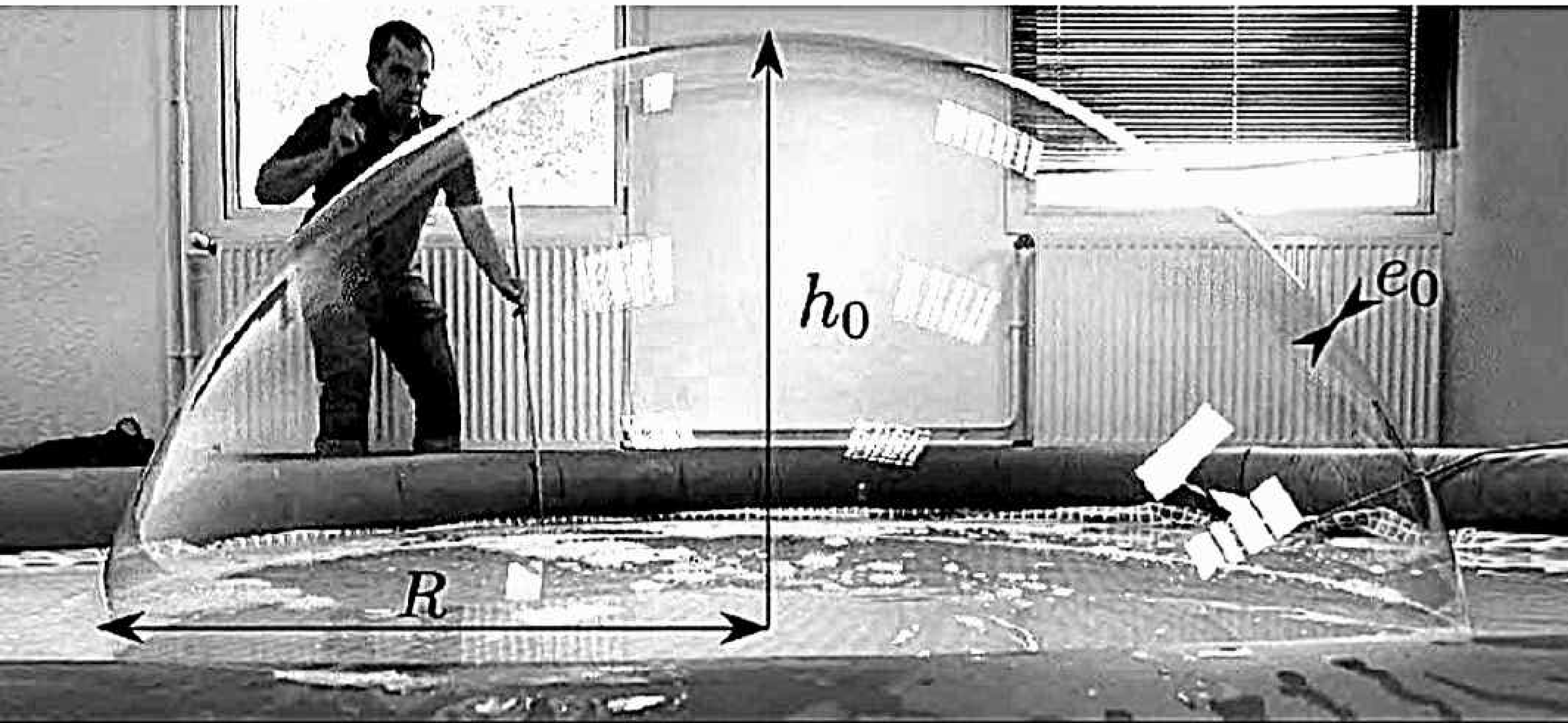
6 Applications of Derivatives



The sight of soap bubbles produced using a bubble wand is very exciting! One application of derivatives is finding the rate of increase in the size of the bubble (dV) due to its increasing radius, where V is the volume of the spherical bubble and r is its radius. This can be calculated by knowing the rate of increase of radius with time (dr/dt).

Topic Notes

- *Rate of Change of Quantities*
- *Increasing and Decreasing Functions*
- *Maxima and Minima*

RATE OF CHANGE OF QUANTITIES 1

TOPIC 1

AVERAGE AND INSTANTANEOUS RATE OF CHANGE

We have learnt to find derivative of various functions like composite, inverse trigonometric, implicit and logarithmic. The concept of derivative is applicable in engineering, basic sciences, social sciences and many other fields. In this chapter, we shall discuss a few examples as why the derivative is such a useful tool. We shall use differentiation to determine rate of change of quantities, turning points on the graph of a function and maximum and minimum values of a function.

Let us first study the physical meaning of the derivative and then apply it in some real-life situations.

Let $y = f(x)$ be a function of x. Let Δx be a small change in x and Δy be the corresponding change in y (Δx and Δy are called *increments*).

Then, $y + \Delta y = f(x + \Delta x)$,

or $\Delta y = f(x + \Delta x) - f(x)$

Now, average rate of change of y w.r.t. x is denoted by $\frac{\Delta y}{\Delta x}$; and instantaneous rate of change of y w.r.t. x is denoted by $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

Since,
$$\frac{dy}{dx} = f'(x)$$
$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Therefore, $\frac{dy}{dx}$ is the instantaneous rate of change of y w.r.t. x.

Caution

➥ *We now onwards, drop the term instantaneous and write "rate of change of y w.r.t. x" instead of "instantaneous rate of change of y w.r.t. x."*

The rate of change of y w.r.t. x is denoted by $\frac{dy}{dx}$.

The value of $\frac{dy}{dx}$ at $x = a$, i.e., $\left(\frac{dy}{dx}\right)_{x=a}$ represents the rate of change of y w.r.t. x at $x = a$.

Important

➥ *If the value of y increases with increase in the value of x, then Δy and Δx are both positive and hence $\frac{dy}{dx}$ is positive.*

➥ *If the value of y decreases with increase in the value of x, then Δy is negative and Δx is positive. So, $\frac{dy}{dx}$ is negative.*

Further, if two variables x and y both are varying with respect to another variable t, i.e., $x = f(t)$ and $y = f(t)$, then by chain rule

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \text{ provided } \frac{dx}{dt} \neq 0$$

With the help of an illustration, we explain the method of finding the rate of change.

Illustration: The radius of a circle is increasing uniformly at the rate of 2 cm/s. Find the rate at which area of the circle is increasing when the radius is 6 cm.

Step	Thinking Process	Applications
(1)	Identify the quantities (variables) whose rate is given, whose rate is to be obtained.	Rate of change of radius $\frac{dr}{dt} = 2$ cm/s is given and rate of change of area $\frac{dA}{dt}$ is to be determined, when radius (r) = 6 cm.
(2)	Write a relation involving the variables whose rate is given and whose rate is to be determined.	$A = \pi r^2$
(3)	Eliminate all the variables except those which are mentioned in Step 1.	$A = \pi r^2$ (here no other variable is appearing)
(4)	Differentiate the relation in Step 3 w.r.t. independent variable 't'.	Differentiate A w.r.t. t $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$
(5)	Put the given values and obtain the desired rate of change	$\left(\frac{dA}{dt}\right)_{r=6} = 2\pi(6) \times (2) = 24\pi$ cm^2/s

Example 1.1: An edge of a variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long? [NCERT]

Ans. Let x be the length of an edge of the cube and V be the volume of the cube at any time t.

Then, $\frac{dx}{dt} = 3$ cm/sec and $x = 10$ cm

Now, $V = x^3$

So, $\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

$$\Rightarrow \left(\frac{dV}{dt}\right)_{x=10} = 3(10)^2 \times 3$$

$$= 900$$

Thus, the rate of change in volume of the cube is 900 cm^3/s.

Example 1.2: A balloon, which always remains spherical, has a variable radius. Find the rate at which its volume is increasing with the radius when the latter is 10 cm. [NCERT]

Ans. Let r be the radius of the balloon and V be its volume. Then, $r = 10$ cm.

Now, $V = \frac{4}{3}\pi r^3$

$$\Rightarrow \frac{dV}{dr} = \frac{4}{3}\pi \times (3r^2) = 4\pi r^2$$

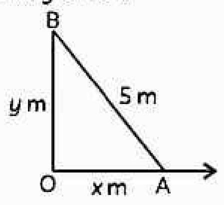
$$\Rightarrow \left(\frac{dV}{dr}\right)_{r=10} = 4\pi(10)^2$$

$$= 400\pi$$

Thus, the rate of change in volume of the balloon is 400π cm^3/cm.

Example 1.3: A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall? [NCERT]

Ans. Let the foot of the ladder be x m away from the wall and let the ladder reach at a height of y m on the wall at any time t.



Then, OA = x m, OB = y m and AB = 5 m.

$$\therefore \frac{dx}{dt} = 2 \text{ cm/s} = 0.02 \text{ m/s}$$

Applying Pythagoras theorem in ΔAOB, we have

$$x^2 + y^2 = 5^2 \quad ...(i)$$

Differentiating both sides w.r.t. t, we have

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow 2x(0.02) + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = -\frac{0.02x}{y} \quad ...(ii)$$

Now on putting $x = 4$ in (i), we get $y = 3$

So, putting these values of x and y in (ii), we get

$$\frac{dy}{dt} = -\frac{0.02(4)}{(3)}$$

$$= -\frac{0.08}{3} \text{ m/s, or } -\frac{8}{3} \text{ cm/s}$$

Here, negative sign denotes that the height is decreasing.

Thus, rate of decrease of height of the ladder on the wall is $\frac{8}{3}$ cm/s.

Marginal Cost

Marginal cost represents the instantaneous rate of change of the total cost with respect to the number of items produced at that instant.

If C(x) represents the cost function for x units produced, then marginal cost, denoted by MC, is given by

$$MC = \frac{d}{dx}\{C(x)\}$$

Example 1.4: The total cost C(x) of producing x items in a firm is given by

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 6000$$

Find the marginal cost when 4 units are produced.

Ans. We have, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 6000$

$\therefore$ Marginal cost,

$$MC = \frac{dC}{dx}$$

$$= 0.005 \times 3x^2 - 0.02 \times 2x + 30$$

$$\Rightarrow [MC]_{x=4} = (0.005 \times 3 \times 16) - (0.02 \times 2 \times 4) + 30$$

$$= 0.24 - 0.16 + 30 = 30.08$$

Hence, the required marginal cost is ₹ 30.08.

Marginal Revenue

Marginal revenue represents the rate of change of total revenue with respect to the number of items sold at an instant.

If R(x) represents the revenue function for x units sold, then marginal revenue, denoted by MR, is given by

$$MR = \frac{d}{dx}\{R(x)\}$$

Important

➥ *Total cost = Fixed cost + Variable cost*

i.e., $C(x) = f(c) + v(x)$

Example 1.5: The total revenue (in ₹) received from the sale of x units of a product is given by $R(x) = 3x^2 + 6x + 5$. Find the marginal revenue, when $x = 5$. **[NCERT]**

Ans. Given, $R(x) = 3x^2 + 6x + 5$.

We know that marginal revenue is the rate of change of total revenue with respect to the number of units sold.

$\therefore$ Marginal Revenue, $MR = \frac{dR}{dx} = 6x + 6$

When $x = 5$, then $MR = 6(5) + 6 = 36$.

Hence, the required marginal revenue is ₹ 36.

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. If $V = \frac{4}{3}\pi r^3$, then at what rate in cubic units is V increasing when $r = 20$ and $\frac{dr}{dt} = 0.02$ unit/time?

(a) 2π unit^3/time (b) 32π unit^3/time
(c) π unit^3/time (d) 16π unit^3/time

2. The total cost C(x) associated with the production of x units of an item is given by

$$C(x) = 0.008x^3 - 0.04x^2 + 20x + 6000$$

Find the marginal cost when 5 units are sold.

(a) 20.6 (b) 20.2
(c) 20.8 (d) 21

Ans. *(b) 20.2*

Explanation: Given:

$$C(x) = 0.008x^3 - 0.04x^2 + 20x + 6000$$

Marginal cost (MC) $= \frac{d}{dx}(C(x))$

$$= \frac{d}{dx}(0.008x^3 - 0.04x^2 + 20x + 6000)$$

$$= 0.024x^2 - 0.08x + 20$$

When $x = 5$,

$$MC = 0.024(5)^2 - 0.08(5) + 20$$
$$= 0.6 - 0.40 + 20$$
$$= 20.2$$

3. The rate of change of the area of a circle w.r.t. its radius r at $r = 6$ cm is:

(a) 10π cm^2/cm (b) 12π cm^2/cm
(c) 8π cm^2/cm (d) 11π cm^2/cm

4. The total revenue received from the sale of x units of a product is given by

$$R(x) = 3x^2 + 36x + 5$$

The marginal revenue, when $x = 15$ is:

(a) 116 (b) 96
(c) 90 (d) 126

Ans. *(d) 126*

Explanation: Marginal revenue (MR) is given by $\frac{dR}{dx}$.

Here, $\frac{dR}{dx} = 6x + 36$

$\Rightarrow \left(\frac{dR}{dx}\right)_{x=15} = 6(15) + 36$, *i.e.,* 126

5. The point on the parabola $y^2 = 8x$ at which the ordinate increases as much as the abscissa is:

(a) (2, 4) (b) (2, – 4)
(c) (8, 8) (d) $(1, 2\sqrt{2})$

6. The surface area of a ball is increasing at the rate of 2π sq. cm/sec. The rate at which the radius is increasing when the surface area is 16π sq. cm, is:

(a) 0.125 cm/sec (b) 0.25 cm/sec
(c) 0.6 cm/sec (d) 1 cm/sec

7. A stone is thrown up vertically and the height 'x' cm reached by it in time 't' seconds is given by $x = 80t - 16t^2$. The stone reaches the maximum height in time

(a) 2 seconds (b) 2.5 seconds
(c) 3 seconds (d) 3.5 seconds

Ans. *(b) 2.5 seconds*

Explanation: Here, $x = 80t - 16t^2$

$$\Rightarrow \quad \frac{dx}{dt} = 80 - 32t$$

Maximum height is attained when $\frac{dx}{dt} = 0$

So, $\quad 80 - 32t = 0$ gives $t = 2.5$

8. **A spherical balloon is being inflated at the rate of 35 cu. cm/sec. How fast is the surface area of the balloon increases when its diameter is 14 cm?**

(a) 5 sq. cm/sec (b) 6 sq. cm/sec

(c) 8 sq. cm/sec (d) 10 sq. cm/sec

9. If the rate of decrease of $\frac{1}{2}x^2 - 2x + 5$ is twice the rate of decrease of x, then the value of x is:

(a) 1 (b) 2

(c) 3 (d) 4

Ans. *(d) 4*

Explanation: Here,

$$\frac{d}{dt}\left[\frac{1}{2}x^2 - 2x + 5\right] = x\frac{dx}{dt} - \frac{2\,dx}{dt}$$

But, $$2\frac{dx}{dt} = x\frac{dx}{dt} - 2\frac{dx}{dt}$$

$$\left\{\because \frac{d}{dt}\left[\frac{1}{2}x^2 - 2x + 5\right] = 2\frac{dx}{dt}, \text{given}\right\}$$

$$\Rightarrow \quad 2 = x - 2, \text{ i.e., } x = 4$$

10. **The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. The rate at which the area increases, when side is 10 cm is:**

(a) 10 cm^2/s (b) $\sqrt{3}$ cm^2/s

(c) $10\sqrt{3}$ cm^2/s (d) $\frac{10}{3}$ cm^2/s

[NCERT Exemplar]

11. An insect moves along the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$. When the rate of change of abscissa is 4 times that of the ordinate, then the insect lies in the quadrant:

(a) I or II (b) II or III

(c) III or IV (d) II or IV

Ans. *(b) II or IV*

Explanation: Given $\frac{x^2}{16} + \frac{y^2}{4} = 1$, we have

$$\Rightarrow \quad \frac{1}{8}x\frac{dx}{dt} + \frac{1}{2}y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad \frac{1}{8}x\left(4\frac{dy}{dt}\right) + \frac{1}{2}y\frac{dy}{dt} = 0$$

$$\left[\because \frac{dx}{dt} = 4\frac{dy}{dt}, \text{given}\right]$$

$\Rightarrow x + y = 0$, or $x = -y$

Thus, insect lies in II or IV quadrant.

12. **A ladder, 5 metre long, standing on a horizontal floor, leans against a vertical wall. If the top of the ladder slides downwards at the rate of 10 cm/sec, then the rate at which the angle between the floor and the ladder is decreasing when lower end of ladder is 2 metres from the wall is:**

(a) $\frac{1}{10}$ radian/sec (b) $\frac{1}{20}$ radian/sec

(c) 20 radian/ sec (d) 10 radian/sec

[NCERT Exemplar]

13. **The altitude of an equilateral triangle is increasing at the rate of $\sqrt{3}$ cm/sec. The rate of increase of the area of the triangle, when the altitude is $5\sqrt{3}$ cm, is:**

(a) $\frac{10}{\sqrt{3}}$ sq. cm/sec (b) 10 sq. cm/sec

(c) $5\sqrt{3}$ sq. cm/sec (d) $10\sqrt{3}$ sq. cm/sec

Ans. *(d) $10\sqrt{3}$ sq. cm/sec*

Explanation: Area (A) of an equilateral triangle in terms of its altitude (p) is given by

$$A = \frac{p^2}{\sqrt{3}}$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{2}{\sqrt{3}}p\frac{dp}{dt} \quad ...(i)$$

We are given that $\frac{dp}{dt} = \sqrt{3}$. So, by (i)

$$\left(\frac{dA}{dt}\right)_{p=5\sqrt{3}} = \left(\frac{2}{\sqrt{3}}p\frac{dp}{dt}\right)_{p=5\sqrt{3}}$$

$$\Rightarrow \quad \left(\frac{dA}{dt}\right)_{p=5\sqrt{3}} = \frac{2}{\sqrt{3}}(5\sqrt{3})(\sqrt{3}) = 10\sqrt{3}$$

14. **A cylindrical tank of radius 10 m is being filled with wheat at the rate iof 314 cubic metre per hour. Then, the depth of the wheat is increasing at the rate of**

(a) 1 m/h (b) 0.1 m/h

(c) 1.1 m/h (d) 0.5 m/h

15. **Cylindrical water storage tanks are installed on the roofs of all multi-storeyed buildings. These come in different capacities starting from 100 litres. Raman went to the terrace of his building to check the water flow in his tank as he suspected some leakage.**

Water is flowing out of a cylindrical tank at the rate of π cu. cm/sec. If the height of the water level is decreasing at the rate of 0.01 cm/sec, then the radius of that tank is

(a) 1 cm (b) 10 cm

(c) 20 cm (d) 100 cm

Ans. *(b) 10 cm*

Explanation: Here, volume (V) of the tank

$$= \pi r^2 h$$

$$\Rightarrow \quad \frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \quad (r \text{ is constant})...(i)$$

It is given that $\frac{dV}{dt} = -\pi; \frac{dh}{dt} = -0.01$

Put the values in (i),

$$-\pi = \pi r^2(-0.01)$$

$$\Rightarrow \quad r = 10 \text{ cm}$$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

16. **If a particle moves in a straight line such that the distance travelled in time t is given by $S = 3t^3 - 5t^2 + 8t + 7$. Find the initial velocity of the particle.**

Ans. Given, $S = 3t^3 - 5t^2 + 8t + 7$

On differentiating with respect to t, we get

$$\frac{dS}{dt} = 9t^2 - 10t + 8$$

Initially, $t = 0$, therefore

$$\frac{dS}{dt} = 9(0)^2 - 10(0) + 8$$

$$= 8 \text{ units/time}$$

17. **A cylinder vessel of radius 0.8 m is filled with oil at the rate of 0.30π m^3/minute. Find the rate at which the surface of the oil is rising.**

18. **The side of a square is increasing at the rate 0.5 cm/s. Find the rate of increase of its perimeter.**

19. **A balloon, which always remains spherical, has a variable diameter $(3x + 4)$. Determine the rate of change of volume with respect to x at $x = 4$.**

Ans. Given: $d = 3x + 4$

$$\Rightarrow \quad r = \frac{3x+4}{2}$$

Let volume of sphere be V cubic units.

Then, $$V = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi\left(\frac{3x+4}{2}\right)^3$$

On differentiating with respect to x, we get

$$\frac{dV}{dx} = \frac{4}{3}\pi \times 3\left(\frac{3x+4}{2}\right)^2 \frac{d}{dx}\left(\frac{3x+4}{2}\right)$$

$$= \frac{4\pi(3x+4)^2}{4} \times \frac{1}{2}(3)$$

$$= \frac{3\pi}{2}(3x+4)^2$$

At $x = 4$,

$$\frac{dV}{dx} = \frac{3\pi}{2}(3 \times 4 + 4)^2$$

$$= \frac{3\pi}{2} \times (16)^2$$

$$= 384\pi \text{ unit}^3/\text{unit}$$

20. **The radius of a circle is increasing at the uniform rate of 3 cm/sec. At the instant when the radius of the circle is 2 cm, find its area increases at the rate of cm^2/sec.**

[CBSE 2020]

21. **Find an angle θ, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, which increases twice as fast as its cosine.**

Ans. Given that θ increases twice as fast as its cosine.

$$\therefore \quad \theta = 2\cos\theta$$

On differentiating with respect to 't', we get

$$\frac{d\theta}{dt} = -2\sin\theta\frac{d\theta}{dt}$$

$$\Rightarrow \quad 1 = -2\sin\theta$$

$$\Rightarrow \quad \sin\theta = -\frac{1}{2}$$

$$\Rightarrow \quad \theta = -\frac{\pi}{6}$$

22. **The total cost C(x) with the production of x units of an item is given by**

$$C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$$

Find the marginal cost when 17 units are produced. [NCERT]

23. **The radius of balloon is increasing at the rate of 20 m/s. At what rate is the surface area of the balloon increasing when the radius is 30 m?**

24. **Employee welfare entails everything from services, facilities and benefits that are provided or done by an employer for the advantage or comfort of an employee. It is undertaken in order to motivate employees and raise the productivity levels. In most cases, employee welfare comes in monetary form, but it doesn't always bend that way. Other forms of employee welfare include housing, health insurance, stipends, transportation and provision of food. An employer may also cater for employees' welfare by monitoring their working conditions.**

The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when $x = 5$.

