CHAPTER

02

Theory of Equations

Session 1

Polynomial in One Variable, Identity, Linear Equation, Quadratic Equations, Standard Quadratic Equation

Polynomial in One Variable

An algebraic expression containing many terms of the form cx^n, n being a non-negative integer is called a polynomial,

i.e., $$f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \dots + a_{n-1} \cdot x + a_n,$$

where x is a variable, $a_0, a_1, a_2, \dots, a_n$ are constants and $a_0 \neq 0$.

1. Real Polynomial

Let $a_0, a_1, a_2, \dots, a_n$ be real numbers and x is a real variable. Then,

$f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \dots + a_{n-1} \cdot x + a_n$

is called a real polynomial of real variable (x) with real coefficients.

For example, $5x^3 - 3x^2 + 7x - 4$, $x^2 - 3x + 1$, etc., are real polynomials.

2. Complex Polynomial

Let $a_0, a_1, a_2, \dots, a_n$ are complex numbers and x is a varying complex number.

Then $f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \dots + a_{n-1} \cdot x + a_n$ is called a complex polynomial or a polynomial of complex variable with complex coefficients.

For example, $x^3 - 7ix^2 + (3 - 2i)x + 13$, $3x^2 - (2 + 3i)x + 5i$, etc. (where $i = \sqrt{-1}$) are complex polynomials.

3. Rational Expression or Rational Function

An expression of the form $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x, is called a rational expression. As a particular case when $Q(x)$ is a non-zero constant, $\dfrac{P(x)}{Q(x)}$ reduces to a polynomial.

Thus, every polynomial is a rational expression but a rational expression may or may not be a polynomial.

For example,

(i) $x^2 - 7x + 8$ (ii) $\dfrac{2}{x-3}$

(iii) $\dfrac{x^3 - 6x^2 + 11x - 6}{(x-4)}$ (iv) $x + \dfrac{3}{x}$ or $\dfrac{x^2 + 3}{x}$

4. Degree of Polynomial

The highest power of variable (x) present in the polynomial is called the degree of the polynomial.

For example, $f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \dots + a_{n-1} \cdot x + a_n$ is a polynomial in x of degree n.

Remark

A polynomial of degree one is generally called a linear polynomial. Polynomials of degree 2, 3, 4 and 5 are known as quadratic, cubic, biquadratic and pentic polynomials, respectively.

5. Polynomial Equation

If $f(x)$ is a polynomial, real or complex, then $f(x) = 0$ is called a polynomial equation.

(i) A polynomial equation has atleast one root.

(ii) A polynomial equation of degree n has n roots.

Remarks

1. A polynomial equation of degree one is called a **linear equation** i.e. $ax + b = 0$, where $a, b \in C$, set of all complex numbers and $a \neq 0$.
2. A polynomial equation of degree two is called a **quadratic equation** i.e., $ax^2 + bx + c$, where $a, b, c \in C$ and $a \neq 0$.
3. A polynomial equation of degree three is called a **cubic equation** i.e., $ax^3 + bx^2 + cx + d = 0$, where $a, b, c, d \in C$ and $a \neq 0$.
4. A polynomial equation of degree four is called a **biquadratic equation** i.e., $ax^4 + bx^3 + cx^2 + dx + e = 0$, where $a, b, c, d, e \in C$ and $a \neq 0$.
5. A polynomial equation of degree five is called a **pentic equation** i.e., $ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$, where $a, b, c, d, e, f \in C$ and $a \neq 0$.

6. Roots of an Equation

The values of the variable for which an equation is satisfied are called the roots of the equation.

If $x = \alpha$ is a root of the equation $f(x) = 0$, then $f(\alpha) = 0$.

Remark

The real roots of an equation $f(x) = 0$ are the values of x, where the curve $y = f(x)$ crosses X-axis.

7. Solution Set

The set of all roots of an equation in a given domain is called the solution set of the equation.

For example, The roots of the equation $x^3 - 2x^2 - 5x + 6 = 0$ are $1, -2, 3$, the solution set is $\{1, -2, 3\}$.

