in the given equation, then

$$\frac{6}{y^3}-\frac{11}{y^2}+\frac{6}{y}-1=0$$

$$\Rightarrow \quad y^3-6y^2+11y-6=0 \quad \text{...(i)}$$

Now, roots of Eq. (i) are in AP.

Let the roots be $\alpha-\beta, \alpha, \alpha+\beta$.

Then, sum of roots $=\alpha-\beta+\alpha+\alpha+\beta=6$

$$\Rightarrow \quad 3\alpha=6$$

$$\therefore \quad \alpha=2$$

Product of roots $=(\alpha-\beta)\cdot\alpha\cdot(\alpha+\beta)=6$

$$\Rightarrow \quad (2-\beta)2(2+\beta)=6 \Rightarrow 4-\beta^2=3$$

$$\therefore \quad \beta=\pm 1$$

$\therefore$ Roots of Eqs. (i) are 1, 2, 3 or 3, 2, 1.

Hence, roots of the given equation are $1, \frac{1}{2}, \frac{1}{3}$ or $\frac{1}{3}, \frac{1}{2}, 1$.

Example 44. If α, β, γ are the roots of the equation $x^3-px^2+qx-r=0$, find

(i) $\Sigma\alpha^2$. (ii) $\Sigma\alpha^2\beta$. (iii) $\Sigma\alpha^3$.

Sol. Since, α, β, γ are the roots of $x^3-px^2+qx-r=0$.

$$\therefore \quad \Sigma\alpha=p, \Sigma\alpha\beta=q \text{ and } \alpha\beta\gamma=r$$

(i) $\because \Sigma\alpha\cdot\Sigma\alpha=p\cdot p$

$$\Rightarrow \quad (\alpha+\beta+\gamma)(\alpha+\beta+\gamma)=p^2$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)=p^2$$

or $\quad \Sigma\alpha^2+2\Sigma\alpha\beta=p^2$

or $\quad \Sigma\alpha^2=p^2-2q$

(ii) $\because \Sigma\alpha\cdot\Sigma\alpha\beta=p\cdot q$

$$\Rightarrow \quad (\alpha+\beta+\gamma)\cdot(\alpha\beta+\beta\gamma+\gamma\alpha)=pq$$

$$\Rightarrow \alpha^2\beta+\alpha\beta\gamma+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\alpha\beta\gamma + \gamma^2\beta+\gamma^2\alpha=pq$$

$$\Rightarrow (\alpha^2\beta+\alpha^2\alpha+\beta^2\gamma+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta) + 3\alpha\beta\gamma=pq$$

or $\quad \Sigma\alpha^2\beta+3r=pq$

or $\quad \Sigma\alpha^2\beta=pq-3r$

(iii) $\because \Sigma\alpha^2\cdot\Sigma\alpha=(p^2-2q)\cdot p$ [from result (i)]

$$\Rightarrow \quad (\alpha^2+\beta^2+\gamma^2)(\alpha+\beta+\gamma)=p^3-2pq$$

$$\Rightarrow \quad \alpha^3+\beta^3+\gamma^3+(\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma + \gamma^2\alpha+\gamma^2\beta)=p^3-2pq$$

$$\Rightarrow \quad \Sigma\alpha^3+\Sigma\alpha^2\beta=p^3-2pq$$

$$\Rightarrow \quad \Sigma\alpha^3+pq-3r=p^3-2pq \quad \text{[from result (ii)]}$$

or $\quad \Sigma\alpha^3=p^3-3pq+3r$

Example 45. If α, β, γ are the roots of the cubic equation $x^3+qx+r=0$, then find the equation whose roots are $(\alpha-\beta)^2, (\beta-\gamma)^2, (\gamma-\alpha)^2$.

Sol. $\because \alpha, \beta, \gamma$ are the roots of the cubic equation

$$x^3+qx+r=0 \quad \text{...(i)}$$

Then, $\quad \Sigma\alpha=0, \Sigma\alpha\beta=q, \alpha\beta\gamma=-r \quad$ ...(ii)

If y is a root of the required equation, then

$$y=(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$$

$$=(\alpha+\beta+\gamma-\gamma)^2-\frac{4\alpha\beta\gamma}{\gamma}$$

$$= (0-\gamma)^2 + \frac{4r}{\gamma} \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow \quad y = \gamma^2 + \frac{4r}{\gamma}$$

[replacing γ by x which is a root of Eq. (i)]

$$\therefore \quad y = x^2 + \frac{4r}{x}$$

$$\text{or} \quad x^3 - yx + 4r = 0 \quad \text{...(iii)}$$

The required equation is obtained by eliminating x between Eqs. (i) and (iii).

Now, subtracting Eq. (iii) from Eq. (i), we get

$$(q+y)\,x - 3r = 0$$

$$\text{or} \quad x = \frac{3r}{q+y}$$

On substituting the value of x in Eq. (i), we get

$$\left(\frac{3r}{q+y}\right)^3 + q\left(\frac{3r}{q+y}\right) + r = 0$$

Thus, $y^3 + 6qy^2 + 9q^2y + (4q^3 + 27r^2) = 0$

which is the required equation.

Remark

$\Sigma(\alpha-\beta)^2 = -6q$, $\quad \Pi(\alpha-\beta)^2 = -(4q^3 + 27r^2)$

Some Results on Roots of a Polynomial Equation

1. **Remainder Theorem** If a polynomial $f(x)$ is divided by a linear function $x-\lambda$, then the remainder is $f(\lambda)$,

   i.e. Dividend = Divisor × Quotient + Remainder

   Let $Q(x)$ be the quotient and R be the remainder, thus

   $$f(x) = (x-\lambda)\,Q(x) + R$$

   $$\Rightarrow \quad f(\lambda) = (\lambda-\lambda)\,Q(\lambda) + R = 0 + R = R$$

Example 46. If the expression $2x^3 + 3px^2 - 4x + p$ has a remainder of 5 when divided by $x+2$, find the value of p.

Sol. Let $f(x) = 2x^3 + 3px^2 - 4x + p$

$\because \quad f(x) = (x+2)Q(x) + 5$

$\Rightarrow \quad f(-2) = 5$

$\Rightarrow \quad 2(-2)^3 + 3p(-2)^2 - 4(-2) + p = 5$ or $13p = 13$

$\therefore \quad p = 1$

2. **Factor Theorem** Factor theorem is a special case of Remainder theorem.

   Let $f(x) = (x-\lambda)\,Q(x) + R = (x-\lambda)\,Q(x) + f(\lambda)$

   If $f(\lambda) = 0$, $f(x) = (x-\lambda)\,Q(x)$, therefore $f(x)$ is exactly divisible by $x-\lambda$.

   or

   If λ is a root of the equation $f(x) = 0$, then $f(x)$ is exactly divisible by $(x-\lambda)$ and conversely, if $f(x)$ is exactly divisible by $(x-\lambda)$, then λ is a root of the equation $f(x) = 0$ and the remainder obtained is $f(\lambda)$.

Example 47. ***If $x^2 + ax + 1$ is a factor of $ax^3 + bx + c$, find the conditions.***

Sol. $\because ax^3 + bx + c = (x^2 + ax + 1)Q(x)$

Let $\quad Q(x) = Ax + B,$

then $\quad ax^3 + bx + c = (x^2 + ax + 1)(Ax + B)$

On comparing coefficients of x^3, x^2, x and constant on both sides, we get

$$a = A, \quad \text{...(i)}$$

$$0 = B + aA, \quad \text{...(ii)}$$

$$b = aB + A, \quad \text{...(iii)}$$

$$\text{and} \quad c = B \quad \text{...(iv)}$$

From Eqs. (i) and (iv), we get

$$A = a \text{ and } B = c$$

From Eqs. (ii) and (iii), $a^2 + c = 0$ and $b = ac + a$ are the required conditions.

Example 48. A certain polynomial $f(x), x \in R$, when divided by $x-a, x-b$ and $x-c$ leaves remainders a, b and c, respectively. Then, find the remainder when $f(x)$ is divided by $(x-a)(x-b)(x-c)$, where a, b, c are distinct.

Sol. By Remainder theorem $f(a) = a$, $f(b) = b$ and $f(c) = c$

Let the quotient be $Q(x)$ and remainder is $R(x)$.

$\therefore \quad f(x) = (x-a)(x-b)(x-c)Q(x) + R(x)$

$\therefore \quad f(a) = 0 + R(a) \Rightarrow R(a) = a$

$f(b) = 0 + R(b) \Rightarrow R(b) = b$ and $f(c) = 0 + R(c)$

$\Rightarrow \quad R(c) = c$

So, the equation $R(x) - x = 0$ has three roots a, b and c. But its degree is atmost two. So, $R(x) - x$ must be zero polynomial (or identity).

Hence, $R(x) = x$.

3. Every equation of an odd degree has atleast one real root, whose sign is opposite to that of its last term, provided that the coefficient of the first term is positive.
4. Every equation of an even degree has atleast two real roots, one positive and one negative, whose last term is negative, provided that the coefficient of the first term is positive.
5. If an equation has no odd powers of x, then all roots of the equation are complex provided all the coefficients of the equation have positive sign.