[CBSE 2013]

Ans. Total revenue is given by

$$R(x) = 3x^2 + 36x + 5$$

$$\therefore \text{Marginal revenue} = \frac{dR}{dx} = 6x + 36$$

$$\text{Now, } \left(\frac{dR}{dx}\right)_{x=5} = 6\times5+36 = 66$$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

25. **The side of an equilateral triangle is increasing at the rate of 3 cm/sec. At what rate is its area increasing when the side of the triangle is 30 cm?**

Ans. Let 'a' be the side of an equilateral triangle

Then $\frac{da}{dt} = 3$ cm/sec

Let 'A' be the area of an equilateral triangle. Then,

$$A = \frac{\sqrt{3}}{4}a^2$$

$$\Rightarrow \quad \frac{dA}{dt} = \frac{\sqrt{3}}{4}\left(2a\frac{da}{dt}\right)$$

$$= \frac{\sqrt{3}}{2}a\frac{da}{dt}$$

$$\therefore \quad \left(\frac{dA}{dt}\right)_{a=30} = \frac{\sqrt{3}}{2}\times30\times3$$

$$= 45\sqrt{3}\ \text{cm}^2/\text{sec}$$

26. **The total revenue received from the sale of x units of a product is given by $P(x) = 3x^2 + 36x + 5$ in rupees. Find the marginal revenue when $x = 5$, where by marginal revenue we mean the rate of change of total revenue with respect to the number of items sold at an instant. [CBSE 2018]**

27. **A stone is dropped into a quiet lake and waves move in a circle at a speed of 3.5 cm/sec. At the instant when the radius of the circular wave is 7.5 cm, how fast is the enclosed area increasing?**

Ans. Let A and r be the area and radius of the circular wave at any time t.

Then, $A = \pi r^2$ and $\frac{dr}{dt} = 3.5$ cm/sec

$$\therefore \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

When $r = 7.5$ cm/sec

$$\frac{dA}{dt} = 2\pi \times 7.5 \times 3.5$$

$$= 52.5\pi \text{ cm}^2/\text{sec}$$

Hence, enclosing area is increasing at the rate of 52.5π cm^2/sec.

28. **x and y are the sides of two squares such that $y = x - x^2$. Find the rate of change of the area of second square with respect to the area of first square. [NCERT Exemplar]**

29. **The total cost C(x) associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$.**

Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output. [CBSE 2018]

Ans. Given: $C(x) = 0.005x^3 - 0.002x^2 + 30x + 5000$

Now marginal cost, $MC(x) = \frac{d}{dx}\{C(x)\}$

$$= \frac{d}{dx}(0.005x^3 - 0.02x^2 + 30x + 5000)$$

$$= 0.005 \times 3x^2 - 0.02 \times 2x + 30 + 0$$

$$= 0.015x^2 - 0.04x + 30$$

When production is 3 units, then marginal cost is

$$MC(3) = 0.015\,(3)^2 - 0.04(3) + 30$$

$$= 0.135 - 0.12 + 30$$

$$= 30.015$$

Hence, the marginal cost is ₹ 30.015.

30. **The volume of a cube is increasing at a constant rate. Prove that the increase in surface area varies inversely as the length of the edge of the cube. [DIKSHA]**

31. **A spherical ball of salt is dissolving in water in such a manner that the rate of decrease of the volume at any instant is proportional to the surface. Prove that the radius is decreasing at a constant rate.**

[NCERT Exemplar]

Ans. Let the radius of the spherical ball be r.

Then, surface area at any time t, $S = 4\pi r^2$

And volume at any time t, $V = \frac{4}{3}\pi r^3$

According to the question,

$$\frac{dV}{dt} \propto S$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \times 4\pi r^2$$

$$\Rightarrow \quad \frac{4}{3}\pi \times 3r^2 \times \frac{dr}{dt} = k \times 4\pi r^2$$

$$\Rightarrow \quad \frac{dr}{dt} = k$$

Hence, the radius of ball is decreasing at a constant rate.

Hence, proved.

Concept Applied

➥ *First consider the volume of ball (V) and surface area of ball (S), then by using $\frac{dV}{dt} \propto S$, prove the desired result.*

32. **The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm. [CBSE 2017]**

33. **Find an angle θ, where $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine.**

[NCERT Exemplar]

Ans. It it given that

$$\theta = 2\sin\theta$$

Differentiating w.r.t. t, we get

$$\frac{d\theta}{dt} = 2\left(\cos\theta\frac{d\theta}{dt}\right)$$

$$\Rightarrow \quad \cos\theta = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

34. **A particle moves along the curve $8y = 2x^3 + 5$. Find the points on the curve at which the y coordinate is changing 12 times as fast as the x-coordinate.**

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

35. A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall? [CBSE 2019]

Ans. Let foot of the ladder be at a distance x from the wall and height on the wall be y.

Here, $x^2 + y^2 = (13)^2$

On differentiating with respect to t, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

When $x = 5$ cm, $y^2 = (13)^2 - (5)^2 = 169 - 25 = 144$

$\therefore \quad y = 12$ cm

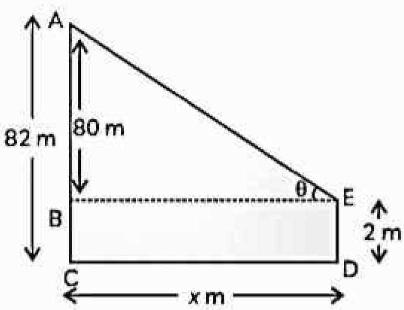
Also, $\frac{dx}{dt} = 2\text{ cm}^2/\text{sec}$

$\therefore \quad \frac{dy}{dt} = \frac{-5}{12} \times 2 = \frac{-5}{6}$ cm/sec

Hence, height on the wall is decreasing at the rate of $\frac{5}{6}$ cm/sec.

36. A man is moving away from a tower 82 m high at the rate of 10 m/s. Find the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 60 m from the foot of the tower. Assume that the eye level of the man is 2 m from the ground.

Ans. Let height of the tower be AC = 82 m and CD = x m be the horizontal distance from tower. Let DE = 2 m be the position of eye level of the man.



Let $\angle BEA = \theta$ be the angle of elevation.

Given $\frac{dx}{dt} = 10$ m/s

In ΔABE, $\quad \tan\theta = \frac{80}{x} \quad$ ...(i)

$\Rightarrow \quad x = 80\cot\theta$

On differentiating with respect to t, we get

$$\frac{dx}{dt} = 80\left(-\text{cosec}^2\theta\frac{d\theta}{dt}\right)$$

$\Rightarrow \quad 10 = -80\,\text{cosec}^2\theta\frac{d\theta}{dt}$

$\Rightarrow \quad \frac{d\theta}{dt} = -\frac{1}{8\,\text{cosec}^2\theta} \quad$ ...(ii)

From Eq. (i), when $x = 60$ m, then

$$\tan\theta = \frac{80}{60} = \frac{4}{3}$$

$\Rightarrow \quad \cot\theta = \frac{3}{4}$

Now $\quad \text{cosec}\,\theta = \sqrt{1 + \cot^2\theta} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$

$\therefore$ From Eq. (ii), we get

$$\frac{d\theta}{dt} = -\frac{1}{8 \times \frac{25}{16}} = \frac{-2}{25} \text{ rad/sec}$$

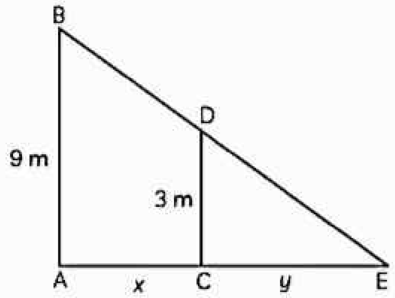
Hence angle of elevation of the top of the tower is decreasing at the rate of $\frac{2}{25}$ rad/sec.

37. A kite is moving horizontally at a height of 151.5 m. If the speed of kite is 10 m/s, how fast is the string being let out, when the kite is 250 away from the boy who is flying the kite. The height of boy is 1.5 m.

[NCERT Exemplar]

38. A man, 3 m tall, walks at a uniform speed of 9 km/h away from a lamp post 9 m high. At what rate, the length of the shadow increases?

Ans. Let AB be the lamp post and a man CD be at a distance x from the lamp post and let CE = y be his shadow.



Given $\frac{dx}{dt} = 9 \text{km/h}$, $AB = 9 \text{ m} = \frac{9}{1000}$ km

and $CD = 3\text{m} = \frac{3}{1000}$ km

[$\because$ 1 km = 1000 m]

Here, $\Delta ABE \sim \Delta CDE$

$$\therefore \quad \frac{AB}{CD} = \frac{AE}{CE}$$

$$\Rightarrow \quad \frac{9/1000}{3/1000} = \frac{x+y}{y}$$

$$\Rightarrow \quad 3 = \frac{x+y}{y}$$

$$\Rightarrow \quad x + y = 3y$$

$$\Rightarrow \quad 2y = x$$

On differentiating with respect to t, we get

$$2\frac{dy}{dt} = \frac{dx}{dt}$$

$$\Rightarrow \quad 2\frac{dy}{dt} = 9$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{9}{2} = 4.5 \text{km/h}$$

Hence, the shadow of the man increases at the rate of 4.5 km/h.

39. A swimming pool is to be drained for cleaning. If L represents the number of litres of water in the pool t seconds after the pool has been plugged off to drain and $L = 200\ (10 - t)^2$. How fast is the water running out at the end of 5 seconds? What is the average rate at which the water flows out during the first 5 seconds?

[NCERT Exemplar]

Ans. Here, L represents the number of litres of water in the pool 't' seconds after the pool has been plugged off to drain.

Also, $L = 200\,(10 - t)^2$

$\therefore$ Rate at which water is running out $= \frac{-dL}{dt}$

$$\Rightarrow \quad \frac{-dL}{dt} = 200.\,2(10-t)\,(-1)$$

$$\Rightarrow \quad \frac{dL}{dt} = 400\,(10-t)$$

Rate at which the water is running out at the end of 5 seconds

$= 400\,(10 - 5)$

$= 2000$ L/s = final rate

Since, initial rate $= \left(\frac{-dL}{dt}\right)_{t=0} = 4000$ L/s

$\therefore$ Average rate during 5 seconds

$$= \frac{4000+2000}{2}$$

$$= \frac{6000}{2} = 3000 \text{ L/s}$$

40. The volume of a cube is increasing at the rate of 8 cm^3/s. How fast is the surface area increasing when the length of its edge is 12 cm? [CBSE 2019]

41. Sand is pouring from a pipe at the rate of 12 cm^2/sec. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand-cone increasing when the height is 4 cm?

Ans. Let V, r and h be the volume, radius and height, respectively, of the sand-cone at any time t.

Now, volume of sand-cone,

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(6h)^2 h \quad \left[\because h = \frac{1}{6}r\right]$$

$$\therefore \quad V = 12\,\pi h^3$$

On differentiating with respect to t, we get

$$\frac{dV}{dt} = 36\pi h^2 \frac{dh}{dt}$$

$$\Rightarrow \quad 12 = 36\pi h^2 \frac{dh}{dt}$$

$$\left[\because \frac{dV}{dt} = 12 \text{ given}\right]$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{1}{3\pi h^2}$$

When $h = 4$ cm, then

$$\frac{dh}{dt} = \frac{1}{3\pi(4)^2}$$

$$= \frac{1}{48\pi} \text{ cm/sec.}$$

Hence, the height of the sand-cone is increasing at the rate of $\frac{1}{48\pi}$ cm/sec.

42. The length x of a rectangle is decreasing at the rate of 5 cm/minute and the width y is increasing at the rate of 4 cm/minute.

When $x = 8$ cm and $y = 6$ cm, find the rate of change of

(A) the perimeter

(B) area of rectangle [CBSE 2017]

Ans. We have given that length of a rectangle is decreasing at the rate of 5 cm/minute.

$$\therefore \quad \frac{dx}{dt} = -5 \text{ cm/minute}$$

Also, we have given that breadth of a rectangle is increasing at the rate of 4 cm/minute

$$\therefore \quad \frac{dy}{dt} = 4 \text{ cm/minute}$$

(A) Now, perimeter of rectangle,

$$P = 2(x + y)$$

On differentiating with respect to t, we get

$$\frac{dA}{dt} = 2\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$$

$$= 2(-5 + 4)$$

$$= -2 \text{ cm/minute}$$

Here, rate of change of perimeter have negative sign, it means that perimeter of a rectangle is decreasing at the rate of 2 cm/minute.

(B) Now, area of rectangle, $A = xy$

On differentiating with respect to t, we get

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

When, $x = 8$ cm and $y = 6$ cm, then

$$\frac{dA}{dt} = 8 \times 4 + 6 \times (-5)$$

$$= 32 - 30 = 2 \text{ cm}^2\text{/minute}$$

Hence, the area of rectangle is increasing at the rate of 2 cm^2/minute.

43. Two men A and B start with velocities v at the same time from the junction of two roads inclined at 45° to each other. If they travel by different roads, then find the rate at which they are being separated.

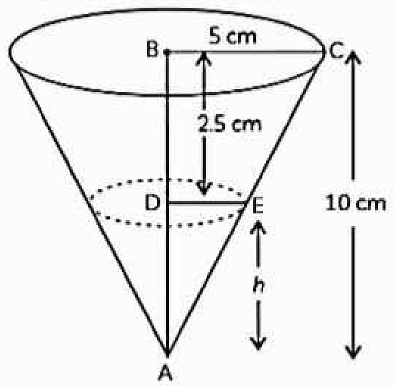
[NCERT Exemplar]

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

44. Water is leaking from a conical funnel at the rate of 5 cm^3/s. If the radius of the base of funnel is 5 cm and height is 10 cm, then find the rate of which the water level is dropping, when it is 2.5 cm from the top.

Ans. Let r and h be the radius and height of conical funnel and V be the volume of water.



Given $r = 5$ cm and $h = 10$ cm.

Since, water is leaking from a conical funnel, so we take negative sign for rate of change of volume.

$$\therefore \quad \frac{dV}{dt} = -5 \text{ cm}^3\text{/s}$$

and $h = 10 - 2.5 = 7.5$ cm

Since $\quad \Delta ABC \sim \Delta ADE$

$$\therefore \quad \frac{AB}{AD} = \frac{BC}{DE}$$

$$\Rightarrow \quad \frac{10}{h} = \frac{5}{r}$$

$$\Rightarrow \quad 10r = 5h$$

$$\Rightarrow \quad r = \frac{h}{2}$$

Since, volume of cone $= \frac{1}{3}\pi r^2 h$

$\therefore$ Volume of water in conical funnel,

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

On differentiating with respect to t, we get

$$\frac{dV}{dt} = \frac{\pi}{12} \times 3h^2 \times \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{\pi}{4}h^2 \times \frac{dh}{dt}$$

$$\Rightarrow \quad -5 = \frac{\pi}{4} \times (7.5)^2 \times \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{-5 \times 4}{\pi} \times \frac{1}{56.25}$$

$$= \frac{-20 \times 100}{\pi \times 5625}$$

$$= \frac{-16}{45\pi} \text{ cm/s.}$$

Hence, the rate at which the water level is dropping, is $\frac{16}{45\pi}$ cm/s.

45. **A man, 2 m tall, walks at the rate of $1\frac{2}{3}$ m/s towards a street light which is $5\frac{1}{3}$ m above the ground. At what rate is the tip of his shadow moving? At what rate is the length of the shadow changing when he is $3\frac{1}{3}$ m from the base of the light?[NCERT Exemplar]**

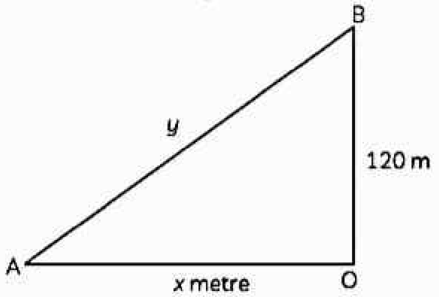
46. **An inverted cone has a depth of 20 cm and a base of radius 10 cm. Water is poured into it at the rate of $\frac{3}{2}$ cm^3/minute. Find the rate at which the level of water in the cone is rising when the depth is 8 cm.**

47. **(A) A man is walking at the rate of 6.5 km/hr towards the foot of a tower, 120 m high. At what rate is he approaching the top of the tower when he is 50 m away from the tower?**

(B) A stone is dropped into a quiet lake and waves move in a circle at a speed of 4.5 cm/sec. At the instant when the radius of the circular wave is 9.5 cm, how fast is the enclosed area increasing?

Ans. (A) Let at any time t, the man be at distance of x and y metres from the foot and top of the tower respectively. Then

$$y^2 = x^2 + (120)^2 \quad \text{...(i)}$$



On differentiating with respect to t, we get

$$2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$$

Given that, $\frac{dx}{dt} = -6.5$ km/hr.

$$\therefore \quad \frac{dy}{dt} = -\frac{6.5x}{y} \quad \text{...(ii)}$$

Put $x = 50$ in Eqn. (i), we get

$$y^2 = (50)^2 + (120)^2$$

$$\Rightarrow \quad = \sqrt{2500 + 14400}$$

$$= \sqrt{16900} = 130$$

Put $x = 50$, $y = 130$ in Eq. (ii), we get

$$\frac{dy}{dt} = -\frac{6.5 \times 50}{130}$$

$$= -2.5$$

Hence, the man is approaching the top of the tower at the rate of 2.5 km/hr.

(B) Let r be the radius and A be the area of the circular wave at any time t. Then

$A = \pi r^2$ and $\frac{dr}{dt} = 4.5$ cm/sec

$$\text{Now,} \quad \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dA}{dt} = 2\pi r\,(4.5)$$

$$\Rightarrow \quad \left(\frac{dA}{dt}\right)_{r=9.5} = 9\pi \times 9.5$$

$$= 85.5\,\pi \text{ cm}^2\text{/sec}$$

Hence, the enclosed area is increasing at the rate of 85.5π cm^2/sec.

INCREASING AND DECREASING FUNCTIONS 2

TOPIC 1

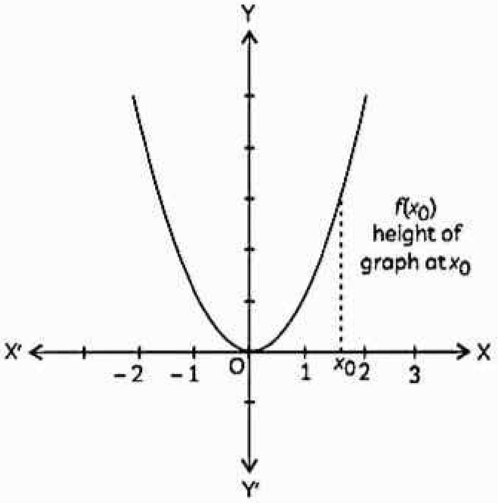
INCREASING AND DECREASING FUNCTIONS ON AN INTERVAL

Let $y = f(x)$ be any function. A change in value of x, either increased or decreased, certainly will make a change in the value of $f(x)$ [except when $f(x)$ is a constant function], either it will be increased or decreased depending on the function. Now, we will study this behaviour of a function in an interval and how to use derivative of a function to determine whether it is increasing or decreasing.

Values left to origin:

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
$-\frac{1}{2}$	$\frac{1}{4}$
0	0

As we move from left to right, the height of the graph decreases



Values right to origin

x	$f(x) = x^2$
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4

As we move from left to right, the height of the graph increases

Let f be a real valued function and let I be any interval in the domain of f. Then, f is said to be:

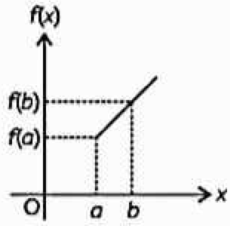
(1) strictly increasing on I, if for all $x_1, x_2 \in$ I, we have $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

(2) increasing on I, if for all $x_1, x_2 \in$ I, we have $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.

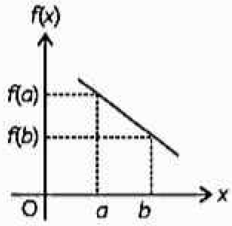
(3) strictly decreasing on I, if for all $x_1, x_2 \in$ I, we have $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

(4) decreasing on I, if for all $x_1, x_2 \in$ I, we have $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

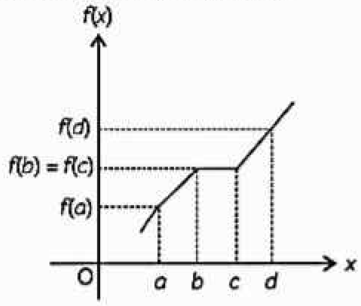
Let us now consider the following graphs to understand the above terms:



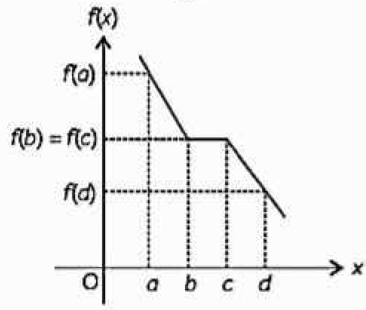
(a)



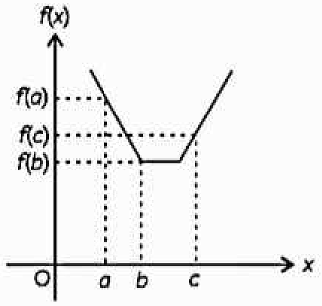
(b)



(c)



(d)



(e)

We observe the following:

(1) The function shown in Fig. (a) is strictly increasing on I. We see that $a, b \in I$ and $a < b \Rightarrow f(a) < f(b)$.

(2) The function shown in Fig. (b) is strictly decreasing on I. We see that $a, b \in I$ and $a < b \Rightarrow f(a) > f(b)$.

(3) The function shown in Fig. (c) is increasing on I. We see that $a, b \in I$ and $a < b \Rightarrow f(a) \leq f(b)$ for all $a, b \in I$.

(4) The function shown in Fig. (d) is decreasing on I. We see that $a, b, c, d \in I$ and $a < b \Rightarrow f(a) \geq f(b)$ for all $a, b \in I$.

(5) The function shown in Fig. (e) is neither increasing nor decreasing on I. We see that $a, b \in I$ and $a < b \Rightarrow f(a) > f(b)$; $b < c \Rightarrow f(b) < f(c)$.

Important

- *Constant function is both increasing and decreasing functions.*
- *Every strictly increasing function is an increasing function, but an increasing function may not be strictly increasing function.*
- *Every strictly decreasing function is a decreasing function, but a decreasing function may not be strictly decreasing function.*

Example 2.1: Show that the function given by $f(x) = 3x + 17$ is increasing on R. [NCERT]

Ans. Given, $f(x) = 3x + 17$ on R

Let $x_1, x_2 \in R$ such that $x_1 < x_2$.

Now, $x_1 < x_2 \Rightarrow 3x_1 < 3x_2 \Rightarrow 3x_1 + 17 < 3x_2 + 17$, *i.e.*, $f(x_1) < f(x_2)$.