Remark

Solve or solving an equation means finding its solution set or obtaining all its roots.

Identity

If two expressions are equal for all values of x, then the statement of equality between the two expressions is called an identity.

For example, $(x+1)^2 = x^2 + 2x + 1$ is an identity in x.

or

If $f(x) = 0$ is satisfied by every value of x in the domain of $f(x)$, then it is called an identity.

For example, $f(x) = (x+1)^2 - (x^2 + 2x + 1) = 0$ is an identity in the domain C, as it is satisfied by every complex number.

or

If $f(x) = a_0 \cdot x^n + a_1 \cdot x^{n-1} + a_2 \cdot x^{n-2} + \ldots + a_{n-1} \cdot x + a_n = 0$ have more than n distinct roots, it is an identity, then

$$a_0 = a_1 = a_2 = \ldots = a_{n-1} = a_n = 0$$

For example, If $ax^2 + bx + c = 0$ is satisfied by more than two values of x, then $a = b = c = 0$.

or

In an identity in x coefficients of similar powers of x on the two sides are equal.

For example, If $ax^4 + bx^3 + cx^2 + dx + e = 5x^4 - 3x^3 + 4x^2 - 7x - 9$ be an identity in x, then

$$a = 5, b = -3, c = 4, d = -7, e = -9.$$

Thus, an identity in x satisfied by all values of x, where as an equation in x is satisfied by some particular values of x.

Example 1. ***If equation $(\lambda^2 - 5\lambda + 6)x^2 + (\lambda^2 - 3\lambda + 2)x + (\lambda^2 - 4) = 0$ is satisfied by more than two values of x, find the parameter λ.***

Sol. If an equation of degree two is satisfied by more than two values of unknown, then it must be an identity. Then, we must have

$$\lambda^2 - 5\lambda + 6 = 0, \lambda^2 - 3\lambda + 2 = 0, \lambda^2 - 4 = 0$$

$\Rightarrow \quad \lambda = 2, 3$ and $\lambda = 2, 1$ and $\lambda = 2, -2$

Common value of λ which satisfies each condition is $\lambda = 2$.

Example 2. ***Show that***

$$\frac{(x+b)(x+c)}{(b-a)(c-a)} + \frac{(x+c)(x+a)}{(c-b)(a-b)} + \frac{(x+a)(x+b)}{(a-c)(b-c)} = 1$$

is an identity.

Sol. Given relation is

$$\frac{(x+b)(x+c)}{(b-a)(c-a)} + \frac{(x+c)(x+a)}{(c-b)(a-b)} + \frac{(x+a)(x+b)}{(a-c)(b-c)} = 1 \quad \ldots(i)$$

When $x = -a$, then LHS of Eq. (i) $= \dfrac{(b-a)(c-a)}{(b-a)(c-a)} = 1$

= RHS of Eq. (i)

When $x = -b$, then LHS of Eq. (i)

$= \dfrac{(c-b)(a-b)}{(c-b)(a-b)} = 1 =$ RHS of Eq. (i)

and when $x = -c$, then LHS of Eq. (i) $= \dfrac{(a-c)(b-c)}{(a-c)(b-c)} = 1$

= RHS of Eq. (i).

Thus, highest power of x occurring in relation of Eq. (i) is 2 and this relation is satisfied by three distinct values of $x\ (= -a, -b, -c)$. Therefore, it cannot be an equation and hence it is an identity.

Example 3. ***Show that $x^2 - 3|x| + 2 = 0$ is an equation.***

Sol. Put $x = 0$ in $x^2 - 3|x| + 2 = 0$

$\Rightarrow \quad 0^2 - 3|0| + 2 = 2 \neq 0$

Since, the relation $x^2 - 3|x| + 2 = 0$ is not satisfied by $x = 0$.

Hence, it is an equation.

Linear Equation

An equation of the form

$$ax + b = 0 \quad \ldots(i)$$

where $a, b \in R$ and $a \neq 0$, is a linear equation.