6. If $x = \alpha$ is root repeated m times in $f(x) = 0$
($f(x) = 0$ is an nth degree equation in x), then
$f(x) = (x - \alpha)^m g(x)$
where, $g(x)$ is a polynomial of degree $(n - m)$ and the root $x = \alpha$ is repeated $(m - 1)$ time in $f'(x) = 0$, $(m - 2)$ times in $f''(x) = 0, \ldots, (m - (m - 1))$ times in $f^{m-1}(x) = 0$.

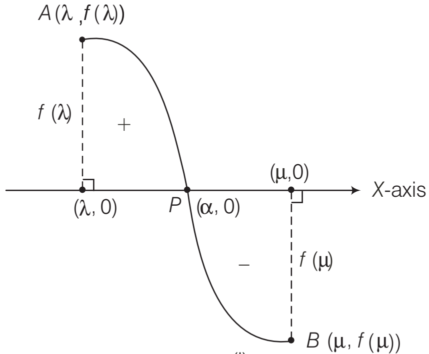
7. Let $f(x) = 0$ be a polynomial equation and λ, μ are two real numbers.

Then, $f(x) = 0$ will have atleast one real root or an odd number of roots between λ and μ, if $f(\lambda)$ and $f(\mu)$ are of opposite signs.

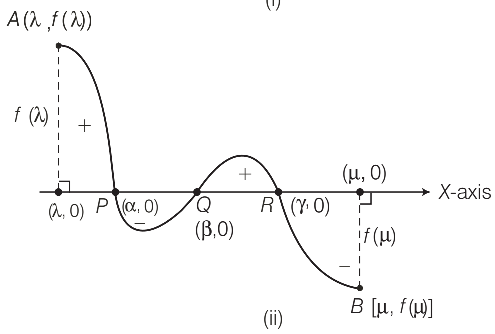
But if $f(\lambda)$ and $f(\mu)$ are of same signs, then either $f(x) = 0$ has no real roots or an even number of roots between λ and μ.

Illustration by Graphs

Since, $f(x)$ be a polynomial in x, then graph of $y = f(x)$ will be continuous in every interval.



(i)



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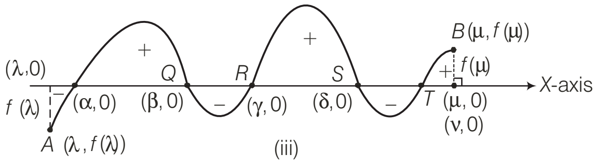

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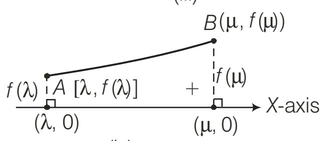

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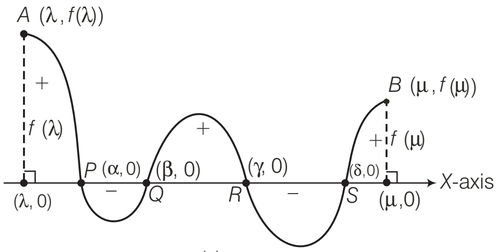

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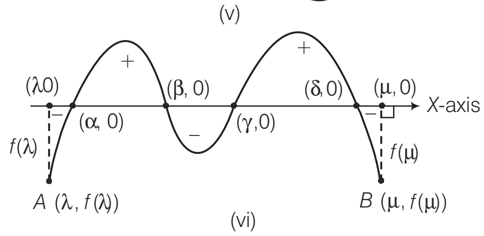


(vi)

(a) In figure (i), (ii) and (iii), $f(\lambda)$ and $f(\mu)$ have opposite signs and equation $f(x) = 0$, has one, three, five roots between λ and μ, respectively.

(b) In figure (iv), (v) and (vi), $f(\lambda)$ and $f(\mu)$ have same signs and equation $f(x) = 0$, has no, four and four roots between λ and μ, respectively.

Example 49. If a, b, c are real numbers, $a \neq 0$. If α is root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, show that the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies $\alpha < \gamma < \beta$.

Sol. Since, α is a root of $a^2x^2 + bx + c = 0$.

Then, $\quad a^2\alpha^2 + b\alpha + c = 0 \quad$ ...(i)

and β is a root of $\quad a^2x^2 - bx - c = 0,$

then $\quad a^2\beta^2 - b\beta - c = 0 \quad$ ...(ii)

Let $\quad f(x) = a^2x^2 + 2bx + 2c$

$\therefore \quad f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = a^2\alpha^2 - 2a^2\alpha^2$ [from Eq. (i)]

$= -a^2\alpha^2$

$\Rightarrow \quad f(\alpha) < 0$ and $f(\beta) = a^2\beta^2 + 2b\beta + 2c$

$= a^2\beta^2 + 2a^2\beta^2$ [from Eq. (ii)]

$= 3a^2\beta^2$

$\Rightarrow \quad f(\beta) > 0$

Since, $f(\alpha)$ and $f(\beta)$ are of opposite signs, then it is clear that a root γ of the equation $f(x) = 0$ lies between α and β.

Hence, $\quad \alpha < \gamma < \beta \quad [\because \alpha < \beta]$

Example 50. If $a < b < c < d$, then show that $(x - a)(x - c) + 3(x - b)(x - d) = 0$ has real and distinct roots.

Sol. Let $\quad f(x) = (x - a)(x - c) + 3(x - b)(x - d)$

Then, $f(a) = 0 + 3(a - b)(a - d) > 0$ $[\because a - b < 0, a - d < 0]$
and $f(b) = (b - a)(b - c) + 0 < 0$ $[\because b - a > 0, b - c < 0]$
Thus, one root will lie between a and b.
and $f(c) = 0 + 3(c - b)(c - d) < 0$ $[\because c - b > 0, c - d < 0]$
and $f(d) = (d - a)(d - c) + 0 > 0$ $[\because d - a > 0, d - c > 0]$
Thus, one root will lie between c and d. Hence, roots of equation are real and distinct.

8. Let $f(x) = 0$ be a polynomial equation then

(a) the number of positive roots of a polynomial equation $f(x) = 0$ (arranged in decreasing order of the degree) cannot exceed the number of changes of signs in $f(x) = 0$ as we move from left to right.
For example, Consider the equation
$$2x^3 - x^2 - x + 1 = 0.$$
The number of changes of signs from left to right is 2 (+ to –, then – to +). Then, number of positive roots cannot exceed 2.

(b) The number of negative roots of a polynomial equation $f(x) = 0$ cannot exceed the number of changes of signs in $f(-x)$.
For example, Consider the equation
$$5x^4 + 3x^3 - 2x^2 + 5x - 8 = 0$$
Let $f(x) = 4x^4 + 3x^3 - 2x^2 + 5x - 8$
$\therefore$ $f(-x) = 5x^4 - 3x^3 - 2x^2 - 5x - 8$
The number of changes of signs from left to right is (+ to –). Then number of negative roots cannot exceed 1.

(c) If equation $f(x) = 0$ have atmost r positive roots and atmost t negative roots, then equation $f(x) = 0$ will have atmost $(r + t)$ real roots, i.e. it will have atleast $n - (r + t)$ imaginary roots, where n is the degree of polynomial.
For example, Consider the equation
$$5x^6 - 8x^3 + 3x^5 + 5x^2 + 8 = 0$$
The given equation can be written as
$$5x^6 + 3x^5 - 8x^3 + 5x^2 + 8 = 0$$
Let $f(x) = 5x^6 + 3x^5 - 8x^3 + 5x^2 + 8$
Here, $f(x)$ has two changes in signs.
So, $f(x)$ has atmost two positive real roots
and $f(-x) = 5x^6 - 3x^5 + 8x^3 + 5x^2 + 8$
Here, $f(-x)$ has two changes in signs.
So, $f(x)$ has atmost two negative real roots.
and $x = 0$ cannot be root of $f(x) = 0$.
Hence, $f(x) = 0$ has atmost four real roots, therefore atleast two imaginary roots.

9. **Rolle's Theorem** If $f(x)$ is continuous function in the interval $[a, b]$ and differentiable in interval (a, b) and $f(a) = f(b)$, then equation $f'(x) = 0$ will have atleast one root between a and b. Since, every polynomial $f(x)$ is always continuous and differentiable in every interval. Therefore, Rolle's theorem is always applicable to polynomial function in every interval $[a, b]$ if $f(a) = f(b)$.

Example 51. If $2a + 3b + 6c = 0$; $a, b, c \in R$, then show that the equation $ax^2 + bx + c = 0$ has atleast one root between 0 and 1.

Sol. Given, $2a + 3b + 6c = 0$
$$\Rightarrow \quad \frac{a}{3} + \frac{b}{2} + c = 0 \quad \text{...(i)}$$
Let $f'(x) = ax^2 + bx + c$,
Then, $f(x) = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx + d$
Now, $f(0) = d$ and $f(1) = \dfrac{a}{3} + \dfrac{b}{2} + c + d$
$= 0 + d$ [from Eq. (i)]
Since, $f(x)$ is a polynomial of three degree, then $f(x)$ is continuous and differentiable everywhere and $f(0) = f(1)$, then by Rolle's theorem $f'(x) = 0$ i.e., $ax^2 + bx + c = 0$ has atleast one real root between 0 and 1.