Hence, the given function is increasing (rather *strictly increasing*) on R.

Example 2.2: Show that the function given by $f(x) = \sin x$ is:

(A) increasing in $\left(0, \frac{\pi}{2}\right)$

(B) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(C) neither increasing nor decreasing in $(0, \pi)$

[NCERT]

Ans. (A) Let $x_1, x_2 \in \left(0, \frac{\pi}{2}\right)$ such that $x_1 < x_2$.

Now, $x_1 < x_2 \Rightarrow \sin x_1 < \sin x_2$, *i.e.*, $f(x_1) < f(x_2)$. Hence, the given function is increasing (rather *strictly increasing*) in $\left(0, \frac{\pi}{2}\right)$.

(B) Let $x_1, x_2 \in \left(\frac{\pi}{2}, \pi\right)$ such that $x_1 < x_2$.

Now, $x_1 < x_2 \Rightarrow \sin x_1 > \sin x_2$,

i.e., $f(x_1) > f(x_2)$

This is because value of $\sin x$ decreases, as x increases from $\frac{\pi}{2}$ to π.

Hence, the given function is decreasing (rather *strictly decreasing*) in $\left(\frac{\pi}{2}, \pi\right)$.

(C) When $x \in (0, \pi)$, we find that $f(x) = \sin x$ is increasing in $\left(0, \frac{\pi}{2}\right)$ and decreasing in $\left(\frac{\pi}{2}, \pi\right)$. So, $f(x) = \sin x$ is neither increasing nor decreasing in $(0, \pi)$.

TOPIC 2

INCREASING AND DECREASING FUNCTIONS, USING DERIVATIVES

We shall now use differentiation to determine the intervals on which a given function is *increasing, strictly increasing, decreasing or strictly decreasing.* In this regard, we shall state and prove an important theorem, known as First Derivative Test.

Theorem: Let f be a continuous function on $[a, b]$ and differentiable in (a, b). Then,

(1) If $f'(x) > 0$ for each $x \in (a, b)$, then f is strictly increasing in $[a, b]$.

(2) If $f'(x) < 0$ for each $x \in (a, b)$, then f is strictly decreasing in $[a, b]$.

(3) If $f'(x) = 0$ for each $x \in (a, b)$, f is constant in $[a, b]$.

Proof: (1) Let $f'(x) > 0$ for each $x \in (a, b)$.

Let $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$.

Clearly, f is continuous in $[x_1, x_2]$ and differentiable in (x_1, x_2).

So, by the Lagrange's Mean Value Theorem, there exists a real number $c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

Since $f'(x) > 0$ for each $x \in (a, b)$ so in particular, $f'(c) > 0$.

$$\Rightarrow \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

$\Rightarrow f(x_2) - f(x_1) > 0$, or $f(x_1) < f(x_2)$.

Thus, f is strictly increasing in $[a, b]$.

(2) Let $f'(x) < 0$ for each $x \in (a, b)$.

Let $x_1, x_2 \in (a, b)$ such that $x_1 < x_2$.

Clearly, f is continuous in $[x_1, x_2]$ and differentiable in (x_1, x_2)

So, by the Lagrange's Mean Value Theorem, there exists a real number $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Since $f'(x) < 0$ for each $x \in (a, b)$ so in particular, $f'(c) < 0$.

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$$

$\Rightarrow f(x_2) - f(x_1) < 0$, or $f(x_1) > f(x_2)$

Thus, f is strictly decreasing in $[a, b]$.

(3) Let $f'(x) = 0$ for each $x \in (a, b)$.

Let $x_1, x_2 \in (a, b)$.

Clearly, $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b).

So, by Lagrange's mean value theorem, there exists a real number $c \in (x_1, x_2)$ such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Since, $f'(x) = 0$ for each $x \in (a, b)$, so in particular $f'(c) = 0$.

$$\Rightarrow \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$

$\Rightarrow f(x_2) - f(x_1) = 0$

$\Rightarrow f(x_2) = f(x_1)$

Since for different values of x, we get same value of $f(x)$, so we can say that $f(x)$ is constant function.

Important

➥ *The above theorem only gives the sufficient condition for the function f to be strictly increasing or strictly decreasing or constant.*

➥ *The converse of the theorem need not be true. In other words, if f is strictly increasing on [a, b], then the condition that for each x may not be true.*
For example, the function $f(x) = x^3$ defined in [-1, 1] is strictly increasing in (-1, 1), but $f'(0) = 0$, where $0 \in (-1, 1)$.

➥ *Similar examples can be given for (2) and (3).*

For Applying the First Derivative Test, we suggest the following Working Procedure :

Working Procedure:

Step 1: Write the given function and the given interval.

Step 2: Find the discrete values of x in the given interval for which $f'(x) = 0$.

Step 3: Draw a number line and mark on it the given interval. Mark the discrete values obtained in Step 2 with black dots '•'

Step 4: Mark the sub-intervals of the given interval as (A), (B), (C), ...

Step 5: Draw a table consisting of columns 'sub-interval', 'Test Point', 'Test value', 'sign of $f'(x)$' and 'conclusion'.

The sub-intervals (A), (B), (C), ... are to be written in the first column. The sub-interval should be open at the point(s) marked with '•'

'Test Point' is any point in the sub-interval and 'Test Value' is the value of $f'(x)$ at the test point. 'Sign of $f'(x)$' is same as the sign of the corresponding Test value.

Step 6: To find the sub-interval in which the function is increasing/strictly increasing: Take the sub-interval in which the conclusion is $f'(x) > 0$ and include the end-points marked with '•'.

To find the sub-interval in which the function is decreasing/strictly decreasing: Take the sub-interval in which the conclusion is $f'(x) < 0$ and include the end-points marked with '•'.

Example 2.3: Find the intervals in which the following functions are strictly increasing or decreasing:

(A) $f(x) = x^2 + 2x - 5$

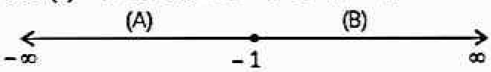
(B) $f(x) = -2x^3 - 9x^2 - 12x + 1$ **[NCERT]**

Ans. (A) Given: $f(x) = x^2 + 2x - 5$ on R.

Then, $f'(x) = 2x + 2$

To find discrete values of x,

Put $f'(x) = 0 \Rightarrow 2x + 2 = 0 \Rightarrow x = -1$



	Sub-interval	Test point	Test value	Sign of $f'(x)$	Conclusion
(A)	$(-\infty, -1)$	-2	-2	(–)	f is strictly decreasing
(B)	$(-1, \infty)$	0	2	(+)	f is strictly increasing

Hence, $f(x)$ is strictly decreasing in $(-\infty, -1)$ and strictly increasing in $(-1, \infty)$.

(B) Given, $f(x) = -2x^3 - 9x^2 - 12x + 1$ on R.

Then, $f'(x) = -6x^2 - 18x - 12$

$= -6(x^2 + 3x + 2)$
$= -6(x + 2)(x + 1)$

Now, $f'(x) = 0$

$\Rightarrow -6(x + 2)(x + 1) = 0$

$\Rightarrow \quad x = -1, -2$

	Sub-interval	Test point	Test value	Sign of $f'(x)$	Conclusion
(A)	$(-\infty, -2)$	-3	-12	$(-)$	f is strictly decreasing
(B)	$(-2, -1)$	-1.5	1.5	$(+)$	f is strictly increasing
(C)	$(-1, \infty)$	0	-12	$(-)$	f is strictly decreasing

Hence $f(x)$ is strictly decreasing in $(-\infty, -2) \cup (-1, \infty)$ and strictly increasing in $(-2, -1)$.

⚠ Caution

➥ *To save time while solving these type of questions, skip the calculations done to find the value of $f'(x)$ at test points. Instead, focus on sign on $f'(x)$ only.*

For example, in part (A),

$f'(x) = -6(x + 2)(x + 1)$

So, for any value in the sub-interval $(-\infty, -2)$,

$f'(x) = (-6)(-)(-)$
$= (-)$

Example 2.4: Show that $y = \dfrac{4\sin\theta}{(2 + \cos)} - \theta$, is an increasing function of θ in $\left[0, \dfrac{\pi}{2}\right]$. **[NCERT]**

Ans. Given: $y = \dfrac{4\sin\theta}{(2+\cos\theta)} - \theta$ in $\left[0, \dfrac{\pi}{2}\right]$.

Then, $$\frac{dy}{d\theta} = \frac{4(2+\cos\theta)\cos\theta - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4(\sin^2\theta + \cos^2\theta) - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - 4 - 4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$= \frac{\cos\theta(4 - \cos\theta)}{(2+\cos\theta)^2}$$

Since $(4 - \cos\theta)$ and $(2 + \cos\theta)^2$ are always positive, when $\theta \in \left(0, \dfrac{\pi}{2}\right)$

So, $\dfrac{dy}{d\theta} > 0$

$\Rightarrow y = \dfrac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function of θ in $\left[0, \dfrac{\pi}{2}\right]$.

OBJECTIVE Type Questions

[**1** mark]

Multiple Choice Questions

1. The interval in which $y = x^2e^{-x}$ is increasing, is:

(a) $(-\infty, \infty)$ **(b)** $(-2, 0)$

(c) $(2, \infty)$ **(d)** $(0, 2)$

[CBSE Term-1 2021]

Ans. *(d)* $(0, 2)$

Explanation: $y = x^2e^{-x}$ gives $\dfrac{dy}{dx} = 2xe^{-x} - x^2e^{-x}$

$= (2 - x)xe^{-x}$

Now, $\dfrac{dy}{dx} > 0$, when $(2 - x)x > 0$,

i.e., when $0 < x < 2$.

2. The interval in which $y = -x^2 + 6x - 3$ is increasing, is:

(a) $(7, 8)$ **(b)** $(5, 6)$

(c) $(-\infty, 3)$ **(d)** $(3, \infty)$

3. The interval in which $y = x^{\frac{1}{x}}$ is increasing, is:

(a) $(0, e^2)$ **(b)** $(1, 2e)$

(c) $(0, e)$ **(d)** $(1, e^2)$

Ans. *(c)* $(0, e)$

Explanation: $y = x^{\frac{1}{x}}$

gives $\frac{dy}{dx} = x^{\frac{1}{x}}\left[\frac{1-\log x}{x^2}\right]$

The function is increasing when $\frac{dy}{dx} > 0$,

i.e., when $\frac{1-\log x}{x^2} > 0$, or, when $1 - \log x > 0$

$\Rightarrow \quad \log x < 1$

$\Rightarrow \quad \log x < \log e$

$\Rightarrow \quad x < e$

4. The real function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is:

(a) Strictly increasing in $(-\infty, -2)$ and strictly decreasing in $(-2, \infty)$.

(b) Strictly decreasing in $(-2, 3)$.

(c) Strictly decreasing in $(-\infty, 3)$ and strictly increasing in $(3, \infty)$.

(d) Strictly decreasing in $(-\infty, -2) \cup (3, \infty)$.

[CBSE Term-1 SQP 2021]

Ans. *(b) Strictly decreasing in (–2, 3).*

$f'(x) = 6(x^2 - x - 6) = 6(x-3)(x+2)$

$-\infty$ (+) -2 (–) 3 (+) ∞

As $f'(x) < 0 \ \forall\ x \in (-2, 3)$

$\Rightarrow f(x)$ is strictly decreasing in $(-2, 3)$

[CBSE Marking Scheme Term-1 SQP 2021]

Explanation: We have,

$$f(x) = 2x^3 - 3x^2 - 36x + 7$$

To find critical points,

Put $\quad f'(x) = 0$

$\Rightarrow \quad 6x^2 - 6x - 36 = 0$

$\Rightarrow \quad 6(x^2 - x - 6) = 0$

$\Rightarrow \quad 6(x-3)(x+2) = 0$

$\Rightarrow \quad x = -2, 3$

$-\infty$ ←—— -2 —— 3 ——→ ∞

For $x \in (-\infty, -2)$, $f'(x) > 0$

For $x \in (-2, 3)$, $f'(x) < 0$

For $x \in (3, \infty)$, $f'(x) > 0$

$\therefore f(x)$ is strictly increasing in $(-\infty, -2) \cup (3, \infty)$ and strictly decreasing in $(-2, 3)$.

5. Ⓐ**The interval in which $y = \frac{x^2 - x + 1}{x^2 + x + 1}$ is decreasing, is:**

(a) $(-1, 1)$ **(b) $(-\infty, 1)$**

(c) $(1, \infty)$ **(d) $(-2, 2)$**

6. The interval in which $y = -x^3 + 3x^2 + 2021$ is increasing, is:

(a) $(-\infty, \infty)$ **(b) $(0, 2)$**

(c) $(2, \infty)$ **(d) $(-2, 0)$**

[Delhi Gov. 2022]

Ans. *(b) (0, 2)*

Explanation: We have,

$$y = -x^3 + 3x^2 + 2021$$

So, $\quad \frac{dy}{dx} = -3x^2 + 6x$

$\quad = -3x(x-2)$

To find critical points,

put $\quad \frac{dy}{dx} = 0$

$\Rightarrow \quad -3x(x-2) = 0$

$\Rightarrow \quad x = 0, 2$

$-\infty$ (–) 0 (+) 2 (–) ∞

Clearly $\frac{dy}{dx} > 0$ for $x \in (0, 2)$

$\therefore y$ is increasing in $(0, 2)$.

7. Ⓐ**If the function $f(x) = 2x^2 - kx + 5$ is increasing on $[1, 2]$, then k lies in the interval:**

(a) $(-\infty, 4)$ **(b) $(4, \infty)$**

(c) $(-\infty, 8)$ **(d) $(8, \infty)$** **[DIKSHA]**

8. The value of b for which the function $f(x) = x + \cos x + b$ is strictly decreasing over R is :

(a) $b < 1$

(b) No value of b exists

(c) $b \le 1$

(d) $b \ge 1$

[CBSE Term-1 SQP 2021]

Ans. *(b) No value of b exists*

$f'(x) = 1 - \sin x \Rightarrow f'(x) > 0 \ \forall\ x \in R$

no value of b exists

[CBSE Marking Scheme Term-1 SQP 2021]

Explanation: For $f(x)$ to be decreasing,

$$f'(x) \le 0$$

$\Rightarrow \quad 1 - \sin x \le 0$

$\Rightarrow \quad 1 \le \sin x$

But $\sin x \ge 1$ is not possible.

Thus, for no value of b, $f(x)$ is decreasing.

9. If the function $f(x) = \sin x - ax + b$, is decreasing on $x \in R$, then a belongs to:

(a) $(1, \infty)$ **(b) $[0, \infty)$**

(c) $(0, \infty)$ **(d) $[1, \infty)$**

[Delhi Gov. 2022]

Ans. *(d)* $[1, \infty)$

Explanation: We have,

$$f(x) = \sin x - ax + b$$
$$f'(x) = \cos x - a$$

$\therefore$ For $f(x)$ to be decreasing,

$$f'(x) \le 0$$
$$\Rightarrow \quad \cos x - a \le 0$$
$$\Rightarrow \quad \cos x \le a$$

$\because$ The maximum value of $\cos x$ is 1

$$\therefore \quad a \ge 1$$
$$\Rightarrow \quad a = [1, \infty).$$

10. **The function $f(x) = \tan x - x$:**

(a) always increases
(b) always decreases
(c) never increases
(d) sometimes increases and sometimes decreases **[NCERT Exemplar]**

11. Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is strictly increasing.

(a) $(-\infty, 2) \cup (2, \infty)$ (b) $(2, \infty)$
(c) $(-\infty, 2)$ (d) $(-\infty, 2] \cup (2, \infty)$

[CBSE Term-1 SQP 2021]

Ans. *(b)* $(2, \infty)$

$$f(x) = x^2 - 4x + 6$$
$$f'(x) = 2x - 4$$
$$\text{let } f'(x) = 0$$
$$\Rightarrow \quad x = 2$$

as $f'(x) > 0 \ \forall \ x \in (2, \infty)$

$\Rightarrow f(x)$ is strictly increasing in $(2, \infty)$

[CBSE Marking Scheme Term-1 SQP 2021]

Explanation: For $f(x)$ to be strictly increasing,

$$f'(x) > 0$$
$$\Rightarrow \quad 2x - 4 > 0$$
$$\Rightarrow \quad x > 2$$

$\therefore$ Required interval $= (2, \infty)$.

12. If the function $f(x) = \dfrac{k\sin x + 2\cos x}{\sin x + \cos x}$ is an increasing for all values of x, then

(a) $k < 1$ (b) $1 < k$
(c) $k < 2$ (d) $k > 2$

Ans. *(d)* $k > 2$

Explanation: $f(x) = \dfrac{k\sin x + 2\cos x}{\sin x + \cos x}$ gives

$$f'(x) = \frac{k-2}{(\sin x + \cos x)^2}$$

The function is increasing for all values of x, if $k - 2 > 0$, *i.e.*, $k > 2$

13. The length of one of the intervals on which the function $f(x) = \sin 3x$ is increasing, is

(a) $\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{2}$
(c) $\dfrac{3\pi}{2}$ (d) π

Ans. *(a)* $\dfrac{\pi}{3}$

Explanation: $f(x) = \sin 3x$ gives $f'(x) = 3\cos 3x$

The function is increasing if $f'(x) > 0$,

i.e., if $\cos 3x > 0$

As $\cos 3x$ is a periodic function with period $\dfrac{2\pi}{3}$.

One of the intervals on which $\cos 3x$ is positive is $\left[-\dfrac{\pi}{6}, \dfrac{\pi}{6}\right]$, whose length is $\dfrac{\pi}{3}$.

14. The interval on which the function $f(x) = 2x^3 - 3x^2 - 36x + 10$ is decreasing, is

(a) $(-\infty, -2)$ (b) $(-2, 3)$
(c) $(2, 3)$ (d) $(3, \infty)$

[Delhi Gov. 2022]

Ans. *(b)* $(-2, 3)$

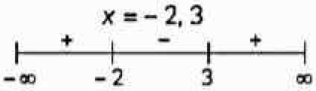
Explanation: We have,

$$f(x) = 2x^3 - 3x^2 - 36x + 10$$
$$\therefore \quad f'(x) = 6x^2 - 6x - 36$$
$$= 6(x^2 - x - 6)$$
$$= 6(x - 3)(x + 2)$$

To find critical points,

Put $f'(x) = 0$

$$\Rightarrow \quad 6(x - 3)(x + 2) = 0$$
$$\Rightarrow \quad x = -2, 3$$



Clearly, $f'(x) < 0$ for $x \in (-2, 3)$

$\therefore f(x)$ is decreasing in $(-2, 3)$.

15. **The function $f(x) = 5 + 36x + 3x^2 - 2x^3$ is increasing in the interval**

(a) $(-2, 3)$ (b) $(2, 3)$
(c) $[2, 3)$ (d) $(2, 3]$

[NCERT Exemplar]

16. Which of the following functions is decreasing on $\left(0, \dfrac{\pi}{2}\right)$?

(a) $\sin 2x$ (b) $\tan x$
(c) $\cos x$ (d) $\cos 3x$

Ans. (c) $\cos x$

Explanation : We have,

$f_a(x) = \sin 2x$ increases from 0 to 1 in $\left(0, \frac{\pi}{2}\right)$.

$f_b(x) = \tan x$ is an increasing function in each quadrant.

$f_c(x) = \cos x$, decreases from 1 to 0 in $\left(0, \frac{\pi}{2}\right)$.

$f_d(x) = \cos 3x$, decreases if $3x \in \left(0, \frac{\pi}{2}\right)$ or $x \in \left(0, \frac{\pi}{6}\right)$

17. **On which of the following intervals is the function $f(x) = x^{100} + \sin x - 1$ decreasing?**

(a) $(0, 1)$ (b) $\left(\frac{\pi}{2}, \pi\right)$

(c) $\left(0, \frac{\pi}{2}\right)$ (d) None of these

18. **$y = x(x - 3)^2$ decreases for the values of x given by:**

(a) $1 < x < 3$ (b) $x < 0$

(c) $x > 0$ (d) $0 < x < \frac{3}{2}$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

19. **Prove that the function given by $f(x) = 3x + 5$ is strictly increasing on R.**

Ans. Given: $f(x) = 3x + 5$

$\Rightarrow$ $f'(x) = 3 > 0$

So, $f(x)$ is strictly increasing on R.

20. **Show that $f(x) = 3x^2$ is strictly decreasing in $(-\infty, 0)$.**

21. **Find the interval in which $f(x) = \sin x$; $x \in \left[0, \frac{3\pi}{2}\right]$ is increasing.**

Ans. We have: $f(x) = \sin x$

On differentiating with respect to x, we get

$f'(x) = \cos x$

For $f(x)$ to be increasing, $f'(x) \geq 0$

$\cos x \geq 0$

$\Rightarrow$ $x \in \left[0, \frac{\pi}{2}\right]$

22. **Show that the function given by $f(x) = e^{3x}$ is strictly increasing on R.**

23. **Find the interval in which the function $f(x) = 2x^2 - 3x$ is strictly decreasing.**

Ans. Given: $f(x) = 2x^2 - 3x$

$\Rightarrow$ $f'(x) = 4x - 3$

For $f(x)$ to be strictly decreasing,

$f'(x) < 0$

$\Rightarrow$ $4x - 3 < 0$

$x < \frac{3}{4}$

Hence, $f(x)$ is strictly decreasing in $\left(-\infty, \frac{3}{4}\right)$.

24. **Prove that the logarithmic function is strictly increasing on $(0, \infty)$.**

25. **Show that $f(x) = |x^3|$ is strictly increasing function in $(0, \infty)$.**

Ans. We have, $f(x) = |x^3|$

For $x \in (0, \infty)$

$f(x) = x^3$

$\Rightarrow$ $f'(x) = 3x^2$

$\therefore$ $f'(x) > 0$

Hence, $f(x)$ is strictly increasing function in $(0, \infty)$.

26. **Show that $y = \cos^{-1} x$ is strictly decreasing function in $(-1, 1)$.**

27. **Find the least value of a such that the function f given by $f(x) = x^2 + ax + 1$ is strictly increasing on $(1, 2)$.** **[NCERT]**

Ans. We have, $f(x) = x^2 + ax + 1$

$\therefore$ $f'(x) = 2x + a$

Since, given interval is $(1, 2)$

$\therefore$ $1 < x < 2$

$\Rightarrow$ $2 < 2x < 4$

$\Rightarrow$ $(2 + a) < 2x + a < (4 + a)$

$\Rightarrow$ $(2 + a) < f'(x) < (4 + a)$

For $f(x)$ to be strictly increasing,

$f'(x) > 0$

$\Rightarrow$ $(2 + a) > 0$

$\Rightarrow$ $a > -2$

28. **Show that $f(x) = \log_a x$, $0 < a < 1$ is a decreasing function for all $x > 0$.**

29. **Show that $f(x) = \tan x$ is an increasing function on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.**

30. **Penguins can't fly, but they can jump! Seriously. They can jump over 9 feet (or up to 3 meters), depending on their species. Sumit found out that Penguin feathers collect tiny bubbles when they're swimming, creating a layer of air around the animal which means that they can swim fast enough to fly out of the sea and into the air! The trajectory of a penguin in air can be approximated by a quadratic function.**

Find the interval in which the function f given by $f(x) = 7 - 4x - x^2$ is strictly increasing.