Eq. (i) has an unique root equal to $-\dfrac{b}{a}$.

Example 4. ***Solve the equation*** $\frac{x}{2}+\frac{(3x-1)}{6}=1-\frac{x}{2}$

Sol. We have, $\frac{x}{2}+\frac{(3x-1)}{6}=1-\frac{x}{2}$

or $\frac{x}{2}+\frac{x}{2}+\frac{x}{2}=1+\frac{1}{6}$

or $\frac{3x}{2}=\frac{7}{6}$

or $x=\frac{7}{9}$

Example 5. ***Solve the equation*** $(a-3)x+5=a+2$.

***Sol. Case* I** For $a \neq 3$, this equation is linear, then

$(a-3)\,x=(a-3)$

$\therefore \quad x=\frac{(a-3)}{(a-3)}=1$

***Case* II** For $a=3$,

$0 \cdot x+5=3+2$

$\Rightarrow \quad 5=5$

Therefore, any real number is its solution.

Quadratic Equations

An equation in which the highest power of the unknown quantity is 2, is called a quadratic equation.

Quadratic equations are of two types :

1. Purely Quadratic Equation

A quadratic equation in which the term containing the first degree of the unknown quantity is absent, is called a purely quadratic equation.

i.e., $ax^2+c=0$,

where $a, c \in C$ and $a \neq 0$.

2. Adfected Quadratic Equation

A quadratic equation in which it contains the terms of first as well as second degrees of the unknown quantity, is called an adfected (or complete) quadratic equation.

i.e., $ax^2+bx+c=0$,

where $a, b, c \in C$ and $a \neq 0, b \neq 0$.

Standard Quadratic Equation

An equation of the form

$$ax^2+bx+c=0 \quad \text{...(i)}$$

where $a, b, c \in C$ and $a \neq 0$, is called a standard quadratic equation.

The numbers a, b, c are called the coefficients of this equation.

A root of the quadratic Eq. (i) is a complex number α, such that $a\alpha^2+b\alpha+c=0$. Recall that $D=b^2-4ac$ is the discriminant of the Eq. (i) and its roots are given by the following formula.

$$x=\frac{-b \pm \sqrt{D}}{2a} \quad \text{[Shridharacharya method]}$$

Nature of Roots

1. If $a, b, c \in R$ and $a \neq 0$, then

(i) If $D<0$, then Eq. (i) has non-real complex roots.

(ii) If $D>0$, then Eq. (i) has real and distinct roots, namely

$$x_1=\frac{-b+\sqrt{D}}{2a},\ x_2=\frac{-b-\sqrt{D}}{2a} \text{ and then}$$

$$ax^2+bx+c=a(x-x_1)(x-x_2). \quad \text{...(ii)}$$

(iii) If $D=0$, then Eq. (i) has real and equal roots, then $x_1=x_2=-\frac{b}{2a}$ and then

$$ax^2+bx+c=a(x-x_1)^2. \quad \text{...(iii)}$$

To represent the quadratic ax^2+bx+c in form Eqs. (ii) or (iii), is to expand it into linear factors.

(iv) If $D \geq 0$, then Eq. (i) has real roots.

(v) If D_1 and D_2 be the discriminants of two quadratic equations, then

(a) If $D_1+D_2 \geq 0$, then

- atleast one of D_1 and $D_2 \geq 0$.
- if $D_1<0$, then $D_2>0$ and if $D_1>0$, then $D_2<0$.

(b) If $D_1+D_2<0$, then

- atleast one of D_1 and $D_2<0$.
- If $D_1<0$, then $D_2>0$ and if $D_1>0$, then $D_2<0$.

2. If $a, b, c \in Q$ and D is a perfect square of a rational number, the roots are rational and in case it is not a perfect square, the roots are irrational.

3. If $a, b, c \in R$ and $p+iq$ is one root of Eq. (i) $(q \neq 0)$, then the other must be the conjugate $(p-iq)$ and *vice-versa* (where, $p, q \in R$ and $i=\sqrt{-1}$).