Reciprocal Equation of the Standard Form can be Reduced to an Equation of Half Its Dimensions

Let the equation be
$$ax^{2m} + bx^{2m-1} + cx^{2m-2} + \ldots + kx^m + \ldots + cx^2 + bx + a = 0$$
On dividing by x^m, then
$$ax^m + bx^{m-1} + cx^{m-2} + \ldots + k + \ldots + \frac{c}{x^{m-2}} + \frac{b}{x^{m-1}} + \frac{a}{x^m} = 0$$
On rearranging the terms, we have
$$a\left(x^m + \frac{1}{x^m}\right) + b\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + c\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \ldots + k = 0$$
Now, $$x^{p+1} + \frac{1}{x^{p+1}} = \left(x^p + \frac{1}{x^p}\right)\left(x + \frac{1}{x}\right) - \left(x^{p-1} + \frac{1}{x^{p-1}}\right)$$

Hence, writing z for $x + \frac{1}{x}$ and given to p succession the values $1, 2, 3, \ldots$, we obtain

$$x^2 + \frac{1}{x^2} = z^2 - 2$$

$$x^3 + \frac{1}{x^3} = z(z^2 - 2) - z = z^3 - 3z$$

$$x^4 + \frac{1}{x^4} = z(z^3 - 3z) - (z^2 - 2) = z^4 - 4z^2 + 2$$

and so on and **generally $x^m + \frac{1}{x^m}$ is of m dimensions in z and therefore the equation in z is of m dimensions**.

Example 52. Solve the equation $2x^4 + x^3 - 11x^2 + x + 2 = 0$.

Sol. Since, $x = 0$ is not a solution of the given equation.

On dividing by x^2 in both sides of the given equation, we get

$$2\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 11 = 0 \qquad \ldots\text{(i)}$$

Put $x + \frac{1}{x} = y$ in Eq. (i), then Eq. (i) reduce in the form

$$2(y^2 - 2) + y - 11 = 0$$

$$\Rightarrow \quad 2y^2 + y - 15 = 0$$

$$\therefore \quad y_1 = -3 \text{ and } y_2 = \frac{5}{2}$$

Consequently, the original equation is equivalent to the collection of equations

$$\left\{\begin{array}{l} x + \frac{1}{x} = -3 \\ x + \frac{1}{x} = \frac{5}{2} \end{array}\right.,$$

we find that, $x_1 = \frac{-3 - \sqrt{5}}{2}, x_2 = \frac{-3 + \sqrt{5}}{2}, x_3 = \frac{1}{2}, x_4 = 2$

Equations which can be Reduced to Linear, Quadratic and Biquadratic Equations

***Type* I An equation of the form**

$$(x - a)(x - b)(x - c)(x - d) = A$$

where, $a < b < c < d$, $b - a = d - c$, can be solved by a change of variable.

i.e. $$y = \frac{(x - a) + (x - b) + (x - c) + (x - d)}{4}$$

$$y = x - \frac{(a + b + c + d)}{4}$$

Example 53. Solve the equation $(12x - 1)(6x - 1)(4x - 1)(3x - 1) = 5$.

Sol. The given equation can be written as

$$\left(x - \frac{1}{12}\right)\left(x - \frac{1}{6}\right)\left(x - \frac{1}{4}\right)\left(x - \frac{1}{3}\right) = \frac{5}{12 \cdot 6 \cdot 4 \cdot 3} \qquad \ldots\text{(i)}$$

Since, $\frac{1}{12} < \frac{1}{6} < \frac{1}{4} < \frac{1}{3}$ and $\frac{1}{6} - \frac{1}{12} = \frac{1}{3} - \frac{1}{4}$

We can introduced a new variable,

$$y = \frac{1}{4}\left[\left(x - \frac{1}{12}\right) + \left(x - \frac{1}{6}\right) + \left(x - \frac{1}{4}\right) + \left(x - \frac{1}{3}\right)\right]$$

$$y = x - \frac{5}{24}$$

On substituting $x = y + \frac{5}{24}$ in Eq. (i), we get

$$\left(y + \frac{3}{24}\right)\left(y + \frac{1}{24}\right)\left(y - \frac{1}{24}\right)\left(y - \frac{3}{24}\right) = \frac{5}{12 \cdot 6 \cdot 4 \cdot 3}$$

$$\Rightarrow \quad \left[y^2 - \left(\frac{1}{24}\right)^2\right]\left[y^2 - \left(\frac{3}{24}\right)^2\right] = \frac{5}{12 \cdot 6 \cdot 4 \cdot 3}$$

Hence, we find that

$$y^2 = \frac{49}{24^2}$$

i.e. $$y_1 = \frac{7}{24} \text{ and } y_2 = -\frac{7}{24}$$

Hence, the corresponding roots of the original equation are $-\frac{1}{12}$ and $\frac{1}{2}$.

***Type* II** An equation of the form

$$(x - a)(x - b)(x - c)(x - d) = Ax^2$$

where, $ab = cd$ can be reduced to a collection of two quadratic equations by a change of variable $y = x + \frac{ab}{x}$.

Example 54. Solve the equation $(x + 2)(x + 3)(x + 8)(x + 12) = 4x^2$.

Sol. Since, $(-2)(-12) = (-3)(-8)$, so we can write given equation as

$$(x + 2)(x + 12)(x + 3)(x + 8) = 4x^2$$

$$\Rightarrow \quad (x^2 + 14x + 24)(x^2 + 11x + 24) = 4x^2 \qquad \ldots\text{(i)}$$

Now, $x = 0$ is not a root of given equation.

On dividing by x^2 in both sides of Eq. (i), we get

$$\left(x + \frac{24}{x} + 14\right)\left(x + \frac{24}{x} + 11\right) = 4 \qquad \ldots\text{(ii)}$$

Put $x + \frac{24}{x} = y$, then Eq. (ii) can be reduced in the form

$$(y + 14)(y + 11) = 4 \quad \text{or} \quad y^2 + 25y + 150 = 0$$

$$\therefore \quad y_1 = -15 \text{ and } y_2 = -10$$

Thus, the original equation is equivalent to the collection of equations

$$\left[\begin{array}{l} x + \dfrac{24}{x} = -15, \\ x + \dfrac{24}{x} = -10, \end{array}\right.$$

i.e.

$$\left[\begin{array}{l} x^2 + 15x + 24 = 0 \\ x^2 + 10x + 24 = 0 \end{array}\right.$$

On solving these collection, we get

$$x_1 = \frac{-15 - \sqrt{129}}{2},\ x_2 = \frac{-15 + \sqrt{129}}{2},\ x_3 = -6,\ x_4 = -4$$

***Type* III An equation of the form $(x-a)^4 + (x-b)^4 = A$** can also be solved by a change of variable, i.e. making a substitution $y = \dfrac{(x-a)+(x-b)}{2}$.

Example 55. ***Solve the equation*** $(6-x)^4 + (8-x)^4 = 16$.

Sol. After a change of variable,

$$y = \frac{(6-x)+(8-x)}{2}$$

$$\therefore \quad y = 7 - x \quad \text{or} \quad x = 7 - y$$

Now, put $x = 7 - y$ in given equation, we get

$$(y-1)^4 + (y+1)^4 = 16$$

$$\Rightarrow \quad y^4 + 6y^2 - 7 = 0$$

$$\Rightarrow \quad (y^2+7)(y^2-1) = 0$$

$$y^2 + 7 \neq 0$$

[y gives imaginary values]

$$\therefore \quad y^2 - 1 = 0$$

Then, $\quad y_1 = -1$ and $y_2 = 1$

Thus, $x_1 = 8$ and $x_2 = 6$ are the roots of the given equation.

Rational Algebraic Inequalities

Consider the following types of rational algebraic inequalities

$$\frac{P(x)}{Q(x)} > 0,\ \frac{P(x)}{Q(x)} < 0,$$

$$\frac{P(x)}{Q(x)} \geq 0,\ \frac{P(x)}{Q(x)} \leq 0$$

If $P(x)$ and $Q(x)$ can be resolved in linear factors, then use *Wavy curve method*, otherwise we use the following statements for solving inequalities of this kind.

(1) $$\frac{P(x)}{Q(x)} > 0 \Rightarrow \{P(x)\,Q(x) > 0 \Rightarrow \left\{\begin{array}{c} P(x) > 0, Q(x) > 0 \\ \text{or} \\ P(x) < 0, Q(x) < 0 \end{array}\right.$$

(2) $$\frac{P(x)}{Q(x)} < 0 \Rightarrow \{P(x)\,Q(x) < 0 \Rightarrow \left\{\begin{array}{c} P(x) > 0, Q(x) < 0 \\ \text{or} \\ P(x) < 0, Q(x) > 0 \end{array}\right.$$

(3) $$\frac{P(x)}{Q(x)} \geq 0 \Rightarrow \left\{\begin{array}{l} P(x)\,Q(x) \geq 0 \\ Q(x) \neq 0 \end{array}\right. \Rightarrow \left\{\begin{array}{c} P(x) \geq 0, Q(x) > 0 \\ \text{or} \\ P(x) \leq 0, Q(x) < 0 \end{array}\right.$$

(4) $$\frac{P(x)}{Q(x)} \leq 0 \Rightarrow \left\{\begin{array}{l} P(x)\,Q(x) \leq 0 \\ Q(x) \neq 0 \end{array}\right. \Rightarrow \left\{\begin{array}{c} P(x) \geq 0, Q(x) < 0 \\ \text{or} \\ P(x) \leq 0, Q(x) > 0 \end{array}\right.$$

Example 56. ***Find all values of a for which the set of all solutions of the system***

$$\left\{\begin{array}{l} \dfrac{x^2 + ax - 2}{x^2 - x + 1} < 2 \\ \dfrac{x^2 + ax - 2}{x^2 - x + 1} > -3 \end{array}\right.$$

is the entire number line.