[CBSE 2020]

Ans. Let $y = f(x) = 7 - 4x - x^2$

On differentiating with respect to x, we get

$$\frac{dy}{dx} = -4 - 2x$$

For strictly increasing, $\frac{dy}{dx} > 0$

$\Rightarrow -4 - 2x > 0 \Rightarrow x < -2$

$\therefore$ Required interval is $(-\infty, -2)$.

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

31. **Prove that the function $f(x) = x^2 - x + 1$ is neither increasing nor decreasing strict on $(-1, 1)$.**

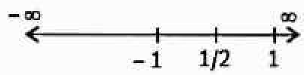
Ans. We have $\quad f(x) = x^2 - x + 1$

$\therefore \quad f'(x) = 2x - 1$

Put $f'(x) = 0 \Rightarrow 2x - 1 = 0$

$\Rightarrow \quad x = \frac{1}{2}$

Point $x = \frac{1}{2}$ divides the line in intervals $\left(-1, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right)$



When $x \in \left(-1, \frac{1}{2}\right)$

$f'(x) < 0$, so it is strictly decreasing.

When $x \in \left(\frac{1}{2}, 1\right)$

$f'(x) > 0$, so it is strictly increasing.

Hence, $f(x)$ is neither increasing nor decreasing on $(-1, 1)$.

32. **Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R.**

[CBSE 2017]

33. **Find the value(s) of k for which $f(x) = kx^3 - 9kx^2 + 9x + 3$ is increasing on R.**

Ans. It is given that $f(x)$ is increasing on R.

Therefore, $f'(x) > 0$, for all $x \in R$.

$\Rightarrow 3kx^2 - 18kx + 9 > 0$, for all $x \in R$

$\Rightarrow kx^2 - 6kx + 3 > 0$, for all $x \in R$

$\Rightarrow k > 0$ and Discriminant $= 36k^2 - 12k < 0$

$\Rightarrow k > 0$ and $12k(3k - 1) < 0$

$\Rightarrow \quad 3k - 1 < 0$

$\Rightarrow \quad k < \frac{1}{3}$

$\Rightarrow k > 0$ and $k < \frac{1}{3}$

$\Rightarrow k \in \left(0, \frac{1}{3}\right)$

Caution

➡ *Remember the condition for increasing function.*

34. **Show that the function $f(x) = \frac{5}{x} + 8$ is decreasing for $x \in R - \{0\}$.**

35. **In which interval the function $f(x) = 2x^3 - 9x^2 + 12x + 15$ is increasing?**

Ans. We have,

$$f(x) = 2x^3 - 9x^2 + 12x + 15$$

$\therefore \quad f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

For, $f(x)$ to be increasing, we have

$f'(x) > 0$

$\Rightarrow \quad 6(x^2 - 3x + 2) > 0$

$\Rightarrow \quad (x - 1)(x - 2) > 0$

$\Rightarrow \quad x < 1$ or $x > 2$

$\therefore \quad x \in (-\infty, 1) \cup (2, \infty)$

36. Find the interval in which $f(x) = \log(x + 3)$ is a decreasing function.

Ans. We have $\quad f(x) = \log(x + 3)$

Here $f(x)$ is defined for $x > -3$

On differentiating with respect to x, we get

$$f'(x) = \frac{1}{x+3} \text{ and } x > -3$$

For $f(x)$ to be decreasing function,

$f'(x) < 0$ and $x > -3$

$\frac{1}{x+3} < 0$ and $x > -3$

$\Rightarrow \quad x < -3$ and $x > -3$

Hence, for no value of x, $f(x)$ is decreasing.

37. Show that the function $f(x) = \cos^2(2x)$ is strictly decreasing on $\left(0, \frac{\pi}{4}\right)$.

38. Show that $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in $\left(0, \frac{\pi}{4}\right)$.

[NCERT Exemplar]

Ans. We have, $f(x) = \tan^{-1}(\sin x + \cos x)$

$$\Rightarrow \quad f'(x) = \frac{1}{1+(\sin x+\cos x)^2}(\cos x - \sin x)$$

$$= \frac{\cos x - \sin x}{1+(\sin x+\cos x)^2}$$

For $x \in \left(0, \frac{\pi}{4}\right)$, $\cos x > \sin x$

$\therefore \quad f'(x) > 0$,

$\therefore f(x)$ is increasing in $\left(0, \frac{\pi}{4}\right)$.

Hence, proved.

39. Find the interval(s) in which the function $f(x) = \frac{2x}{\log x}$ is increasing.

Ans. Given, $\quad f(x) = \frac{2x}{\log x}$

$$f'(x) = \frac{\log x \frac{d}{dx}(2x) - 2x\frac{d}{dx}(\log x)}{(\log x)^2}$$

$$= \frac{(\log x)2 - \frac{2x}{x}}{(\log x)^2}$$

$$= \frac{2(\log x) - 2}{(\log x)^2}$$

For $f(x)$ to be increasing,

$f'(x) \geq 0$

$$\Rightarrow \quad \frac{2\log x - 2}{(\log x)^2} \geq 0$$

Here $(\log x)^2$ is always positive.

$\therefore \quad 2\log x - 2 \geq 0$

$\Rightarrow \quad \log x \geq 1$

$\Rightarrow \quad x \geq e$

Hence, $f(x)$ is increasing in $[e, \infty)$.

40. Show that $f(x) = (x - 1)e^x + 1$ is an increasing function for all $x > 0$.

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

41. Prove that the function f defined by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$. Hence, find the intervals in which $f(x)$ is

(A) strictly increasing

(B) strictly decreasing **[CBSE 2014]**

Ans. Here, $f(x) = x^2 - x + 1; x \in (-1, 1)$

On differentiating with respect to x, we get

$$f'(2x) = 2x - 1$$

Now $f'(x) = 2\left(x - \frac{1}{2}\right) > 0$ for $\frac{1}{2} < x < 1$

$\Rightarrow f(x)$ is strictly increasing in $\left(\frac{1}{2}, 1\right)$

Also $f'(x) = 2\left(x - \frac{1}{2}\right) < 0$ for $-1 < x < \frac{1}{2}$

So, $f(x)$ is strictly decreasing in $\left(-1, \frac{1}{2}\right)$

Thus $f(x)$ is neither increasing nor decreasing in $(-1, 1)$.

42. Prove that $y = \frac{4\sin\theta}{2+\cos\theta} - \theta$ is an increasing function in $\left[0, \frac{\pi}{2}\right]$. **[CBSE 2011]**

Ans. We have, $y = \frac{4\sin\theta}{2+\cos\theta} - \theta,\ \theta \in \left[0, \frac{\pi}{2}\right]$

$$\frac{dy}{d\theta} = 4\left\{\frac{(2+\cos\theta)\cos\theta - \sin\theta(-\sin\theta)}{(2+\cos\theta)^2}\right\} - 1$$

$$= \frac{4(2\cos\theta+1)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2} = \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

Now, $\cos\theta \geq 0$ in $\left[0, \frac{\pi}{2}\right]$, so $4 - \cos\theta \geq 0$ in $\left[0, \frac{\pi}{2}\right]$ and $(2+\cos\theta)^2 > 0$ in $\left[0, \frac{\pi}{2}\right]$

$\therefore \frac{dy}{d\theta} > 0$ for all $\theta \in \left[0, \frac{\pi}{2}\right]$

Hence, y is an increasing function in $\left[0, \frac{\pi}{2}\right]$.

43. Show that $f(x) = 2x + \cot^{-1} x + \log\left(\sqrt{1+x^2} - x\right)$ is increasing in R.

[NCERT Exemplar]

44. Find the interval in which the function

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x}$$

is either increasing or decreasing. **[NCERT]**

Ans. Given,

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2+\cos x}$$

$$= \frac{4\sin x - x(2+\cos x)}{2+\cos x}$$

$$= \frac{4\sin x}{2+\cos x} - x$$

On differentiating with respect to x, we get $f'(x)$

$$= \frac{(2+\cos x)(4\cos x) - 4\sin x(0-\sin x)}{(2+\cos x)^2} - 1$$

$$= 4\left\{\frac{2\cos x + \cos^2 x + \sin^2 x}{(2+\cos x)^2}\right\} - 1$$

$$= \frac{4\{2\cos x + 1\}}{(2+\cos x)^2} - 1$$

$$= \frac{8\cos x + 4 - (2+\cos x)^2}{(2+\cos x)^2}$$

$$= \frac{8\cos x + 4 - (4 + \cos^2 x + 4\cos x)}{(2+\cos x)^2}$$

$$= \frac{4\cos x - \cos^2 x}{(2+\cos x)^2}$$

$$= \frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

Since, $-1 \leq \cos x \leq 1$,

$\therefore 4 - \cos x \geq 0$ and $(2+\cos x)^2 \geq 0$

So, increasing or decreasing nature of $f(x)$, depends only on $\cos x$ of $f'(x)$.

Now, for $f(x)$ to be increasing, $f'(x) \geq 0$

This is possible when $\cos x \geq 0$, it means $\cos x$ lies in Ist or IVth quadrants.

$$\Rightarrow x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$$

Hence, $f(x)$ is increasing in $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$

For $f(x)$ to be decreasing.

$$f'(x) \leq 0$$

This is possible when $\cos x \leq 0$, it means $\cos x$ lies in IInd or IIIrd quadrants.

$$\Rightarrow x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

Hence, $f(x)$ is decreasing in $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

45. Find the interval in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is

(A) strictly increasing

(B) strictly decreasing **[CBSE 2018]**

Ans. We have, $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$

$f(x)$ being polynomial function is continuous and derivable on R.

$$f'(x) = x^3 - 3x^2 - 10x + 24$$
$$= (x-2)(x^2 - x - 12)$$
$$= (x-2)(x-4)(x+3)$$

So, the critical points are

$$x = -3, 2, 4$$

Now, for $x \in (-\infty, -3)$

$$f'(x) = (-)\,(-)\,(-)$$
$$= (-)$$

$\therefore$ $f(x)$ is decreasing in $(-\infty, -3)$.

For $x \in (-3, 2)$

$$f'(x) = (-)\,(-)\,(+)$$
$$= (+)$$

$\therefore$ $f(x)$ is increasing in $(-3, 2)$.

For $x \in (2, 4)$

$$f'(x) = (+)\,(-)\,(+)$$
$$= (-)$$

$\therefore$ $f(x)$ is decreasing in $(2, 4)$.

For $x \in (4, \infty)$

$$f'(x) = (+)\,(+)\,(+)$$
$$= (+)$$

$\therefore$ $f(x)$ is increasing in $(4, \infty)$.

(A) $f(x)$ is strictly increasing in $(-3, 2) \cup (4, \infty)$.

(B) $f(x)$ is strictly decreasing in $(-\infty, -3) \cup (2, 4)$.

46. Ⓐ**Find the intervals in which the function $f(x) = (x+1)^3(x-3)^3$ is strictly increasing or strictly decreasing.**

47. Show that for $a \geq 1$, $f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$ is decreasing in R. [NCERT Exemplar]

Ans. We have, $\quad f(x) = \sqrt{3}\sin x - \cos x - 2ax + b$

$$\therefore \quad f'(x) = \sqrt{3}\cos x + \sin x - 2a$$

$$= 2\left[\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x\right] - 2a$$

$$= 2\left[\cos\frac{\pi}{6}\cos x + \sin\frac{\pi}{6}\sin x\right] - 2a$$

$$= 2\cos\left(\frac{\pi}{6} - x\right) - 2a$$

$$= 2\left[\cos\left(\frac{\pi}{6} - x\right) - a\right]$$

For $f(x)$ to be decreasing,

$$f'(x) \leq 0 \text{ for real } x$$

So, $2\left[\cos\left(\frac{\pi}{6} - x\right)\right] - a \leq 0$

$\Rightarrow \quad a \geq \cos\left(\frac{\pi}{6} - x\right)$ for all real x

$\Rightarrow \quad a \geq$ max. value of $\cos\left(\frac{\pi}{6} - x\right)$

$\Rightarrow \quad a \geq 1$

Hence, for $a \geq 1$, given function is decreasing in R.

Hence, proved.

Concept Applied

➥ *For a function to be decreasing, $f'(x) \leq 0$. Use this condition to get the result.*

48. Show that $f(x) = \log\cos x$ is

(A) Strictly decreasing in $\left(0, \frac{\pi}{2}\right)$

(B) Strictly increasing in $\left(\frac{\pi}{2}, \pi\right)$

Ans. (A) We have, $\quad f(x) = \log\cos x$

On differentiating with respect to x, we get

$$f'(x) = \frac{1}{\cos x}\frac{d}{dx}(\cos x)$$

$$= \frac{-\sin x}{\cos x}$$

$$= -\tan x \quad \text{...(i)}$$

For $x \in \left(0, \frac{\pi}{2}\right)$, $\tan x > 0$

$\Rightarrow \quad -\tan x < 0$

$\Rightarrow \quad f'(x) < 0,$

So it is strictly decreasing in $\left(0, \frac{\pi}{2}\right)$.

(B) From Eq. (i),

$$f'(x) = -\tan x$$

When $x \in \left(\frac{\pi}{2}, \pi\right)$, $\tan x < 0$

$\Rightarrow \quad -\tan x > 0$

$\Rightarrow \quad f'(x) > 0,$

So it is strictly increasing in $\left(\frac{\pi}{2}, \pi\right)$.

49. Find the intervals on which the function $g(x) = 5 + 36x + 3x^2 - 2x^3$ is

(A) decreasing

(B) increasing

Ans. Given: $g(x) = -2x^3 + 3x^2 + 36x + 5$

On differentiating with respect to x, we get

$$g'(x) = -6x^2 + 6x + 36$$

$$= -6(x^2 - x - 6)$$
$$= -6(x + 2)(x - 3)$$

So, critical points are

$$x = -2, 3$$

For $x \in (-\infty, -2)$

$$g'(x) = (-)(-)(-)$$
$$= (-)$$

$\therefore$ $g(x)$ is decreasing in $(-\infty, -2)$.

For $x \in (-2, 3)$

$$g'(x) = (-)(+)(-)$$
$$= (+)$$

$\therefore$ $g(x)$ is increasing in $(-2, 3)$.

For $x \in (3, \infty)$

$$g'(x) = (-)(+)(+)$$
$$= (-)$$

$\therefore$ $g(x)$ is decreasing in $(3, \infty)$.

(A) $g(x)$ is decreasing in $(-\infty, -2) \cup (3, \infty)$

(B) $g(x)$ is increasing in $(-2, 3)$.

50. Find the interval in which the function given by $f(x) = \sin x + \cos x$, $0 \le x \le 2\pi$ is

(A) increasing

(B) decreasing **[CBSE 2012]**

Ans. The given function is

$$f(x) = \sin x + \cos x; \ 0 \le x \le 2\pi$$

$$\Rightarrow \quad f'(x) = \cos x - \sin x$$

Put $\quad f'(x) = 0 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4} \in [0, 2\pi]$

Now, $f'(x) > 0$ in $\left[0, \frac{\pi}{4}\right)$ is, $f'(x) < 0$ in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

and $f'(x) > 0$ in $\left(\frac{5\pi}{4}, 2\pi\right]$

Thus, function f is decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ and it is increasing in $\left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$.

LONG ANSWER Type Questions **(LA)**

[**4** & **5** marks]

51. For which values of x, the function $g(x) = \dfrac{2x}{x^2 + 1}$ is increasing and for which values of x, it is decreasing?

Ans. We have, $\quad g(x) = \dfrac{2x}{x^2 + 1}$

On differentiating with respect to x, we get

$$g'(x) = 2\left[\frac{(x^2+1)\frac{d}{dx}(x) - x\frac{d}{dx}(x^2+1)}{(x^2+1)^2}\right]$$

$$= 2\left[\frac{(x^2+1)\times 1 - x \times 2x}{(x^2+1)^2}\right]$$

$$= 2\left[\frac{1 - x^2}{(x^2+1)^2}\right]$$

For critical points,

Put $\quad g'(x) = 0$

$\Rightarrow \quad 1 - x^2 = 0$

$\Rightarrow \quad x = \pm 1$

For $x \in (-\infty, -1)$,

$$1 - x^2 < 0$$

$\Rightarrow \quad g'(x) < 0$

For $x \in (-1, 1)$

$$1 - x^2 > 0$$

$\Rightarrow \quad g'(x) > 0$

For $x \in (1, \infty)$

$$1 - x^2 < 0$$

$\Rightarrow \quad g'(x) < 0$

Thus, $g(x)$ is increasing in $(-1, 1)$ and is decreasing in $(-\infty, -1) \cup (1, \infty)$.

52. Find the interval in which $f(x) = \sin 3x - \cos 3x$, $0 < x < \pi$, is strictly increasing or strictly decreasing. **[CBSE 2016]**

53. Determine the intervals of x in which $f(x) = 2 \log (x - 2) - x^2 + 4x + 1$ is increasing or decreasing.

Ans. We have, $f(x) = 2 \log (x - 2) - x^2 + 4x + 1$

Here, $f(x)$ is defined for $x > 2$.

Now, $$f'(x) = \frac{2}{(x-2)} - 2x + 4$$

$$= \frac{2 - 2x(x-2) + 4(x-2)}{(x-2)}$$

$$= \frac{2 - 2x^2 + 4x + 4x - 8}{(x-2)}$$

$$= \frac{-2x^2 + 8x - 6}{(x-2)}$$

$$= \frac{-2(x-1)(x-3)}{(x-2)}$$

For $f(x)$ to be increasing, we have $f'(x) > 0$

$$\Rightarrow \frac{-2(x-1)(x-3)}{(x-2)} > 0$$

$$\Rightarrow \frac{(x-1)(x-3)}{(x-2)} < 0$$

We have domain $x > 2$, therefore $(x - 1) > 0$ and, $(x - 3) < 0 \Rightarrow x < 3$

Hence, $f(x)$ is increasing in $(2, 3)$.

For $f(x)$ to be decreasing, we have $f'(x) < 0$

$$\Rightarrow \frac{-2(x-1)(x-3)}{x-2} < 0$$

$$\Rightarrow \frac{(x-1)(x-3)}{(x-2)} > 0$$

$$\Rightarrow (x-3) > 0 \qquad \begin{bmatrix} \because \text{domain of } f(x) \text{ is } x > 2 \\ \therefore (x-1) > 0 \end{bmatrix}$$

$$\Rightarrow x > 3$$

Hence, $f(x)$ is decreasing in $(3, \infty)$.

54. **For which values of x, $f(x) = (x - 1)^3 (x - 2)^2$ is increasing and for which values of x it is decreasing?**

MAXIMA AND MINIMA 3

TOPIC 1

MAXIMUM AND MINIMUM VALUES OF A FUNCTION

The maximum and minimum (*Plural* : Maxima and Minima) of a function are the greatest and smallest values that a function attains at a point either within a given neighbourhood (*local* or *relative* extremum) or on the entire domain of the function (*global* or *absolute* extremum). Application of derivatives is a very powerful tool for finding maxima and minima of a function.

Let $y = f(x)$ be a real function defined on an interval I and a be any point in I. Then, f is said to have a maximum value in I at $x = a$ if

$$f(a) \geq f(x),\ \forall\ x \in I$$

Here, $f(a)$ is called the maximum value of f in I and a is called the point of maximum value of f in I.

Again, let $y = f(x)$ be a real function defined on an interval I and b be any point in I. Then, f is said to have a minimum value in I at $x = b$ if

$$f(b) \leq f(x),\ \forall\ x \in I$$

So, $f(b)$ is called the minimum value of f in I and b is called the point of minimum value of f in I.

Extreme Value of a Function

Let $y = f(x)$ be a real function defined on an interval I and c be any point in I. Then, f is said to have an extreme value in I, if $f(c)$ is either maximum or minimum value of f in I. Here, $f(c)$ is called the extreme value and c is called one of the extreme points.

Illustration:

(1) Let $f(x) = (2x - 1)^2 + 3$ **[NCERT]**

Then, $f(x) \geq 3$, as $(2x - 1)^2 \geq 0$ for any real number x.

$\Rightarrow \qquad (2x - 1)^2 + 3 \geq 0 + 3$

Thus, minimum value of $f(x)$ is 3, which occurs at $x = \frac{1}{2}$.

Also, $f(x)$ has no maximum value as $f(x) \to \infty$ as $|x| \to \infty$.

(2) Let $g(x) = -(x - 1)^2 + 10$ **[NCERT]**

Then, $g(x) = 10 - (x - 1)^2 \leq 10,\ \forall\ x \in R$, as $(x - 1)^2$ is always greater than or equal to zero.

Thus, maximum value of $g(x)$ is 10, which occurs at $x = 1$.

Also, $g(x)$ has no minimum value as $f(x) \to -\infty$ as $|x| \to \infty$

(3) Let $h(x) = |\sin 4x + 3|$ **[NCERT]**

For all $x \in R$, $\qquad -1 \leq \sin 4x \leq 1$

$\Rightarrow \qquad 2 \leq \sin 4x + 3 \leq 4$

$\Rightarrow \qquad 2 \leq |\sin 4x + 3| \leq 4$

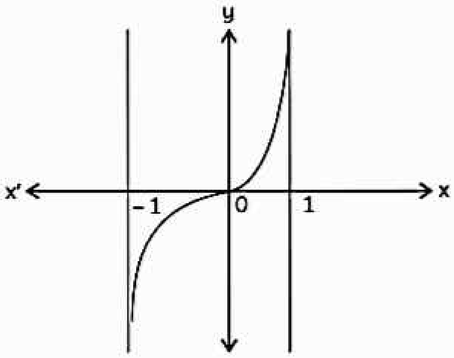
$\Rightarrow \qquad 2 \leq h(x) \leq 4$

Thus, maximum value of $h(x)$ is 4, when $\sin 4x = 1$; and minimum value of $h(x)$ is 2, when $\sin 4x = -1$.

Neither Maximum nor Minimum Values of a Function

Illustration: Let us consider a function $f(x) = x^3$, $x \in (-1, 1)$.

