10 Vector Algebra

> “ Vector algebra is used to make programming video games much easier. Vectors are used to represent the velocity of players, control where they are aiming, or what they can see (where they are facing). We also need a point to keep track of the player's position at all times in that case the player's position will be the origin for our velocity and rotation vector. All this can be calculated using vector algebra.

Topic Notes

- *Basic Concepts of Vectors*
- *Product of Two Vectors*

BASIC CONCEPTS OF VECTORS 1

TOPIC 1

TERMINOLOGY AND REPRESENTATION OF VECTORS

In our day-to-day life, we use terms like length, mass, time, distance, speed , area, volume, density, temperature etc., whose measurement can be expressed in simple numbers together with appropriate units of measurements. However, some quantities are tricky to deal with. For example, if you want to travel from one place to another, you will not only want to know how far apart the two places are (*i.e.*, the distance between them), but you will also need to know the direction that leads from one place to another. The physical quantity that combines distance and direction is called displacement. There are many more physical quantities that also require direction like velocity, acceleration, force etc. Consider the following two statements:

(1) The mass of a body is 4 kg.

(2) The displacement of a body is 4 m.

In statement (1), we have complete information. To ask its direction is a meaningless question. In statement (2), we have incomplete information because the direction of displacement is not given. However, the statement that "The displacement of a body is 4 m towards North" is meaningful and gives us complete information.

So, we say mass is a scalar, and displacement is a vector.

In this chapter, we are going to study about various forms of vectors, their properties, scalar product, vector product etc.

Scalars

The quantities which have only magnitude but no direction are called scalars (or scalar quantities). For example, length, mass, area, volume, density and temperature, are scalars. There are represented by single letters such as *a*, *b*, *c*, etc.

Vectors

The quantities which have magnitude as well as direction are called vectors (or vector quantities). For example, displacement, velocity, acceleration, force etc., are vectors. These are represented by single letters with an arrow on them such as $\vec{a}, \vec{b}, \vec{c}$, etc.

Important

➥ *A vector has both magnitude and direction and thus can be represented by a real number and a specified direction.*

Geometric Concepts of a Vector

Though vectors can be dealt both with analytically and geometrically, but we shall confine ourselves to a detailed study of geometric concept of vectors.

Directed line segment

An ordered pair of points (A, B) in a plane or space and written as AB is called a directed line segment. A is called its initial point and B is called its terminal point. Its direction is from A and B. It is therefore, represented by $\overline{AB}$ (read as 'vector AB').

A ⟶ B

The length of the line segment AB indicates the magnitude of the vector $\overline{AB}$ and the direction of arrow head indicates the direction of the vector $\overline{AB}$.

TOPIC 2

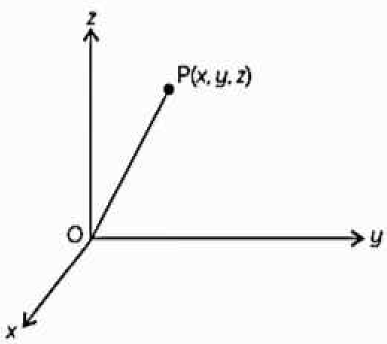
TYPES OF VECTORS

Position Vector

Let O(0, 0, 0) be the origin and P be a point in space having coordinates (*x*, *y*, *z*) with respect to the origin O. Then, the vector $\overline{OP}$ is called position vector of the point P with respect to O.

Using distance formula (from Class XI), the magnitude of $\overline{OP}$ is given by

$$|\overline{OP}| = \sqrt{x^2+y^2+z^2}$$



(Here, O is the initial point and P is the terminal point.)

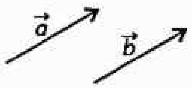
Important

➥ *In practice, the position vectors of points A, B, C, etc., with respect to the origin O are denoted by $\vec{a}$, $\vec{b}$, $\vec{c}$, etc., respectively.*

Equal Vectors

Two vectors are said to be equal if they have (i) the same length and (ii) the same direction. (The starting (or initial) points of two vectors is immaterial)

In the following figure, $\vec{a}$ and $\vec{b}$ are equal vectors.



Equal vectors $\vec{a}$ and $\vec{b}$ are written as $\vec{a} = \vec{b}$.

Example 1.1: Find the values of x and y so that the vectors $2\hat{i} + 3\hat{j}$ and $x\hat{i} + y\hat{j}$ are equal. [NCERT]

Ans. Two vectors are equal if and only if their corresponding components are equal.

Thus, $2\hat{i} + 3\hat{j} = x\hat{i} + y\hat{j}$

$\Rightarrow$ $x = 2$ and $y = 3$

Zero Vector or Null Vector

A vector, whose length is zero and having any direction, is called a zero vector. It is denoted by $\vec{0}$.

The vectors $\overline{AA}$, $\overline{BB}$, $\overline{TT}$, etc., are also zero vectors. Thus, initial and terminal points of a zero vector coincide.

Unit Vector

A vector, whose magnitude is unity (*i.e.*, 1 unit), is called a unit vector.

A unit vector in the direction of a vector $\vec{a}$ is denoted by $\hat{a}$.

where, $$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

Important

➥ *$\vec{a}$ can also be represented by $|\vec{a}|\,\hat{a}$.*

Example 1.2: Find a unit vector in the direction of the vector $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$.

Ans. Unit vector in the direction of $\vec{a}$

$$= \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{1^2 + 1^2 + 2^2}}$$

$$= \frac{\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}\hat{i} + \frac{1}{\sqrt{6}}\hat{j} + \frac{2}{\sqrt{6}}\hat{k}$$

Coinitial Vectors

Two or more vectors having the same initial point are called coinitial vectors.

Collinear or Parallel Vectors

Two or more vectors are said to be collinear or parallel if they are parallel to the same line, irrespective of their magnitudes and directions.

i.e., Vectors $\vec{a}$ and $\vec{b}$ are collinear or parallel if $\vec{a} = \lambda\vec{b}$.

Example 1.3: Show that the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $-4\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear. [NCERT]

Ans. Two vectors $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

are collinear if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Here, $\frac{2}{-4} = \frac{-3}{6} = \frac{4}{-8} = -\frac{1}{2}$.

So, the given vectors are collinear.

Like Vectors

Two vectors are said to be like vectors if they have the same direction, irrespective of their magnitudes.

Unlike Vectors

Two vectors are said to be unlike vectors if they have the opposite directions, irrespective of their magnitudes.

Negative of a Vector

A vector whose magnitude is the same as that of a given vector, but direction is opposite to that of it, is called negative of the given vector. For example, vector $\overline{BA}$ is negative of the vector $\overline{AB}$ and is written as $\overline{BA} = -\overline{AB}$

Free Vector

A vector, which is independent of its position, is called a free vector. Its initial point is arbitrary.

Caution

➥ *Throughout this chapter, we will be dealing with free vectors only.*

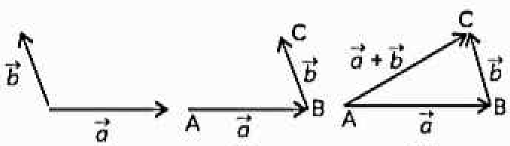
TOPIC 3

ADDITION AND SUBTRACTION OF VECTORS

Triangle Law of Vector Addition

Triangle law of vector addition states that if two vectors, say $\vec{a}$ and $\vec{b}$, are positioned in such a way that the initial point of $\vec{b}$ is at the terminal point of $\vec{a}$ and then the vector joining the initial point of $\vec{a}$ to the terminal point of $\vec{b}$ gives of the sum of the two vectors. If $\vec{c}$ (say) is the sum of the two vectors, then $\vec{c}$ is written as $\vec{c} = \vec{a} + \vec{b}$.

In general, if we have two vectors $\vec{a}$ and $\vec{b}$ [Fig. (i)], then to add them, they are positioned so that the initial point of vector $\vec{b}$ coincides with the terminal point of vectors $\vec{a}$ [Fig. (ii)]. Then, the vector $\vec{a} + \vec{b}$ represented by the third side AC of the triangle ABC, gives us the sum (or, resultant) of the vectors $\vec{a}$ and $\vec{b}$ [Fig. (iii)].



(i) (ii) (iii)

i.e., in ΔABC, we have

$$\vec{AB} + \vec{BC} = \vec{AC}$$

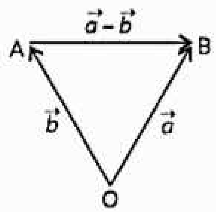
Now, again, since $\vec{AC} = -\vec{CA}$, from the above equation, we have

$$\vec{AB} + \vec{BC} = -\vec{CA}$$

$$\Rightarrow \quad \vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$$

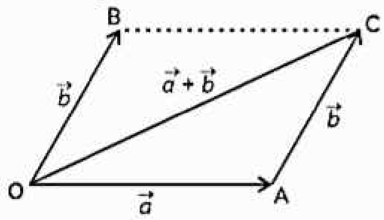
This means that when the sides of a triangle are taken in order, it leads to zero resultant as the initial and terminal points get coincided.

The difference of vectors $\vec{a}$ and $\vec{b}$, represented by $\vec{a} - \vec{b}$ is that vector which when added to $\vec{b}$ gives $\vec{a}$. Equivalently, $\vec{a} - \vec{b}$ may be defined as $\vec{a} + (-\vec{b})$.



Parallelogram Law of Vector Addition

Consider a boat in a river going from one bank of the river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by the two velocity vectors – one is the velocity imparted to the boat by its engine and other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, boat will travel with a different velocity. To get idea about the effective speed and direction (*i.e.*, the resultant velocity) of the boat, we have the parallelogram law of vector addition.



Let $\vec{a}$ and $\vec{b}$ be two vectors represented by the two adjacent sides of a parallelogram in magnitude and direction (see figure), then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram through their common vertex.

Important

➥ *From the figure above, using the triangle law, we have*

$$\vec{OA} + \vec{AC} = \vec{OC}$$

Or $$\vec{OA} + \vec{OB} = \vec{OC}$$

which is parallelogram law. Thus, we may say that the two laws of vectors addition are equivalent to each other.

Properties of Vector Addition

(1) Commutative property : For any two vectors $\vec{a}$ and $\vec{b}$, we have

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

(2) Associative property : For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

(3) Additive Identity : For any vector $\vec{a}$, we have

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$$

(4) Additive Inverse : For any vector $\vec{a}$, there exists $-\vec{a}$ such that

$$\vec{a} + (-\vec{a}) = \vec{0} = (-\vec{a}) + \vec{a}$$

Here, $(-\vec{a})$ is called the additive inverse of $\vec{a}$.

TOPIC 4

MULTIPLICATION OF A VECTOR BY A SCALAR

When we multiply a vector $\vec{a}$ by a scalar λ, then the product $\lambda\vec{a}$ is defined to be a vector whose magnitude is $|\lambda|$ times the magnitude of $\vec{a}$, *i.e.*, $|\lambda\vec{a}| = |\lambda||\vec{a}|$ and whose direction is the same or opposite to that of $\vec{a}$ according to as λ is positive or negative.

Thus, $2\vec{a}$ is a vector whose magnitude is twice and direction is the same as that of $\vec{a}$, and $-\vec{a}$ is a vector which has the same magnitude but direction is opposite to that of $\vec{a}$.

Note that if $\lambda = \dfrac{1}{|\vec{a}|}$, provided $\vec{a} \neq \vec{0}$,

then $|\lambda\vec{a}| = |\lambda||\vec{a}| = \dfrac{1}{|\vec{a}|}|\vec{a}| = 1$.

So, $\lambda\vec{a}$ represents the unit vector in the direction of $\vec{a}$. We write it as $\hat{a} = \dfrac{1}{|\vec{a}|}\vec{a}$

⚠ Caution

➥ *For any scalar* λ, $\lambda\vec{0} = \vec{0}$

The addition of vectors and the multiplication of a vector by a scalar together give the following distributive laws:

Let $\vec{a}$ and $\vec{b}$ be any two vectors, and k and m be any two scalars. Then,

(1) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$;

(2) $(k + m)\vec{a} = k\vec{a} + m\vec{a}$;

(3) $(km)\vec{a} = k(m\vec{a}) = m(k\vec{a})$

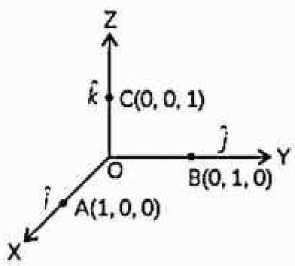
TOPIC 5

COMPONENTS OF A VECTOR

Orthogonal Unit Vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$

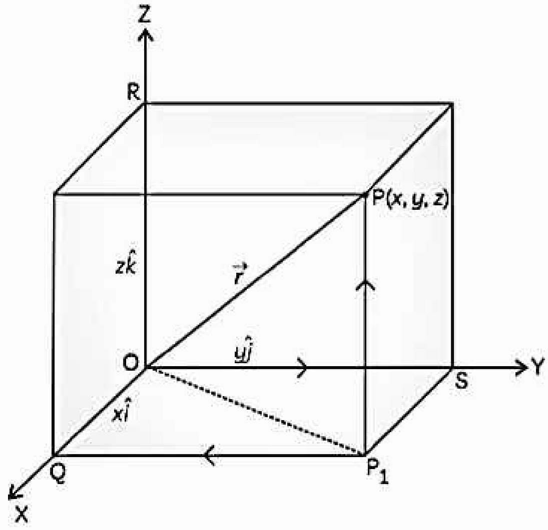
Let us take the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on the x-axis, y-axis and z-axis respectively. Then, clearly $|\overline{OA}| = 1$; $|\overline{OB}| = 1$ and $|\overline{OC}| = 1$.

The vectors $|\overline{OA}|$, $|\overline{OB}|$ and $|\overline{OC}|$, each having magnitude 1, are called unit vectors along the axes OX, OY and OZ respectively, and denoted by $\hat{i}, \hat{j}$ and $\hat{k}$, respectively (see figure). Clearly, $\hat{i}, \hat{j}$ and $\hat{k}$ are orthogonal unit vectors.



Components of a Vector

Consider the position vector $\overline{OP}$ of a point P(x, y, z). Let P_1 be the foot of the perpendicular from P on the plane XOY.