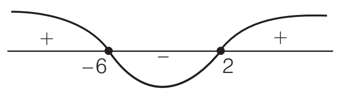
Sol. The system is equivalent to

$$\left\{\begin{array}{l} \dfrac{x^2 - (a+2)x + 4}{x^2 - x + 1} > 0 \\ \dfrac{4x^2 + (a-3)x + 1}{x^2 - x + 1} > 0 \end{array}\right.$$

Since, $x^2 - x + 1 = \left(x - \dfrac{1}{2}\right)^2 + \dfrac{3}{4} > 0$, this system is equivalent to $\left\{\begin{array}{l} x^2 - (a+2)x + 4 > 0 \\ 4x^2 + (a-3)x + 1 > 0 \end{array}\right.$

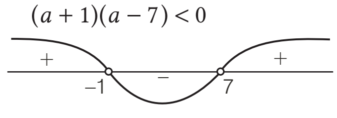
Hence, the discriminants of the both equations of this system are negative.

i.e., $$\left\{\begin{array}{l} (a+2)^2 - 16 < 0 \\ (a-3)^2 - 16 < 0 \end{array}\right. \Rightarrow \quad (a+6)(a-2) < 0$$



i.e., $\quad x \in (-6, 2)$ ...(i)

$\Rightarrow \quad (a+1)(a-7) < 0$



i.e. $\quad x \in (-1, 7)$ ...(ii)

Hence, from Eqs. (i) and (ii), we get

$$x \in (-1, 2)$$

Equations Containing Absolute Values

By definition, $|x| = x$, if $x \geq 0$ $|x| = -x$, if $x < 0$

Example 57. ***Solve the equation*** $x^2 - 5|x| + 6 = 0$.

Sol. The given equation is equivalent to the collection of systems

$$\begin{cases} x^2 - 5x + 6 = 0, \text{ if } x \geq 0 \\ x^2 + 5x + 6 = 0, \text{ if } x < 0 \end{cases} \Rightarrow \begin{cases} (x-2)(x-3) = 0, \text{ if } x \geq 0 \\ (x+2)(x+3) = 0, \text{ if } x < 0 \end{cases}$$

Hence, the solutions of the given equation are

$$x_1 = 2, x_2 = 3, x_3 = -2, x_4 = -3$$

Example 58. ***Solve the equation***

$$\left| \frac{x^2 - 8x + 12}{x^2 - 10x + 21} \right| = -\frac{x^2 - 8x + 12}{x^2 - 10x + 21}.$$

Sol. This equation has the form $|f(x)| = -f(x)$

when, $f(x) = \dfrac{x^2 - 8x + 12}{x^2 - 10x + 21}$

such an equation is equivalent to the collection of systems

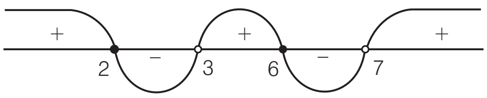
$$\begin{cases} f(x) = -f(x), \text{ if } f(x) \geq 0 \\ f(x) = f(x), \text{ if } f(x) < 0 \end{cases}$$

The first system is equivalent to $f(x) = 0$ and the second system is equivalent to $f(x) < 0$ the combining both systems, we get

$$f(x) \leq 0$$

$$\therefore \quad \frac{x^2 - 8x + 12}{x^2 - 10x + 21} \leq 0$$

$$\Rightarrow \quad \frac{(x-2)(x-6)}{(x-3)(x-7)} \leq 0$$



Hence, by *Wavy curve method,*

$$x \in [2, 3) \cup [6, 7)$$

Example 59. Solve the equation $|x - |4 - x|| - 2x = 4$.

Sol. This equation is equivalent to the collection of systems

$$\begin{cases} |x - (4 - x)| - 2x = 4, \text{ if } 4 - x \geq 0 \\ |x + (4 - x)| - 2x = 4, \text{ if } 4 - x < 0 \end{cases}$$

$$\Rightarrow \begin{cases} |2x - 4| - 2x = 4, & \text{if } x \leq 4 \\ 4 - 2x = 4, & \text{if } x > 4 \end{cases} \quad \text{...(i)}$$

The second system of this collection

gives $\quad x = 0$

but $\quad x > 4$

Hence, second system has no solution.

The first system of collection Eq. (i) is equivalent to the system of collection

$$\begin{cases} 2x - 4 - 2x = 4, \text{ if } 2x \geq 4 \\ -2x + 4 - 2x = 4, \text{ if } 2x < 4 \end{cases}$$

$$\Rightarrow \begin{cases} -4 = 4, \text{ if } x \geq 2 \\ -4x = 0, \text{ if } x < 2 \end{cases}$$

The first system is failed and second system gives $x = 0$. Hence, $x = 0$ is unique solution of the given equation.

Important Forms Containing Absolute Values

Form **1 The equation of the form**

$$|f(x) + g(x)| = |f(x)| + |g(x)|$$

is equivalent of the system

$$f(x)\, g(x) \geq 0.$$

Example 60. Solve the equation

$$\left| \frac{x}{x-1} \right| + |x| = \frac{x^2}{|x-1|}.$$

Sol. Let $f(x) = \dfrac{x}{x-1}$ and $g(x) = x$,

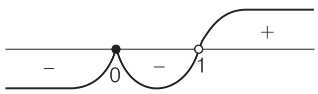
Then, $\quad f(x) + g(x) = \dfrac{x}{x-1} + x = \dfrac{x^2}{x-1}$

$\therefore$ The given equation can be reduced in the form

$$|f(x)| + |g(x)| = |f(x) + g(x)|$$

Hence, $\quad f(x) \cdot g(x) \geq 0$

$$\Rightarrow \quad \frac{x^2}{x-1} \geq 0$$



From *Wavy curve method,* $x \in (1, \infty) \cup \{0\}$.

Form **2 The equation of the form**

$$|f_1(x)| + |f_2(x)| + \ldots + |f_n(x)| = g(x) \quad \text{...(i)}$$

where, $f_1(x), f_2(x), \ldots, f_n(x), g(x)$ are functions of x and $g(x)$ may be constant.

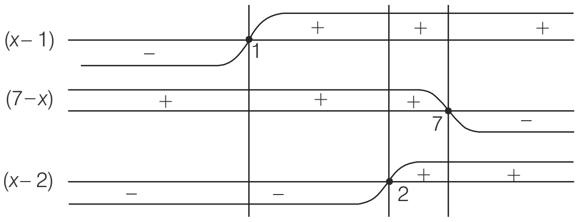
Equations of this form solved by the **method of intervals**. We first find all critical points of $f_1(x), f_2(x), \ldots, f_n(x)$, if coefficient of x is positive, then graph start with positive sign (+) and if coefficient of x is negative, then graph start with negative sign (–). Then, using the definition of the absolute value, we pass from Eq. (i) to a collection of systems which do not contain the absolute value symbols.

Example 61. ***Solve the equation***

$|x-1|+|7-x|+2|x-2|=4.$

Sol. Here, critical points are 1, 2, 7 using the method of intervals, we find intervals when the expressions $x-1, 7-x$ and $x-2$ are of constant signs.

i.e. $\quad x<1, 1<x<2, 2<x<7, x>7$



Thus, the given equation is equivalent to the collection of four systems,

$$\left[\begin{array}{l}\begin{cases}x<1\\-(x-1)+(7-x)-2(x-2)=4\end{cases}\\\begin{cases}1\le x<2\\(x-1)+(7-x)-2(x-2)=4\end{cases}\\\begin{cases}2\le x<7\\(x-1)+(7-x)+2(x-2)=4\end{cases}\\\begin{cases}x\ge 7\\(x-1)-(7-x)+2(x-2)=4\end{cases}\end{array}\right. \Rightarrow \left[\begin{array}{l}\begin{cases}x<1\\x=2\end{cases}\\\begin{cases}1\le x<2\\x=3\end{cases}\\\begin{cases}2\le x<7\\x=1\end{cases}\\\begin{cases}x\ge 7\\x=4\end{cases}\end{array}\right.$$

From the collection of four systems, the given equation has no solution.