Since this function is an increasing function in $(-1, 1)$, it should have minimum value at a point nearest to -1 and maximum value at a point nearest to 1.



But we cannot locate such points (see figure).

So, $f(x) = x^3$ has neither maximum nor minimum value in $(-1, 1)$.

But, if we extend the domain of f to $[-1, 1]$, then the function $f(x) = x^3$ has maximum value 1 at $x = 1$ and minimum value -1 at $x = -1$.

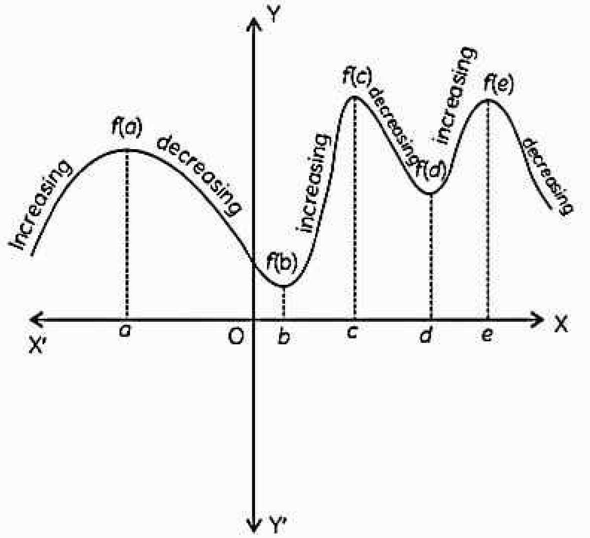
Important

➥*Every continuous function on an closed interval has a maximum and a minimum value.*

TOPIC 2

LOCAL MAXIMA AND MINIMA

Consider the following graph of a function $y = f(x)$.



We observe that at $(a, f(a))$, $(b, f(b))$, $(c, f(c))$, $(d, f(d))$, $(e, f(e))$, the function changes its nature from increasing to decreasing or vice-versa. These points are called turning points or stationary points. At these turning points, the graph either has a peak or trough (cavity). We observe that the function has minimum value in some neighbourhood of $x = b$, $x = d$ which are at the lowest of their respective troughs. Similarly, the function has maximum value in some neighbourhood of $x = a$, $x = c$ and $x = e$ which are at the top of their respective peaks.

On the basis of this observation, the points $x = a$, $x = c$ and $x = e$ are points of local maximum value (or *relative maximum value*) and the points $x = b$ and $x = d$ are points of local minimum value (or *relative minimum value*) of the function.

Let f be a real valued function and let c be an interior point in the domain of f.

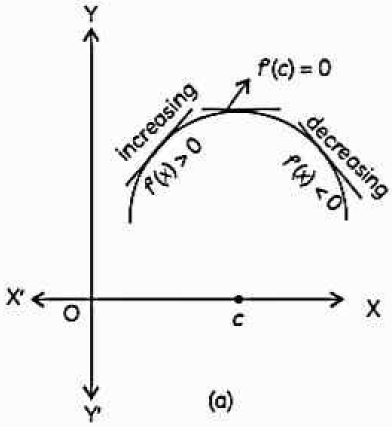
Then,

(1) c is called a point of *local maxima* if there is an $h > 0$ such that $f(c) \geq f(x)$, $\forall\, x \in (c - h, c + h)$, $x \neq c$.
The value $f(c)$ is called the *local maximum value* of f.

(2) c is called a point of *local minima* if there is an $h > 0$ such that $f(c) \leq f(x)$, $\forall\, x \in (c - h, c + h)$, $x \neq c$.
The value $f(c)$ is called the *local minimum value* of f.

Geometrical Interpretation

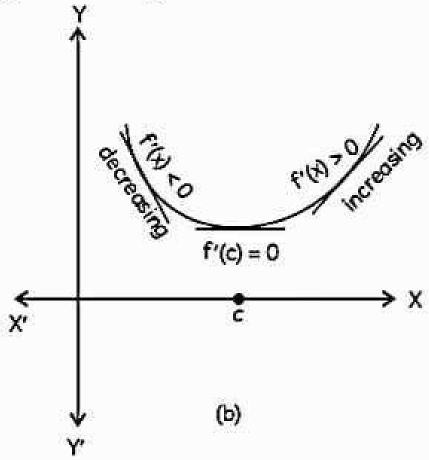
Geometrically, the above definition states that if $x = c$ is a point of local maxima of f, then the graph of f around c will be as shown in Fig(a). Note that the function f is increasing (*i.e.*, $f'(x) > 0$ in the interval $(c - h, c)$ and decreasing (*i.e.*, $f'(x) < 0$) in the interval $(c, c + h)$. This suggests that $f'(c) = 0$.



(a)

Similarly, if $x = c$ is a point of local minima of f, then the graph of f around c will be as shown in Fig(b). Note that the function f is decreasing (*i.e.*, $f(x) < 0$) in the interval $(c - h, c)$ and increasing (*i.e.*, $f'(x) > 0$) in the interval $(c, c + h)$.

This suggests that $f'(c) = 0$.



(b)

After observing these two facts, we can state the following necessary condition for points of local maxima and local minima.

Theorem: Let $y = f(x)$ be a function defined on an open interval I. If f has a local maxima or local minima at $x = c$, where $c \in$ I, then either $f(c) = 0$ or f is not differentiable at $x = c$.

Important

➥ *The converse of this theorem need not to be true, i.e., it does not imply that $x = c$ is a point of local maxima or local minima.*
For example, $f(x) = x^3 \Rightarrow f'(x) = 3x^2 \Rightarrow f'(0) = x = 0$. But 0 is neither a point of local maxima nor a point of local minima. If $f'(c) = 0$ and c is neither a point of local maxima nor a point of local minima, then c is called a point of inflexion.

TOPIC 3

FIRST DERIVATIVE TEST

(1) A critical point $x = c$ is a point of local maxima if $f'(x)$ changes its sign from +ve to –ve in the neighbourhood of c.

(2) A critical point $x = c$ is a point of local minima if $f'(x)$ changes its sign from –ve to +ve in the neighbourhood of c.

(3) A critical point $x = c$ is a point of inflexion if $f'(x)$ does not changes its sign in the neighbourhood of c.

Here is a **working rule** for finding points of local maxima or points of local minima, using only the **first derivative test**.

Step 1: Write down the given function $f(x)$ with its domain and compute $f'(x)$.

Step 2: Find the *critical points, i.e.*, the values of x for which either $f'(x) = 0$ or $f'(x)$ does not exist.

Step 3: The critical points (except the end points) are the possible points, where f can attain local maximum value or local minimum value .

Step 4: At each point c obtained above, check the signs of $f'(x)$ for the values of x slightly less and slightly more than c and follow the below criterion to get the result:

(1) If $f'(x) > 0$, for the values of x slightly less than c and $f'(x) < 0$, for the values of x slightly more than c, then f has *local maximum* at the point $x = c$. Also, the point $x = c$ is called a *point of local maximum* and $f(c)$ is *local maximum value of f*.

(2) If $f'(x) < 0$, for the values of x slightly less than c and $f'(x) > 0$, for the values of x slightly more than c, then f has *local minimum* at the point $x = c$. Also, the point $x = c$ is called a *point of local minimum* and $f(c)$ is *local minimum value of f*.

(3) If $f'(x)$ keeps the same sign for the values of x slightly less than c and for the values of x slightly more than c, then f has *neither local maximum nor local minimum* at the point $x = c$. And, the point $x = c$ is called a *point of inflexion*.

Let us make use of this working rule to determine the local maximum or local minimum value of a function.

Example 3.1: Find the local maxima and local minima, if any, of the following functions. Also, find the local maximum and the local minimum values, as the case may be.

(A) $f(x) = x^3 - 6x^2 + 9x + 15$

(B) $f(x) = \sin x - \cos x, x \in (0, 2\pi)$

(C) $f(x) = 2x^3 - 6x^2 + 6x + 5$ [NCERT]

Ans. (A) Given: $f(x) = x^3 - 6x^2 + 9x + 15$

$\Rightarrow \quad f'(x) = 3x^2 - 12x + 9$

Now, $f'(x) = 0$ gives $3x^2 - 12x + 9 = 0$,

or $\quad 3(x^2 - 4x + 3) = 0$

or $\quad 3(x - 3)(x - 1) = 0$

Thus, the critical points are $x = 1, x = 3$.

At $x = 1$:

For values of x slightly less than 1, we have $f'(x) = (-)(-) = (+)$

For values of x slightly more than 1, we have $f'(x) = (-)(+) = (-)$

Hence, by first derivative test, $f(x)$ has local maximum at $x = 1$, and local maximum value of $f(x)$ is $(1)^3 - 6(1)^2 + 9(1) + 15$, *i.e.*, 19.

At $x = 3$:

For values of x slightly less than 3, we have $f'(x) = (-)(+) = (-)$

For values of x slightly more than 3, we have $f'(x) = (+)(+) = (+)$

Hence, by first derivative test, $f(x)$ has local minimum at $x = 3$, and local minimum value of $f(x)$ is $(3)^3 - 6(3)^2 + 9(3) + 15$, *i.e.*, 15

(B) Given: $f(x) = \sin x - \cos x \Rightarrow f'(x) = \cos x + \sin x$

Now, $f'(x) = 0$ gives $\cos x + \sin x = 0$, *i.e.*, $\tan x = -1$

$\Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4} \qquad \{\because x \in (0, 2\pi)\}$

Thus, the critical points are $x = \frac{3\pi}{4}, x = \frac{7\pi}{4}$.

At $x = \frac{3\pi}{4}$:

For values of x slightly less than $\frac{3\pi}{4}$, we have $f'(x) = (+)$

For values of x slightly less than $\frac{7\pi}{4}$, we have $f'(x) = (-)$

Hence, by first derivative test, $f(x)$ has local maximum at $x = \frac{3\pi}{4}$, and local maximum value of $f(x)$ is $\sin \frac{3\pi}{4} - \cos \frac{3\pi}{4} = \frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2}$

At $x = \frac{7\pi}{4}$:

For values of x slightly less than $\frac{7\pi}{4}$, we have

$f'(x) = (-)$

For values of x slightly more than $\frac{7\pi}{4}$, we have

$f'(x) = (+)$

Hence, by first derivative test, $f(x)$ has local minimum at $x = \frac{7\pi}{4}$, and local minimum value of $f(x)$ is $\sin\frac{7\pi}{4}$

$- \cos\frac{7\pi}{4} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2}$

(C) Given: $f(x) = 2x^3 - 6x^2 + 6x + 5$

$\Rightarrow \quad f'(x) = 6x^2 - 12x + 6$

Now, $f'(x) = 0$ gives $6x^2 - 12x + 6 = 0$,
or $6(x^2 - 2x + 1) = 0$, or $6(x - 1)^2 = 0$

Thus, the only critical point is $x = 1$.

At $x = 1$,

For values of x slightly less than 1, we have $f'(x) = (+)$

For values of x slightly more than 1, we have $f'(x) = (+)$

Hence, by first derivative test, $f(x)$ has *neither local maximum or local minimum* at the point $x = 1$. And, the point $x = 1$ is a *point of inflexion.*

We shall now give another test to examine local maxima and local minima of a given function. This test is often easier to apply than the first derivative test.

TOPIC 4

SECOND DERIVATIVE TEST

(1) A critical point $x = c$ is said to be a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$.

(2) A critical point $x = c$ is said to be a point of local minima if $f'(c) = 0$ and $f''(c) > 0$.

Here is a working rule for finding points of local maxima or points of local minima, using the second derivative test.

Step 1: Write down the given function $f(x)$ with its domain and compute $f'(x)$.

Step 2: Find the critical points, *i.e.*, the values of x for which either $f'(x) = 0$.

Important

➥ *$f'(x)$ must exist for all values of x in the domain of f, otherwise apply the first derivative test.*

Step 3: The critical points (except the end points) are the possible points, where f can attain local maximum value or local minimum value.

Step 4: Compute $f''(x)$.

Step 5: At each critical point c, follow the below criterion to get the result:

(1) If $f''(c) < 0$, then f has *local maximum* at the point $x = c$. Also, the point $x = c$ is the *point of local maxima* and $f(c)$ is the *local maximum value* of f.

(2) If $f''(x) > 0$, then f has *local minimum* at the point $x = c$. Also, the point $x = c$ is the *point of local minima* and $f(c)$ is the *local minimum value* of f.

(3) If $f''(c) = 0$, then we have no conclusion from the second derivative test. We need to go back to the first derivative test to find whether $x = c$ is a point of local maxima or local minima or a point of inflexion.

Let us make use of this working rule to determine the local maximum or local minimum of a function.

Example 3.2: Find the local maxima and local minima, if any, of the following functions. Also, find the local maximum and the local minimum values, as the case may be.

(A) $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25, x \in (0, 3)$

(B) $f(x) = x\sqrt{1-x}, x \in (0, 1)$ [NCERT]

Ans. (A) Given: $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$

$\Rightarrow \quad f'(x) = 12x^3 - 24x^2 + 24x - 48$

Now, $f'(x) = 0$ gives $12x^3 - 24x^2 + 24x - 48 = 0$

or $\quad 12(x^3 - 2x^2 + 2x - 4) = 0$

or $\quad 12(x^2 + 2)(x - 2) = 0$

Thus, the critical point is $x = 2$.

$[\because x^2 + 2 \neq 0 \text{ for } x \in (0, 3)]$

Now, $f''(x) = 36x^2 - 48x + 24$

$= 12(3x^2 - 4x + 2)$

At $x = 2$, $f''(2) = 12[3(2)^2 - 4(2) + 2] = 72 > 0$

Hence, by second derivative test, $f(x)$ has local minimum at $x = 2$.

And local minimum value is

$f(2) = 3(2)^4 - 8(2)^3 + 12(2)^2 - 48(2) + 25$

$= -39$

(B) Given: $f(x) = x\sqrt{1-x}$

$\Rightarrow \quad f'(x) = \sqrt{1-x} + x\left[\frac{1}{2}(1-x)^{-1/2}(-1)\right]$

$= \frac{2-3x}{2\sqrt{1-x}}$

Now, $f'(x) = 0$ gives $\frac{2-3x}{2\sqrt{1-x}} = 0$,

or $2 - 3x = 0$, or $x = \frac{2}{3}$

Thus, the critical point is $x = \frac{2}{3}$.

Now, $f''(x) = \frac{d}{dx}\left\{\frac{2-3x}{2\sqrt{1-x}}\right\}$

$$= \frac{1}{2}\left\{\frac{\sqrt{1-x}(-3) - (2-3x)\left(-\frac{1}{2\sqrt{1-x}}\right)}{1-x}\right\}$$

At $x = \frac{2}{3}$

$$f''\left(\frac{2}{3}\right) = \frac{1}{2}\left\{\frac{\sqrt{1-\frac{2}{3}}(-3) - 0}{1-\frac{2}{3}}\right\} = -\frac{9}{2\sqrt{3}} < 0$$

Hence, by second derivative test, $f(x)$ has local maximum at $x = \frac{2}{3}$.

And local maximum value is $f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{1-\frac{2}{3}} = \frac{2}{3\sqrt{3}}$

TOPIC 5

ABSOLUTE MAXIMUM AND MINIMUM VALUES OF A FUNCTION

A function may have a number of local maxima or local minima in a given interval. A local maximum value may not be the greatest and a local minimum may not even be the least value of the given function in any given interval.

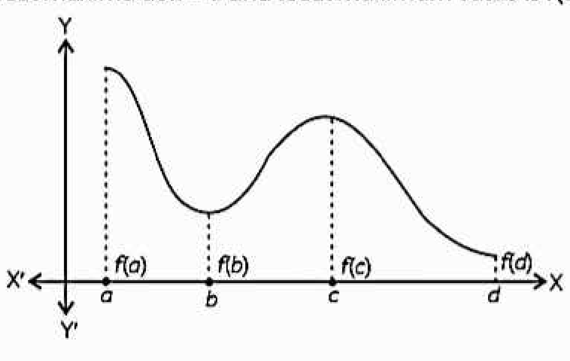
Illustration: Let us consider a function f given by

$f(x) = x + 2, x \in (0, 1)$

Now, observe that this function is continuous on (0, 1) and neither has a maximum value nor has minimum value. Further, the function even has neither local maximum value nor local minimum value.

However, if we extend the domain of f to the closed interval [0, 1], then f still may not have a local maximum (minimum) values but it certainly does have maximum value $3 = f(1)$ and minimum value $2 = f(0)$. The maximum value 3 of f at $x = 1$ is called absolute maximum value (*global maximum* or *greatest value*). Similarly, the minimum value 2 of f at $x = 0$ is called absolute minimum value (*global minimum* or *least value*) of f on [0, 1].

Consider the graph given in fig. below of a continuous function defined on a closed interval $[a, d]$. Observe that the function f has a local minima at $x = b$ and local minimum value is $f(b)$. The function also has a local maxima at $x = c$ and local maximum value is $f(c)$.



Also from the graph, it is evident that f has absolute maximum value $f(a)$ and absolute minimum value $f(d)$. Further note that the absolute maximum (minimum) value of f is different from local maximum (minimum) value of f.

We will now state two results (without proof) regarding absolute maximum and absolute minimum values of a function on a closed interval I.

Theorem: Let f be a continuous function on an interval $I = [a, b]$. Then f has the absolute maximum value and f attains it at least once in I. Also, f has the absolute minimum value and attains it at least once in I.

Theorem: Let f be a differentiable function on a closed interval I and let c be any interior point of I. Then

(1) $f'(c) = 0$, if f attains its absolute maximum value at c.

(2) $f'(c) = 0$, if f attain its absolute minimum value at c.

Here is a working rule for finding the absolute maximum value and absolute minimum value of a function on a closed interval, using the derivatives.

Step 1: Write down the given function $f(x)$ with the given closed interval and compute $f'(x)$.

Step 2: Find the critical points, *i.e.*, the values of x for which $f'(x) = 0$.

Step 3: The critical points and the end points are the possible points, where f can attain absolute maximum value or absolute minimum value.

Step 4: Compute $f(x)$ at all these points. The maximum of all the values will be the absolute maximum value of f, and minimum of all the values will be the absolute minimum value of f.

Important

- *Let f be an increasing function on a closed interval [a, b], then f attains its absolute maximum value at b and absolute minimum value at a.*
- *Let f be a decreasing function on a closed interval [a, b], then f attains its absolute minimum value at b and absolute maximum value at a.*
- *Every continuous function on a closed interval [a, b] has at least one absolute maximum value and at least one absolute minimum value.*

Let us make use of this working rule to determine the absolute maximum value or absolute minimum value of a function.

Example 3.3: Find the absolute maximum value or absolute minimum value of the following functions on the given interval:

(A) $f(x) = \sin x + \cos x, x \in [0, \pi]$

(B) $f(x) = (x-1)^2 + 3, x \in [-3, 1]$ [NCERT]

Ans. (A) Here, $f(x) = \sin x + \cos x$ gives

$$f'(x) = \cos x - \sin x$$

For maximum or minimum value,

Put $f'(x) = 0 \Rightarrow \cos x - \sin x = 0$

$$\Rightarrow \tan x = 1, \text{ or } x = \frac{\pi}{4} \qquad \{\because x \in [0, \pi]\}$$

Now we shall find the values of $f(x)$ at $x = 0$, $x = \frac{\pi}{4}$ and $x = \pi$

$f(0) = \sin 0 + \cos 0 = 1;$

$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \sqrt{2};$

$f(\pi) = \sin\pi + \cos\pi = -1$

Thus, the absolute maximum value of f is $\sqrt{2}$ at $x = \frac{\pi}{4}$; and absolute minimum value of f is -1 at $x = \pi$.

(B) Here, $f(x) = (x-1)^2 + 3$ gives $f'(x) = 2(x-1)$

For maximum or minimum value, put $f'(x) = 0$

$\Rightarrow \quad 2(x-1) = 0$

$\Rightarrow \quad x = 1 \qquad \{\because x \in [-3, 1]\}$

Now we shall find the values of $f(x)$ at $x = -3$ and $x = 1$

$f(-3) = ((-3)-1)^2 + 3 = 19;$

$f(1) = ((1)-1)^2 + 3 = 3$

Thus, the absolute maximum value of f is 19 at $x = -3$; and absolute minimum value of f is 3 at $x = 1$.

Example 3.4: Find the absolute maximum and minimum values of a function f given by

$f(x) = 12x^{4/3} - 6x^{1/3}, x \in [-1, 1]$ [NCERT]

Ans. We have, $f(x) = 12x^{4/3} - 6x^{1/3}$

On differentiating both sides w.r.t. x, we get

$$f'(x) = 12 \times \frac{4}{3}x^{1/3} - 6 \times \frac{1}{3}x^{-2/3}$$

$$= 16x^{1/3} - \frac{2}{x^{2/3}} = \frac{2(8x-1)}{x^{2/3}}$$

For critical points, put $f'(x) = 0$

$\Rightarrow \quad \frac{2(8x-1)}{x^{2/3}} = 0$

$\Rightarrow \quad 8x - 1 = 0$

$\Rightarrow \quad x = \frac{1}{8}$

Also, $f'(x)$ is not defined at $x = 0$, so critical points are $x = 0$ and $x = \frac{1}{8}$ and end points of interval are -1 and 1.

Now, $f(-1) = 12(-1)^{4/3} - 6(-1)^{1/3}$

$= 12 + 6 = 18$

$f(0) = 12(0) - 6(0) = 0$

$f\left(\frac{1}{8}\right) = 12\left(\frac{1}{8}\right)^{4/3} - 6\left(\frac{1}{8}\right)^{1/3}$

$= 12\left(\frac{1}{16}\right) - 6\left(\frac{1}{2}\right)$

$= \frac{-9}{4}$

and $f(1) = 12(1)^{4/3} - 6(1)^{1/3} = 6$

Hence, absolute maximum value of function f is 18 at $x = -1$ and absolute minimum value of function f is $\frac{-9}{4}$ at $x = \frac{1}{8}$.