We, thus, see that P_1P is parallel to z-axis. As $\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors along the x, y and z axes, respectively and by the definition of the coordinates of P, we have

$$\overrightarrow{P_1P} = \overrightarrow{OR} = z\hat{k}.$$

Similarly, $\overrightarrow{QP_1} = \overrightarrow{OS} = y\hat{j}$ and $\overrightarrow{SP_1} = \overrightarrow{OQ} = x\hat{i}$.

Therefore, it follows that

$$\overrightarrow{OP_1} = \overrightarrow{OQ} + \overrightarrow{OS} = x\hat{i} + y\hat{j}$$

and

$$\overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = x\hat{i} + y\hat{j} + z\hat{k}$$

Hence, the position vector of P with reference to O is given by

$$\overrightarrow{OP} \text{ (or } \vec{r}\text{)} = x\hat{i} + y\hat{j} + z\hat{k}$$

This form of any vector is called its component form. Here, x, y and z are called as the scalar components of $\vec{r}$, and $x\hat{i}, y\hat{j}$ and $z\hat{k}$ are called as the vector components of $\vec{r}$ along the respective axes. The length of $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is given by

$$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}| = \sqrt{x^2 + y^2 + z^2}$$

Some Important Results

If $\vec{a}$ and $\vec{b}$ are any two vectors given in the component form $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, respectively, then,

(1) The sum (or resultant) of the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

(2) The difference of the vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$$

(3) The vectors $\vec{a}$ and $\vec{b}$ are equal if and only if

$$a_1 = b_1;\ a_2 = b_2;\ a_3 = b_3$$

(4) The multiplication of vector $\vec{a}$ by any scalar λ is given by

$$\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$$

(5) Vectors $\vec{a}$ and $\vec{b}$ are collinear (or parallel) if there exists a non-zero scalar λ such that $\vec{b} = \lambda\vec{a}$

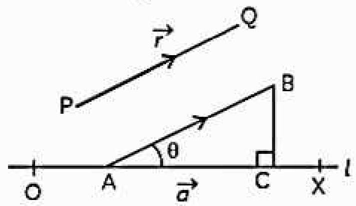
$$\Rightarrow \quad b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = \lambda(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$\Rightarrow \quad b_1 = \lambda a_1,\ b_2 = \lambda a_2,\ b_3 = \lambda a_3$$

$$\Rightarrow \quad \frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Components of a Vector Along a Given Direction

Let $\vec{r} = \overrightarrow{PQ}$ be any vector and $\overrightarrow{OX} = \vec{a}$ along the line l (say) as shown in the figure.



Take a point A on the line l. Through A, draw a line segment AB parallel and equal to $|\overrightarrow{PQ}|$. Therefore,

$$\overrightarrow{AB} = \overrightarrow{PQ} = \vec{r}$$

Let θ be the angle between $\vec{r}$ and $\vec{a}$.
From B, draw BC ⊥ OX. Then, ∠CAB = θ.

Directed line segment $\overrightarrow{AC}$ is defined as the component of vector $\vec{r}$ along $\vec{a}$.
Length of the component of vector $\vec{r}$ along $\vec{a}$

$$= AC = AB\cos\theta = |\vec{r}|\cos\theta$$

Hence, the length of the component of vector $\vec{r}$ along $\vec{a}$ is $|\vec{r}||\cos\theta|$.

⚠ Caution

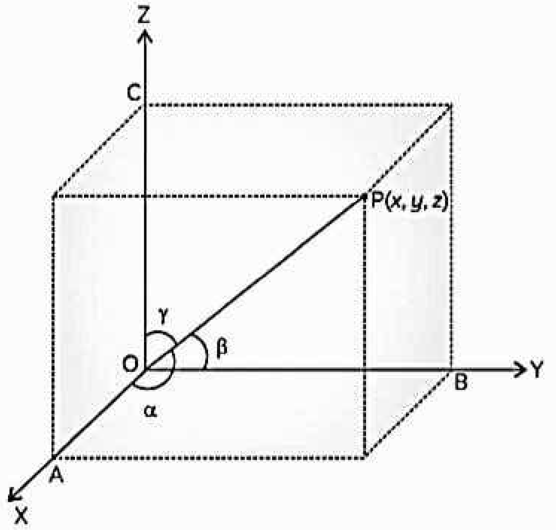
➥ *If the direction of $\vec{a}$ is reversed, the angle between $\vec{r}$ and $\vec{a}$ is 180° − θ and the length of the component is* $|\vec{r}||\cos(180° - \theta)|$.

TOPIC 6

DIRECTION COSINES AND DIRECTION RATIOS OF A VECTOR

Let P(x, y, z) be any point in space whose position vector is $\vec{OP}$.

If $\vec{OP}$ makes angles α, β and γ with positive directions of the coordinates axes OX, OY and OZ respectively, then cos α, cos β and cos γ are called direction cosines of $\vec{OP}$ and usually denoted by l, m and n respectively. The angles α, β and γ are called direction angles and lies between – π and π.



As x-axis makes angles of 0, $\frac{\pi}{2}$, $\frac{\pi}{2}$ with OX, OY and OZ respectively, therefore, direction cosines of x-axis are $\left(\cos 0, \cos\frac{\pi}{2}, \cos\frac{\pi}{2}\right)$ *i.e.,* (1, 0, 0). Similarly, the direction cosines of y-axis and z-axis are (0, 1, 0) and (0, 0, 1) respectively.

Some Important Results

Let P(x, y, z) be any point in space such that $\vec{OP} = \vec{r}$ and direction cosines of $\vec{OP}$ are l, m, n.

Then,

(1) $x = l|\vec{r}|; y = m|\vec{r}|; z = n|\vec{r}|$

(2) $\vec{r} = l\hat{i} + m\hat{j} + n\hat{k}$

(3) $l^2 + m^2 + n^2 = 1$

Let l, m, n be the direction cosines of a vector $\vec{r}$ and a, b, c are three numbers such that $l = \lambda a$, $m = \lambda b$ and $n = \lambda c$ for some non-zero real number λ. The triplet (a, b, c) are called direction ratios of $\vec{r}$.

Relation between Direction Cosines and Direction Ratios

Let a, b, c be the direction ratios of any vector $\vec{r}$. whose direction cosines are l, m, n. From the definition, we have,

$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \lambda$, then $l = \lambda a$, $m = \lambda b$ and $n = \lambda c$.

Since $l^2 + m^2 + n^2 = 1$, we have $\lambda^2(a^2 + b^2 + c^2) = 1$

$\Rightarrow \lambda = \pm\frac{1}{\sqrt{a^2 + b^2 + c^2}}$

If we consider the positive value of λ, then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Thus, to obtain the direction cosines of a vector from the given direction ratios a, b, c, divide each number a, b, c by the positive or negative square root of the sum of the squares of direction ratios *i.e.,* $\sqrt{a^2 + b^2 + c^2}$.

⚠ Caution

- *The sum of the squares of the direction cosines is always equal to 1, but the sum of the squares of the direction ratios need not be 1.*

Example 1.4: Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$. [NCERT]

Ans. We know that the direction ratios a, b, c of a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ are just the components x, y and z of the vector.

Further, the direction cosines l, m, n are given by

$$l = \frac{a}{|\vec{r}|},\ m = \frac{b}{|\vec{r}|} \text{ and } l = \frac{c}{|\vec{r}|}$$

Here, $a = 1$, $b = 2$ and $c = 3$ and

$|\vec{r}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

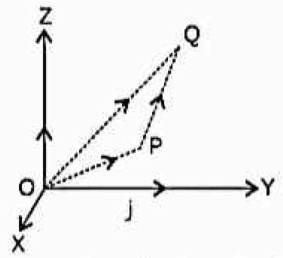
Thus, $l = \frac{1}{\sqrt{14}}$, $m = \frac{2}{\sqrt{14}}$ and $n = \frac{3}{\sqrt{14}}$

TOPIC 7

POSITION VECTOR OF A VECTOR JOINING TWO POINTS

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points.

Then the vector joining P and Q is the vector $\overline{PQ}$ (see figure).



Joining the points P and Q with the origin and applying triangle law to Δ OPQ, we have

$$\vec{OP} + \vec{PQ} = \vec{OQ}$$

Using the properties of vector addition, the above equation becomes

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

i.e., $\vec{PQ} = (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

Also, the magnitude of $\vec{PQ}$ is given by

$$|\vec{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example 1.5: Find the direction cosines of the vector joining the points A(1, 2, –3) and B(–1, –2, 1) directed from A to B. [NCERT]

Ans. Here, $\vec{AB}$ = Position vector of B – Position of vector of A

$\Rightarrow \quad \vec{AB} = (-\hat{i} - 2\hat{j} + \hat{k}) - (\hat{i} + 2\hat{j} - 3\hat{k})$

$\Rightarrow \quad \vec{AB} = -2\hat{i} - 4\hat{j} + 4\hat{k}$

So, the direction ratios of $\vec{AB}$ are – 2, – 4 and 4. Also,

$$|\overline{AB}| = \sqrt{(-2)^2 + (-4)^2 + 4^2}$$

$$= \sqrt{36} = 6.$$

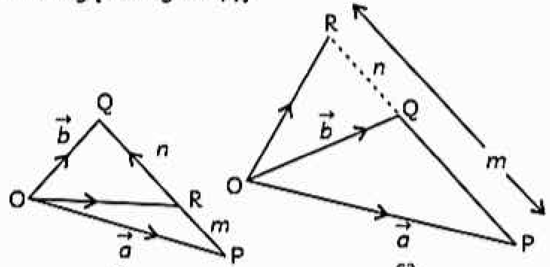
So, the direction cosines of $\vec{AB}$ are

$$\frac{-2}{6}, \frac{-4}{6} \text{ and } \frac{4}{6}, \text{ or } -\frac{1}{3}, -\frac{2}{3} \text{ and } \frac{2}{3}.$$

TOPIC 8

SECTION FORMULA

Let P and Q be two points represented by the position vectors $\overline{OP}(=\vec{a})$ and $\overline{OQ}(=\vec{b})$, respectively with respect to the origin O. Then the line segment joining the points P and Q may be divided by a third point, say R, in two ways, internally [see figure (i)] and externally [see figure (ii)].



(i) (ii)

Here, we intend to find the position vector $\vec{OR}(=\vec{r})$ for the point R with respect to O.

Case 1: When R divides PQ internally in the ratio $m:n$, then $\vec{OR}(=\vec{r}) = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

Case 2: When R divides PQ externally in the ratio $m:n$, then $\vec{OR}(=\vec{r}) = \dfrac{m\vec{b} - n\vec{a}}{m-n}$

Important

- *If R is the mid-point of PQ, then* $\overline{OR}(=\vec{r}) = \dfrac{\vec{a}+\vec{b}}{2}$
- *If $\vec{a}, \vec{b}$ and $\vec{c}$ be the position vectors of the vertices A, B and C of ΔABC, then the position vector of centroid of the triangle is* $\dfrac{\vec{a}+\vec{b}+\vec{c}}{3}$.

When R divides PQ internally in the ratio $m:n$, then

$\Rightarrow \quad \vec{r} = \left(\dfrac{m}{m+n}\right)\vec{b} + \left(\dfrac{n}{m+n}\right)\vec{a}$

or $\quad \vec{r} = \lambda\vec{b} + \mu\vec{a},$

where $\lambda + \mu = \dfrac{m}{m+n} + \dfrac{n}{m+n} = 1$

Thus, position vector of any point R on the line PQ can be taken as $\lambda\vec{b} + \mu\vec{a}$.

Example 1.6: Find the position of a point R which divide the line joining two points P and Q whose positions are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1 (A) internally (B) externally.

Ans. (A) $\overrightarrow{OR} = \frac{2(-\hat{i}+\hat{j}+\hat{k})+1(\hat{i}+2\hat{j}-\hat{k})}{2+1}$

$= \frac{-\hat{i}+4\hat{j}+\hat{k}}{3}$

(B) $\overrightarrow{OR} = \frac{2(-\hat{i}+\hat{j}+\hat{k})-1(\hat{i}+2\hat{j}-\hat{k})}{2-1}$

$= \frac{-3\hat{i}+0\hat{j}+3\hat{k}}{1}$

$= -3\hat{i}+3\hat{k}$

Example 1.7: Show that the points A, B and C with position vectors $\vec{a} = 3\hat{i}-4\hat{j}-4\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}-3\hat{j}-5\hat{k}$ respectively form the vertices of a right angled triangle. [NCERT]

Ans. Here,

$\overrightarrow{AB}$ = P.V. of B – P.V. of A

$= \vec{b}-\vec{a}$

$= (2\hat{i}-\hat{j}+\hat{k})-(3\hat{i}-4\hat{j}-4\hat{k})$

$= -\hat{i}+3\hat{j}+5\hat{k}$

So, $|\overrightarrow{AB}| = \sqrt{(-1)^2+3^2+5^2} = \sqrt{35}$ ...(i)

$\overrightarrow{BC}$ = P.V. of C – P.V. of B

$= \vec{c}-\vec{b}$

$= (\hat{i}-3\hat{j}-5\hat{k})-(2\hat{i}-\hat{j}+\hat{k})$

$= -\hat{i}-2\hat{j}-6\hat{k}$

So, $|\overrightarrow{BC}| = \sqrt{(-1)^2+(-2)^2+(-6)^2} = \sqrt{41}$ ...(ii)

$\overrightarrow{CA}$ = P.V. of A – P.V. of C

$= \vec{a}-\vec{c}$

$= (3\hat{i}-4\hat{j}-4\hat{k})-(\hat{i}-3\hat{j}-5\hat{k})$

$= 2\hat{i}-\hat{j}+\hat{k}$

So, $|\overrightarrow{CA}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$ ...(iii)

We can see $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA} = \vec{0}$. So A, B, C are the vertices of a triangle.

From (i) to (iii), we have

$|\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2 = 35+6 = 41 = |\overrightarrow{BC}|^2$

Hence, the points A, B and C with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ respectively form the vertices of a right angled triangle.