Inequations Containing Absolute Values

By definition, $\quad |x|<a \Rightarrow -a<x<a\,(a>0)$

$|x|\le a \Rightarrow -a\le x\le a$

$|x|>a \Rightarrow x<-a$ and $x>a$

and $\quad |x|\ge a \Rightarrow x\le -a$ and $x\ge a.$

Example 62. ***Solve the inequation*** $\left|1-\dfrac{|x|}{1+|x|}\right|\ge\dfrac{1}{2}.$

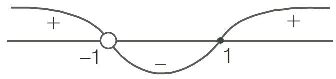
Sol. The given inequation is equivalent to the collection of systems

$$\left\{\begin{array}{l}\left|1-\dfrac{x}{1+x}\right|\ge\dfrac{1}{2},\text{ if } x\ge 0\\\left|1+\dfrac{x}{1-x}\right|\ge\dfrac{1}{2},\text{ if } x<0\end{array}\right. \Rightarrow \left\{\begin{array}{l}\dfrac{1}{|1+x|}\ge\dfrac{1}{2},\text{ if } x\ge 0\\\dfrac{1}{|1-x|}\ge\dfrac{1}{2},\text{ if } x<0\end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l}\dfrac{1}{1+x}\ge\dfrac{1}{2},\text{ if } x\ge 0\\\dfrac{1}{1-x}\ge\dfrac{1}{2},\text{ if } x<0\end{array}\right. \Rightarrow \left\{\begin{array}{l}\dfrac{1-x}{1+x}\ge 0,\text{ if } x\ge 0\\\dfrac{1+x}{1-x}\ge 0,\text{ if } x<0\end{array}\right.$$

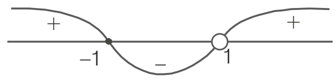
$$\Rightarrow \left\{\begin{array}{l}\dfrac{x-1}{x+1}\le 0,\text{ if } x\ge 0\\\dfrac{x+1}{x-1}\le 0,\text{ if } x<0\end{array}\right.$$

For $\quad \dfrac{x-1}{x+1}\le 0$, if $x\ge 0$



$\therefore \qquad 0\le x\le 1 \qquad$ ...(i)

For $\quad \dfrac{x+1}{x-1}\le 0$, if $x<0$



$\therefore \qquad -1\le x<0 \qquad$ ...(ii)

Hence, from Eqs. (i) and (ii), the solution of the given equation is $x\in[-1, 1]$.

Aliter

$$\left|1-\frac{|x|}{1+|x|}\right|\ge\frac{1}{2} \Rightarrow \left|\frac{1}{1+|x|}\right|\ge\frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{1+|x|}\ge\frac{1}{2} \Rightarrow 1+|x|\le 2 \text{ or } |x|\le 1$$

$$\therefore \quad -1\le x\le 1 \text{ or } x\Rightarrow[-1, 1]$$

Equations Involving Greatest Integer, Least Integer and Fractional Part

1. Greatest Integer

$[x]$ denotes the greatest integer less than or equal to x i.e., $[x]\le x$. It is also known as **floor** of x.

Thus, $\quad [3.5779]=3, [0.89]=0, [3]=3$

$[-8.7285]=-9$

$[-0.6]=-1$

$[-7]=-7$

In general, if n is an integer and x is any real number between n and $n+1$.

i.e. $\quad n\le x<n+1$, then $[x]=n$

Properties of Greatest Integer

(i) $[x\pm n]=[x]\pm n, n\in I$

(ii) $[-x]=-[x], x\in I$

(iii) $[-x]=-1-[x], x\notin I$

(iv) $[x]-[-x]=2n$, if $x=n, n\in I$

(v) $[x]-[-x]=2n+1$, if $x=n+\{x\}, n\in I$ and $0<\{x\}<1$

(vi) $[x]\ge n \Rightarrow x\ge n, n\in I$

(vii) $[x]>n \Rightarrow x\ge n+1, n\in I$

(viii) $[x]\le n \Rightarrow x<n+1, n\in I$

(ix) $[x]<n \Rightarrow x<n, n\in I$

(x) $n_2\le[x]\le n_1 \Rightarrow n_2\le x<n_1+1, n_1, n_2\in I$

(xi) $[x+y]\ge[x]+[y]$

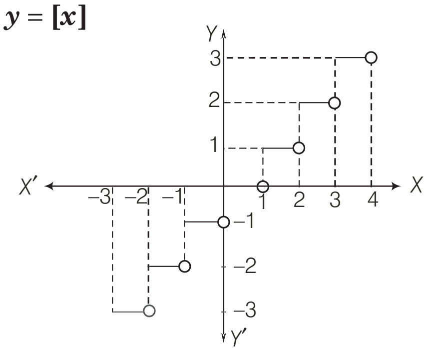
(xii) $\left[\dfrac{[x]}{n}\right] = \left[\dfrac{x}{n}\right], n \in N$

(xiii) $\left[\dfrac{n+1}{2}\right] + \left[\dfrac{n+2}{4}\right] + \left[\dfrac{n+4}{8}\right] + \left[\dfrac{n+8}{16}\right] + \ldots = n, n \in N$

(xiv) $[x] + \left[x + \dfrac{1}{n}\right] + \left[x + \dfrac{2}{n}\right] + \ldots + \left[x + \dfrac{n-1}{n}\right] = [nx],$

$n \in N$

Graph of $y = [x]$



Remark

Domain and Range of $[x]$ are R and I, respectively.

Example 63. If $[x]$ denotes the integral part of x for real x, then find the value of

$$\left[\frac{1}{4}\right] + \left[\frac{1}{4} + \frac{1}{200}\right] + \left[\frac{1}{4} + \frac{1}{100}\right] + \left[\frac{1}{4} + \frac{3}{200}\right] + \ldots + \left[\frac{1}{4} + \frac{199}{200}\right].$$

Sol. The given expression can be written as

$$\left[\frac{1}{4}\right] + \left[\frac{1}{4} + \frac{1}{200}\right] + \left[\frac{1}{4} + \frac{2}{200}\right] + \left[\frac{1}{4} + \frac{3}{200}\right] + \ldots + \left[\frac{1}{4} + \frac{199}{200}\right]$$

$$= \left[200 \cdot \frac{1}{4}\right] = [50] = 50 \qquad \text{[from property (xiv)]}$$

Example 64. ***Let $[a]$ denotes the larger integer not exceeding the real number a. If x and y satisfy the equations $y = 2[x] + 3$ and $y = 3[x-2]$ simulaneously, determine $[x+y]$.***

Sol. We have, $\quad y = 2[x] + 3 = 3[x-2] \qquad$ ...(i)

$\Rightarrow \quad 2[x] + 3 = 3([x] - 2) \qquad$ [from property (i)]

$\Rightarrow \quad 2[x] + 3 = 3[x] - 6$

$\Rightarrow \quad [x] = 9$

From Eq. (i), $y = 2 \times 9 + 3 = 21$

$\therefore \quad [x+y] = [x+21] = [x] + 21 = 9 + 21 = 30$

Hence, the value of $[x+y]$ is 30.

2. Least Integer

(x) or $\lceil x \rceil$ denotes the least integer greater than or equal to x i.e., $(x) \geq x$ or $\lceil x \rceil \geq x$. It is also known as **ceilling** of x.

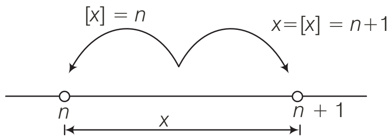
Thus, $\quad (3.578) = 4, (0.87) = 1,$

$(4) = 4$

$\lceil -8.239 \rceil = -8, \lceil -0.7 \rceil = 0$

In general, if n is an integer and x is any real number between n and $n+1$

i.e., $\quad n < x \leq n+1$, then $(x) = n+1$



Relation between Greatest Integer and Least Integer

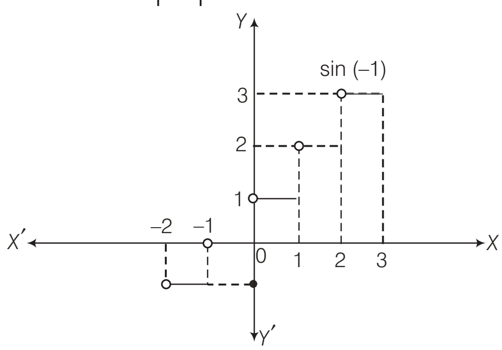
$$(x) = \begin{cases} [x], & x \in I \\ [x]+1, & x \notin I \end{cases}$$

i.e. If $x \in I$, then $x = [x] = (x)$. $\qquad$ [remember]

Remark

If $(x) = n$, then $(n-1) < x \leq n$

Graph of $y = (x) = \lceil x \rceil$



Remark

Domain and Range of (x) are R and $[x] + 1$, respectively.

Example 65. ***If $[x]$ and (x) are the integral part of x and nearest integer to x, then solve $(x)[x] = 1$.***

Sol. ***Case* I** If $x \in I$, then $x = [x] = (x)$

$\therefore$ Given equation convert in $x^2 = 1$.

$\therefore \qquad x = (\pm 1)$

***Case* II** If $x \notin I$, then $(x) = [x] + 1$

$\therefore$ Given equation convert in

$$([x]+1)[x]=1 \Rightarrow [x]^2+[x]-1=0$$

or $$[x]=\frac{-1\pm\sqrt{5}}{2} \quad \text{[impossible]}$$

Then, final answer is $x=\pm 1$.