4. If $a, b, c \in Q$ and $p+\sqrt{q}$ is one root of Eq. (i), then the other must be the conjugate $p-\sqrt{q}$ and *vice-versa* (where, p is a rational and $\sqrt{q}$ is a surd).

5. If $a=1$ and $b, c \in I$ and the roots of Eq. (i) are rational numbers, these roots must be integers.

6. If $a+b+c=0$ and a, b, c are rational, 1 is a root of the Eq. (i) and roots of the Eq. (i) are rational.

7. $a^2+b^2+c^2-ab-bc-ca=\frac{1}{2}$

$\{(a-b)^2+(b-c)^2+(c-a)^2\}$

$=-\{(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)\}$

Example 6. Find all values of the parameter a for which the quadratic equation

$$(a+1)x^2+2(a+1)x+a-2=0$$

(i) has two distinct roots.

(ii) has no roots.

(iii) has two equal roots.

Sol. By the hypothesis, this equation is quadratic and therefore $a \neq -1$ and the discriminant of this equation,

$$D = 4(a+1)^2 - 4(a+1)(a-2)$$
$$= 4(a+1)(a+1-a+2)$$
$$= 12(a+1)$$

(i) For $a > (-1)$, then $D > 0$, this equation has two distinct roots.

(ii) For $a < (-1)$, then $D < 0$, this equation has no roots.

(iii) This equation cannot have two equal roots. Since, $D=0$ only for $a=-1$ and this contradicts the hypothesis.

Example 7. ***Solve for*** x,

$$(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10.$$

Sol. $\because \quad (5+2\sqrt{6})(5-2\sqrt{6})=1$

$\therefore \quad (5-2\sqrt{6})=\frac{1}{(5+2\sqrt{6})}$

$\therefore \quad (5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$

reduces to $(5+2\sqrt{6})^{x^2-3}+\left(\frac{1}{5+2\sqrt{6}}\right)^{x^2-3}=10$

Put $(5+2\sqrt{6})^{x^2-3}=t$, then $t+\frac{1}{t}=10$

$\Rightarrow \quad t^2-10t+1=0$

or $\quad t=\frac{10 \pm \sqrt{(100-4)}}{2}=(5 \pm 2\sqrt{6})$

$\Rightarrow \quad (5+2\sqrt{6})^{x^2-3}=(5 \pm 2\sqrt{6})=(5+2\sqrt{6})^{\pm 1}$

$\therefore \quad x^2-3=\pm 1$

$\Rightarrow \quad x^2-3=1$ or $x^2-3=-1$

$\Rightarrow \quad x^2=4$ or $x^2=2$

Hence, $\quad x=\pm 2, \ \pm\sqrt{2}$

Example 8. Show that if p, q, r and s are real numbers and $pr=2(q+s)$, then atleast one of the equations $x^2+px+q=0$ and $x^2+rx+s=0$ has real roots.

Sol. Let D_1 and D_2 be the discriminants of the given equations $x^2+px+q=0$ and $x^2+rx+s=0$, respectively.

Now, $D_1+D_2=p^2-4q+r^2-4s=p^2+r^2-4(q+s)$

$=p^2+r^2-2pr \qquad$ [given, $pr=2(q+s)$]

$=(p-r)^2 \geq 0 \qquad$ [$\because$ p and q are real]

or $\quad D_1+D_2 \geq 0$

Hence, atleast one of the equations $x^2+px+q=0$ and $x^2+rx+s=0$ has real roots.

Example 9. ***If*** α, β ***are the roots of the equation*** $(x-a)(x-b)=c, c \neq 0$***. Find the roots of the equation*** $(x-\alpha)(x-\beta)+c=0$.

Sol. Since, α, β are the roots of

$(x-a)(x-b)=c$

or $\quad (x-a)(x-b)-c=0,$

Then $\quad (x-a)(x-b)-c=(x-\alpha)(x-\beta)$

$\Rightarrow \quad (x-\alpha)(x-\beta)+c=(x-a)(x-b)$

Hence, roots of $(x-\alpha)(x-\beta)+c=0$ are a, b.