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. The value of x if $x(\hat{i}+\hat{j}+\hat{k})$ is a unit vector, is:

(a) $\pm\sqrt{3}$ (b) $\pm\frac{1}{3}$

(c) ± 3 (d) $\pm\frac{1}{\sqrt{3}}$

Ans. *(d)* $\pm\frac{1}{\sqrt{3}}$

Explanation: $x(\hat{i}+\hat{j}+\hat{k})$ is a unit vector, if

$\sqrt{x^2+x^2+x^2} = 1$

$\Rightarrow \sqrt{3x^2} = 1$

$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$

2. The value of p for which $p(\hat{i}+\hat{j}+\hat{k})$ is a unit vector is:

(a) 0 (b) $\frac{1}{\sqrt{3}}$

(c) 1 (d) $\sqrt{3}$ [CBSE 2020]

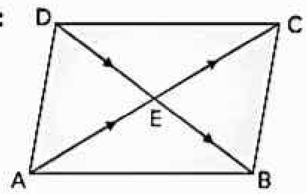
3. ABCD is a rhombus, whose diagonals intersect at E. Then $\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}+\overrightarrow{ED}$ equals to:

(a) $\vec{0}$ (b) $\overrightarrow{AD}$

(c) $2\overrightarrow{BC}$ (d) $2\overrightarrow{AD}$ [CBSE 2020]

Ans. *(a)* $\vec{0}$

Explanation:



$\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}+\overrightarrow{ED}$

$= \overrightarrow{EA}+\overrightarrow{EB}-\overrightarrow{EA}-\overrightarrow{EB}$

[As diagonals of rhombus bisect each other]

$= \vec{0}$

4. **The position vector of the point which divides the join of points $2\vec{a} - 3\vec{b}$ and $\vec{a}+\vec{b}$ in the ratio 3 : 1 internally, is:**

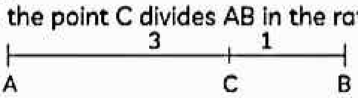
(a) $\dfrac{3\vec{a}-2\vec{b}}{2}$ (b) $\dfrac{7\vec{a}-8\vec{b}}{2}$

(c) $\dfrac{3\vec{a}}{4}$ (d) $\dfrac{5\vec{a}}{4}$

Ans. *(d)* $\dfrac{5\vec{a}}{4}$

Explanation: Let given points be $A(2\vec{a}-3\vec{b})$ and $B(\vec{a}+\vec{b})$.

Also let, the point C divides AB in the ratio 3 : 1.



∴ Position vector of point C

$$= \frac{3(\vec{a}+\vec{b})+1(2\vec{a}-3\vec{b})}{3+1}$$

$$= \frac{5\vec{a}}{4}$$

5. **If $|\vec{a}| = 4$ and $-3 \le \lambda \le 2$, then the range of $|\lambda\vec{a}|$ is:**

(a) [0, 8] (b) [–12, 8]

(c) [0, 12] (d) [8, 12]

6. **If ABCD is a quadrilateral, then $\vec{CD} + \vec{BA} + \vec{DA} + \vec{BC}$ is equals to:**

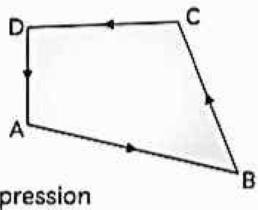
(a) $2\vec{AB}$ (b) $2\vec{BA}$

(c) $2\vec{AC}$ (d) $2\vec{BC}$

Ans. *(b)* $2\vec{BA}$

Explanation: In a quadrilateral ABCD, we have

$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DA} = \vec{0}$ ...(i)



So, given expression

$\vec{AB} + \vec{BC} = \vec{AC}$

$\vec{CD} + \vec{DA} = \vec{AC}$

$= \vec{BA} + (\vec{BC} + \vec{CD} + \vec{DA})$

$= \vec{BA} + \vec{AC} + \vec{AB} + \vec{AC}$

$= \vec{BA} - \vec{AB}$ [Using (i)]

$= 2\vec{BA}$ $[\because \vec{AB} = -\vec{BA}]$

7. **The vector having initial and terminal points as (2, 5, 0) and (–3, 7, 4), respectively is:**

(a) $-\hat{i} + 12\hat{j} + 4\hat{k}$ (b) $5\hat{i} + 2\hat{j} - 4\hat{k}$

(c) $-5\hat{i} + 2\hat{j} + 4\hat{k}$ (d) $\hat{i} + \hat{j} + \hat{k}$

8. **If cos α, cos β, cos γ are the direction cosines of the vector $\vec{a}$, then cos 2α + cos 2β + cos 2γ is equal to:**

(a) 3 (b) 0

(c) 2 (d) –1

Ans. *(d)* – 1

Explanation: Here, cos 2α + cos 2β + cos 2γ

$= (2\cos^2\alpha - 1) + (2\cos^2\beta - 1) + (2\cos^2\gamma - 1)$

$= 2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3$

$= 2(1) - 3$, *i.e.*, -1

9. **If the vectors $\vec{a}$ and $\vec{b}$ are the diagonals of a parallelogram, then its adjacent sides are:**

(a) $(\vec{a} + \vec{b}), (\vec{a} - \vec{b})$

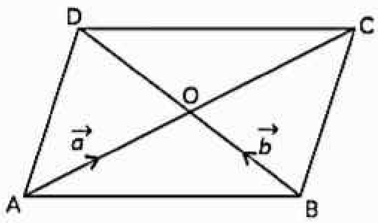
(b) $(\vec{a} + \vec{b}), (\vec{b} - \vec{a})$

(c) $\frac{1}{2}(\vec{a} + \vec{b}), \frac{1}{2}(\vec{a} - \vec{b})$

(d) $\frac{1}{2}(\vec{a} + \vec{b}), \frac{1}{2}(\vec{b} - \vec{a})$

Ans. *(c)* $\frac{1}{2}(\vec{a} + \vec{b}), \frac{1}{2}(\vec{a} - \vec{b})$

Explanation: Let $\vec{AC} = \vec{a}$ and $\vec{BD} = \vec{b}$ be the diagonals of the parallelogram ABCD and O be the point of their intersection as shown in the figure.



In ΔAOB,

$\vec{AB}$ = P.V. of B – P.V. of A

$= -\frac{1}{2}\vec{b} - \left(-\frac{1}{2}\vec{a}\right)$

$= \frac{1}{2}(\vec{a} - \vec{b})$

In ΔBOC,

$$\vec{BC} = \text{P.V. of C} - \text{P.V. of B}$$
$$= \frac{1}{2}\vec{a} - \left(-\frac{1}{2}\vec{b}\right)$$
$$= \frac{1}{2}(\vec{a} + \vec{b})$$

10. If $\vec{\ }$ is a non-zero vector of magnitude a and λ a non-zero scalar, then $\lambda\vec{a}$ is a unit vector if:

(a) $\lambda = 1$ **(b) $a = |\lambda|$**

(c) $\lambda = -1$ **(d) $a = \frac{1}{|\mu|}$**

Ans. *(d)* $a = \frac{1}{|\mu|}$

Explanation: We have, $\lambda\vec{a}$ is a unit vector.

$\because \quad |\lambda\vec{a}| = 1$

$\Rightarrow \quad |\lambda||\vec{a}| = 1$

$\Rightarrow \quad |\lambda| a = 1$

$\Rightarrow \quad a = \frac{1}{|\lambda|}$

11. The figure formed by four points $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 3\hat{j}$, $3\hat{i} + 5\hat{j} - 2\hat{k}$, $\hat{k} - \hat{j}$ is a:

(a) parallelogram **(b) rectangle**

(c) trapezium **(d) square**

Ans. *(c) trapezium*

Explanation: Consider the vertices as

$$\vec{OA} = \hat{i} + \hat{j} + \hat{k}$$
$$\vec{OB} = 2\hat{i} + 3\hat{j}$$
$$\vec{OC} = 3\hat{i} + 5\hat{j} - 2\hat{k}$$
$$\vec{OD} = \hat{k} - \hat{j}$$

$\therefore \quad \vec{AB} = \hat{i} + 2\hat{j} - \hat{k},$

$$\vec{BC} = \hat{i} + 2\hat{j} - 2\hat{k}$$
$$\vec{CD} = -3\hat{i} - 6\hat{j} + 3\hat{k}$$
$$= -3(\hat{i} + 2\hat{j} - \hat{k})$$
$$\vec{DA} = \hat{i} + 2\hat{j}$$

It is clear that,

$|\vec{AB}| \neq |\vec{BC}| \neq |\vec{CA}| \neq |\vec{DA}|$

Also, $\vec{AB} \parallel \vec{CD}$

Hence, figure formed by four points is a trapezium.

12. In ΔABC, which of the following is not true?

(a) $\vec{AB} \oplus \vec{BC} + \vec{CA} = 0$

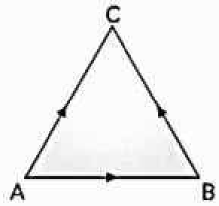
(b) $\vec{AB} \oplus \vec{BC} - \vec{AC} = 0$

(c) $\vec{AB} + \vec{BC} - \vec{CA} = 0$

(d) $\vec{AB} - \vec{CB} + \vec{CA} = 0$

Ans. *(c)* $\vec{AB} + \vec{BC} - \vec{CA} = 0$

Explanation: By triangle law of vector addition,



$$\vec{AB} + \vec{BC} = \vec{AC}$$

or $\vec{AB} + \vec{BC} = -\vec{CA}$

13. The magnitude of vector $6\hat{i} + 2\hat{j} + 3\hat{k}$ is:

(a) 5 **(b) 7**

(c) 12 **(d) 1**

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

14. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with x-axis, $\frac{\pi}{2}$ with y-axis and an acute angle θ with z-axis.

[CBSE 2014]

Ans. Given, $l = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $m = \cos\frac{\pi}{2} = 0$,

$n = \cos\theta$

As we know that,

$$l^2 + m^2 + n^2 = 1$$

$\therefore \left(\frac{1}{\sqrt{2}}\right)^2 + (0)^2 + \cos^2\theta = 1$

$\Rightarrow \quad \cos^2\theta = 1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$\therefore \quad n = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore$ Vector $\vec{a} = 5\sqrt{2}(l\hat{i} + m\hat{j} + n\hat{k})$

$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{i} + 0\hat{j} + \frac{1}{\sqrt{2}}\hat{k}\right)$

$= 5(\hat{i} + \hat{k})$

15. Find the sum of the vectors $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$. **[CBSE 2012]**

Ans. The given vectors are $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = -2\hat{i} + 4\hat{j} + 5\hat{k}$ and $\vec{c} = \hat{i} - 6\hat{j} - 7\hat{k}$.

$\therefore$ Sum of vectors $= \vec{a} + \vec{b} + \vec{c}$

$= \hat{i} - 2\hat{j} + \hat{k} - 2\hat{i} + 4\hat{j} + 5\hat{k} + \hat{i} - 6\hat{j} - 7\hat{k}$

$= -4\hat{j} - \hat{k}$

16. Find a unit vector in the direction of $\vec{PQ}$, where P and Q have co-ordinates (5, 0, 8) and (3, 3, 2) respectively. **[NCERT Exemplar]**

Ans. Given, coordinates of P and Q are (5, 0, 8) and (3, 3, 2) respectively.

$\therefore \quad \vec{PQ} = \vec{OQ} - \vec{OP}$

$= (3\hat{i} + 3\hat{j} + 2\hat{k}) - (5\hat{i} + 0\hat{j} + 8\hat{k})$

$= -2\hat{i} + 3\hat{j} - 6\hat{k}$

$\therefore$ Unit vector in the direction of $\vec{PQ}$

$= \frac{\vec{PQ}}{|\vec{PQ}|} = \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{(-2)^2 + 3^2 + (-6)^2}}$

$= \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{49}}$

$= \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{7}$

17. Find a vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9.

Ans. Let, $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

Any vector in the direction of vector $\vec{a}$

$= \frac{\vec{a}}{|\vec{a}|}$

$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + 2^2 + 2^2}}$

$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$

$\therefore$ Vector in the direction of $\vec{a}$ with magnitude 9

$= 9 \cdot \frac{(\hat{i} - 2\hat{j} + 2\hat{k})}{3}$

$= 3(\hat{i} - 2\hat{j} + 2\hat{k})$

18. Find a unit vector in the direction of sum of vectors $\bar{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\bar{b} = 2\hat{j} + \hat{k}$. **[NCERT Exemplar]**

Ans. Let $\bar{c}$ be the sum of $\bar{a}$ and $\bar{b}$.

$\therefore \quad \bar{c} = \bar{a} + \bar{b}$

$= 2\hat{i} - \hat{j} + \hat{k} + 2\hat{j} + \hat{k}$

$= 2\hat{i} + \hat{j} + 2\hat{k}$

$\therefore$ Unit vector in the direction of $\bar{c}$

$= \frac{\bar{c}}{|\bar{c}|}$

$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}}$

$= \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{9}}$

$= \frac{1}{3}(2\hat{i} + \hat{j} + 2\hat{k})$

⚠ Caution

- *In this, we will first find the sum of vectors and then the unit vector.*

19. Find a vector in the direction of $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$, which has magnitude of 6 units. **[Delhi Gov. 2022]**

Ans. Given, $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$

So, $\quad |\vec{a}| = \sqrt{2^2 + (-1)^2 + 2^2}$

$= \sqrt{4 + 1 + 4} = \sqrt{9} = 3$

$\therefore$ Required vector

$$= 6 \times \frac{\vec{a}}{|\vec{a}|}$$

$$= 6 \times \frac{(2\hat{i} - \hat{j} + 2\hat{k})}{3}$$

$$= 2(2\hat{i} - \hat{j} + 2\hat{k})$$

$$= 4\hat{i} - 2\hat{j} + 4\hat{k}$$

20. If P(1, 5, 4) and Q(4, 1, –2), then find the direction ratios of $\vec{PQ}$.

Ans. Let, O be the origin of reference.

Then, $\vec{OP} = \hat{i} + 5\hat{j} + 4\hat{k}$

and $\vec{OQ} = 4\hat{i} + \hat{j} - 2\hat{k}$

Now $\vec{PQ} = \vec{OQ} - \vec{OP}$

$$= 4\hat{i} + \hat{j} - 2\hat{k} - \hat{i} - 5\hat{j} - 4\hat{k}$$

$$= 3\hat{i} - 4\hat{j} - 6\hat{k}$$

So, the d.r's of $\vec{PQ}$ are 3, –4, –6.