Example 66. Find the solution set of $(x)^2+(x+1)^2=25$, where (x) is the least integer greater than or equal to x.

Sol. ***Case*** **I** If $x \in I$, then $x=(x)=[x]$

Then, $(x)^2+(x+1)^2=25$ reduces to

$$x^2+\overline{x+1}^2=25 \Rightarrow 2x^2+2x-24=0$$

$$\Rightarrow \quad x^2+x-12=0 \Rightarrow (x+4)(x-3)=0$$

$$\therefore \quad x=-4, 3 \quad \text{...(i)}$$

Case **II** If $x \notin I$, then $(x)=[x]+1$

Then, $(x)^2+(x+1)^2=25$ reduces to

$$\{[x]+1\}^2+\{[x+1]+1\}^2=25$$

$$\Rightarrow \quad \{[x]+1\}^2+\{[x]+2\}^2=25$$

$$\Rightarrow \quad 2[x]^2+6[x]-20=0$$

$$\Rightarrow \quad [x]^2+3[x]-10=0$$

$$\Rightarrow \quad \{[x]+5\}\{[x]-2\}=0$$

$$\therefore \quad [x]=-5 \text{ and } [x]=2$$

$$\Rightarrow \quad x \in [-5,-4) \cup [2,3)$$

$$\because \quad x \notin I,$$

$$\therefore \quad x \in (-5,-4) \cup (2,3) \quad \text{...(ii)}$$

On combining Eqs. (i) and (ii), we get

$$x \in (-5,-4] \cup (2,3]$$

3. Fractional Part

$\{x\}$ denotes the fractional part of x, i.e. $0 \le \{x\} < 1$.

Thus, $\{2 \cdot 7\}=0.7$, $\{5\}=0$, $\{-3.72\}=0.28$

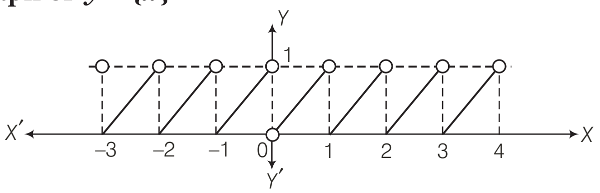
If x is a real number, then $x=[x]+\{x\}$

i.e., $x=n+f$, where $n \in I$ and $0 \le f < 1$

Properties of Fractional Part of x

(i) $\{x \pm n\}=\{x\}, n \in I$ (ii) If $0 \le x < 1$, then $\{x\}=x$

Graph of $y=\{x\}$



Remark

1. For proper fraction $0 < \{x\} < 1$.
2. Domain and range of $\{x\}$ are R and $[0, 1)$, respectively.
3. $\{-5.238\}=\{-5-0.238\}=\{-5-1+1-0.238\}$
$=\{-6+0.762\}=\{\bar{6}.762\}=0.762$

Example 67. If $\{x\}$ and $[x]$ represent fractional and integral part of x respectively, find the value of $[x]+\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}$.

Sol. $$[x]+\sum_{r=1}^{2000}\frac{\{x+r\}}{2000}=[x]+\sum_{r=1}^{2000}\frac{\{x\}}{2000} \quad \text{[from property (i)]}$$

$$=[x]+\frac{\{x\}}{2000}\sum_{r=1}^{2000}1=[x]+\frac{\{x\}}{2000}\times 2000=[x]+\{x\}=x$$

Example 68. ***If $\{x\}$ and $[x]$ represent fractional and integral part of x respectively, then solve the equation*** $x-1=(x-[x])(x-\{x\})$.

Sol. $\because \quad x=[x]+\{x\}, 0 \le \{x\} < 1$

Thus, given equation reduces to

$$[x]+\{x\}-1=\{x\}[x]$$

$$\Rightarrow \quad \{x\}[x]-[x]-\{x\}+1=0$$

$$\Rightarrow \quad ([x]-1)(\{x\}-1)=0$$

Now, $$\{x\}-1 \ne 0 \quad [\because 0 \le \{x\} < 1]$$

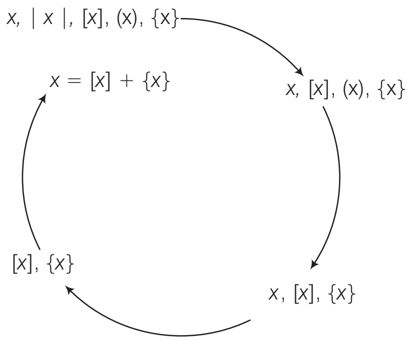
$$\therefore \quad [x]-1=0$$

$$\Rightarrow \quad [x]=1$$

$$\therefore \quad x \in [1, 2)$$

Problem Solving Cycle

If a problem has $x, |x|, [x], (x), \{x\}$, then first solve $|x|$, then problem convert in $x, [x], (x), \{x\}$.



Secondly, solve $(x)=\begin{cases}[x], & x \in I\\ [x]+1, & x \notin I\end{cases}$

Then, problem convert in $x, [x], \{x\}$.

Now, put $x=[x]+\{x\}$

Then, problem convert in $[x]$ and $\{x\}$. ...(i)

Since, $0 \le \{x\} < 1$, then we get $[x]$

From Eq. (i), we get $\{x\}$

Hence, final solution is $x=[x]+\{x\}$.

Example 69. Let $\{x\}$ and $[x]$ denotes the fractional and integral parts of a real number x, respectively. Solve $4\{x\} = x + [x]$.

Sol. $\because \quad x = [x] + \{x\}$ ...(i)

Then, given equation reduces to

$$4\{x\} = [x] + \{x\} + [x]$$

$\Rightarrow \quad \{x\} = \frac{2}{3}[x]$ ...(ii)

$\because \quad 0 \le \{x\} < 1 \Rightarrow 0 \le \frac{2}{3}[x] < 1 \text{ or } 0 \le [x] < \frac{3}{2}$

$\therefore \quad [x] = 0, 1$

From Eq. (ii), $\{x\} = 0, \frac{2}{3}$

From Eq. (i), $x = 0, 1 + \frac{2}{3}$ i.e., $x = 0, \frac{5}{3}$

Example 70. Let $\{x\}$ and $[x]$ denotes the fractional and integral part of a real number (x), respectively. Solve $|2x - 1| = 3[x] + 2\{x\}$.

Sol. ***Case* I** $2x - 1 \ge 0$ or $x \ge \frac{1}{2}$

Then, given equation convert to

$2x - 1 = 3[x] + 2\{x\}$ ...(i)

$\because \quad x = [x] + \{x\}$ ...(ii)

From Eqs. (i) and (ii), we get

$$2([x] + \{x\}) - 1 = 3[x] + 2\{x\}$$

$\therefore \quad [x] = -1$

$\therefore \quad -1 \le x < 0$

No solution $\left[\because x \ge \frac{1}{2}\right]$

***Case* II** $2x - 1 < 0$ or $x < \frac{1}{2}$

Then, given equation reduces to

$1 - 2x = 3[x] + 2\{x\}$ ...(iii)

$\because \quad x = [x] + \{x\}$ ...(iv)

From Eqs. (iii) and (iv), we get

$$1 - 2([x] + \{x\}) = 3[x] + 2\{x\}$$

$\Rightarrow \quad 1 - 5[x] = 4\{x\}$

$\therefore \quad \{x\} = \frac{1 - 5[x]}{4}$ ...(v)

Now, $\quad 0 \le \{x\} < 1$

$\Rightarrow \quad 0 \le \frac{1 - 5[x]}{4} < 1$

$\Rightarrow \quad 0 \le 1 - 5[x] < 4$

$\Rightarrow \quad 0 \ge -1 + 5[x] > -4$

$\Rightarrow \quad 1 \ge 5[x] > -3 \text{ or } -\frac{3}{5} < [x] \le \frac{1}{5}$

$\therefore \quad [x] = 0$

From Eq. (v), $\{x\} = \frac{1}{4}$

$\therefore \quad x = 0 + \frac{1}{4} = \frac{1}{4}$

Example 71. Solve the equation

$$(x)^2 = [x]^2 + 2x$$

where, $[x]$ and (x) are integers just less than or equal to x and just greater than or equal to x, respectively.

Sol. ***Case* I** If $x \in I$ then

$$x = [x] = (x)$$

The given equation reduces to

$$x^2 = x^2 + 2x$$

$\Rightarrow \quad 2x = 0 \text{ or } x = 0$ ...(i)

***Case* II** If $x \notin I$, then $(x) = [x] + 1$

The given equation reduces to

$$([x] + 1)^2 = [x]^2 + 2x$$

$\Rightarrow \quad 1 = 2(x - [x]) \text{ or } \{x\} = \frac{1}{2}$

$\therefore \quad x = [x] + \frac{1}{2} = n + \frac{1}{2}, n \in I$ ...(ii)

Hence, the solution of the original equation is $x = 0, n + \frac{1}{2}$, $n \in I$.

Example 72. Solve the system of equations in x, y and z satisfying the following equations:

$$x + [y] + \{z\} = 3\cdot 1$$
$$\{x\} + y + [z] = 4\cdot 3$$
$$[x] + \{y\} + z = 5\cdot 4$$

where, $[\cdot]$ and $\{\cdot\}$ denotes the greatest integer and fractional parts, respectively.