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. The maximum value of $\left(\frac{1}{x}\right)^x$ is:

(a) $e^{1/e}$ **(b) e**

(c) $\left(\frac{1}{e}\right)^{1/e}$ **(d) e^e**

[CBSE Term-1 2021]

Ans. *(a)* $e^{1/e}$

Explanation: Let, $y = \left(\frac{1}{x}\right)^x$

Taking log on both sides,

$$\log y = x.\log\left(\frac{1}{x}\right) = x.(0 - \log x)$$

$$\Rightarrow \quad \log y = -x.\log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{d}{dx}(\log y) = -\frac{d}{dx}(x \log x)$$

$$\frac{1}{y}.\frac{dy}{dx} = -\left[x.\frac{1}{x} + \log x.1\right]$$

$$\frac{dy}{dx} = -y(1 + \log x)$$

For critical points

Put $\quad \frac{dy}{dx} = 0$

$$\Rightarrow \quad 1 + \log x = 0$$

$$\Rightarrow \quad \log e + \log x = 0 = \log 1$$

$$\Rightarrow \quad \log(e.x) = \log 1$$

$$\Rightarrow \quad ex = 1 \Rightarrow x = \frac{1}{e}$$

Also, $$f'(x) = -\left[\frac{dy}{dx}(1+\log x) + y\left(0+\frac{1}{x}\right)\right]$$

$$= -y(1+\log x)^2 - \frac{y}{x}$$

$$= -x^{\frac{1}{x}}\left[(1+\log x)^2 + \frac{1}{x}\right]$$

$$\Rightarrow \quad f''(x)_{x=\frac{1}{e}} = -\left(\frac{1}{e}\right)^e\left[\left(1+\log\frac{1}{e}\right)^2 + e\right]$$

which is less than zero.

Hence, $f(x)$ is maximum at $x = \frac{1}{e}$.

So, maximum value $= f\left(\frac{1}{e}\right) = \left(\frac{1}{1/e}\right)^{1/e}$

$$= (e)^{1/e}.$$

2. For real values of x, the minimum value of $\frac{1 - x + x^2}{1 + x + x^2}$ is:

(a) 0 **(b)** 1

(c) 3 **(d)** $\frac{1}{3}$

Ans. *(d)* $\frac{1}{3}$

Explanation: Let $y = \frac{1 - x + x^2}{1 + x + x^2}, x \in R$. Then,

$$x^2(y - 1) + x(y + 1) + y - 1 = 0 \quad ...(i)$$

Since x is real, discriminant of equation (i) should be greater than or equal to zero.

$$\Rightarrow \quad (y + 1)^2 - 4(y - 1)(y - 1) \geq 0$$

i.e., $\quad -3y^2 + 10y - 3 \geq 0$

or $\quad -3(y - 3)\left(y - \frac{1}{3}\right) \geq 0$

or $\quad (y - 3)\left(y - \frac{1}{3}\right) \leq 0$

or $\quad \frac{1}{3} \leq y \leq 3$

Hence, minimum value of y is $\frac{1}{3}$.

3. The maximum value of the function $f(x)$ = 4.sin x. cos x is:

(a) 2 **(b)** 4

(c) 1 **(d)** 8

Ans. *(a)* 2

Explanation: Since $f(x) = 2 \sin 2x$, so values of $\sin 2x$ lies between -1 to 1, so maximum value of $f(x)$ is 2.

4. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has:

(a) two points of local maximum

(b) two points of local minimum

(c) one maxima and one minima

(d) no maxima or minima

[NCERT Exemplar]

Ans. *(c) one maxima and one minima*

Explanation: Given,

$$f(x) = 2x^3 - 3x^2 - 12x + 4$$

Then, $\quad f'(x) = 6x^2 - 6x - 12$

Put, $\quad f'(x) = 0$

$$\Rightarrow \quad 6x^2 - 6x - 12 = 0$$

$$\Rightarrow \quad x^2 - x - 2 = 0$$

$$\Rightarrow \quad (x + 1)(x - 2) = 0$$

$$\Rightarrow \quad x = -1, 2$$

Now, $\quad f''(x) = 12x - 6$

$$\therefore \quad f''(x)_{x=-1} = 12(-1) - 6 = -18 < 0$$

So, $x = -1$ is a point of local maxima.

Also, $f''(x)_{x=2} = 12(2) - 6 = 18 > 0$

So, $x = 2$ is a point of local minima.

5. The maximum value of $[x(x-1)+1]^{1/3}$, $0 \le x \le 1$ is:

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) $\left(\frac{1}{3}\right)^{1/3}$

[CBSE Term-1 SQP 2021]

6. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is:

(a) $(2\sqrt{2}, 4)$ (b) $(2\sqrt{2}, 0)$

(c) (0, 0) (d) (2, 2)

Ans. *(a)* $(2\sqrt{2}, 4)$

Explanation: Let d be the distance of the point (x, y) on $x^2 = 2y$ from the point (0, 5). Then,

$$d = \sqrt{(x-0)^2 + (y-5)^2}$$

$$= \sqrt{2y + (y-5)^2} \qquad [\because x^2 = 2y]$$

$$= \sqrt{y^2 - 8y + 25}$$

$$= \sqrt{(y-4)^2 + 9}$$

$\Rightarrow d$ is least when $y = 4$

For $y = 4$, $x = \sqrt{2 \times 4} = 2\sqrt{2}$

Thus, the required point is $(2\sqrt{2}, 4)$.

7. The least value of the function $f(x) = 2\cos x + x$ in the closed interval $\left[0, \frac{\pi}{2}\right]$ is:

(a) 2

(b) $\frac{\pi}{6} + \sqrt{3}$

(c) $\frac{\pi}{2}$

(d) The least value does not exist.

[CBSE Term-1 SQP 2021]

8. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is:

(a) 0 (b) 12

(c) 16 (d) 32

[NCERT Exemplar]

Ans. *(b)* 12

Explanation:

Given, $y = -x^3 + 3x^2 + 9x - 27$

$\therefore$ $y' = -3x^2 + 6x + 9$ = Slope of curve

Now, $y'' = -6x + 6 = -6(x-1)$

Put $y'' = 0$

$\Rightarrow$ $-6(x-1) = 0$

$\Rightarrow$ $x = 1$

And $y''' = -6 < 0$

So, maximum slope of curve is at $x = 1$.

$\therefore$ maximum slope $= \left(\frac{dy}{dx}\right)_{x=1}$

$= -3(1)^2 + 6 \times 1 + 9 = 12$

Caution

- *Here, y' is the slope of the curve. So y'' is the first differentiation and y''' is the second differentiation of the function y'.*

9. The area of a trapezium is defined by function f and is given by

$f(x) = (10 + x)\sqrt{100 - x^2}$,

Then the area when it is maximised is:

(a) 75 cm^2 (b) $7\sqrt{3}\ cm^2$

(c) $75\sqrt{3}\ cm^2$ (d) 5 cm^2

[CBSE Term-1 SQP 2021]

Ans. *(c)* $75\sqrt{3}\ cm^2$

$$f'(x) = \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}}$$

$\Rightarrow$ $f'(x) = 0 \Rightarrow -10$ or 5, But $x > 0$
$x = 5$

$$f''(x) = \frac{2x^3 - 300x - 1000}{(100 - x)^{\frac{3}{2}}}$$

$$\Rightarrow \quad f''(5) = \frac{-30}{\sqrt{75}} < 0$$

$\Rightarrow$ Maximum area of trapezium is $75\sqrt{3}$ cm_2 when $x = 5$

[CBSE Marking Scheme Term-1 SQP 2021]

Explanation: We have,

$$f(x) = (10 + x)\sqrt{100 - x^2}$$

$\therefore$ $f'(x) = (10 + x)$

$$= \left|\frac{1}{\sqrt{100 - x^2}} \times (-2x)\right| + \sqrt{100 - x^2}(1)$$

$$= \frac{-x(10 + x) + 100 - x^2}{\sqrt{100 - x^2}}$$

$$= \frac{-2x^2 - 10x + 100}{\sqrt{100 - x^2}}$$

To find maximum/minimum value,

Put $f'(x) = 0$

$$\Rightarrow \frac{-2x^2-10x+100}{\sqrt{100-x^2}} = 0$$

$$\Rightarrow \quad -2x^2 - 10x + 100 = 0$$

$$\Rightarrow \quad x^2 + 5x - 50 = 0$$

$$\Rightarrow \quad (x+10)(x-5) = 0$$

$$\Rightarrow \quad x = -10, 5$$

Also, $f''(x)$

$$= \frac{\sqrt{100-x^2}(-4x-10) - (-2x^2-10x+100)\left(\frac{-2x}{2\sqrt{100-x^2}}\right)}{(100-x^2)^{3/2}}$$

$$= \frac{(100-x^2)(-4x-10)-(2x^3+10x^2-100x)}{(100-x^2)^2}$$

For $x = -10$, $f''(x)$ is not defined.

And, for $x = 5$,

$$f''(5) = \frac{[100-(5)^2][-4(5)-10]-2(5)^3 - 10(5)^2+100(5)}{(100-(5)^2)^2}$$

$$= \frac{-2250-250-250+500}{(75)^2}$$

$$= -\frac{2250}{(75)^2} < 0$$

$\therefore$ $f(x)$ is maximum at $x = 5$.

$\therefore$ Maximum value of $f(x)$

= Maximum area of trapezium

$$= (10+5)\sqrt{100-(5)^2}$$

$$= 15\sqrt{75} = 75\sqrt{3} \text{ cm}^2$$

10. The maximum value of slope of the curve $y = -x^3 + 3x^2 + 12x - 5$ is:

(a) 15 (b) 12

(c) 9 (d) 0 [CBSE 2020]

Ans. *(a)* 15

Explanation: Given $y = -x^3 + 3x^2 + 12x - 5$

On differentiating with respect to x, we get

$$\frac{dy}{dx} = -3x^2 + 6x + 12$$

$$= -3(x^2 - 2x - 4)$$

$$= -3[(x-1)^2 - 5]$$

$$= 15 - 3(x-1)^2$$

Now, the value of $\frac{dy}{dx}$ will be maximum, when $3(x-1)^2 = 0$ *i.e.,* $x = 1$

Hence, the maximum value of slope at $x = 1$ is 15.

11. The function $f(x) = x^4 - 6x^2 + 8x + 11$ has a minimum at:

(a) $x = 1$ (b) $x = -2$

(c) $x = 3$ (d) $x = 4$

12. The values of a and b such that $f(x) = \frac{a}{x} + bx$ has a minimum at (1, 6) are, respectively:

(a) 3, 3 (b) 2, 2

(c) 2, 4 (d) 4, 2

13. The minimum value of $x \log x$ is equal to:

(a) e (b) $\frac{1}{e}$

(c) $\frac{-1}{e}$ (d) $2e$ [DIKSHA]

Ans. (c) $\frac{-1}{e}$

Explanation: Let $y = x \log x$

$$\therefore \quad \frac{dy}{dx} = \log x + \frac{x}{x}$$

$$= 1 + \log x \quad \text{...(i)}$$

Now, to find minimum value,

Put $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 1 + \log x = 0$$

$$\Rightarrow \quad \log x = -1 = -\log e \quad [\because \log e = 1]$$

$$\Rightarrow \quad \log x = \log e^{-1}$$

$$\Rightarrow \quad x = e^{-1} = \frac{1}{e}$$

$$\because \quad \left(\frac{d^2y}{dx^2}\right)_{x=1/e} = e > 0$$

$\therefore$ Minimum value of $x \log x$

$$= \frac{1}{e}\log\frac{1}{e}$$

$$= \frac{1}{e}\log e^{-1}$$

$$= -\frac{1}{e}$$

⚠ Caution

➥ *Find $\frac{d^2y}{dx^2}$ to confirm whether the value obtained is minimum or not.*

14. 20 is divided into two parts such that the product of one part and the cube of the other part is maximum. The two parts are:

(a) 12, 8 (b) 15, 5

(c) 10, 10 (d) 2, 18

15. **The absolute minimum value of the function $f(x) = x^3 - 12x$ on the interval [0, 3] is** **[CBSE Term-1 2021]**

(a) 0 (b) – 9

(c) – 16 (d) – 19

Ans. *(c)* -16

Explanation: Here $f(x) = x^3 - 12x$.

Then, $f'(x) = 3x^2 - 12$

$= 3(x-2)(x+2)$

Put $f'(x) = 0$

$\Rightarrow \quad 3(x-2)(x+2) = 0$

$\Rightarrow x = 2$ and -2

Now, we shall find $f(0)$, $f(3)$, $f(2)$ and $f(-2)$.

$f(0) = 0$; $f(3) = -9$; $f(2) = -16$ and $f(-2) = 16$

Thus, the absolute minimum value is -16.

16. **The greatest fraction whose denominator exceeds the square of its numerator by 16, is**

(a) $\frac{1}{8}$ (b) $\frac{1}{10}$

(c) $\frac{1}{17}$ (d) $\frac{3}{25}$

CASE BASED Questions (CBQs)

[**4** & **5** marks]

Read the following passages and answer the questions that follow:

17. *The coordinated efforts of the local community and district administration have transformed the poor hygiene and diseases driven village Jakhi district in Bundelkhand into a clean water village.*

In continuation, to get rid of sewage and waste water, the local department wish to construct an underground septic tank with open top and a square base which can hold a given quantity V of water and sludge with a metal sheet in minimum cost. Assume the side of the square be 'x' and height of the tank be 'h'.

(A) The relation among x, h and V is:

(a) $h = \frac{x^2}{V}$ (b) $h = \frac{V}{x^2}$

(c) $x = \frac{V}{h^2}$ (d) $x = \frac{h^2}{V}$

(B) The surface area S(x) of the tank is given by:

(a) $S(x) = x^2 + \frac{V}{4x}$

(b) $S(x) = x + \frac{4V}{x^2}$

(c) $S(x) = x^2 + xh$

(d) $S(x) = x^2 + \frac{4V}{x}$

(C) The cost of construction will be the least when:

(a) $h = \frac{x}{2}$ (b) $x = \frac{h}{2}$

(c) $hx = 2$ (d) $h + x = 2$

(D) If V = 13600, the value of x is:

(a) 10 (b) 20

(c) 30 (d) 40

(E) If x = 20, the value of V is:

(a) 1000 (b) 2000

(c) 3000 (d) 4000

Ans. (B) *(d)* $S(x) = x^2 + \frac{4V}{x}$

Explanation: Here, $S(x) = x^2 + 4xh$

$$\Rightarrow \quad S(x) = x^2 + 4x\left(\frac{V}{x^2}\right)$$

[From part (A)]

$$= x^2 + \frac{4V}{x}$$

(D) *(c)* 30

Explanation: From part (A), we have $h = \frac{V}{x^2}$

And from part (C), we have $h = \frac{x}{2}$

$$\Rightarrow \quad \frac{V}{x^2} = \frac{x}{2}$$

or $\quad x^3 = 2V$

For V = 13500, $x^3 = 27000$

$\Rightarrow \quad x = 30$

18. *A wire of length 34 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into rectangle whose length is twice its breadth.*

Another piece of wire of the same length is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. Assume that the length of the piece which is made into a square is 'x' m.

(A) **For the combined area of the square and the rectangle to be minimum, find the value of x.**

(B) **For the combined area of the square and the circle to be minimum, find the value of x.**

19. *Saregama Carvaan mini is a pocket sized player that comes with 5000 handpicked high quality songs inside. Ravi, Saregamma's (the parent company) financial officer is trying to create a simple profit function to identify how small distributors can be provided a higher discount during Diwali time without eating into the profit margin (EBITDA) too much, keeping the variable of quantity (pieces) ordered.*

Ravi has come up with the following idea:

"Saregamma will produce 'x' radios per day at a cost of ₹$\left(\dfrac{x^2}{4}+35x+25\right)$ *and sells them to distributor at a price of* ₹$\dfrac{1}{2}(100-x)$ *per piece."*

(A) **What is the profit function P(x)?**

(B) **What must be the number of toys produced per day to give the maximum profit?**

Ans. (A) Selling price of 'x' radios = $\dfrac{1}{2}x(100-x)$

Cost price of 'x' radios = $\dfrac{x^2}{4}+35x+25$

So, profit function

$$P(x)=\left[\frac{1}{2}x(100-x)\right]-\left[\frac{x^2}{4}+35x+25\right]$$

$$=15x-\frac{3x^2}{4}-25,$$

(B) For $P(x)=15x-\dfrac{3x^2}{4}-25,$

$$P'(x)=15-\frac{3x}{2};$$

and $P''(x)=-\dfrac{3}{2}<0$

So, P(x) is maximum

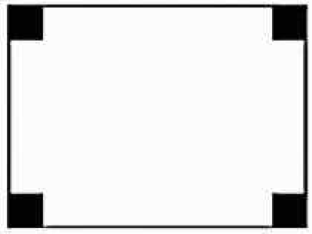
Now, to find maximum value,

Put $P'(x)=0\Rightarrow\left(15-\dfrac{3x}{2}\right)=0.$

i.e., $x=10$

Thus, profit will be maximum when 10 radios are produced per day.

20. *In a festival season, a manufacturing company wishes to prepare card board boxes (without top) with a rectangular card board sheet of dimensions 45 cm × 24 cm by cutting off squares of side 'x' from each corner and folding up the flaps.*



(A) **What is the volume of the box, in terms of x?**

(a) **(45 × 24 × x) cu. cm**

(b) **[x(45 − x)(24 − x)] cu. cm**

(c) **[2x (45 − 2x) (24 − 2x)] cu. cm**

(d) **[x (45 − 2x)(24 − 2x)] cu. cm**

(B) **For the maximum volume of the box, the value of x is:**

(a) **18** (b) **11**

(c) **7** (d) **5**

(C) **What is the volume of the box?**

(a) **2450 cu. cm** (b) **2170 cu. cm**

(c) **2870 cu. cm** (d) **1560 cu. cm**

(D) **If the whole box is completely covered with a white sheet of paper, what is total area of the sheet required?**

(a) **980 sq. cm** (b) **1450 sq. cm**

(c) **1960 sq. cm** (d) **2940 sq. cm**

(E) **If the cost of white paper sheet is ₹ 12 per square metre, what is the cost of sheet required for 100 boxes?**

(a) **₹ 117.60** (b) **₹ 174**

(c) **₹ 235.20** (d) **₹ 252.80**

Ans. (A) *(d) [x (45 − 2x)(24 − 2x)] cu. cm*

Explanation: Length of the box = 45 − 2x; breadth of the box = 24 − 2x; and height of the box = x

So, the volume of the box

$$=x(45-2x)(24-2x) \text{ cu. cm}$$

(B) *(a) 5*

Explanation: Here,

$$V(x)=x(45-2x)(24-2x)$$
$$=4x^3-138x^2+1080x$$

$\Rightarrow$ $V'(x)=12x^2-276x+1080$

and $V''(x)=24x-276$

Equating V'(x) to zero, we have

$$12x^2-276x+1080=0,$$

or $12(x^2-23x+90)=0$

or $12(x-5)(x-18)=0$

or $\quad x = 5 \quad [\because x \neq 18]$

Thus, $V''(5) = 24(5) - 276 < 0$

So, volume of the box is maximum when $x = 5$.

21. *A magazine seller has 500 subscribers and collects annual subscription charges of ₹ 300. He proposes to increase the annual subscription charges and it is believed that for every increase of ₹ 1, one subscriber will discontinue. Suppose that the magazine seller increases the annual subscription charges by ₹ x per subscriber.*

(A) For what value of 'x', the total revenue function $f(x)$ attains the maximum or minimum?

(B) Find the maximum increase in the income of the magazine seller.

Ans. (A) It is given that for every increase of ₹ 1, one subscriber will discontinue.

So, for increase of ₹ x, x subscribers will discontinue the service.

$\therefore$ New number of subscribers

$= (500 - x)$

And, New annual charges per subscriber

$=$ ₹ $(300 + x)$

So, Total revenue after increment,

$R(x)$ = Number of subscribers × Annual charges

$= (500 - x)(300 + x)$

$= 150000 + 200x - x^2$

Now, to find maximum/minimum value,

Put $\quad R'(x) = 0$

$\Rightarrow \quad 0 + 200 - 2x = 0$

$\Rightarrow \quad x = 100$

So, $R(x)$ will attain maximum or minimum or minimum value when $x = 100$.

(B) Maximum increase in the income of the magazine seller

= New Revenue − Old Revenue

$= (150000 + 200x - x^2) - (500 \times 300)$

$= 200x - x^2$

$= 200 \times 100 - (100)^2 = 10{,}000$

22. *The fuel cost per hour for running a train is proportional to the square of the speed it generates in km per hour. If the fuel costs ₹ 48 per hour at speed 16 km per hour and the fixed charges to run the train amount to ₹ 1200 per hour.*

Assume the speed of the train as v km/h.

(A) Given that the fuel cost per hour is k times the square of the speed the train generates in km/h, the value of $16k$ is:

(a) 1 (b) 2

(c) 3 (d) 4

(B) If the train has travelled a distance of 1000 km, then the total cost of running the train is given by function:

(a) $\frac{375}{4}v + \frac{60000}{v}$ (b) $\frac{375}{8}v + \frac{60000}{v^{1/2}}$

(b) $\frac{375}{2}v + \frac{60000}{v}$ (d) $\frac{375}{2}v + \frac{1200000}{v}$

(C) The most economical speed to run the train (in km/hr) is:

(a) 50 (b) 80

(c) 400 (d) 800

(D) The fuel cost (in ₹) for the train to travel 1000 km at the most economical speed is:

(a) 15000 (b) 75000

(c) 100000 (d) 150000

(E) The total cost of the train to travel 1000 km at the most economical speed is:

(a) 15000 (b) 30000

(c) 100000 (d) 150000

Ans. (A) *(c) 3*

Explanation: Let C be the cost of the fuel, consumed by running the train for time t.

Then, $\frac{C}{t} = kv^2$

Also, $48 = k(16)^2$

$\Rightarrow \quad 3 = 16k$

(B) *(d)* $\frac{375}{2}v + \frac{1200000}{v}$

Explanation: Let C' be the total cost of running the train.

Then, according to the question,

$$\frac{C'}{t} = kv^2 + 1200$$

$$\Rightarrow \quad \frac{C'}{t} = \frac{3}{16}v^2 + 1200 \quad \text{[From part (A)]}$$

$$\text{or,} \quad C' = \frac{3}{16}v^2t + 1200t$$

$\because$ Train has travelled a distance of 1000 km,

$$\therefore \quad v = \frac{1000}{t},$$

$$\text{or} \quad t = \frac{1000}{v}$$

$$\therefore \quad C' = \frac{3}{16}v^2\left(\frac{1000}{v}\right) + 1200\left(\frac{1000}{v}\right)$$

$$= \frac{375}{2}v + \frac{1200000}{v}$$

(C) *(b) 80*

Explanation: From part (B), we have

$$C' = \frac{375}{2}v + \frac{1200000}{v}$$

$$\therefore \quad \frac{dC'}{dv} = \frac{375}{2} + 1200000\left(-\frac{1}{v^2}\right)$$

To find economical speed,

$$\text{Put} \quad \frac{dC'}{dv} = 0$$

$$\Rightarrow \frac{375}{2} - \frac{1200000}{v^2} = 0$$

$$\Rightarrow \quad v^2 = \frac{2}{375} \times 1200000$$

$$= 2 \times 3200$$

$$= 6400$$

$$\Rightarrow \quad v = 80$$

$$\text{Also, } \frac{d^2C'}{dv^2} = 0 + \frac{2400000}{v^3} > 0$$

$\therefore$ C' is minimum.