21. Find the scalar components of $\vec{AB}$ with initial point A(2, 1) and terminal point B(– 5, 7). [CBSE 2012]

Ans. $\vec{AB}$ = Position vector of B – Position vector of A

$$= (-5\hat{i} + 7\hat{j}) - (2\hat{i} + \hat{j})$$

$$= (-5 - 2)\hat{i} + (7 - 1)\hat{j}$$

$$= -7\hat{i} + 6\hat{j}$$

$\therefore$ The scalar components are (– 7, 6, 0).

22. What is the value of λ, if $\lambda(3\hat{i} + 2\hat{j} - 6\hat{k})$ is a unit vector?

Ans. We have,

$$|(3\hat{i} + 2\hat{j} - 6\hat{k})| = \sqrt{3^2 + 2^2 + (-6)^2}$$

$$= \sqrt{9 + 4 + 36}$$

$$= \sqrt{49} = 7$$

Since, $\lambda(3\hat{i} + 2\hat{j} - 6\hat{k})$ is a unit vector

$$\therefore \quad \lambda = \pm\frac{1}{|3\hat{i} + 2\hat{j} - 6\hat{k}|} = \pm\frac{1}{7}$$

23. Find the ratio in which $\hat{i} + 2\hat{j} + 3\hat{k}$ divides the join of $-2\hat{i} + 3\hat{j} + 5\hat{k}$ and $7\hat{i} - \hat{k}$.

Ans. Let, the required ratio be λ : 1, then on applying section formula, we get

$$\hat{i} + 2\hat{j} + 3\hat{k} = \frac{\lambda(7\hat{i} - \hat{k}) + (-2\hat{i} + 3\hat{j} + 5\hat{k})}{\lambda + 1}$$

$$\Rightarrow \quad \hat{i} + 2\hat{j} + 3\hat{k} = \left(\frac{7\lambda - 2}{\lambda + 1}\right)\hat{i} + \left(\frac{3}{\lambda + 1}\right)\hat{j} + \left(\frac{5 - \lambda}{\lambda + 1}\right)\hat{k}$$

On equating the coeffiicient of $\hat{j}$ both sides, we get

$$2 = \frac{3}{\lambda + 1}$$

$$\Rightarrow \quad \lambda + 1 = \frac{3}{2} \Rightarrow \lambda = \frac{1}{2}$$

Hence, the required ratio is $\frac{1}{2}$: 1 *i.e.*, 1 : 2.

24. Shown below are two vectors in their component forms.

$$\vec{u} = 3\hat{i} - p\hat{j} + 5\hat{k}$$

$$\vec{v} = -6\hat{i} + 14\hat{j} + q\hat{k}$$

For what values of *p* and *q* are the vectors collinear?

Ans. Since the vectors $\vec{u}$ and $\vec{v}$ are collinear

$$\therefore \quad \frac{3}{-6} = \frac{-p}{14} = \frac{5}{q}$$

$$\Rightarrow \quad -\frac{1}{2} = \frac{-p}{14}; -\frac{1}{2} = \frac{5}{q}$$

$$\Rightarrow \quad p = 7; q = -10$$

25. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} - \hat{k}$ are two equal vectors, then write the value of $x + y + z$. [CBSE 2013]

26. Find the position vector of a point which divides the join of points with position vectors $\vec{a} - 2\vec{b}$ and $2\vec{a} + \vec{b}$ externally in the ratio 2 : 1. [CBSE 2016]

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

27. If $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}-2\hat{k}$, find the unit vector in the direction of:

(A) $6\vec{b}$ **(B) $2\vec{a}-\vec{b}$**

[NCERT Exemplar]

Ans. Here, $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}-2\hat{k}$

(A)
$$6\vec{b} = 6(2\hat{i}+\hat{j}-2\hat{k})$$
$$= 12\hat{i}+6\hat{j}-12\hat{k}$$

$\therefore$ Unit vector in the direction of $6\vec{b}$

$$= \frac{6\vec{b}}{|6\vec{b}|} = \frac{12\hat{i}+6\hat{j}-12\hat{k}}{\sqrt{12^2+6^2+(-12)^2}}$$
$$= \frac{12\hat{i}+6\hat{j}-12\hat{k}}{\sqrt{324}}$$
$$= \frac{6(2\hat{i}+\hat{j}-2\hat{k})}{18}$$
$$= \frac{2\hat{i}+\hat{j}-2\hat{k}}{3}$$

(B)
$$2\vec{a}-\vec{b} = 2(\hat{i}+\hat{j}+2\hat{k})-(2\hat{i}+\hat{j}-2\hat{k})$$
$$= 2\hat{i}+2\hat{j}+4\hat{k}-2\hat{i}-\hat{j}+2\hat{k}$$
$$= \hat{j}+6\hat{k}$$

$\therefore$ Unit vector in the direction of $2\vec{a}-\vec{b}$

$$= \frac{2\vec{a}-\vec{b}}{|2\vec{a}-\vec{b}|} = \frac{\hat{j}+6\hat{k}}{\sqrt{1^2+6^2}}$$
$$= \frac{1}{\sqrt{37}}(\hat{j}+6\hat{k})$$

28. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B, respectively, find the position vector of a point C in BA produced such that $\overline{BC} = 1.5\ \overline{BA}$. **[NCERT Exemplar]**

Ans. Here, $\overline{OA} = \vec{a}, \overline{OB} = \vec{b}$

$\therefore$ $\overline{BA} = \overline{OA} - \overline{OB} = \vec{a}-\vec{b}$

So, $1.5\,\overline{BA} = 1.5(\vec{a}-\vec{b})$

But, $\overline{BC} = 1.5\,\overline{BA}$ [Given]

$\therefore$ $\overline{OC}-\overline{OB} = 1.5(\vec{a}-\vec{b})$

$\Rightarrow$ $\overline{OC}-\vec{b} = 1.5\vec{a} - 1.5\vec{b}$

$\Rightarrow$ $\overline{OC} = 1.5\vec{a}-0.5\vec{b}$

$$= \frac{3}{2}\vec{a}-\frac{\vec{b}}{2}$$
$$= \frac{3\vec{a}-\vec{b}}{2}$$

Hence, the position vector of the point C is $\frac{3\vec{a}-\vec{b}}{2}$.

29. Write the position vector of mid-point of the vector joining points P(2, 3, 4) and Q(4, 1, – 2). **[CBSE 2011]**

Ans. Given points P(2, 3, 4) and Q(4, 1, – 2) whose position vectors are $\vec{OP} = 2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{OQ} = 4\hat{i}+\hat{j}-2\hat{k}$.

Now, position vector of mid-point of vector joining points P(2, 3, 4) and Q(4, 1, – 2) is

$$\vec{OR} = \frac{\vec{OP}+\vec{OQ}}{2}$$
$$= \frac{(2\hat{i}+3\hat{j}+4\hat{k})+(4\hat{i}+\hat{j}-2\hat{k})}{2}$$
$$= \frac{6\hat{i}+4\hat{j}+2\hat{k}}{2}$$
$$= 3\hat{i}+2\hat{j}+\hat{k}$$

30. If $\vec{PO}+\vec{OQ} = \vec{QO}+\vec{OR}$, then show that the points P, Q and R are collinear.

Ans. Given, $\vec{PO}+\vec{OQ} = \vec{QO}+\vec{OR}$

$\Rightarrow$ $\vec{OQ}-\vec{OP} = \vec{OR}-\vec{OQ}$

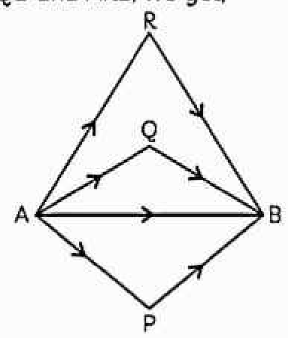
[$\because \vec{OP} = -\vec{PO}, \vec{OQ} = -\vec{QO}$]

$\Rightarrow$ $\vec{PQ} = \vec{QR}$

As Q is a common point for both vectors. Hence, the poins P, Q, R are collinear.

31. **If A, B, P, Q and R be the five points in a plane, then show that the sum of vectors $\vec{AP}, \vec{AQ}, \vec{AR}, \vec{PB}, \vec{QB}$ and $\vec{RB}$ is $3\vec{AB}$.**

Ans. Applying triangle law of addition of vectors in Δs APB, AQB and ARB, we get,



$$\vec{AP} + \vec{PB} = \vec{AB},$$

$$\vec{AQ} + \vec{QB} = \vec{AB},$$

$$\vec{AR} + \vec{RB} = \vec{AB}$$

On adding all these, we get

$$\vec{AP} + \vec{PB} + \vec{AQ} + \vec{QB} + \vec{AR} + \vec{RB} = 3\vec{AB}$$

32. **Find the value of p for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel.**

33. **Write the value of cosine of the angle which the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ makes with the Y-axis.** **[CBSE 2014]**

Ans. Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

Now, unit vector in the direction of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{1^2 + 1^2 + 1^2}}$$

$$= \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

Hence, the cosine of angle which the given vector makes with Y-axis is $\frac{1}{\sqrt{3}}$.

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

34. **A vector $\vec{r}$ has magnitude 14 and direction ratios 2, 3, –6. Find the direction cosines and components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with x-axis.** **[NCERT Exemplar]**

Ans. Here, $|\vec{r}| = 14$

Let $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$

Since, 2, 3, – 6 are direction ratios of $\vec{r}$,

$\therefore \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{-6} = \lambda$ (say)

$\Rightarrow a = 2\lambda, b = 3\lambda$ and $c = -6\lambda$.

$\therefore \quad \vec{r} = \lambda(2\hat{i} + 3\hat{j} - 6\hat{k})$

Since, $|\ | = 14$

$\therefore \ 4\lambda^2 + 9\lambda^2 + 36\lambda^2 = 196$

$\Rightarrow \quad 49\lambda^2 = 196$

$\Rightarrow \quad \lambda^2 = 4$

$\Rightarrow \quad \lambda = \pm 2$

Since, $\vec{r}$ makes an acute angle with x-axis,

$\therefore \quad \lambda = 2$

$\therefore \quad \vec{r} = 4\hat{i} + 6\hat{j} - 12\hat{k}$

Now, $\hat{r} = \frac{\vec{r}}{|\vec{r}|}$

$$= \frac{4\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{4^2 + 6^2 + (-12)^2}}$$

$$= \frac{4\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{196}}$$

$$= \frac{4\hat{i} + 6\hat{j} - 12\hat{k}}{14}$$

$$= \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}$$

Hence, the direction cosines of $\vec{r}$ are $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$

and its components are $4\hat{i}, 6\hat{j}, -12\hat{k}$.

35. A vector $\vec{r}$ **is inclined at equal angles to the three axes. If the magnitude of** $\vec{r}$ **is** $2\sqrt{3}$ **units, find** $\vec{r}$. **[NCERT Exemplar]**

36. Show that the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$ **and** $\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ **form a right angled triangle.**

Ans. Given vectors are $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$

Then,
$$|\vec{a}| = \sqrt{3^2 + (-2)^2 + (1)^2}$$
$$= \sqrt{9+4+1} = \sqrt{14}$$
$$|\vec{b}| = \sqrt{(1)^2 + (-3)^2 + (5)^2}$$
$$= \sqrt{1+9+25} = \sqrt{35}$$
$$|\vec{c}| = \sqrt{2^2 + 1^2 + (-4)^2}$$
$$= \sqrt{4+1+16} = \sqrt{21}$$

Now, $|\vec{a}|^2 + |\vec{c}|^2 = (\sqrt{14})^2 + (\sqrt{21})^2$
$$= 35 = |\vec{b}|^2$$
$$\therefore \quad |\vec{a}|^2 + |\vec{c}|^2 = |\vec{b}|^2$$

Hence, the vectors form a right-angled triangle.

37. If the position vectors of the points A, B, C and D are $2\hat{i} + 4\hat{k}$, $5\hat{i} + 3\sqrt{3}\hat{j} + 4\hat{k}$, $-2\sqrt{3}\hat{j} + \hat{k}$ **and** $2\hat{i} + \hat{k}$ **respectively, then prove that** $\vec{CD}$ **is parallel to** $\vec{AB}$ **and** $\vec{CD} = \frac{2}{3}\vec{AB}$.

Ans. We have,

$\vec{AB}$ = Position vector of B – Position vector of A
$$= (5\hat{i} + 3\sqrt{3}\hat{j} + 4\hat{k}) - (2\hat{i} + 4\hat{k})$$
$$= 3\hat{i} + 3\sqrt{3}\hat{j}$$

$\vec{CD}$ = Position vector of D – Position vector of C
$$= (2\hat{i} + \hat{k}) - (-2\sqrt{3}\hat{j} + \hat{k})$$
$$= 2\hat{i} + 2\sqrt{3}\hat{j}$$
$$= 2(\hat{i} + \sqrt{3}\hat{j})$$
$$= \frac{2}{3}(3\hat{i} + 3\sqrt{3}\hat{j})$$
$$= \frac{2}{3}\vec{AB}$$

Hence, $\vec{CD}$ is parallel to $\vec{AB}$ and $\vec{CD} = \frac{2}{3}\vec{AB}$.

38. If a unit vector $\vec{a}$ **makes angle** $\frac{\pi}{4}$ **with** $\hat{i}$, $\frac{\pi}{3}$ **with** $\hat{j}$ **and an acute angle θ with** $\hat{k}$, **then find the components of** $\vec{a}$ **and the angle θ.**

Ans. Consider, vector $\vec{a}$ makes angle α, β and γ with $\hat{i}, \hat{j}$ and $\hat{k}$ respectively, then
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

It is given that $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{3}$ and $\gamma = \theta$ is an acute angle.
$$\therefore \cos^2\frac{\pi}{4} + \cos^2\frac{\pi}{3} + \cos^2\theta = 1$$
$$\Rightarrow \frac{1}{2} + \frac{1}{4} + \cos^2\theta = 1$$
$$\Rightarrow \cos^2\theta = 1 - \frac{1}{2} - \frac{1}{4}$$
$$= \frac{4-2-1}{4} = \frac{1}{4}$$
$$\Rightarrow \cos\theta = \frac{1}{2}$$
$$\therefore \quad \theta = \frac{\pi}{3} \text{ or } \gamma = \frac{\pi}{3}$$

Now, $\vec{a} = |\vec{a}|\{(\cos\alpha)\hat{i} + (\cos\beta)\hat{j} + (\cos\gamma)\hat{k}\}$
$$= \left(\cos\frac{\pi}{4}\right)\hat{i} + \left(\cos\frac{\pi}{3}\right)\hat{j} + \left(\cos\frac{\pi}{3}\right)\hat{k}$$
$$\Rightarrow \vec{a} = \frac{1}{\sqrt{2}}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k}$$

Thus, the components of $\vec{a}$ are $\frac{1}{\sqrt{2}}\hat{i}, \frac{1}{2}\hat{j}, \frac{1}{2}\hat{k}$.