Sol. $\because [x] + \{x\} = x, \; [y] + \{y\} = y$ and $[z] + \{z\} = z$,

On adding all the three equations, we get

$$2(x + y + z) = 12.8$$

$\Rightarrow \quad x + y + z = 6.4$ ...(i)

Now, adding first two equations, we get

$$x + y + z + [y] + \{x\} = 7.4$$

$\Rightarrow \quad 6.4 + [y] + \{x\} = 7.4$ [from Eq. (i)]

$\Rightarrow \quad [y] + \{x\} = 1$

$\therefore \quad [y] = 1 \text{ and } \{x\} = 0$ ...(ii)

On adding last two equations, we get

$$x + y + z + \{y\} + [z] = 9.7$$

$\{y\} + [z] = 3.3$ [from Eq. (ii)]

$\therefore \quad [z] = 3 \text{ and } \{y\} = 0.3$ ...(iii)

On adding first and last equations, we get

$$x + y + z + [x] + \{z\} = 8.5$$

$\Rightarrow \quad [x] + \{z\} = 2.1$ [from Eq. (i)]

$\therefore \quad [x] = 2, \; \{z\} = 0.1$ ...(iv)

From Eqs. (i), (ii) and (iii), we get

$$x = [x] + \{x\} = 2 + 0 = 2$$
$$y = [y] + \{y\} = 1 + 0.3 = 1.3$$

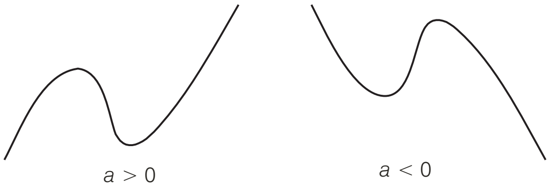
and $\quad z = [z] + \{z\} = 3 + 0.1 = 3.1$

Roots of Equation with the Help of Graphs

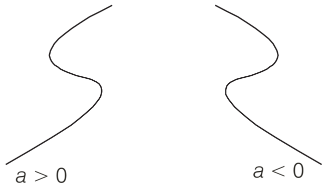
Here, we will discuss some examples to find the roots of equations with the help of graphs.

Important Graphs

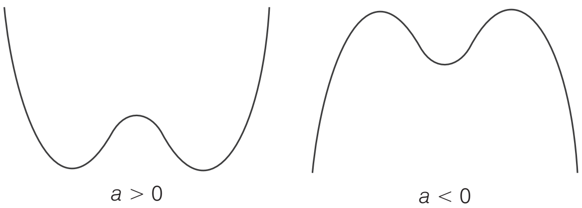
1. $y = ax^3 + bx^2 + cx + d$



2. $x = ay^3 + by^2 + cy + d$



3. $y = ax^4 + bx^3 + cx^2 + dx + e$



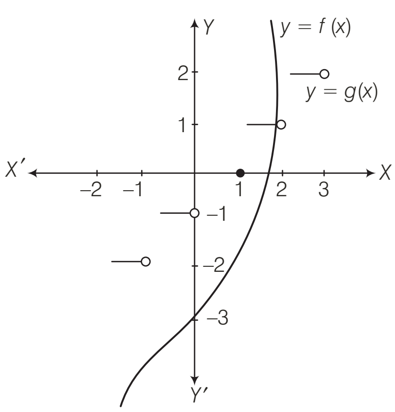
Example 73. Solve the equation $x^3 - [x] = 3$, where $[x]$ denotes the greatest integer less than or equal to x.

Sol. We have, $x^3 - [x] = 3$

$\Rightarrow \quad x^3 - 3 = [x]$

Let $\quad f(x) = x^3 - 3$ and $g(x) = [x]$.

It is clear from the graphs, the point of intersection of two curves $y = f(x)$ and $y = g(x)$ lies between (1, 0) and (2, 0).



$\therefore \quad 1 < x < 2$

We have, $f(x) = x^3 - 3$ and $g(x) = 1$

or $\quad x^3 - 3 = 1 \Rightarrow x^3 = 4$

$\therefore \quad x = (4)^{1/3}$

Hence, $x = 4^{1/3}$ is the solution of the equation $x^3 - [x] = 3$.

Aliter

$\because \quad x = [x] + f, 0 \le f < 1,$

Then, given equation reduces to

$$x^3 - (x - f) = 3 \Rightarrow x^3 - x = 3 - f$$

Hence, it follows that

$$2 < x^3 - x \le 3$$

$\Rightarrow \quad 2 < x(x+1)(x-1) \le 3$

Further for $x \ge 2$, we have $x(x+1)(x-1) \ge 6 > 3$

For $x < -1$, we have $x(x+1)(x-1) < 0 < 2$

For $x = -1$, we have $x(x+1)(x-1) = 0 < 2$

For $-1 < x \le 0$, we have $x(x+1)(x-1) \le -x < 1$

and for $0 < x \le 1$, we have $x(x+1)(x-1) < x < x^3 \le 1$

Therefore, x must be $1 < x < 2$

$\therefore \quad [x] = 1$

Now, the original equation can be written as

$$x^3 - 1 = 3 \Rightarrow x^3 = 4$$

Hence, $x = 4^{1/3}$ is the solution of the given equation.

Example 74. Solve the equation $x^3 - 3x - a = 0$ for different values of a.

Sol. We have, $x^3 - 3x - a = 0 \Rightarrow x^3 - 3x = a$

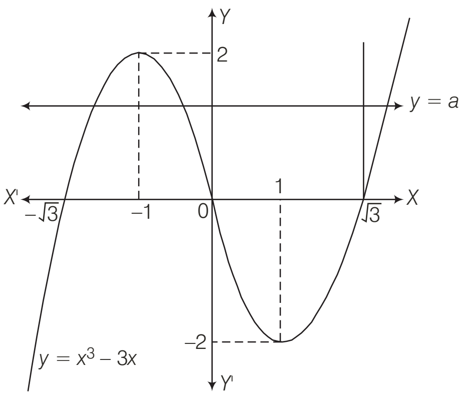
Let $f(x) = x^3 - 3x$ and $g(x) = a$

$\because \quad f'(x) = 0$

$\Rightarrow \quad 3x^2 - 3 = 0$

$\Rightarrow \quad x = -1, 1$

$$f''(x) = 6x$$



$\therefore \quad f''(-1) = -6 < 0$ and $f''(1) = 6 > 0$

$\therefore f(x)$ local maximum at $x = (-1)$ and local minimum at $x = 1$ and $f(-1) = 2$ and $f(1) = -2$ and $y = g(x) = a$ is a straight line parallel to X-axis.

Following cases arise

***Case* I** When $a > 2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at only one point, so $x^3 - 3x - a = 0$ has only one real root.

***Case* II** When $a = 2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at two points, so $x^3 - 3x - a = 0$ has three real roots, two are equal and one different.

***Case* III** When $-2 < a < 2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at three points, so $x^3 - 3x - a = 0$ has three distinct real roots.

***Case* IV** When $a = -2$,

In this case $y = f(x)$ and $y = g(x)$ touch at one point and intersect at other point, so $x^3 - 3x - a = 0$ has three real roots, two are equal and one different.

***Case* V** When $a < -2$,

In this case $y = f(x)$ and $y = g(x)$ intersects at only one point, so $x^3 - 3x - a = 0$ has only one real root.

Example 75. Show that the equation $x^3 + 2x^2 + x + 5 = 0$ has only one real root, such that $[\alpha] = -3$, where $[x]$ denotes the integral point of x.

Sol. We have, $x^3 + 2x^2 + x + 5 = 0$

$$\Rightarrow \quad x^3 + 2x^2 + x = -5$$

Let $f(x) = x^3 + 2x^2 + x$ and $g(x) = -5$

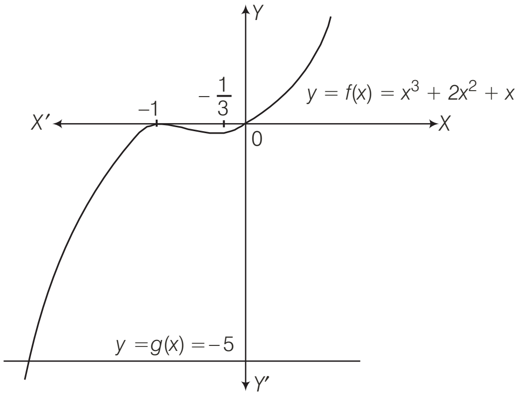
$\because \quad f'(x) = 0 \Rightarrow 3x^2 + 4x + 1 = 0$

$\Rightarrow \quad x = -1, -\frac{1}{3}$ and $f''(x) = 6x + 4$

$\therefore \quad f''(-1) = -2 < 0$ and $f''\left(-\frac{1}{3}\right) = -2 + 4 = 2 > 0$

$\therefore f(x)$ local maximum at $x = -1$ and local minimum at $x = -\frac{1}{3}$

and $\quad f(-1) = 0, f\left(-\frac{1}{3}\right) = -\frac{4}{27}$



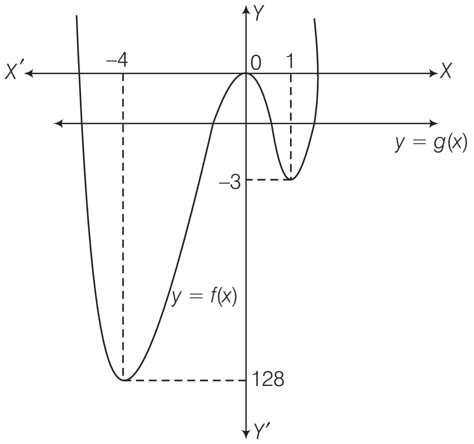
and $\quad f(-2) = -2$ and $f(-3) = -12$

Therefore, x must lie between (-3) and (-2).

i.e. $\quad -3 < \alpha < -2 \Rightarrow [\alpha] = -3$

Example 76. Find all values of the parameter k for which all the roots of the equation $x^4 + 4x^3 - 8x^2 + k = 0$ are real.