Hence, the most economical speed of train is 80 km/hr.

(D) *(a) 15000*

Explanation: Since fuel cost,

$$C = \frac{375}{2}v$$

$\therefore$ At $v = 80$,

$$C = \frac{375}{2} \times 80$$

$$= ₹\,15,000$$

(E) *(b) 30,000*

Explanation: Total cost of train to travel 1000 km,

$$C' = \frac{375}{2}v + \frac{1200000}{v}$$

$\therefore$ At $v = 80$,

$$C' = \frac{375}{2} \times 80 + \frac{1200000}{80}$$

$$= 15,000 + 15,000$$

$$= ₹\,30,000$$

23. *The relation between the height of the plant (y in cm) with respect to exposure to sunlight is governed by the equation $y = 4x - \frac{1}{2}x^2$, where x is the number of days exposed to sunlight.*

(A) The rate of growth of the plant with respect to sunlight is ....

(a) $4x - \frac{1}{2}x^2$ (b) $4 - x$

(c) $x - 4$ (d) $x - \frac{1}{2}x^2$

(B) What is the number of days it will take for the plant to grow to the maximum height?

(a) 4 (b) 6

(c) 7 (d) 10

(C) What is the maximum height of the plant?

(a) 12 cm (b) 10 cm

(c) 8 cm (d) 6 cm

(D) What will be the height of the plant after 2 days?

(a) 4 cm (b) 6 cm

(c) 8 cm (d) 10 cm

(E) If the height of the plant be $\frac{7}{2}$ cm, the number of days it has been exposed to the sunlight is......

(a) 2 (b) 3

(c) 4 (d) 1

[CBSE Question Bank 2021]

Ans. (A) *(b)* $4 - x$

Explanation: Given, equation is

$$y = 4x - \frac{1}{2}x^2$$

Then rate of growth, $\frac{dy}{dx} = 4 - x$

(B) *(a)* 4

Explanation: From part (A), we have

$$\frac{dy}{dx} = 4 - x$$

Put $\frac{dy}{dx} = 0$

$\Rightarrow$ $x = 4$

Also $\frac{d^2y}{dx^2} = -1 < 0$

$\therefore$ Maximum height will be maximum in 4 days.

24. $P(x) = -5x^2 + 125x + 37500$ is the total profit function of a company, where x is the production of the company.

(A) What will be the production when the profit is maximum?

(B) What will be the maximum profit?

[CBSE Question Bank 2021]

Ans. (A) Given,

$$P(x) = -5x^2 + 125x + 37500$$

$\therefore$ $\frac{dP(x)}{dx} = -10x + 125$

For maximum profit,

Put $\frac{dP(x)}{dx} = 0$

$-10x + 125 = 0$

$\Rightarrow$ $x = 12.5$

Now, $\frac{d^2P(x)}{dx^2} = -10 < 0$

$\therefore$ For $x = 12.5$, profit is maximum.

Hence, profit will be maximum when production of the company is 12.5.

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

25. Find the minimum value of $f(x) = |x|$.

Ans. We have, $f(x) = |x|$

This is an absolute function, so it is always non-negative.

$\therefore$ $f(x) = |x| \geq 0$

Hence, minimum value of $f(x)$ is 0.

26. Find the maximum and minimum values of the function $h(x) = \sin 2x + 5$.

Ans. We have, $h(x) = \sin 2x + 5$

We know that

$$-1 \leq \sin x \leq 1$$

$\therefore$ $-1 \leq \sin 2x \leq 1$

$\Rightarrow$ $-1 + 5 \leq \sin 2x + 5 \leq 1 + 5$

$\Rightarrow$ $4 \leq h(x) \leq 6$

Hence, minimum value of $h(x)$ is 4 and maximum value of $h(x)$ is 6.

27. Find the maximum and minimum values of $f(x) = x + 3$ in $x \in [-2, 2]$.

28. Find the total extremum points of the function $f(x) = \sec x$ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

29. Find the points in which the function $f(x) = (x - 1)(x + 3)^2$ is local extrema.

Ans. Given, $f(x) = (x - 1)(x + 3)^2$

On differentiating with respect to x, we get

$$f'(x) = (x - 1)2(x + 3)(1) + 1.(x + 3)^2$$

For local extremum, put $f'(x) = 0$

$\Rightarrow$ $2(x - 1)(x + 3) + (x + 3)^2 = 0$

$\Rightarrow$ $(x + 3)[2(x - 1) + (x + 3)] = 0$

$\Rightarrow$ $(x + 3)(3x + 1) = 0$

$\Rightarrow$ $x + 3 = 0$ or $3x + 1 = 0$

$\Rightarrow$ $x = -3$ or $x = -\frac{1}{3}$

Hence, the points of local extrema are $x = -3$ and $x = -\frac{1}{3}$.

30. Find the absolute minimum value of $f(x) = 2\sin x$ in $\left[0, \frac{3\pi}{2}\right]$. [CBSE 2020]

31. It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval [0, 2]. Find the value of a. [NCERT]

Ans. Let $f(x) = x^4 - 62x^2 + ax + 9$

On differentiating with respect to x, we get

$$f'(x) = 4x^3 - 124x + a$$

Since, it is given that the $f(x)$ is maximum at $x = 1$.

$\therefore \quad f'(1) = 0$

$\Rightarrow \quad 4(1)^3 - 124(1) + a = 0$

$\Rightarrow \quad 4 - 124 + a = 0$

$\Rightarrow \quad a = 120$

32. Find the least value of the function $f(x) = ax + \frac{b}{x}$ $(a > 0, b > 0, x > 0)$. [CBSE 2020]

33. Find the minimum value of $f(x) = x^2$.

34. Find the point on the curve $y = \sin x$, in which y is maximum in the interval $[0, \pi]$.

Ans. Given: $y = \sin x$

$\therefore \quad \frac{dy}{dx} = \cos x$

For maximum value, Put $\frac{dy}{dx} = 0$

$\Rightarrow \quad \cos x = 0$

$\Rightarrow \quad x = \frac{\pi}{2} \in [0, \pi]$

Now, $\frac{d^2y}{dx^2} = -\sin x$

At $x = \frac{\pi}{2}$,

$$\frac{d^2y}{dx^2} = \sin\left(\frac{\pi}{2}\right) = -1 < 0$$

Hence, y is maximum at $x = \frac{\pi}{2}$.

35. Prove that the function $f(x) = \log 2x$ is neither maxima nor minima.

Ans. Given: $f(x) = \log 2x$

On differentiating with respect to x, we get

$$f'(x) = \frac{1}{2x} \times 2$$

$$= \frac{1}{x}$$

Since, $\log 2x$ is defined only for $x > 0$.

$\therefore f'(x) > 0 \; \forall \; x > 0$

So, $f(x)$ does not exist for any $x \in R$ such that $f'(x) = 0$.

Hence, $f(x)$ is neither maxima nor minima.

36. Jasmine was collecting information on famous buildings and monuments shaped like an arch, or mathematically speaking, like a parabola which could be represented by a quadratic function. Moreover, using her knowledge of maxima and minima, Jasmine could also find the maximum and minimum values of the function.

Find the maximum and minimum values of $f(x) = -(x - 1)^2 + 10$, if any of the function has.

Ans. Given: $f(x) = -(x - 1)^2 + 10$

We see that $(x - 1)^2 \geq 0$

$\Rightarrow \quad -(x - 1)^2 \leq 0$

$\Rightarrow \quad 10 - (x - 1)^2 \leq 10$

$\Rightarrow \quad f(x) \leq 10$

$[\because f(x) = -(x - 1)^2 + 10]$

Thus, maximum value of $f(x)$ is 10 and there is minimum value of $f(x)$.

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

37. Find the least value of the function $f(x) = 2x + \frac{3}{x}, x > 0$.

Ans. We have, $f(x) = 2x + \frac{3}{x}$

$\therefore \quad f'(x) = 2 - \frac{3}{x^2}$

Put $\quad f'(x) = 0$

$\Rightarrow \quad 2 - \frac{3}{x^2} = 0$

$\Rightarrow \quad x^2 = \frac{3}{2}$

$\Rightarrow \quad x = \pm\sqrt{\frac{3}{2}}$

Since, $x > 0$, so we consider only $x = \sqrt{\frac{3}{2}}$.

Now, $\quad f''(x) = 0 + \frac{6}{x^3}$

At $x = \sqrt{\frac{3}{2}}$,

$$f''(x) = \frac{6}{\left(\sqrt{\frac{3}{2}}\right)^3} = \frac{6 \times (\sqrt{2})^3}{(\sqrt{3})^3}$$

$$= \frac{6 \times 2\sqrt{2}}{3\sqrt{3}} = \frac{4\sqrt{2}}{\sqrt{3}} > 0,$$

$\therefore f(x)$ is minimum at $x = \sqrt{\frac{3}{2}}$.

$\therefore$ The least value of $f(x)$ at $x = \sqrt{\frac{3}{2}}$ is

$$f\left(\sqrt{\frac{3}{2}}\right) = 2 \times \sqrt{\frac{3}{2}} + \frac{3}{\sqrt{3/2}}$$

$$= \sqrt{6} + \sqrt{6} = 2\sqrt{6}$$

38. Of all the rectangles each of which has perimeter 40 metres, find one which has maximum area. Find the area also. [DIKSHA]

Ans. Let x, y be the sides of a rectangle.

Given, $\quad 2(x + y) = 40$

$\Rightarrow \quad x + y = 20$

Now, $\quad$ area, $A = xy$

$= x(20 - x)$

$= 20x - x^2$

$\therefore \quad \frac{dA}{dx} = 20 - 2x$

Put $\quad \frac{dA}{dx} = 0$

$\Rightarrow \quad 20 - 2x = 0$

$\Rightarrow \quad x = 10$

$\because \quad \frac{d^2A}{dx^2} = -2 < 0$

$\therefore$ Area is maximum at $x = 10, y = 10$.

$\therefore$ Area is maximum, when rectangle is a square.

Also, maximum area = 100 m^2

39. Find the local maxima or local minima of $f(x) = \sin x + \cos x$, $0 < x < \frac{\pi}{2}$. [NCERT]

40. Find all the points of local maxima and local minima of the function $f(x) = x^3 - 6x^2 + 12x - 8$.

Ans. Given: $f(x) = x^3 - 6x^2 + 12x - 8$

On differentiating with respect to x, we get

$$f'(x) = 3x^2 - 12x + 12$$
$$= 3(x-2)^2$$

The critical points of $f(x)$ are given by $\frac{dy}{dx} = 0$.

$\therefore \quad 3(x-2)^2 = 0$

$\Rightarrow \quad x = 2$

Here, we see that $f'(x) = 3(x - 2)^2 > 0$ for all $x \neq 2$.

So, $\frac{dy}{dx}$ does not change sign as x changes through $x = 2$.

Hence, $x = 2$ is neither a point of local maximum nor a point of local minimum.

So, it is a point of inflexion.

41. Find the points of local maxima or local minima of the function $f(x) = 2\sin x - x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

Ans. Given: $\quad f(x) = 2\sin x - x$

On differentiating twice with respect to x, we get

$$f'(x) = 2\cos x - 1$$

and $\quad f''(x) = -2\sin x$

For maxima or minima, put $f'(x) = 0$.

$\therefore \quad 2\cos x - 1 = 0$

$\Rightarrow \quad \cos x = \frac{1}{2}$

$\Rightarrow \quad x = \frac{\pi}{3}, -\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

When $x = \frac{\pi}{3}$,

$$f''\left(\frac{\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3} < 0$$

Thus, $x = \frac{\pi}{3}$ is a point of local maxima of $f(x)$.

When $x = -\frac{\pi}{3}$,

$$f\left(-\frac{\pi}{3}\right) = -2\sin\left(-\frac{\pi}{3}\right)$$
$$= 2\sin\frac{\pi}{3}$$
$$= 2\times\frac{\sqrt{3}}{2} = \sqrt{3} > 0.$$

Thus, $x = -\frac{\pi}{3}$ is a point of local maxima of $f(x)$.

42. ⓐ**Find the absolute maximum and the absolute minimum of $f(x) = 4x - \frac{x^2}{2}, x \in \left[-2, \frac{9}{2}\right]$. [NCERT]**

43. Show that the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$.

Ans. Let $$y = \left(\frac{1}{x}\right)^x = (x)^{-x}$$

On taking log both sides, we get

$$\log y = -x \log x$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx} = -\left(1\times\log x + x\times\frac{1}{x}\right)$$
$$= -(\log x + 1)$$
$$\Rightarrow \quad \frac{dy}{dx} = -y(\log x + 1)$$
$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{dy}{dx}(\log x + 1) - y\left(\frac{1}{x}\right)$$
$$= y(1+\log x)^2 - \frac{y}{x}$$
$$\Rightarrow \quad \frac{d^2y}{dx^2} = x^{-x}(1+\log x)^2 - \frac{x^{-x}}{x}$$
$$= x^{-x}\left((1+\log x)^2 - \frac{1}{x}\right)$$

To find critical points, put $\frac{dy}{dx} = 0$

$$-y(1+\log x) = 0$$
$$\Rightarrow \quad 1 + \log x = 0$$
$$\Rightarrow \quad \log x = -1 = -\log e = \log e^{-1}$$
$$\Rightarrow \quad x = e^{-1} = \frac{1}{e}$$

At $x = \frac{1}{e}$,

$$\frac{d^2y}{dx^2} = \left(\frac{1}{e}\right)^{-\frac{1}{e}}\left(\left(1+\log\frac{1}{e}\right)^2 - \frac{1}{1/e}\right)$$
$$= \left(\frac{1}{e}\right)^{-1/e}((1-1)^2 - e)$$
$$= \left(\frac{1}{e}\right)^{-1/e}(0 - e)$$
$$= -e\left(\frac{1}{e}\right)^{-1/e} < 0$$

So, at $x = \frac{1}{e}$, y is maximum and the maximum value is

$$y\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{-1/e} = e^{1/e}$$

Hence, proved.

44. Prove that the largest rectangle with given perimeter is a square.

Ans. Let x and y be the length and breadth of the rectangle. Then,

Perimeter of a rectangle, $p = 2(x + y)$

Now, area of rectangle,

$$A = xy$$
$$= x\left(\frac{p}{2} - x\right)$$
$$= \frac{p}{2}x - x^2$$

On differentiating with respect to x, we get

$$\frac{dA}{dx} = \frac{p}{2} - 2x$$

For largest rectangle, put $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{p}{2} - 2x = 0$$
$$x = \frac{p}{4}$$

Now $\frac{d^2A}{dx^2} = -2 < 0$,

$\therefore$ Area is maximum when $x = \frac{p}{4}$.

When $x = \frac{p}{4}$,

$$p = 2\left(\frac{p}{4} + y\right)$$
$$p = \frac{p}{2} + 2y$$

$\Rightarrow \quad 2y = \frac{p}{2}$

$\Rightarrow \quad y = \frac{p}{4}$

$\therefore \quad x = y$

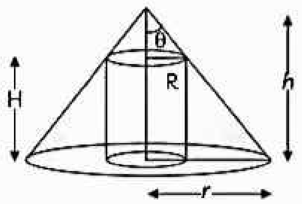
Hence $A = x \times x = x^2$, which is the area of square.

Hence, proved.

45. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right circular cone of height *h* and radius *r* is one-third of the height of the cone and greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone.

[CBSE 2020]

Ans. Let R, H respectively be the radius, height of the cylinder incribed in a cone of semi-vertical angle θ.



We have, $\tan\theta = \frac{r}{h} = \frac{R}{h-H}$

$\Rightarrow \quad R = r\left(\frac{h-H}{h}\right)$

Now volume of cylinder $(V) = \pi R^2 H$

$= \pi r^2\left(1-\frac{H}{h}\right)^2 H$

For maximum volume, Put $\frac{dV}{dH} = 0$

$\Rightarrow \pi r^2\left[H \times 2\left(1-\frac{H}{h}\right) \times \left(-\frac{1}{h}\right) + \left(1-\frac{H}{h}\right)^2\right] = 0$

$\Rightarrow \pi r^2\left[-\frac{4H}{h} + \frac{3H^2}{h^2} + 1\right] = 0$

$\Rightarrow \pi r^2\left(\frac{H}{h}-1\right)\left(\frac{3H}{h}-1\right) = 0$

$\Rightarrow \quad H = \frac{h}{3} \qquad [\because H \neq h]$

So, height of cylinder $= \frac{1}{3}$ of height of cone

Also maximum volume of cylinder

$= \pi r^2\left(1-\frac{1}{3}\right)^2 \times \frac{h}{3}$

$= \left(\frac{1}{3}\pi r^2 h\right) \times \frac{4}{9}$

$= \frac{4}{9}$ of volume of cone.

Hence, proved.

SHORT ANSWER Type-II Questions (SA-II)

[**3** marks]

46. Find the absolute maximum and minimum values of $f(x) = 2x^3 - 24x + 57$ in the interval [1, 5].

Ans. Given: $f(x) = 2x^3 - 24x + 57$

On differentiating with respect to x, we get

$f'(x) = 6x^2 - 24$

For extreme values, put $f'(x) = 0$

$\Rightarrow \quad 6x^2 - 24 = 0$

$\Rightarrow \quad x^2 = 4$

$\Rightarrow \quad x = \pm 2$

Now, $\quad f(1) = 2(1)^2 - 24(1) + 57$

$= 2 - 24 + 57 = -35$

$f(2) = 2(2)^3 - 24(2) + 57$

$= 16 - 48 + 57 = 25$

$f(5) = 2(5)^3 - 24(5) + 57$

$= 250 - 120 + 57 = 187$

Hence, absolute maximum value is 187 and absolute minimum value is – 35.

⚠ **Caution**

➥ *Here, $x = -2$ is ignored as we have to find the absolute maximum and minimum values of f(x) in the interval [1.5] and $-2 \notin [1.5]$.*

47. Find the local maximum and local minimum values of $\frac{(x-1)(x-6)}{(x-10)}, x \neq 10$.

Ans. Let $\quad y = \frac{(x-1)(x-6)}{(x-10)}$

$= \frac{x^2 - 7x + 6}{(x-10)}$

On differentiating with respect to x, we get

$\frac{dy}{dx}$

$$= \frac{(x-10)\frac{d}{dx}(x^2-7x+6)-(x^2-7x+6)\frac{d}{dx}(x-10)}{(x-10)^2}$$

$$= \frac{(x-10)(2x-7)-(x^2-7x+6)(1)}{(x-10)^2}$$

$$= \frac{2x^2-27x+70-(x^2-7x+6)}{(x-10)^2}$$

$$= \frac{x^2-20x+64}{(x-10)^2}$$

$$= \frac{(x-16)(x-4)}{(x-10)^2}$$

For local maxima and local minima, put $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{(x-16)(x-4)}{(x-10)^2} = 0$$

$\Rightarrow x = 16, 4$

When $x = 16$

For x slightly less than 16,

$$\frac{dy}{dx} = \frac{(-)(+)}{(+)} = \text{Negative}$$

For x slightly greater than 16,

$$\frac{dy}{dx} = \frac{(+)(+)}{(+)} = \text{Positive}$$

Here, $\frac{dy}{dx}$ changes sign from negative to positive, so $y = 16$ is a point of local minima at $x = 16$.

$\therefore$ The local minimum value of $x = 16$ is

$$y = \frac{(16-1)(16-6)}{16-10}$$

$$= \frac{15\times10}{6}$$

$$= 25$$

When $x = 4$,

For x slightly less than 4,

$$\frac{dy}{dx} = \frac{(-)(-)}{(+)} = \text{Positive}$$

For x slightly greater than 4,

$$\frac{dy}{dx} = \frac{(-)(+)}{(+)} = \text{Negative}$$

Here, $\frac{dy}{dx}$ changes sign from positive to negative, so $x = 4$ is a point of is local maxima.

$\therefore$ Local maximum value of $x = 4$ is

$$y = \frac{(4-1)(4-6)}{4-10}$$

$$= \frac{3\times(-2)}{-6} = 1$$

48. **At what point, the slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is maximum? Also, find the maximum slope. [NCERT Exemplar]**

49. **Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.**

Ans. Let BCDE be a rectangle inscribed in the given circle of radius r having centre at O.

Let $\angle CEB = \theta$

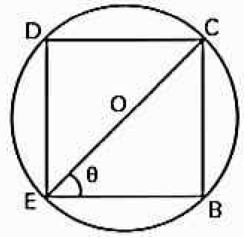
Then, $EC = 2r$, $EB = 2r\cos\theta$ and $BC = 2r\sin\theta$

Let A be the area of rectangle BCDE

Then, $A = EB \times BC$

$= 4r^2 \sin\theta\cos\theta$

$= 2r^2 \sin 2\theta$



Thus, $A = 2r^2 \sin 2\theta$

where, r is constant.

$\therefore \quad \frac{dA}{d\theta} = 4r^2\cos 2\theta,$

and $\quad \frac{d^2A}{d\theta^2} = -8r^2\sin 2\theta$

Now, put $\frac{dA}{d\theta} = 0$

$\Rightarrow \quad 4r^2\cos 2\theta = 0$

$\Rightarrow \quad \cos 2\theta = 0$

$\Rightarrow \quad 2\theta = \frac{\pi}{2}$ i.e., $\theta = \frac{\pi}{4}$

At $\theta = \frac{\pi}{4}, \left(\frac{d^2A}{d\theta^2}\right)_{\theta=\frac{\pi}{4}} = -8r^2\sin\frac{\pi}{2} = -8r^2 < 0$

$\therefore \theta = \frac{\pi}{4}$ is a point of maxima.

Thus, area is maximum when $\theta = \frac{\pi}{4}$.

In this case, EB = $2r\cos\frac{\pi}{4} = r\sqrt{2}$

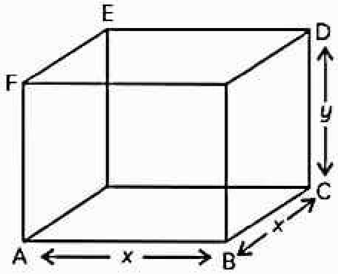
and BC = $2r\sin\frac{\pi}{4} = r\sqrt{2}$

Thus, EB = BC and therefore, BCDE is a square.

Hence, proved.

50. An open tank with a square base and vertical sides is to be construted from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of it width. [CBSE 2018]

Ans. Let x be the length of a side of square base and y be the length of vertical side. Also let V be the given quantity of water.