Example 10. ***Find all roots of the equation*** $x^4+2x^3-16x^2-22x+7=0$, ***if one root is*** $2+\sqrt{3}$.

Sol. All coefficients are real, irrational roots will occur in conjugate pairs.

Hence, another root is $2-\sqrt{3}$.

$\therefore$ Product of these roots $=(x-2-\sqrt{3})(x-2+\sqrt{3})$

$=(x-2)^2-3=x^2-4x+1.$

On dividing $x^4+2x^3-16x^2-22x+7$ by x^2-4x+1, then the other quadratic factor is x^2+6x+7.

Then, the given equation reduce in the form

$(x^2-4x+1)(x^2+6x+7)=0$

$\therefore \quad x^2+6x+7=0$

Then, $\quad x=\frac{-6 \pm \sqrt{36-28}}{2}=-3 \pm \sqrt{2}$

Hence, the other roots are $2-\sqrt{3}, -3 \pm \sqrt{2}$.

Relation between Roots and Coefficients

1. **Relation between roots and coefficients of quadratic equation** If roots of the equation $ax^2+bx+c=0 \ (a \neq 0)$ be real and distinct and $\alpha < \beta$, then $\alpha=\frac{-b+\sqrt{D}}{2a}, \beta=\frac{-b-\sqrt{D}}{2a}$.

(i) Sum of roots

$$= S = \alpha + \beta = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}.$$

(ii) Product of roots

$$= P = \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

(iii) Difference of roots

$$= D' = \alpha - \beta = \frac{\sqrt{D}}{a} = \frac{\sqrt{\text{Discriminant}}}{\text{Coefficient of } x^2}.$$

2. **Formation of an equation with given roots**
A quadratic equation whose roots are α and β, is given by $(x-\alpha)(x-\beta)=0$ or $x^2-(\alpha+\beta)x+\alpha\beta=0$
i.e. $x^2 -$ (Sum of roots) x + Product of roots $= 0$

$\therefore \qquad x^2 - Sx + P = 0.$

3. **Symmetric function of roots** A function of α and β is said to be symmetric function, if it remains unchanged, when α and β are interchanged.
For example, $\alpha^3+3\alpha^2\beta+3\alpha\beta^2+\beta^3$ is a symmetric function of α and β, whereas $\alpha^3-\beta^3+5\alpha\beta$ is not a symmetric function of α and β. In order to find the value of a symmetric function in terms of $\alpha+\beta, \alpha\beta$ and $\alpha-\beta$ and also in terms of *a*, *b* and *c*.

(i) $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

$$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right) = \frac{b^2-2ac}{a^2}.$$

(ii) $\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)$

$$= \left(-\frac{b}{a}\right)\left(\frac{\sqrt{D}}{a}\right) = -\frac{b\sqrt{D}}{a^2}.$$

(iii) $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$

$$= \left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right) = -\left(\frac{b^3-3abc}{a^3}\right).$$

(iv) $\alpha^3-\beta^3=(\alpha-\beta)^3+3\alpha\beta(\alpha-\beta)$

$$= \left(\frac{\sqrt{D}}{a}\right)^3 + 3\left(\frac{c}{a}\right)\left(\frac{\sqrt{D}}{a}\right) = \frac{\sqrt{D}(D+3ac)}{a^3}.$$

(v) $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2$

$$= \left(\frac{b^2-2ac}{a^2}\right)^2 - 2\left(\frac{c}{a}\right)^2 = \frac{b^4+2a^2c^2-4acb^2}{a^4}.$$