Sol. We have, $x^4 + 4x^3 - 8x^2 + k = 0$



$$\Rightarrow \quad x^4 + 4x^3 - 8x^2 = -k$$

Let $\quad f(x) = x^4 + 4x^3 - 8x^2$ and $g(x) = -k$

$\because \quad f'(x) = 0$

$\Rightarrow \quad 4x^3 + 12x^2 - 16x = 0 \Rightarrow x = -4, 0, 1$

and $\quad f''(x) = 12x^2 + 24x - 16$

$\therefore \quad f''(-4) = 80, \ f''(0) = -16, f''(1) = 20$

$\therefore f(x)$ has local minimum at $x = -4$ and $x = 1$ and local maximum at $x = 0$

and $\quad f(-4) = -128, f(0) = 0, f(1) = -3.$

Following cases arise

***Case* I** When $-k > 0$ i.e., $k < 0$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = (-k)$ intersect at two points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two real roots.

***Case* II** When $-k = 0$ and $-k = -3$, i.e. $k = 0, 3$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ intersect at four points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two distinct real roots and two equal roots.

***Case* III** When $-3 < -k < 0$, i.e. $0 < k < 3$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ intersect at four distinct points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has four distinct real roots.

***Case* IV** When $-128 < -k < -3$, i.e. $3 < k < 128$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ intersect at two distinct points, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two distinct real roots.

***Case* V** When $-k = -128$ i.e., $k = 128$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ touch at one point, so $x^4 + 4x^3 - 8x^2 + k = 0$ has two real and equal roots.

***Case* VI** When $-k < -128$, i.e. $k > 128$

In this case $y = x^4 + 4x^3 - 8x^2$ and $y = -k$ do not intersect, so there is no real root.

Example 77. Let $-1 \le p \le 1$, show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it.

Sol. We have, $4x^3 - 3x - p = 0$

$\Rightarrow \quad 4x^3 - 3x = p$

Let $\quad f(x) = 4x^3 - 3x$ and $g(x) = p$

$\therefore \quad f'(x) = 0$

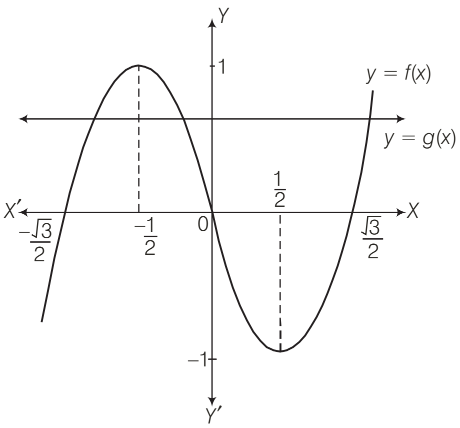
$\Rightarrow \quad 12x^2 - 3 = 0$

$\Rightarrow \quad x = -\frac{1}{2}, -\frac{1}{2}$ and $f''(x) = 24x$

$\therefore \quad f''\left(-\frac{1}{2}\right) = -12 < 0$ and $f''\left(\frac{1}{2}\right) = 12 > 0$

$\therefore f(x)$ has local maximum at $\left(x = -\frac{1}{2}\right)$ and local minimum at $\left(x = \frac{1}{2}\right)$.

Also, $f\left(-\frac{1}{2}\right) = -\frac{4}{8} + \frac{3}{2} = 1$ and $f\left(\frac{1}{2}\right) = \frac{4}{8} - \frac{3}{2} = -1$



We observe that, the line $y = g(x) = p$, where $-1 \le p \le 1$ intersect the curve $y = f(x)$ exactly at point $\alpha \in \left[\frac{1}{2}, 1\right]$.

Hence, $4x^3 - 3x - p = 0$ has exactly one root in the interval $\left[\frac{1}{2}, 1\right]$.

Now, we have to find the value of root α.

Let $\alpha = \cos\theta$, then $4\cos^3\theta - 3\cos\theta - p = 0$

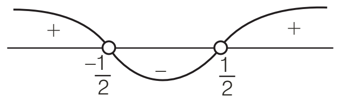
$\Rightarrow \quad \cos 3\theta = p \Rightarrow 3\theta = \cos^{-1}(p)$ or $\theta = \frac{1}{3}\cos^{-1}(p)$

$\therefore \quad \alpha = \cos\theta = \cos\left\{\frac{1}{3}\cos^{-1}(p)\right\}$

Aliter

Let $\quad \phi(x) = 4x^3 - 3x - p$

$\therefore \quad \phi'(x) = 12x^2 - 3 = 12\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)$



Clearly, $\phi'(x) > 0$ for $x \in \left[\frac{1}{2}, 1\right]$.

Hence, $\phi(x)$ can have atmost one root in $\left[\frac{1}{2}, 1\right]$.

Also, $\phi\left(\frac{1}{2}\right) = -1 - p$ and $\phi(1) = 1 - p$

$\therefore \quad \phi\left(\frac{1}{2}\right)\phi(1) = -(1 - p^2) = (p^2 - 1) \le 0 \qquad [\because -1 \le p \le 1]$

Since, $\phi(x)$ being a polynomial, continuous on $\left[\frac{1}{2}, 1\right]$ and $\phi\left(\frac{1}{2}\right)\phi(1) \le 0$. Therefore, by intermediate value theorem $\phi(x)$ has atleast one root in $\left[\frac{1}{2}, 1\right]$.

Hence, $\phi(x)$ has exactly one root in $\left[\frac{1}{2}, 1\right]$.

Exercise for Session 4

1. If α, β, γ are the roots of $x^3 - x^2 - 1 = 0$, the value of $\sum\left(\frac{1+\alpha}{1-\alpha}\right)$, is equal to

(a) -7
(b) -6
(c) -5
(d) -4

2. If r, s, t are the roots of the equation $8x^3 + 1001x + 2008 = 0$. The value of $(r+s)^3 + (s+t)^3 + (t+r)^3$ is

(a) 751
(b) 752
(c) 753
(d) 754

3. If $\alpha, \beta, \gamma, \delta$ are the roots of equation $x^4 + 4x^3 - 6x^2 + 7x - 9 = 0$, the value of $\prod(1+\alpha^2)$ is

(a) 9
(b) 11
(c) 13
(d) 15

4. If a, b, c, d are four consecutive terms of an increasing AP, the roots of the equation $(x-a)(x-c) + 2(x-b)(x-d) = 0$ are

(a) non-real complex
(b) real and equal
(c) integers
(d) real and distinct

5. If $x^2 + px + 1$ is a factor of the expression $ax^3 + bx + c$ then

(a) $a^2 - c^2 = ab$
(b) $a^2 + c^2 = -ab$
(c) $a^2 - c^2 = -ab$
(d) None of these

6. The number of real roots of the equation $x^2 - 3|x| + 2 = 0$ is

(a) 1
(b) 2
(c) 3
(d) 4

7. Let $a \neq 0$ and $p(x)$ be a polynomial of degree greater than 2, if $p(x)$ leaves remainder a and $(-a)$ when divided respectively by $x + a$ and $x - a$, the remainder when $p(x)$ is divided by $x^2 - a^2$, is

(a) $2x$
(b) $-2x$
(c) x
(d) $-x$

8. The product of all the solutions of the equation $(x-2)^2 - 3|x-2| + 2 = 0$ is

(a) 2
(b) -4
(c) 0
(d) None of these

9. If $0 < x < 1000$ and $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right] = \frac{31}{30}x$, where $[x]$ is the greatest integer less than or equal to x, the number of possible values of x is

(a) 32
(b) 33
(c) 34
(d) None of these

10. If $[x]$ is the greatest integer less than or equal to x and (x) be the least integer greater than or equal to x and $[x]^2 + (x)^2 > 25$ then x belongs to

(a) $[3, 4]$
(b) $(-\infty, -4]$
(c) $[4, \infty)$
(d) $(-\infty, -4] \cup [4, \infty)$

Answers

Exercise for Session 4

1.(c) **2.** (c) **3.** (c) **4.** (d) **5.**(a) **6.** (d)
7.(d) **8.** (c) **9.** (b) **10.** (d)