LONG ANSWER Type Questions (LA)

[**4** & **5** marks]

39. Using vectors, find the value of k such that the points (k, –10, 3), (1, –1, 3) and (3, 5, 3) are collinear. [NCERT Exemplar]

Ans. Given: Points are A(k, –10, 3), B(1, –1, 3) and C(3, 5, 4).

$$\therefore \quad \overline{AB} = \overline{OB} - \overline{OA}$$
$$= (\hat{i} - \hat{j} + 3\hat{k}) - (k\hat{i} - 10\hat{j} + 3\hat{k})$$
$$= (1-k)\hat{i} + 9\hat{j}$$

and,
$$\overline{BC} = \overline{OC} - \overline{OB}$$
$$= (3\hat{i} + 5\hat{j} + 3\hat{k}) - (\hat{i} - \hat{j} + 3\hat{k})$$
$$= 2\hat{i} + 6\hat{j}$$

Since, the points A, B, C are collinear, then there exist some scalar λ such that

$$\overline{AB} = \lambda \overline{BC}$$

$$\Rightarrow \quad (1-\hat{k})\hat{i} + 9\hat{j} = \lambda(2\hat{i} + 6\hat{j})$$

On comparing the coeffiicients, we get

$$1 - k = 2\lambda \quad \text{...(i)}$$

and,
$$9 = 6\lambda$$

$$\therefore \quad \lambda = \frac{3}{2}$$

Putting the value of λ in equation (i), we get

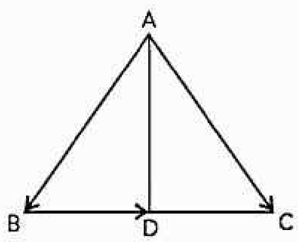
$$1 - k = 2 \times \frac{3}{2}$$
$$\Rightarrow \quad 1 - k = 3$$
$$\Rightarrow \quad k = -2$$

Hence, the value of k is –2.

40. The two vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represent the two sides AB and AC, respectively of a ΔABC. Find the length of the median through A. [CBSE 2016, 15]

Ans. In ΔABC, using the triangle law of vector addition, we have

$$\vec{BC} = \vec{AC} - \vec{AB}$$



Here, $\vec{AC} = 3\hat{i} - \hat{j} + 4\hat{k}$ and $\vec{AB} = \hat{j} + \hat{k}$ (given)

$$\therefore \quad \vec{BC} = (3\hat{i} - \hat{j} + 4\hat{k}) - (\hat{j} + \hat{k})$$
$$= 3\hat{i} - 2\hat{j} + 3\hat{k}$$

Now, AD is a median of ΔABC at point D. So, point D divides BC in two equal parts.

$$\therefore \quad \vec{BD} = \frac{1}{2}\vec{BC}$$
$$= \frac{1}{2}(3\hat{i} - 2\hat{j} + 3\hat{k})$$
$$= \frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}$$

Now, in ΔABD, using the triangle law of vector addition we have

$$\vec{AD} = \vec{AB} + \vec{BD}$$
$$= (\hat{j} + \hat{k}) + \left(\frac{3}{2}\hat{i} - \hat{j} + \frac{3}{2}\hat{k}\right)$$
$$= \frac{3}{2}\hat{i} + 0\hat{j} + \frac{5}{2}\hat{k}$$

$$\therefore \quad |\vec{AD}| = \sqrt{\left(\frac{3}{2}\right)^2 + 0^2 + \left(\frac{5}{2}\right)^2}$$
$$= \sqrt{\frac{9}{4} + \frac{25}{4}}$$
$$= \sqrt{\frac{34}{4}} = \frac{1}{2}\sqrt{34}$$

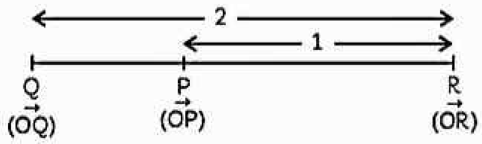
Hence, the length of median through A is $\frac{1}{2}\sqrt{34}$ units.

41. Find the position vector of a point R, which divides the line joining two points P and Q whose position vectors are $2\vec{a} + \vec{b}$ and $\vec{a} - 3\vec{b}$ respectively, externally in the ratio 1 : 2. Also, show that P is the mid-point of line segment RQ. [CBSE 2010]

Ans. Given, $\vec{OP}$ = Position vector of $\vec{P} = 2\vec{a} + \vec{b}$

OQ = Position vector of $\vec{Q} = \vec{a} - 3\vec{b}$

Let $\vec{OR}$ be the position vector of a point R, which divides PQ in the ratio 1 : 2 externally.



$$\therefore \quad \vec{OR} = \frac{1(\vec{a} - 3\vec{b}) - 2(2\vec{a} + \vec{b})}{1 - 2}$$

[by external section formula]

$$= \frac{\vec{a} - 3\vec{b} - 4\vec{a} - 2\vec{b}}{-1}$$

$$= \frac{-3\vec{a} - 5\vec{b}}{-1}$$

$$= 3\vec{a} + 5\vec{b}$$

Hence, $\vec{OR} = 3\vec{a} + 5\vec{b}$

Now, lets prove that P is the mid point of RQ.

i.e., $\vec{OP} = \dfrac{\vec{OR} + \vec{OQ}}{2}$

We have, $\vec{OR} = 3\vec{a} + 5\vec{b}$, $\vec{OQ} = \vec{a} - 3\vec{b}$

$$\therefore \frac{\vec{OR} + \vec{OQ}}{2} = \frac{3\vec{a} + \vec{b} + (\vec{a} - 3\vec{b})}{2}$$

$$= \frac{4\vec{a} - 2\vec{b}}{2} = 2\vec{a} - \vec{b}$$

$$= \vec{OP}$$

Hence, P is the mid-point of line segment RQ.
Hence, proved

42. **Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$ and find a vector of magnitude 6 units, which is parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$.**

Ans. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$.

Now, $2\vec{a} - \vec{b} + 3\vec{c} = 2(\hat{i} + \hat{j} + \hat{k}) - (4\hat{i} - 2\hat{j} + 3\hat{k}) + 3(\hat{i} - 2\hat{j} + \hat{k})$

$= 2\hat{i} + 2\hat{j} + 2\hat{k} - 4\hat{i} + 2\hat{j} - 3\hat{k} + 3\hat{i} - 6\hat{j} + 3\hat{k}$

$= \hat{i} - 2\hat{j} + 2\hat{k}$

Now, a unit vector in the direction of vector $2\vec{a} - \vec{b} + 3\vec{c}$,

$$= \frac{2\vec{a} - \vec{b} + 3\vec{c}}{|2\vec{a} - \vec{b} + 3\vec{c}|}$$

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1^2 + (-2)^2 + 2^2}}$$

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1 + 4 + 4}}$$

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

$$= \frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

So, vector of magnitude 6 units parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c}$

$$= 6\left(\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}\right)$$

$$= 2\hat{i} - 4\hat{j} + 4\hat{k}$$

43. **Points L, M and N divide the sides BC, CA and AB of ΔABC in the ratio 1 : 4, 3 : 2 and 3 : 7, respectively. Prove that $\vec{AL} + \vec{BM} + \vec{CN}$ is a vector parallel to $\vec{CK}$, when $\vec{K}$ divides AB in the ratio 1 : 3.**

PRODUCT OF TWO VECTORS 2

TOPIC 1

SCALAR (OR DOT) PRODUCT

Multiplication of two vectors is defined in two ways, namely, scalar (or dot) product where the result is a scalar, and vector (or cross) product where the result is vector.

The scalar product of two non-zero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}.\vec{b}$, is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$.

⚠ Caution

➥ *If either* $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, *then* θ *is not defined, and in this case* $\vec{a}.\vec{b} = 0$

Some Important Observations

(1) $\vec{a}.\vec{b}$ is a real number.

(2) Let $\vec{a}$ and $\vec{b}$ two non-zero vectors. Then, $\vec{a}.\vec{b} = 0$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular to each other, *i.e.*, $\vec{a}.\vec{b} = 0 \leftrightarrow \vec{a} \perp \vec{b}$.

(3) If $\theta = 0$, the $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$. In particular, $\vec{a}.\vec{a} = |\vec{a}|^2$

(4) If $\theta = \pi$, then $\vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$.

(5) For three mutually perpendicular unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, we have

$\hat{i}.\hat{i} = 1$; $\hat{j}.\hat{j} = 1$; $\hat{k}.\hat{k} = 1$ and $\hat{i}.\hat{j} = \hat{j}.\hat{k} = \hat{k}.\hat{i} = 0$

(6) The angle θ between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by $\cos\theta = \dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$,

or, $$\theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

Geometrical Interpretation of Scalar Product of Two Vectors

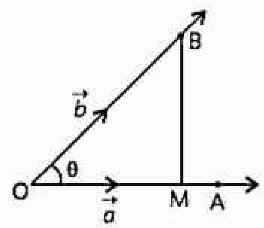
Let $\overline{OA} = \vec{a}$ and $\overline{OB} = \vec{b}$ and $\angle AOB = \theta$

From B, draw BM perpendicular on OA. Then,

$$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$= |\vec{a}|(|\vec{b}|\cos\theta) = |\vec{a}|\vec{OM}$$

$$= (\text{Modulus of } \vec{a})\ (\text{Projection of } \vec{b} \text{ along } \vec{a})$$



Similarly, $\vec{a}.\vec{b}$ = (Modulus of $\vec{b}$) (Projection of $\vec{a}$ along $\vec{b}$)

i.e., Projection of $\vec{a}$ along $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

Properties of Scalar (or Dot) Product

Property 1 (Commutative): For any two vectors $\vec{a}$ and $\vec{b}$, we have

$$\vec{a}.\vec{b} = \vec{b}.\vec{a}$$

Property 2: For any two vectors $\vec{a}$ and $\vec{b}$ and a scalar *m*, we have

$$(m\vec{a}).\vec{b} = m(\vec{a}.\vec{b}) = \vec{a}.(m\vec{b})$$

Property 3: (Distributive over addition/subtraction):

For any two vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, we have

$$\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$$

and $$\vec{a}.(\vec{b} - \vec{c}) = \vec{a}.\vec{b} - \vec{a}.\vec{c}$$

Property 4: $\vec{a}.\vec{a} = |\vec{a}|^2 \ge 0$, where $\vec{a}$ is any vector.

Property 5: For any two vectors $\vec{a}$ and $\vec{b}$, we have

(1) $\vec{a}.(-\vec{b}) = -(\vec{a}.\vec{b}) = (-\vec{a}).\vec{b}$

(2) $(-\vec{a}).(-\vec{b}) = \vec{a}.\vec{b}$

Property 6: For any two vectors $\vec{a}$ and $\vec{b}$, we have,

(1) $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b}$

(2) $|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a}.\vec{b}$

(3) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

Scalar Product in Terms of Components

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$. Then,

$$\vec{a}.\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}).(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1\hat{i}.(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_2\hat{j}.(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + a_3\hat{k}.(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= (a_1b_1)(\hat{i}.\hat{i}) + (a_1b_2)(\hat{i}.\hat{j}) + (a_1b_3)(\hat{i}.\hat{k}) + (a_2b_1)(\hat{j}.\hat{i}) + (a_2b_2)(\hat{j}.\hat{j}) + (a_2b_3)(\hat{j}.\hat{k}) + (a_3b_1)(\hat{k}.\hat{i}) + (a_3b_2)(\hat{k}.\hat{j}) + (a_3b_3)(\hat{k}.\hat{k})$$

$$= a_1b_1(1) + a_1b_2(0) + a_1b_3(0) + a_2b_1(0) + a_2b_2(1) + a_2b_3(0) + a_3b_1(0) + a_3b_2(0) + a_3b_3(1)$$

$$= a_1b_1 + a_2b_2 + a_3b_3$$

Thus, the scalar product of two vectors is equal to the sum of the products of their corresponding components.

Conclusions

(1) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two non-zero vectors inclined at angle θ, then

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^3}.\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

(2) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two non-zero parallel vectors then,

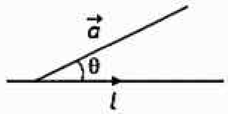
$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$ [Condition of Parallelism]

(3) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two non-zero perpendicular vectors, then $a_1b_1 + a_2b_2 + a_3b_3 = 0$

[Condition of Perpendicularity]

Projection of a Vector on a Line

Suppose a vector $\vec{a}$ makes an angel θ with a directed line, say l, in anti-clockwise direction. Then, the projection of $\vec{a}$ along the line l is a vector $\vec{p}$ (say) whose magnitude is $|\vec{a}|\cos\theta$ and the direction along the line l.



Important

- *$\vec{p}$ is called the projection vector and its magnitude $|\vec{p}|$ is called as the projection of $\vec{a}$ on the directed line l.*

Thus, the projection of $\vec{a}$ on the vector $\vec{b}$ is given by

$$\text{Projection of } \vec{a} \text{ along } \vec{b} = \vec{a}.\hat{b} = \vec{a}.\left(\frac{\vec{b}}{|\vec{b}|}\right) = \frac{\vec{a}.\vec{b}}{|\vec{b}|}.$$

Example 2.1: Find the angle between the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$. [NCERT]

Ans. Let θ be the required angle between the given vectors. Then,

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\Rightarrow \cos\theta = \frac{(\hat{i} - 2\hat{j} + 3\hat{k}).(3\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{1^2 + (-2)^2 + 3^2}\sqrt{3^2 + (-2)^2 + 1^2}}$$

$$= \frac{3+4+3}{\sqrt{14}\sqrt{14}} = \frac{10}{14} = \frac{5}{7}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Example 2.2: **Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.** **[NCERT]**

Ans. Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$.