(vi) $\alpha^4-\beta^4=(\alpha^2+\beta^2)(\alpha^2-\beta^2)$

$$= -\frac{b\sqrt{D}(b^2-2ac)}{a^4}.$$

(vii) $\alpha^5+\beta^5=(\alpha^2+\beta^2)(\alpha^3+\beta^3)-\alpha^2\beta^2(\alpha+\beta)$

$$= \left(\frac{b^2-2ac}{a^2}\right)\left(-\frac{(b^3-3abc)}{a^3}\right) - \frac{c^2}{a^2}\left(-\frac{b}{a}\right)$$

$$= \frac{-(b^5-5ab^3c+5a^2bc^2)}{a^5}.$$

(viii) $\alpha^5-\beta^5=(\alpha^2+\beta^2)(\alpha^3-\beta^3)+\alpha^2\beta^2(\alpha-\beta)$

$$= \left(\frac{b^2-2ac}{a^2}\right)\left(\frac{\sqrt{D}(D+3ac)}{a^3}\right) + \left(\frac{c}{a}\right)^2\left(\frac{\sqrt{D}}{a}\right)$$

$$= \frac{\sqrt{D}(b^4-3acb^2+3a^2c^2)}{a^5}.$$

Example 11. If one root of the equation $x^2-ix-(1+i)=0, (i=\sqrt{-1})$ is $1+i$, find the other root.

Sol. All coefficients of the given equation are not real, then other root $\neq 1-i$.
Let other root be α, then sum of roots $= i$
i.e. $\qquad 1+i+\alpha = i \Rightarrow \alpha = (-1)$
Hence, the other root is (-1).

Example 12. If one root of the equation $x^2-\sqrt{5}\,x-19=0$ is $\dfrac{9+\sqrt{5}}{2}$, then find the other root.

Sol. All coefficients of the given equation are not rational, then other root $\neq \dfrac{9-\sqrt{5}}{2}$.
Let other root be α, sum of roots $=\sqrt{5}$

$$\Rightarrow \quad \frac{9+\sqrt{5}}{2}+\alpha=\sqrt{5} \Rightarrow \alpha = \frac{-9+\sqrt{5}}{2}$$

Hence, other root is $\dfrac{-9+\sqrt{5}}{2}$.

Example 13. If the difference between the corresponding roots of the equations $x^2+ax+b=0$ and $x^2+bx+a=0\ (a\neq b)$ is the same, find the value of $a+b$.

Sol. Let α, β be the roots of $x^2+ax+b=0$ and γ, δ be the roots of $x^2+bx+a=0$, then given

$$\alpha-\beta=\gamma-\delta$$

$$\Rightarrow \quad \frac{\sqrt{a^2-4b}}{1} = \frac{\sqrt{b^2-4a}}{1} \qquad \left[\because \alpha-\beta=\frac{\sqrt{D}}{a}\right]$$

$$\Rightarrow \quad a^2-4b=b^2-4a$$

$$\Rightarrow \quad (a^2-b^2)+4(a-b)=0 \Rightarrow (a-b)(a+b+4)=0$$

$\because \qquad a-b\neq 0$

$\therefore \qquad a+b+4=0$ or $a+b=-4$.

Example 14. *If $a+b+c=0$ and a,b,c are rational. Prove that the roots of the equation*

$$(b+c-a)x^2+(c+a-b)x+(a+b-c)=0$$

are rational.

Sol. Given equation is

$$(b+c-a)x^2+(c+a-b)x+(a+b-c)=0 \quad \text{...(i)}$$

$\because (b+c-a)+(c+a-b)+(a+b-c)=a+b+c=0$

$\therefore$ $x=1$ is a root of Eq. (i), let other root of Eq. (i) is α, then

$$\text{Product of roots} = \frac{a+b-c}{b+c-a}$$

$$\Rightarrow \quad 1\times\alpha = \frac{-c-c}{-a-a} \quad [\because a+b+c=0]$$

$$\therefore \quad \alpha = \frac{c}{a} \quad \text{[rational]}$$

Hence, both roots of Eq. (i) are rational.