Then $V = x^2y$ ...(i)

Clearly the surface area,

$$S = 4xy + x^2$$
$$= 4x \cdot \frac{V}{x^2} + x^2$$

$\Rightarrow$ $$S(x) = \frac{4V}{x} + x^2 \quad ...(ii)$$

On differentiating both sides with respect to x, we get

$$S'(x) = \frac{-4V}{x^2} + 2x \quad ...(iii)$$

For minimum quantity, put $S'(x) = 0$.

$\Rightarrow$ $\frac{-4V}{x^2} + 2x = 0$

$\Rightarrow$ $\frac{-4V}{x^2} = -2x$

$\Rightarrow$ $x^3 = 2V$

$\Rightarrow$ $x = (2V)^{1/3}$

Again on differentiating both sides of eq. (iii) with respect to x, we get

$$S''(x) = \frac{8V}{x^3} + 2$$

and $S''((2V)^{1/3}) = \frac{8V}{2V} + 2 = 4 + 2 = 6 > 0$

$\therefore$ S(x) is minimum when $x = (2V)^3$

From eq. (i), we get

$$y = \frac{V}{x^2} = \frac{\left(\frac{x^3}{2}\right)}{x^2} = \frac{x}{2}$$

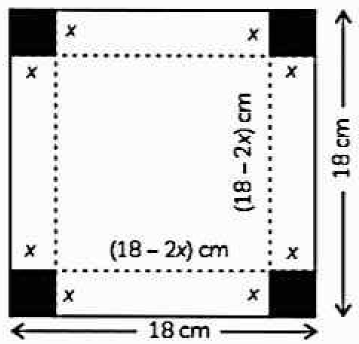
Thus the cost of material will be least when depth of the tank is half of its width.

Hence, proved.

51. A square piece of tin of side 18 cm is to be made into a box without top, by cutting off squares from each corner and folding upto flaps of the box. What should be the side of the square to be cut off so that the volume of the box is maximum possible? [NCERT]

Ans. Let the side of the square to be cut off be x cm $(0 < x < 9)$.

Then, the length and the breadth of the box will be $(18 - 2x)$ cm each and the height of the box is x cm.



Let V be the volume of the open box formed by folding up the flaps, then

$$V = x(18 - 2x)(18 - 2x)$$
$$= 4x(9 - x)^2$$
$$= 4x(81 + x^2 - 18x)$$
$$= 4(x^3 - 18x^2 + 81x)$$

On differentiating twice with respect to x, we get

$$\frac{dV}{dx} = 4(3x^2 - 36x + 81)$$
$$= 12(x^2 - 12x + 27)$$

and $$\frac{d^2V}{dx^2} = 12(2x - 12)$$
$$= 24(x - 6)$$

For maximum value of V, put $\frac{dV}{dx} = 0$

$\Rightarrow 12(x^2 - 12x + 27) = 0$

$\Rightarrow$ $x^2 - 12x + 27 = 0$

$\Rightarrow$ $(x - 3)(x - 9) = 0$

$\Rightarrow \quad x = 3, 9$

But $x = 9$ is not possible

$\because \quad 2x = 2 \times 9 = 18$

which is equal to side of square piece.

At $x = 3$, $\left(\frac{d^2V}{dx^2}\right)_{x=3} = 24(3-6) = -72 < 0$

$\therefore$ By second derivative test, $x = 3$ is the point of maxima.

Hence if we cut off the side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

52. The sum of the perimeters of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of the square is double the radius of the circle. [CBSE 2014]

53. Find all the points of local maxima and minima and the corresponding maximum and minimum values of the function $f(x) = 2x^3 - 21x^2 + 36x - 20$. [DIKSHA]

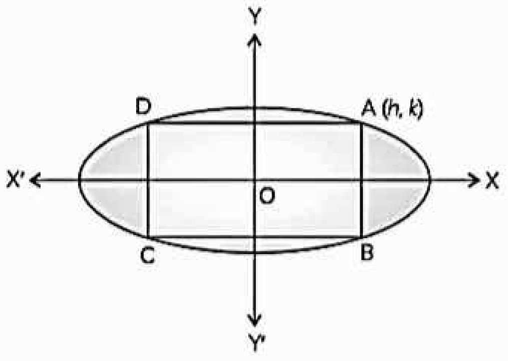
54. Find two positive numbers x and y such that their sum is 35 and the product is x^2y^5 is maximum. [NCERT]

55. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Ans. Let ABCD be a rectangle inscribed in the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

Let, coordinates of one vertex of the rectangle be A(h, k). Then AB = $2k$ and AD = $2h$.



Since, point A also lies on the ellipse, so it must satisfy its equation.

$$\therefore \quad \frac{h^2}{9} + \frac{k^2}{4} = 1$$

$$\Rightarrow \quad k^2 = 4\left(1 - \frac{h^2}{9}\right)$$

Now, area of rectangle ABCD is

$$A_1 = 2h \times 2k$$
$$= 4hk$$
$$\Rightarrow \quad A_1^2 = 16h^2k^2$$
$$= 16h^2 4\left(1 - \frac{h^2}{9}\right)$$
$$\Rightarrow \quad A_1^2 = 64\left(h^2 - \frac{h^4}{9}\right)$$

Now, $\quad \frac{dA_1^2}{dh} = 64\left(2h - \frac{4h^3}{9}\right)$

and $\quad \frac{d^2A_1^2}{d^2h} = 64\left(2 - \frac{12h^2}{9}\right)$

For A_1 to be maximum or minimum,

Put $\quad \frac{dA_1^2}{dh} = 0$

$$\Rightarrow \quad 64\left(2h - \frac{4h^3}{9}\right) = 0$$
$$\Rightarrow \quad 128h\left(1 - \frac{2h^2}{9}\right) = 0$$
$$\Rightarrow \quad h^2 = \frac{9}{2} \qquad [\because h \neq 0]$$

When $h^2 = \frac{9}{2}$,

$$\frac{d^2A_1^2}{d^2h} = 64\left(2 - \frac{12}{9} \times \frac{9}{2}\right)$$
$$= 64\,(2 - 6) = -256 < 0$$

Thus, A_1^2 is maximum, it implies A_1 is maximum.

$\therefore$ The area of the greatest rectangle inscribed in the ellipse

$$= 4hk$$
$$= 4\sqrt{\frac{9}{2}} \cdot \sqrt{4\left(1 - \frac{h^2}{9}\right)}$$
$$= \frac{4 \times 3}{\sqrt{2}} \times \sqrt{4\left(1 - \frac{9}{2 \times 9}\right)}$$
$$= \frac{12}{\sqrt{2}} \times \sqrt{4 \times \frac{1}{2}}$$
$$= 12 \text{ sq. units}$$

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

56. Find the points of local maxima, local minima and the points of inflexion of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$. Also, find the corresponding local maximum and local minimum values. [NCERT Exemplar]

Ans. Here, $f(x) = x^5 - 5x^4 + 5x^3 - 1$

Differentiate $f(x)$ w.r.t. x, we get

$$f'(x) = 5x^4 - 20x^3 + 15x^2$$

For maxima or minima,

Put $f'(x) = 0$

$\Rightarrow \quad 5x^4 - 20x^3 + 15x^2 = 0$

$\Rightarrow \quad 5x^2(x^2 - 4x + 3) = 0$

$\Rightarrow \quad 5x^2(x^2 - 3x - x + 3) = 0$

$\Rightarrow 5x^2 [x(x - 3) - 1(x - 3)] = 0$

$\Rightarrow \quad 5x^2 (x - 1)(x - 3) = 0$

$\Rightarrow \quad x = 0, 1, 3$

Also, $f''(x) = 20x^3 - 60x^2 + 30x$

$= 10x(2x^2 - 6x + 3)$

At $x = 0$,

$f''(x) = 0$

So, $x = 0$ is a point of inflexion.

At $x = 1$,

$f''(x) = 10(1)(2 - 6 + 3)$

$= -10 < 0$

So, $x = 1$ is a point of local maxima.

At $x = 3$,

$f''(x) = 10(3)(18 - 18 + 3)$

$= 90 > 0$

So, $x = 3$ is a point of local minima.

Now, maximum value of $f(x)$ is

$f(x = 1) = (1)^5 - 5(1)^4 + 5(1)^3 - 1$

$= 1 - 5 + 5 - 1$

$= 0$

And, minimum value of $f(x)$ is

$f(x = 3) = (3)^5 - 5(3)^4 + 5(3)^3 - 1$

$= 243 - 405 + 135 - 1$

$= -28.$

57. A wire of length 35 cm is cut into two pieces. One of the pieces is turned in the form of a square and other in the form of a circle. Find the length of each piece, so that the sum of areas of the two be minimum.

Ans. Let x cm and $(35 - x)$ cm be the required lengths. Again, let a be the side of the square formed and r be the radius of the circle formed. Then,

$4a = x$ and $2\pi r = 35 - x$

$\Rightarrow a = \frac{x}{4}$ and $r = \frac{35 - x}{2\pi}$

$\therefore$ Area of the square $= a^2 = \left(\frac{x^2}{16}\right)$ cm^2

and Area of the circle, $A = \pi r^2$

$= \pi\left(\frac{35 - x}{2\pi}\right)^2$

$= \frac{(35 - x)^2}{4\pi}$

Now, combined area,

$$A = \frac{x^2}{16} + \frac{(35 - x)^2}{4\pi}$$

On differentiating with respect to x, we get

$$\frac{dA}{dx} = \frac{2x}{16} + \frac{2(35 - x)(-1)}{4\pi}$$

$$\frac{dA}{dx} = \frac{x}{8} - \frac{(35 - x)}{2\pi}$$

$$= \frac{\pi x - 4(35 - x)}{8\pi}$$

$$= \frac{\pi x - 140 + 4x}{8\pi}$$

$$= \frac{(\pi + 4)x - 140}{8\pi}$$

and $\frac{d^2A}{dx^2} = \frac{(\pi + 4)}{8\pi}$

For maxima or minima, put $\frac{dA}{dx} = 0$

$\Rightarrow \quad \frac{(\pi + 4)x - 140}{8\pi} = 0$

$\Rightarrow \quad x = \frac{140}{\pi + 4}$

At $x = \frac{140}{\pi + 4}$,

$$\frac{d^2A}{dx^2} = \frac{\pi + 4}{8\pi} > 0 \quad \forall x \in R$$

$\therefore$ $x = \frac{140}{\pi + 4}$ is a point of local minima.

Hence, the length of pieces are $\left(\frac{140}{\pi + 4}\right)$ cm and $\left(\frac{35\pi}{\pi + 4}\right)$ cm respectively.

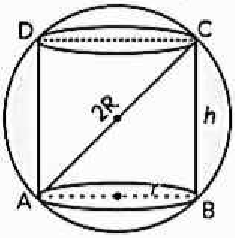
58. AB is a diameter of a circle and C is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles.

[NCERT Exemplar]

59. Show that the height of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the volume of the largest cylinder inscribed in a sphere of radius R.

[CBSE 2019]

Ans. Suppose we have a sphere of radius R and a cylinder is inscribed in it. Let radius and height of the cylinder be r and h, respectively.



In right ΔABC,

$$AC = \sqrt{(AB)^2 + (BC)^2}$$

$$2R = \sqrt{(2r)^2 + (h)^2}$$

$$2R = \sqrt{4r^2 + h^2}$$

$$\Rightarrow \quad 4r^2 = 4R^2 - h^2$$

$$\Rightarrow \quad r^2 = \frac{4R^2 - h^2}{4}$$

Let volume of cylinder, $V = \pi r^2 h$

$$V = \pi\left[\frac{1}{4}(4R^2 - h^2)\right]h$$

$$= \frac{\pi}{4}(4R^2h - h^3)$$

On differentiating with respect to h, we get

$$\frac{dV}{dh} = \frac{\pi}{4}(4R^2 - 3h^2)$$

For finding the largest volume, put $\frac{dV}{dh} = 0$

$$\Rightarrow \quad \frac{\pi}{4}(4R^2 - 3h^2) = 0$$

$$\Rightarrow \quad 4R^2 = 3h^2$$

$$\Rightarrow \quad h = \frac{2R}{\sqrt{3}}$$

($\because$ height cannot be negative, so we consider only positive value)

Now,
$$\frac{d^2V}{dh^2} = -\frac{3}{2}\pi h$$

When $h = \frac{2R}{\sqrt{3}}$,

$$\frac{d^2V}{dh^2} = -\frac{3}{2}\pi \times \frac{2R}{\sqrt{3}}$$

$$= -\sqrt{3}\pi R < 0$$

So, the volume of cylinder is maximum, when

$$h = \frac{2R}{\sqrt{3}}.$$

Consider
$$r^2 = \frac{1}{4}(4R^2 - h^2)$$

$$= \frac{1}{4}\left(4R^2 - \frac{4R^2}{3}\right)$$

$$= \frac{2R^2}{3}$$

$\therefore$ Volume of largest cylinder inscribed in a sphere,

$$V = \pi \times \frac{2R^2}{3} \times \frac{2R}{\sqrt{3}}$$

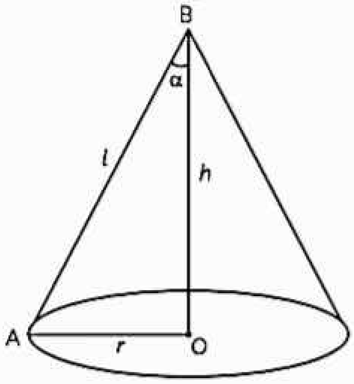
$$= \frac{1}{\sqrt{3}} \times \left(\frac{4}{3}\pi R^3\right)$$

$$= \frac{1}{\sqrt{3}} \times \text{ Volume of sphere}$$

Hence, height of the cylinder of maximum volume is $\frac{2R}{\sqrt{3}}$ and volume of largest cylinder inscribed in sphere is $\frac{1}{\sqrt{3}}$ times of it.

60. Show that the semi-vertical angle of right circular cone of given surface area and maximum volume is $\cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$.

Ans. Let radius and height of the right circular cone be r and h, respectively. And let its slant height be l and semi-vertical angle be α.



In right ΔAOB

$$l^2 = r^2 + h^2$$

$$\Rightarrow \quad h^2 = l^2 - r^2$$

Now, Volume of cone, $V = \frac{1}{3}\pi r^2 h$

$$\Rightarrow \quad V^2 = \frac{1}{9}\pi^2 r^4 h^2$$

$$= \frac{1}{9}\pi^2 r^4 (l^2 - r^2) \quad \text{...(i)}$$

Also, surface area of cone is

$$S = \pi r l + \pi r^2$$

$$\Rightarrow \quad \pi r l = S - \pi r^2$$

$$\Rightarrow \quad l = \frac{S - \pi r^2}{\pi r} \quad \text{...(ii)}$$

Put the value of l in Eq. (i), we get

$$V^2 = \frac{1}{9}\pi^2 r^4 \left(\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right)$$

$$V^2 = \frac{\pi^2 r^4}{9}\left[\frac{S^2 - 2\pi S r^2 + \pi^2 r^4 - \pi^2 r^4}{\pi^2 r^2}\right]$$

$$= \frac{r^2}{9}(S^2 - 2\pi S r^2)$$

$$= \frac{1}{9}(S^2 r^2 - 2\pi S r^4)$$

Consider $V^2 = f(r)$, then V will be maximum or minimum as per the corresponding $f(r)$ is maximum or minimum.

Now $\quad f(r) = \frac{1}{9}(S^2 r^2 - 2\pi S r^4)$

On differentiating with respect to r, we get

$$f'(r) = \frac{1}{9}(2S^2 r - 8\pi S r^3)$$

$$= \frac{2}{9}(S^2 r - 4S\pi r^3)$$

For maximum value, put $f'(r) = 0$

$$\Rightarrow \frac{2\pi}{9}(S^2 r - 4S\pi r^3) = 0$$

$$\Rightarrow \quad rS(S - 4\pi r^2) = 0$$

$$\Rightarrow \quad r = 0 \text{ or } S = 4\pi r^2$$

$$\Rightarrow \quad r = \frac{1}{2}\sqrt{\frac{S}{\pi}} \quad [\because r \neq 0]$$

Now, $\quad f''(r) = \frac{2}{9}(S^2 - 12S\pi r^2)$

When $r = \frac{1}{2}\sqrt{\frac{S}{\pi}}$,

$$f''\left(\frac{1}{2}\sqrt{\frac{S}{r}}\right) = \frac{2}{9}\left(S^2 - 12S\pi \times \frac{S}{4\pi}\right)$$

$$= \frac{2}{9}(S^2 - 3S^2)$$

$$= -\frac{4}{9}S^2 < 0$$

$\therefore$ Volume of cone is maximum, when $r = \frac{1}{2}\sqrt{\frac{S}{\pi}}$.

From Eq. (ii),

$$l = \frac{S - \pi r^2}{\pi r}$$

$$= \frac{4\pi r^2 - \pi r^2}{\pi r}$$

$$= \frac{3\pi r^2}{\pi r}$$

$$= 3r$$

and $\quad h = \sqrt{l^2 - r^2}$

$$= \sqrt{(3r)^2 - r^2}$$

$$= \sqrt{9r^2 - r^2}$$

$$= 2\sqrt{2}r$$

In ΔOBA, $\quad \cos\alpha = \frac{h}{l}$

$$= \frac{2\sqrt{2}r}{3r}$$

$$= \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \quad \alpha = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

Hence, proved.

61. @**If the sum of the surface areas of cube and a sphere is constant, what is the ratio of an edge of the cube to the diameter of the sphere, when the sum of their volumes is minimum? [NCERT Exemplar]**

62. Ⓐ**Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$.** **[CBSE 2016]**

63. Find the point on the curve $y^2 = 4x$, which is nearest to the point (2, – 8). **[CBSE 2019]**

Ans. Given equation of curve is $y^2 = 4x$.

Let P(x, y) be a point on the curve, which is nearest to point A(2, – 8).

Now, distance between the points A and P is given by

$$AP = \sqrt{(x-2)^2+(y+8)^2}$$

$$= \sqrt{\left(\frac{y^2}{4}-2\right)^2+(y+8)^2} \quad \left[\because x = \frac{y^2}{4}\right]$$

$$= \sqrt{\frac{y^4}{16}+4-y^2+y^2+16y+64}$$

$$= \sqrt{\frac{y^4}{16}+16y+68}$$

Let $Z = AP^2 = \frac{y^4}{16}+16y+68$

Now, $\frac{dZ}{dy} = \frac{1}{16}\times 4y^3+16 = \frac{y^3}{4}+16$

For maximum or minimum value of Z,

Put $\frac{dZ}{dy} = 0$

$\Rightarrow \quad \frac{y^3}{4}+16 = 0$

$y = -4$

Now, $\frac{d^2Z}{dy^2} = \frac{3}{4}(-4)^2 = 12 > 0$

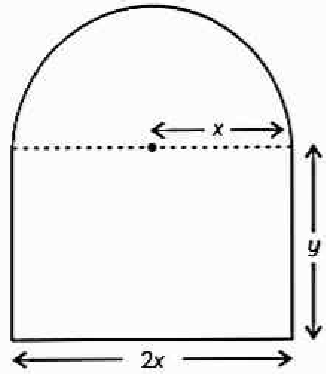
Thus Z is minimum when $y = -4$.

Substituting $y = -4$ in equation of the curve $y^2 = 4x$; we obtain $x = 4$.

Hence, the point (4, –4) on the curve $y^2 = 4x$ is nearest to the point (2, –8).

64. Ⓐ**A window is of the form of a semi-circle with a rectangle on its diameter. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.** **[CBSE 2018]**

Ans. Let $2x$ be the length and y be the width of the window.



Then radius of semicircular opening = x m

Since perimeter of the window is 10 cm.

$$2x+y+y+\frac{2\pi x}{2} = 10$$

$\Rightarrow \quad 2x + 2y + \pi x = 10$

$\Rightarrow \quad x(\pi + 2) + 2y = 10$

$$y = \frac{10-x(\pi+2)}{2} \quad ...(i)$$

Note to admit maximum light the area of window should be maximum.

Here, area of window.

A = Area of rectangle + Area of semi circular region

$$= 2x\times y+\frac{1}{2}\pi x^2$$

$$A = 2x\left(\frac{10-x(\pi+2)}{2}\right)+\frac{1}{2}\pi x^2$$

$\therefore \quad A = 10x - x^2(\pi+2)+\frac{1}{2}\pi x^2$

Then, $\frac{dA}{dx} = 10 - 2x(\pi + 2) + \frac{1}{2}\times 2\pi r$

Put $\frac{dA}{dx} = 0$

$\Rightarrow 10 + x[\pi - 2(\pi + 2)] = 0$

$\Rightarrow \quad x = \frac{-10}{-\pi-4} = \frac{10}{\pi+4}$

Also, $\frac{d^2A}{dx^2} = -2(\pi + 2) + \pi$

$= -(\pi + 4) < 0$

$\therefore$ For $x = \frac{10}{\pi + 4}$, area is maximum.

Now, when $x = \frac{10}{\pi + 4}$

$$y = \frac{10}{2} - \frac{10}{\pi + 4}\frac{(\pi + 2)}{2}$$

$$= 5 - 5\frac{(\pi + 2)}{\pi + 4}$$

$$= \frac{10}{\pi + 4}$$

Dimensions of the window are

Hence, length $= \left(\frac{20}{\pi + 4}\right)$m and breadth $= \left(\frac{10}{\pi + 4}\right)$m

SHORT ANSWER Type-I Questions (SA-I)

[**2** marks]

1. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R.

Ans.

$f(x) = x^3 - 3x^2 + 6x - 100$

$f'(x) = 3x^2 - 6x + 6$

$= 3(x^2 - 2x + 2)$

discriminant of the formed quadratic $= b^2 - 4ac = (-2)^2 - 4(1)(2)$

$= 4 - 8 = -4$

$b^2 - 4ac < 0$

but $a > 0$ $[a = 1]$

Hence $x^2 - 2x + 2 > 0$ for all $x \in R$

$\therefore$ $3(x^2 - 2x + 2) > 0$; $x \in R$

$f'(x) > 0$; $x \in R$

Hence $f(x)$ is increasing on R

[CBSE Topper 2017]

2. The total cost C(x) associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output.

Ans.

$C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$

$C'(x) = 0.005(3x^2) - 0.02(2x) + 30$

$C'(x) = 0.015x^2 - 0.04x + 30$

when $x = 3$.

$C'(3) = 0.015(3)^2 - 0.04(3) + 30$

$= 0.015(9) - 0.04(3) + 30$

$= 0.135 - 0.12 + 30$

$= 30.015$ Ans.

[CBSE Topper 2018]