We know that

$$\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$$

$$= \frac{(\hat{i} - \hat{j}).(\hat{i} + \hat{j})}{\sqrt{1^2 + 1^2}}$$

$$= \frac{1-1}{\sqrt{2}} = 0$$

Example 2.3: **Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.** **[NCERT]**

Ans. Here,

$$(\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = \vec{a}.\vec{a} - \vec{a}.\vec{b} + \vec{b}.\vec{a} - \vec{b}.\vec{b}$$

$$= |\vec{a}|^2 - |\vec{b}|^2$$

$$[\because \vec{a}.\vec{b} = \vec{b}.\vec{a}]$$

$$\Rightarrow \quad |\vec{a}|^2 - |\vec{b}|^2 = 8 \qquad \text{(Given)}$$

$$\Rightarrow \quad 63|\vec{b}|^2 = 8 \qquad [\because |\vec{a}| = 8|\vec{b}|]$$

$$\Rightarrow \quad |\vec{b}| = \sqrt{\frac{8}{63}} = \frac{2\sqrt{2}}{3\sqrt{7}},$$

$$|\vec{a}| = 8\frac{2\sqrt{2}}{3\sqrt{7}} = \frac{16\sqrt{2}}{3\sqrt{7}}$$

Thus, $|\vec{a}| = \frac{16\sqrt{2}}{3\sqrt{7}}$ and $|\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}}$

Example 2.4: **Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$.** **[NCERT]**

Ans. Consider $(|\vec{a}|\vec{b} + |\vec{b}|\vec{a}).(|\vec{a}|\vec{b} - |\vec{b}|\vec{a})$

$$= (|\vec{a}|\vec{b}).(|\vec{a}|\vec{b}) + (|\vec{b}|\vec{a}).(|\vec{a}|\vec{b})$$

$$+ (|\vec{a}|\vec{b}).(-|\vec{b}|\vec{a}) + (|\vec{b}|\vec{a}).(-|\vec{b}|\vec{a})$$

$$= |\vec{a}|^2(\vec{b}.\vec{b}) + |\vec{b}||\vec{a}|(\vec{a}.\vec{b}) - |\vec{a}||\vec{b}|(\vec{b}.\vec{a})$$

$$-|\vec{b}|^2(\vec{a}.\vec{a})$$

$$= |\vec{a}|^2|\vec{b}|^2 + |\vec{a}||\vec{b}|\{(\vec{a}.\vec{b}) - (\vec{b}.\vec{a})\} - |\vec{b}|^2|\vec{a}|^2$$

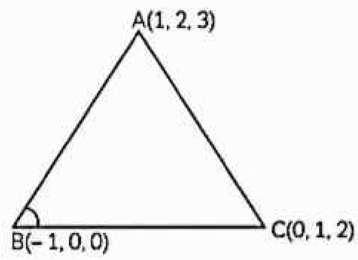
$$= 0 \qquad \{\because \vec{a}.\vec{b} = \vec{b}.\vec{a}\}$$

Thus, $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$.

Hence, proved.

Example 2.5: **If the vertices A, B, C of a triangle ABC are (1, 2, 3), (– 1, 0, 0), (0, 1, 2) respectively, then find ∠ABC.** **[NCERT]**

Ans. Here, ∠ABC is the angle between the vectors $\vec{BA}$ and $\vec{BC}$.



Now,

$$\vec{BA} = \text{P.V. of A} - \text{P.V. of B}$$

$$= (\hat{i} + 2\hat{j} + 3\hat{k}) - (-1\hat{i} + 0\hat{j} + 0\hat{k})$$

$$= 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{BC} = \text{P.V. of C} - \text{P.V. of B}$$

$$= (0\hat{i} + \hat{j} + 2\hat{k}) - (-1\hat{i} + 0\hat{j} + 0\hat{k})$$

$$= \hat{i} + \hat{j} + 2\hat{k}$$

$$\text{Now, } \angle ABC = \cos^{-1}\left(\frac{\vec{BA}.\vec{BC}}{|\vec{BA}||\vec{BC}|}\right)$$

$$= \cos^{-1}\left(\frac{(2\hat{i} + 2\hat{j} + 3\hat{k}).(\hat{i} + \hat{j} + 2\hat{k})}{\sqrt{2^2 + 2^2 + 3^2}\sqrt{1^2 + 1^2 + 2^2}}\right)$$

$$= \cos^{-1}\left(\frac{2+2+6}{\sqrt{17}\sqrt{6}}\right)$$

$$= \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Example 2.6: **Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, – 1) are collinear.** **[NCERT]**

Ans. In order to show that the given points A, B and C are collinear, we shall show that

$$|\vec{AB}| + |\vec{BC}| = |\vec{AC}|$$

Now, $\overrightarrow{AB}$ = P.V. of B – P.V. of A

$= (2\hat{i}+6\hat{j}+3\hat{k})-(\hat{i}+2\hat{j}+7\hat{k})$

$= \hat{i}+4\hat{j}-4\hat{k}$

$\therefore \quad |\overrightarrow{AB}| = \sqrt{1^2+4^2+(-4)^2}$

$= \sqrt{33}$ ...(i)

Similarly, $\overrightarrow{BC}$ = P.V. of C – P.V. of B

$= (3\hat{i}+10\hat{j}-\hat{k})-(2\hat{i}+6\hat{j}+3\hat{k})$

$= \hat{i}+4\hat{j}-4\hat{k}$

$\Rightarrow \quad |\overrightarrow{BC}| = \sqrt{1^2+4^2+(-4)^2}$

$= \sqrt{33}$ ...(ii)

and $\overrightarrow{AC}$ = P.V. of C – P.V. of A

$= (3\hat{i}+10\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+7\hat{k})$

$= 2\hat{i}+8\hat{j}-8\hat{k}$

$\Rightarrow \quad |\overrightarrow{AC}| = \sqrt{1^2+4^2+(-4)^2}$

$= \sqrt{132} = 2\sqrt{33}$ ...(iii)

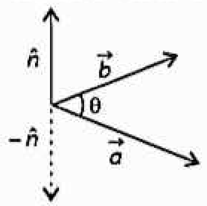
From (i), (ii) and (iii), we find that $|\overrightarrow{AB}|+|\overrightarrow{BC}|=|\overrightarrow{AC}|$.

Hence, the given points are collinear.
Hence, proved.

TOPIC 2

VECTOR (OR CROSS) PRODUCT

The vector (or cross) product of two non-zero vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a}\times\vec{b}$, is defined as $\vec{a}\times\vec{b} = |\vec{a}||\vec{a}|\ \sin\theta\hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both vectors $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system (see figure) *i.e.,* the right handed system rotated from $\vec{a}$ to $\vec{b}$ moves in the direction of $\hat{n}$.



⚠ Caution

➥ *If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then θ is not defined, and in this case, $\vec{a}\times\vec{b}=0$.*

Vector Normal to The Plane of Two Given Vectors

Let $\vec{a}$ and $\vec{b}$ be two non-zero, non-parallel vectors, and let θ be the angle between them. Then

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

where, $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right angled system.

$\therefore \quad \vec{a}\times\vec{b} = |\vec{a}\times\vec{b}|\,\hat{n}$, or $\hat{n} = \dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

Thus, $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ is a unit vector perpendicular to the plane of $\vec{a}$ and $\vec{b}$.

This also give us "a vector of magnitude m perpendicular to the plane of $\vec{a}$ and $\vec{b}$" is given by

$$\pm\frac{m(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}.$$

Some Important Observations

(1) $\vec{a}\times\vec{b}$ is a vector.

(2) Vector product of two non-zero vectors vanishes if and only if they are parallel or collinear.
Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors. Then, $\vec{a}\times\vec{b}=\vec{0}$ if and only if $\vec{a}$ and $\vec{b}$ are parallel to each other, *i.e.,* $\vec{a}\times\vec{b}=\vec{0} \leftrightarrow \vec{a} \parallel \vec{b}$.

(3) If θ = 0, then $\vec{a}\times\vec{b}=\vec{0}$. In particular, $\vec{a}\times\vec{a}=\vec{0}$

(4) If $\theta = \pi$, then $\vec{a}\times\vec{b}=\vec{0}$. In particular, $\vec{a}\times(-\vec{a})=\vec{0}$

(5) Vector product is not commutative, *i.e.,* $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$.

Since, $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\ \sin\theta =$ θ
$=|\vec{b}||\vec{a}|\sin\theta =|\vec{b}\times\vec{a}|$
and
direction of $\vec{a}\times\vec{b}$ is opposite to that of $\vec{b}\times\vec{a}$

So, $(\vec{a}\times\vec{b}=-\vec{b}\times\vec{a})$

(6) For three mutually perpendicular unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, we have

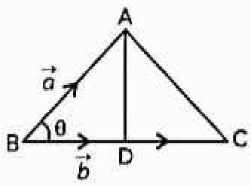
$$\hat{i} \times \hat{i} = 0; \quad \hat{j} \times \hat{j} = 0; \quad \hat{k} \times \hat{k} = 0;$$
$$\hat{i} \times \hat{j} = \hat{k}; \quad \hat{j} \times \hat{k} = \hat{i}; \quad \hat{k} \times \hat{i} = \hat{j};$$
$$\hat{j} \times \hat{i} = -\hat{k}; \quad \hat{k} \times \hat{j} = -\hat{i}; \quad \hat{i} \times \hat{k} = -\hat{j}$$

(7) The angle θ between two non-zero vectors $\vec{a}$ and $\vec{b}$ is given by $\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

or $\qquad \theta = \sin^{-1}\left[\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right]$.

(8) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then its area is given by $\frac{1}{2}|\vec{a} \times \vec{b}|$.

Area of $\Delta ABC = \frac{1}{2} BC.AD$ ...(i)



But, $BC = |\vec{b}|$ and $AD = |\vec{a}|\sin\theta$

Thus, Area of $\Delta ABC = \frac{1}{2} BC.AD$

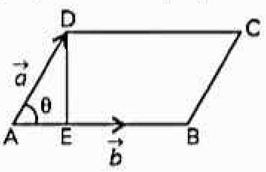
$$= \frac{1}{2}|\vec{b}||\vec{a}|\sin\theta$$

$$= \frac{1}{2}|\vec{a} \times \vec{b}|$$

(9) If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given by $|\vec{a} \times \vec{b}|$.

Area of parallelogram ABCD = AB.DE ...(i)

But, $AB = |\vec{b}|$ and $DE = |\vec{a}| \sin\theta$



Thus, Area of parallelogram ABCD

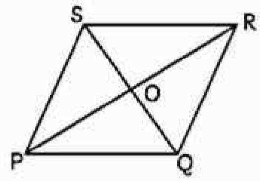
$$= AB.DE$$

$$= |\vec{b}||\vec{a}|\sin\theta$$

$$= |\vec{a} \times \vec{b}|$$

(10) If $\vec{a}$ and $\vec{b}$ represent the diagonals of a then its area is given by $\frac{1}{2}|\vec{a} \times \vec{b}|$.

Let PQRS be a parallelogram with diagonals PR and QS interesting at O(origin).



Let $\vec{PR} = \vec{a}$ and $\vec{QS} = \vec{b}$

Now,

$$\vec{OR} = \frac{1}{2}\vec{a}; \ \vec{OS} = \frac{1}{2}\vec{b}; \ \vec{OP} = -\frac{1}{2}\vec{a}; \ \vec{OQ} = -\frac{1}{2}\vec{b}$$

$\Rightarrow \qquad \vec{PQ}$ = P.V. of Q – P.V. of P

$$= -\frac{1}{2}\vec{b} + \frac{1}{2}\vec{a}$$

$$= \frac{1}{2}(\vec{a} - \vec{b});$$

and $\qquad \vec{PS}$ = P.V. of S – P.V. of P

$$= \frac{1}{2}\vec{b} + \frac{1}{2}\vec{a}$$

$$= \frac{1}{2}(\vec{a} + \vec{b})$$

$\Rightarrow$ Area of ▱PQRS $= |\overline{PQ} \times \overline{PS}|$

$$= \frac{1}{4}|(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})|$$

$$= \frac{1}{4}\left|\vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}\right|$$

$$= \frac{1}{4}\left|0 + \vec{a} \times \vec{b} - (-\vec{a} \times \vec{b}) - 0\right|$$

$$= \frac{1}{4}\left|\vec{a} \times \vec{b} + \vec{a} \times \vec{b}\right|$$

$$= -\left|2(\vec{a} \times \vec{b})\right|$$

$$= \frac{1}{2}|\vec{a} \times \vec{b}|$$

Properties of Vector (or Cross) Product

Property 1: For any two vectors $\vec{a}$ and $\vec{b}$ and a scalar m, we have

$$(m\vec{a})\times\vec{b} = m(\vec{a}\times\vec{b}) = \vec{a}\times(m\vec{b})$$

Property 2: (Distributive over addition/subtraction):

For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$, we have

$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

and $$\vec{a}\times(\vec{b}-\vec{c}) = \vec{a}\times\vec{b}-\vec{a}\times\vec{c}$$

Vector Product in Terms of Components

Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$. Then,

$$\begin{aligned}\vec{a}\times\vec{b} &= (a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\\ &= a_1\hat{i}\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\\ &\quad + a_2\hat{j}\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\\ &\quad + a_3\hat{k}\times(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})\\ &= (a_1b_1)(\hat{i}\times\hat{i})+(a_1b_2)(\hat{i}\times\hat{j})+(a_1b_3)(\hat{i}\times\hat{k})\\ &\quad +(a_2b_1)(\hat{j}\times\hat{i})+(a_2b_2)(\hat{j}\times\hat{j})+(a_2b_3)(\hat{j}\times\hat{k})\\ &\quad +(a_3b_1)(\hat{k}\times\hat{i})+(a_3b_2)(\hat{k}\times\hat{j})+(a_3b_3)(\hat{k}\times\hat{k})\\ &= (a_1b_1)(\vec{0})+(a_1b_2)(\hat{k})+(a_1b_3)(-\hat{j})\\ &\quad +(a_2b_1)(-\hat{k})+(a_2b_2)(\hat{0})+(a_2b_3)(\hat{i})\\ &\quad +(a_3b_1)(\hat{j})+(a_3b_2)(-\hat{i})+(a_3b_3)(\vec{0})\\ &= (a_2b_3-a_3b_2)\hat{i}-(a_1b_3-a_3b_1)\hat{j}\\ &\quad +(a_1b_2-a_2b_1)\hat{k}\\ &= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}\end{aligned}$$