Aliter

Let $b+c-a=A, c+a-b=B, a+b-c=C$

Then, $A+B+C=0$ $\quad [\because a+b+c=0]$ ...(ii)

Now, Eq. (i) becomes

$$Ax^2+Bx+C=0 \quad \text{...(iii)}$$

Discriminant of Eq. (iii),

$$D = B^2-4AC$$
$$=(-C-A)^2-4AC \quad [\because A+B+C=0]$$
$$=(C+A)^2-4AC$$
$$=(C-A)^2=(2a-2c)^2$$
$$=4(a-c)^2 = \text{A perfect square}$$

Hence, roots of Eq. (i) are rational.

Example 15. If the roots of equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ be equal, prove that a,b,c are in HP.

Sol. Given equation is

$$a(b-c)x^2+b(c-a)x+c(a-b)=0 \quad \text{...(i)}$$

Here, coefficient of x^2 + coefficient of x + constant term = 0

i.e., $\quad a(b-c)+b(c-a)+c(a-b)=0$

Then, 1 is a root of Eq. (i).

Since, its roots are equal.

Therefore, its other root will be also equal to 1.

$$\text{Then, product of roots} = 1\times 1 = \frac{c(a-b)}{a(b-c)}$$

$$\Rightarrow \quad ab-ac = ca-bc$$

$$\therefore \quad b = \frac{2ac}{a+c}$$

Hence, a, b and c are in HP.

Example 16. *If α is a root of $4x^2+2x-1=0$. Prove that $4\alpha^3-3\alpha$ is the other root.*

Sol. Let other root is β,

then $\quad \alpha+\beta = -\frac{2}{4} = -\frac{1}{2}$ or $\beta = -\frac{1}{2}-\alpha$ ...(i)

and so $4\alpha^2+2\alpha-1=0$, because α is a root of $4x^2+2x-1=0$.

Now, $\quad \beta = 4\alpha^3-3\alpha = \alpha(4\alpha^2-3)$

$$= \alpha(1-2\alpha-3) \quad [\because 4\alpha^2+2\alpha-1=0]$$
$$= -2\alpha^2-2\alpha$$
$$= -\frac{1}{2}(4\alpha^2)-2\alpha$$
$$= -\frac{1}{2}(1-2\alpha)-2\alpha \quad [\because 4\alpha^2+2\alpha-1=0]$$
$$= -\frac{1}{2}-\alpha = \beta \quad \text{[from Eq. (i)]}$$

Hence, $4\alpha^3-3\alpha$ is the other root.

Example 17. If α,β are the roots of the equation $\lambda(x^2-x)+x+5=0$. If λ_1 and λ_2 are two values of λ for which the roots α,β are related by $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$, find the value of $\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}$.

Sol. The given equation can be written as

$$\lambda x^2-(\lambda-1)x+5=0$$

$\because \alpha,\beta$ are the roots of this equation.

$$\therefore \quad \alpha+\beta = \frac{\lambda-1}{\lambda} \text{ and } \alpha\beta = \frac{5}{\lambda}$$

But, given $\quad \frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$

$$\Rightarrow \quad \frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{4}{5}$$

$$\Rightarrow \quad \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{4}{5} \Rightarrow \frac{\frac{(\lambda-1)^2}{\lambda^2}-\frac{10}{\lambda}}{\frac{5}{\lambda}}=\frac{4}{5}$$

$$\Rightarrow \quad \frac{(\lambda-1)^2-10\lambda}{5\lambda}=\frac{4}{5} \Rightarrow \lambda^2-12\lambda+1=4\lambda$$

$$\Rightarrow \quad \lambda^2-16\lambda+1=0$$

It is a quadratic in λ, let roots be λ_1 and λ_2, then

$$\lambda_1+\lambda_2=16 \text{ and } \lambda_1\lambda_2=1$$

$$\therefore \quad \frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}=\frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}=\frac{(\lambda_1+\lambda_2)^2-2\lambda_1\lambda_2}{\lambda_1\lambda_2}$$

$$=\frac{(16)^2-2(1)}{1}=254$$

Example 18. If α, β are the roots of the equation $x^2 - px + q = 0$, find the quadratic equation the roots of which are $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$ and $\alpha^3\beta^2 + \alpha^2\beta^3$.

Sol. Since, α, β are the roots of $x^2 - px + q = 0$.

$\therefore \quad \alpha + \beta = p, \alpha\beta = q$

$\Rightarrow \quad \alpha - \beta = \sqrt{(p^2 - 4q)}$

Now, $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$

$= (\alpha + \beta)(\alpha - \beta)(\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2)$

$= (\alpha + \beta)(\alpha - \beta)^2 \{(\alpha + \beta)^2 - \alpha\beta\}$

$= p(p^2 - 4q)(p^2 - q)$

and $\alpha^3\beta^2 + \alpha^2\beta^3 = \alpha^2\beta^2(\alpha + \beta) = pq^2$

S = Sum of roots = $p(p^2 - 4q)(p^2 - q) + pq^2$

$= p(p^4 - 5p^2q + 5q^2)$

P = Product of roots = $p^2q^2(p^2 - 4q)(p^2 - q)$

$\therefore$ Required equation is $x^2 - Sx + P = 0$

i.e. $x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 4q)(p^2 - q) = 0$

Exercise for Session 1

1. If $(a^2 - 1)x^2 + (a - 1)x + a^2 - 4a + 3 = 0$ be an identity in x, then the value of a is/are

(a) -1 (b) 1 (c) 3 (d) $-1, 1, 3$

2. The roots of the equation $x^2 + 2\sqrt{3}x + 3 = 0$ are

(a) real and unequal (b) rational and equal
(c) irrational and equal (d) irrational and unequal

3. If $a, b, c \in Q$, then roots of the equation $(b + c - 2a)x^2 + (c + a - 2b)x + (a + b - 2c) = 0$ are

(a) rational (b) non-real (c) irrational (d) equal

4. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, then $P(x)Q(x) = 0$ has atleast

(a) four real roots (b) two real roots
(c) four imaginary roots (d) None of these

5. If roots of the equation $(q - r)x^2 + (r - p)x + (p - q) = 0$ are equal, then p, q, r are in

(a) AP (b) GP (c) HP (d) AGP

6. If one root of the quadratic equation $ix^2 - 2(i + 1)x + (2 - i) = 0, i = \sqrt{-1}$ is $2 - i$, the other root is

(a) $-i$ (b) i (c) $2 + i$ (d $2 - i$

7. If the difference of the roots of $x^2 - \lambda x + 8 = 0$ be 2, the value of λ is

(a) ± 2 (b) ± 4 (c) ± 6 (d) ± 8

8. If $3p^2 = 5p + 2$ and $3q^2 = 5q + 2$ where $p \neq q$, pq is equal to

(a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{3}{2}$

9. If α, β are the roots of the quadratic equation $x^2 + bx - c = 0$, the equation whose roots are b and c, is

(a) $x^2 + \alpha x - \beta = 0$ (b) $x^2 - [(\alpha + \beta) + \alpha\beta]x - \alpha\beta(\alpha + \beta) = 0$
(c) $x^2 + [(\alpha + \beta) + \alpha\beta]x + \alpha\beta(\alpha + \beta) = 0$ (d) $x^2 + [(\alpha + \beta) + \alpha\beta]x - \alpha\beta(\alpha + \beta) = 0$

10. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + 1 = 0$ having real roots, is

(a) 15 (b) 9 (c) 8 (d) 7

11. If α and β are the roots of the equation $ax^2 + bx + c = 0$ ($a \neq 0, a, b, c$ being different), then $(1 + \alpha + \alpha^2)(1 + \beta + \beta^2)$ is equal to

(a) zero (b) positive (c) negative (d) None of these

Answers

Exercise for Session 1

1.(b)	**2.** (c)	**3.** (a)	**4.** (b)	**5.** (a)	**6.** (a)
7.(c)	**8.** (b)	**9.** (c)	**10.** (d)	**11.** (b)	