Example 2.7: Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}-2\hat{k}$. **[NCERT]**

Ans. We know that unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b} = \dfrac{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}$

Now, $$\begin{aligned}\vec{a}+\vec{b} &= (3\hat{i}+2\hat{j}+2\hat{k})+(\hat{i}+2\hat{j}-2\hat{k})\\ &= 4\hat{i}+4\hat{j}\end{aligned}$$

and $$\begin{aligned}\vec{a}-\vec{b} &= (3\hat{i}+2\hat{j}+2\hat{k})-(\hat{i}+2\hat{j}-2\hat{k})\\ &= 2\hat{i}+4\hat{k}\end{aligned}$$

Thus, $$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 4 & 0\\ 2 & 0 & 4\end{vmatrix} = 16\hat{i}-16\hat{j}-8\hat{k}$$

$$\Rightarrow\ |(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})| = \sqrt{576} = \pm 24$$

Hence, unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b} = \dfrac{(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})}{|(\vec{a}+\vec{b})\times(\vec{a}-\vec{b})|}$

$$= \pm\frac{16\hat{i}-16\hat{j}-8\hat{k}}{24},$$

or $$\pm\left(\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}\right)$$

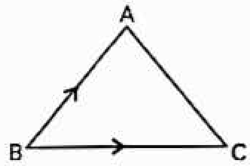
Example 2.8: Find the area of a triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5). **[NCERT]**

Ans. Here,

P.V. of A = $\hat{i}+\hat{j}+2\hat{k}$;

P.V. of B = $2\hat{i}+3\hat{j}+5\hat{k}$;

and P.V. of C = $\hat{i}+5\hat{j}+5\hat{k}$



We know that

$$\text{Area of } \Delta ABC = \frac{1}{2}|\vec{BC}\times\vec{BA}| \quad \text{...(i)}$$

Now, $$\begin{aligned}\vec{BC} &= \text{P.V. of C} - \text{P.V. of B}\\ &= (\hat{i}+5\hat{j}+5\hat{k})-(2\hat{i}+3\hat{j}+5\hat{k})\\ &= -\hat{i}+2\hat{j}\end{aligned}$$

And, $$\begin{aligned}\vec{BA} &= \text{P.V. of A} - \text{P.V. of B}\\ &= (\hat{i}+\hat{j}+2\hat{k})-(2\hat{i}+3\hat{j}+5\hat{k})\\ &= -\hat{i}-2\hat{j}-3\hat{k}\end{aligned}$$

So, $$\vec{BC}\times\vec{BA} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 2 & 0\\ -1 & -2 & -3\end{vmatrix} = -6\hat{i}-3\hat{j}+4\hat{j}$$

$\Rightarrow \quad |\overline{BC} \times \overline{BA}| = \sqrt{(-6)^2 + (-3)^2 + 4^2}$

$= \sqrt{61}$

From (i), we have

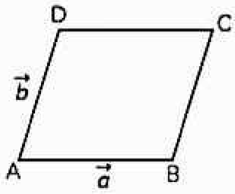
Area of $\Delta ABC = \frac{1}{2} |\overrightarrow{BC} \times \overrightarrow{BA}|$

$= \frac{1}{2}\sqrt{61}$ sq. units

Example 2.9: Find the area of a parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{j}$. [NCERT]

Ans. We know that

Area of paraellelogram ABCD = $|\overrightarrow{AB} \times \overrightarrow{AD}|$ ...(i)



Now, $\overrightarrow{AB} \times \overrightarrow{AD} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$

$= 20\hat{i} + 5\hat{j} - 5\hat{k}$

$\Rightarrow \quad |\overrightarrow{AB} \times \overrightarrow{AD}| = \sqrt{20^2 + 5^2 + (-5)^2}$

$= \sqrt{450} = 15\sqrt{2}$

From (i), we have

Area of parallelogram ABCD

$= |\overrightarrow{AB} \times \overrightarrow{AD}|$

$= 15\sqrt{2}$ sq. units

Example 2.10: If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example. [NCERT]

Ans. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} + 3\hat{j} + 3\hat{k}$.

Here, $\quad |\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3} \neq 0;$

and $\quad |\vec{b}| = \sqrt{3^2 + 3^2 + 3^2} = 3\sqrt{3} \neq 0$

Now, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 3 & 3 & 3 \end{vmatrix}$

$= (3-3)\hat{i} + (3-3)\hat{j} + (3-3)\hat{k}$

$= \vec{0}$

Thus, the converse is not true as $\vec{a} \times \vec{b} = \vec{0}$, but $\vec{a} \neq \vec{0}$ or $\vec{b} \neq \vec{0}$.

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. The value of $\hat{i}.(\hat{j} \times \hat{k}) + \hat{j}.(\hat{i} \times \hat{k}) + \hat{k}.(\hat{i} \times \hat{j})$ is:

(a) 0 **(b) − 1**

(c) 1 **(d) 3**

Ans. *(c) 1*

Explanation: $\hat{i}.(\hat{j} \times \hat{k}) + \hat{j}.(\hat{i} \times \hat{k}) + \hat{k}.(\hat{i} \times \hat{j})$

$= \hat{i}.\hat{i} + \hat{j}.(-\hat{j}) + \hat{k}.\hat{k}$

$= 1 - 1 + 1$, *i.e.*, 1

2. If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to:

(a) 0 **(b) $\frac{\pi}{4}$**

(c) $\frac{\pi}{2}$ **(d) π**

Ans. *(b)* $\frac{\pi}{4}$

Explanation: $|\vec{a}.\vec{b}| = |\vec{a} \times \vec{b}|$

$\Rightarrow \quad |\vec{a}||\vec{b}| \cos\theta = |\vec{a}||\vec{n}| \sin\theta$

$\Rightarrow \quad \cos\theta = \sin\theta$

$\Rightarrow \quad \theta = \frac{\pi}{4}$

3. If $|\vec{a}| = 10$, $|\vec{b}| = 2$ and $\vec{a}.\vec{b} = 12$, then value of $|\vec{a} \times \vec{b}|$ is:

(a) 5 **(b) 10**

(c) 14 **(d) 16**

[NCERT Exemplar]

4. **The value of λ such that the vectors $\vec{a} = 2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}+2\hat{j}+3\hat{k}$ are orthogonal, is:**

(a) 0 (b) 1

(c) $\frac{3}{2}$ (d) $-\frac{5}{2}$

Ans. *(d)* $-\frac{5}{2}$

Explanation: Since, the given vectors are orthogonal, so their dot product is zero as $\theta = \frac{\pi}{2}$, so $\cos\theta = 0$.

$$\therefore \quad \bar{a}.\bar{b} = 0$$

$$\Rightarrow \quad (2\hat{i}+\lambda\hat{j}+\hat{k}).(\hat{i}+2\hat{j}+3\hat{k}) = 0$$

$$\Rightarrow \quad 2 + 2\lambda + 3 = 0$$

$$\Rightarrow \quad \lambda = -\frac{5}{2}$$

Concept Applied

➥ *Two non-zero vectors are orthogonal if their dot product is zero.*

5. **If the projection of $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ on $\vec{b} = 2\hat{i} + \lambda\hat{k}$ is zero, then the value of λ is:**

(a) 0 (b) 1

(c) $-\frac{2}{3}$ (d) $-\frac{3}{2}$

[CBSE 2020]

Ans. *(c)* $-\frac{2}{3}$

Explanation: Here, $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \lambda\hat{k}$

Since, projection of $\vec{a}$ on $\vec{b} = 0$

$$\Rightarrow \quad \frac{\vec{a}.\vec{b}}{|\vec{b}|} = 0$$

$$\Rightarrow \quad \frac{(\hat{i} - 2\hat{j} + 3\hat{k}).(2\hat{i} + \lambda\hat{k})}{\sqrt{2^2 + \lambda^2}} = 0$$

$$\Rightarrow \quad \frac{2 + 3\lambda}{\sqrt{4 + \lambda^2}} = 0$$

$$\Rightarrow \quad 2 + 3\lambda = 0$$

$$\Rightarrow \quad \lambda = -\frac{2}{3}$$

6. **If $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along three mutually perpendicular direction, then:**

(a) $\hat{i}.\hat{j} = 1$ (b) $\hat{i} \times \hat{j} = 1$

(c) $\hat{i} \cdot \hat{k} = 0$ (d) $\hat{i} \times \hat{k} = 0$

[CBSE 2020]

Ans. *(c)* $\hat{i}.\hat{k} = 0$

Explanation: Since, $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular.

$\therefore \hat{i}.\hat{k} = 0$

[Since $\hat{i}.\hat{k} = ik\cos\theta = ik\cos\frac{\pi}{2} = ik \times 0 = 0$]

7. **Ⓐ If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}.\vec{b} \geq 0$ only when:**

(a) $0 < \theta < \frac{\pi}{2}$ (b) $0 \leq \theta \leq \frac{\pi}{2}$

(c) $0 < \theta < \pi$ (d) $0 \leq \theta \leq \pi$

8. **The area of the parallelogram whose adjacent sides are $\hat{i} + \hat{k}$ and $2\hat{i} + \hat{j} + \hat{k}$ is:**

(a) 3 sq. units (b) $\sqrt{2}$ sq. units

(c) 4 sq. units (d) $\sqrt{3}$ sq. units

Ans. *(d)* $\sqrt{3}$ *sq. units.*

Explanation: The area of the parallelogram whose adjacent sides are $\hat{i} + \hat{k}$ and $2\hat{i} + \hat{j} + \hat{k}$ is $|(\hat{i} + \hat{k}) \times (2\hat{i} + \hat{j} + \hat{k})|$ sq. units.

$$\text{Now, } (\hat{i} + \hat{k}) \times (2\hat{i} + \hat{j} + \hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= -\hat{i} + \hat{j} + \hat{k}$$

So, $|(\hat{i} + \hat{k}) \times (2\hat{i} + \hat{j} + \hat{k})|$

$$= \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$$

Hence, the area of parallelogram is $\sqrt{3}$ sq. units.

9. **Ⓐ If $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$, $|\vec{b}| = 5$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then the area of the triangle formed by these two vectors as two sides is:**

(a) $\frac{15}{2}$ sq. units (b) 15 sq. units

(c) $\frac{15}{4}$ sq. units (d) $\frac{15\sqrt{3}}{2}$ sq. units

10. The angle between vectors $\bar{a}$ and $\bar{b}$ with magnitudes $\sqrt{3}$ and 4, respectively and $\bar{a}.\bar{b} = 2\sqrt{3}$ is:

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

[NCERT Exemplar]

Ans. (*b*) $\frac{\pi}{3}$

Explanation: Here,

$$|\bar{a}| = \sqrt{3}, |\bar{b}| = 4$$

and $\bar{a}.\bar{b} = 2\sqrt{3}$

We know, $\bar{a}.\bar{b} = |\bar{a}|\,|\bar{b}| \cos\theta$

$\Rightarrow$ $2\sqrt{3} = \sqrt{3}.4\cos\theta$

$\Rightarrow$ $\cos\theta = \frac{2\sqrt{3}}{4\sqrt{3}} = \frac{1}{2}$

$\Rightarrow$ $\theta = \frac{\pi}{3}$

11. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ be the angle between them. Then, $\vec{a}+\vec{b}$ is a unit vector, if θ is equal to:

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$

Ans. (*d*) $\frac{2\pi}{3}$

Explanation: Here, $|\vec{a}| = |\vec{b}| = 1$ and $|\vec{a}+\vec{b}| = 1$

$\Rightarrow$ $|\vec{a}+\vec{b}|^2 = 1$

$\Rightarrow$ $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$

$\Rightarrow$ $\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b} = 1$

$\Rightarrow$ $|\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 1$ $\{\because \vec{a}.\vec{b} = \vec{b}.\vec{a}\}$

$\Rightarrow$ $2\vec{a}.\vec{b} + 2 = 1$

$\{\because |\vec{a}| = |\vec{b}| = 1\}$

$\Rightarrow$ $|\vec{a}||\vec{b}|\cos\theta = -\frac{1}{2}$

$\Rightarrow$ $\cos\theta = -\frac{1}{2}$

$\{\because |\vec{a}| = |\vec{b}| = 1\}$

$\Rightarrow$ $\theta = \frac{2\pi}{3}$

12. If $\vec{a} = \hat{i} + \lambda\hat{j} + 2\hat{k}$ and $\vec{b} = \mu\hat{i} + \hat{j} - \hat{k}$ are orthogonal and $|\vec{a}| = |\vec{b}|$, then (λ, μ) is:

(a) $\left(\frac{1}{4}, \frac{9}{4}\right)$ (b) $\left(-\frac{1}{4}, \frac{9}{4}\right)$

(c) $\left(\frac{7}{4}, \frac{1}{4}\right)$ (d) $\left(\frac{1}{4}, \frac{7}{4}\right)$

Ans. (*d*) $\left(\frac{1}{4}, \frac{7}{4}\right)$

Explanation: Since $\vec{a} = \hat{i} + \lambda\hat{j} + 2\hat{k}$ and $\vec{b} = \mu\hat{i} + \hat{j} - \hat{k}$ are orthogonal,

$\therefore$ $\vec{a}.\vec{b} = 0$

$\Rightarrow$ $(\hat{i} + \lambda\hat{j} + 2\hat{k}).(\mu\hat{i} + \hat{j} - \hat{k}) = 0$

$\Rightarrow$ $\mu + \lambda - 2 = 0$, or $\mu = 2 - \lambda$ ...(i)

or $\mu = 2 - \lambda$

Also, $|\vec{a}| = |\vec{b}|$ gives

$$\sqrt{1^2 + \lambda^2 + 2^2} = \sqrt{\mu^2 + 1^2 + (-1)^2}$$

$\Rightarrow$ $\lambda^2 + 5 = \mu^2 + 2$ ...(ii)

Solving (i) and (ii), we have $\lambda = \frac{1}{4}, \mu = \frac{7}{4}$

13. If $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$, and $\vec{b} = 5\hat{i} - 3\hat{j} - 4\hat{k}$ then $\frac{\text{Projection of } \vec{a} \text{ on } \vec{b}}{\text{Projection of } \vec{b} \text{ on } \vec{a}}$ is equals to:

(a) 1 (b) $\frac{5\sqrt{2}}{3}$

(c) $\frac{3}{5\sqrt{2}}$ (d) $\frac{3}{2\sqrt{5}}$

14. If $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} \neq 0$ then for any scalar λ:

(a) $\vec{a} + \vec{b} = \lambda(\vec{c} - \vec{b})$ (b) $\vec{b} + \vec{c} = \lambda \vec{a}$

(c) $\vec{a} + \vec{b} = \lambda \vec{c}$ (d) $\vec{a} + \vec{c} = \lambda \vec{b}$

[DIKSHA]

Ans. (d) $\vec{a} + \vec{c} = \lambda \vec{b}$

Explanation:

If $\vec{a} \times \vec{b} = \vec{b} \times \vec{c}$

Then $\vec{a} \times \vec{b} - \vec{b} \times \vec{c} = 0$

$\Rightarrow \vec{a} \times \vec{b} + \vec{c} \times \vec{b} = 0 \quad \left[\because -\vec{b} \times \vec{c} = \vec{c} \times \vec{b}\right]$

$\Rightarrow (\vec{a} + \vec{c}) \times \vec{b} = 0$

This means $(\vec{a} + \vec{c})$ is parallel to $\vec{b}$, so they lie in the same plane.

So vector sum can be represented in scalar multiple as $(\vec{a} + \vec{c}) = \lambda \vec{b}$.

15. If $\vec{a} = 2\hat{i} + 3\hat{j} - 5\hat{k}, \vec{b} = m\hat{i} + n\hat{j} + 12\hat{k}$ and $\vec{a} \times \vec{b} = 0$, then (m, n) is equals to:

(a) $\left(-\frac{24}{5}, -\frac{36}{5}\right)$ (b) $\left(\frac{24}{5}, \frac{36}{5}\right)$

(c) $\left(-\frac{24}{5}, \frac{36}{5}\right)$ (d) $\left(\frac{24}{5}, -\frac{36}{5}\right)$

[DIKSHA]

Ans. (a) $\left(-\frac{24}{5}, -\frac{36}{5}\right)$

Explanation: $\vec{a} = 2\hat{i} + 3\hat{j} - 5\hat{k}$

And $\vec{b} = m\hat{i} + n\hat{j} + 12\hat{k}$

$$\therefore \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -5 \\ m & n & 12 \end{vmatrix}$$

$$= \hat{i}(36 + 5n) - \hat{j}(24 + 5m) + \hat{k}(2n - 3m) = 0$$

$\therefore \quad 36 + 5n = 0$

$\Rightarrow \quad n = -\frac{36}{5}$

And $\quad 24 + 5m = 0$

$m = -\frac{24}{5}$

Thus, $\quad (m, n) = \left(-\frac{24}{5}, -\frac{36}{5}\right)$

16. Few of us would have seen a building whose sectional view resembles a parallelogram. Sneha was speechless when she came across the picture of a building shown below. She wanted to check her friend's knowledge of vector algebra by asking her a very simple question on finding the length of the diagonal of the parallelogram.

If $\hat{i} + \hat{j} - \hat{k}$ and $2\hat{i} - 3\hat{j} + \hat{k}$ are adjacent sides of a parallelogram, then the lengths of its diagonals are:

(a) $\sqrt{3}, \sqrt{14}$ (b) $\sqrt{13}, \sqrt{14}$

(c) $\sqrt{21}, \sqrt{3}$ (d) $\sqrt{21}, \sqrt{13}$

Ans. (d) $\sqrt{21}, \sqrt{13}$

Explanation: Let $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$.

Then,

The lengths of two diagonals are given by $|\vec{a} + \vec{b}|$ and $|\vec{a} - \vec{b}|$

Now, $\quad \vec{a} + \vec{b} = 3\hat{i} - 2\hat{j}$

And $\quad \vec{a} - \vec{b} = -\hat{i} + 4\hat{j} - 2\hat{k}$

$\Rightarrow \quad |\vec{a} + \vec{b}| = \sqrt{3^2 + (-2)^2} = \sqrt{13};$

and $\quad |\vec{a} - \vec{b}| = \sqrt{(-1)^2 + 4^2 + (-2)^2} = 21$

$= \sqrt{21}$

CASE BASED Questions (CBQs)

[4 & 5 marks]

Read the following passages and answer the questions that follow:

17. *A quantity that has magnitude as well as direction is called a vector. The quantities like velocity and displacement are the vector quantities.*

A girl, at a point O, walks 6 km towards East to reach at a point A. Then she walks 4 km in a direction 30° west of North and stops at a point B. Thus, the vector $\vec{OB}$ represents the displacement of the girl from the initial point A to the terminal point B.

Let the direction towards East is represented by positive x-axis and the direction towards North is represented by positive y-axis, with O as origin.

(A) Vector $\vec{OA}$ is given by:

(a) $6\hat{i} + 4\hat{j}$ (b) $6\hat{i} - 4\hat{j}$

(c) $6\hat{i}$ (d) $-6\hat{i}$

(B) Vector $\vec{AB}$ is given by:

(a) $-4\cos 60°\hat{i} + 4\sin 60°\hat{j}$

(b) $4\cos 60°\hat{i} + 4\sin 60°\hat{j}$

(c) $-4\cos 30°\hat{i} + 4\sin 30°\hat{j}$

(d) $4\cos 30°\hat{i} + 4\sin 30°\hat{j}$

(C) Vector $\vec{OB}$ is given by:

(a) $\vec{OA} + \vec{AB}$ (b) $\vec{OA} + \vec{BA}$

(c) $\vec{AB} + \vec{AO}$ (d) $\vec{BA} + \vec{AO}$

(D) The displacement $\vec{OB}$, in terms of its components is:

(a) $-4\hat{i} + 2\sqrt{3}\hat{j}$ (b) $4\hat{i} + 2\sqrt{3}\hat{j}$

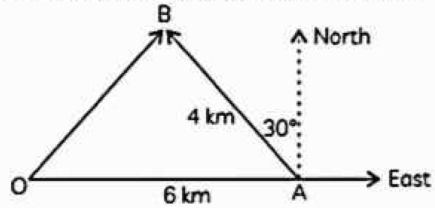
(c) $8\hat{i} + 2\sqrt{3}\hat{j}$ (d) $8\hat{i} - 2\sqrt{3}\hat{j}$

(E) The distance OB (in km) is:

(a) $2\sqrt{7}$ (b) $2\sqrt{19}$

(c) 10 (d) 2

Ans. The movement of the girl is shown here:



(A) *(c)* $6\hat{i}$

(B) *(a)* $-4\cos 60°\hat{i} + 4\sin 60\hat{j}$

Explanation: From the figure,

$$\vec{AB} = -4\cos 60°\hat{i} + 4\cos 30°\hat{j}$$

$$= -4\cos 60°\hat{i} + 4\sin 60°\hat{j}$$

(D) *(b)* $4\hat{i} + 2\sqrt{3}\hat{j}$

Explanation: From part (C), we have

$$\vec{OB} = \vec{OA} + \vec{AB}$$

$$= 6\hat{i} + (-4\cos 60°\hat{i} + 4\sin 60°\hat{j})$$

[From part (B)]

$$= 6\hat{i} - 4 \times \frac{1}{2}\hat{i} + 4 \times \frac{\sqrt{3}}{2}\hat{j}$$

$$= 4\hat{i} + 2\sqrt{3}\hat{j}$$

18. *Solar Panels have to be installed carefully so that the tilt of the roof, and the direction to the sun, produce the largest possible electrical power in the solar panels.*

A surveyor uses his instrument to determine the coordinates of the four corners of a roof where solar panels are to be mounted. In the picture, suppose the points are labelled counter clockwise from the roof corner nearest to the camera in units of meters P_1 (6, 8, 4), P_2(21, 8, 4), P_3 (21, 16, 10) and P_4 (6, 16, 10).

(A) What are the components to the two edge vectors defined by $\vec{A}$ = PV of P_2 - PV of P_1 and $\vec{B}$ = PV of P_4 - PV of P_1? (where PV stands for position vector)

(B) Find the cross product of $\vec{A}$ and $\vec{B}$.

Ans. (B) From part (A), we have

$$\vec{A} = 15\hat{i}$$

and $$\vec{B} = 8\hat{j}+6\hat{k}$$

Now, cross product of $\vec{A}$ and $\vec{B}$

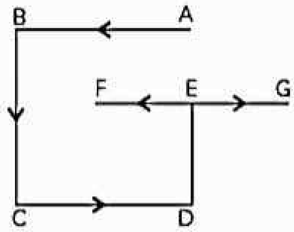
$$= \vec{A}\times\vec{B}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 15 & 0 & 0 \\ 0 & 8 & 6 \end{vmatrix}$$

$$= (0-0)\hat{i}-(90-0)\hat{j}+(120-0)\hat{k}$$

$$= -90\hat{j}+120\hat{k}$$

19. *In the figure, an electric circuit is shown:*



(A) Which two vectors are equal vectors?

(a) $\vec{AB}$ and $\vec{CD}$ (b) $\vec{BC}$ and $\vec{DE}$

(c) $\vec{EF}$ and $\vec{FG}$ (d) None of these

(B) Which two vectors are like vectors?

(a) $\vec{AB}$ and $\vec{CD}$ (b) $\vec{AB}$ and $\vec{EF}$

(c) $\vec{EF}$ and $\vec{EG}$ (d) $\vec{BC}$ and $\vec{CD}$

(C) Which of the following is a negative vector of $\vec{EF}$?

(a) $\vec{ED}$ (b) $\vec{AB}$

(c) $\vec{EG}$ (d) $\vec{DC}$

(D) Which two vectors are parallel vectors?

(a) $\vec{BC}$ and $\vec{ED}$ (b) $\vec{AB}$ and $\vec{BC}$

(c) $\vec{EF}$ and $\vec{EG}$ (d) $\vec{AB}$ and $\vec{DE}$

(E) Which two vectors are unlike vectors?

(a) $\vec{AB}$ and $\vec{BC}$ (b) $\vec{AB}$ and $\vec{CD}$

(c) $\vec{EF}$ and $\vec{ED}$ (d) $\vec{BC}$ and $\vec{EG}$

Ans. (A) *(d) None of these*

(C) *(c)* $\vec{EG}$

(E) *(b)* $\vec{AB}$ *and* $\vec{CD}$

20. *A class XII student appearing for a competitive examination was asked to attempt the following questions.*

Let $\vec{a} = i-2\hat{j}$ *and* $\vec{b} = 2\hat{i}+j+3\hat{k}$

(A) Evaluate $(2\vec{a}+\vec{b}).[(\vec{a}+\vec{b})\times(\vec{a}-2\vec{b})]$.

(B) Find the area of the parallelogram formed by $\vec{a}$ and $\vec{b}$ as diagonals.

Ans. (A) We have $\vec{a} = \hat{i}-2\hat{j}$, $\vec{b} = 2\hat{i}+\hat{j}+3\hat{k}$

So, $$(2\vec{a}+\vec{b}) = 2\hat{i}-4\hat{j}+2\hat{i}+\hat{j}+3\hat{k}$$
$$= 4\hat{i}-3\hat{j}+3\hat{k}$$

Also, $$(\vec{a}+\vec{b}) = 3\hat{i}-\hat{j}+3\hat{k}$$

and, $$(\vec{a}-2\vec{b}) = \hat{i}-2\hat{j}-4\hat{i}-2\hat{j}-6\hat{k}$$
$$= -3\hat{i}-4\hat{j}-6\hat{k}$$

Now, $(\vec{a}+\vec{b})\times(\vec{a}-2\vec{b})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 3 \\ -3 & -4 & -6 \end{vmatrix}$$

$$= \hat{i}(6+12)-\hat{j}(-18+9)+\hat{k}(-12-3)$$

$$= 18\hat{i}+9\hat{j}-15\hat{k}$$

$\therefore$ $(2\vec{a}+\vec{b}).[(\vec{a}+\vec{b})\times(\vec{a}-2\vec{b})]$

$$= (4\hat{i}-3\hat{j}+3\hat{k}).(18\hat{i}+9\hat{j}-15\hat{k})$$
$$= 72-27-45$$
$$= 72-72$$
$$= 0$$

(B) We know, Area of parallelogram

$$= \frac{1}{2}|\vec{a}\times\vec{b}|$$

Here,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 0 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= \hat{i}(-6-0)-\hat{j}(3-0)+\hat{k}(1+4)$$

$$= -6\hat{i}-3\hat{j}+5\hat{k}$$

So, $|\vec{a}\times\vec{b}| = \sqrt{(-6)^2+(-3)^2+(5)^2}$

$= \sqrt{70}$

$\therefore$ Area of parallelogram

$= \frac{1}{2}\times\sqrt{70}$

$= \sqrt{\frac{70}{4}}$ sq. units

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

21. Write the vectors of unit length perpendicular to both the vectors

$\vec{a} = 2\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b} = \hat{j}+\hat{k}$. [CBSE 2016]

Ans. Since we know that there are two such vectors of unit length perpendicular to both the given vectors.

Such vectors are $\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

Then, $\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 2\\ 0 & 1 & 1\end{vmatrix}$

$= \hat{i}(1-2)-\hat{j}(2-0)+\hat{k}(2-0)$

$= -\hat{i}-2\hat{j}+2\hat{k}$

$\Rightarrow \quad |\vec{a}\times\vec{b}| = \sqrt{(-1)^2+(-2)^2+(2)^2}$

$= \sqrt{1+4+4} = 3$ units

$\therefore$ Unit vectors $= \pm\left(\frac{-\hat{i}-2\hat{j}+2\hat{k}}{3}\right)$

$= \pm\left(\frac{-1}{3}\hat{i}+\frac{-2}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$

22. Find the angle between the vectors $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$.

23. Find λ, if the projection of $\vec{a} = \lambda\hat{i}+\hat{j}+4\hat{k}$ on $\vec{b} = 2\hat{i}+6\hat{j}+3\hat{k}$ is 4 units. [CBSE 2012]

Ans. We know,

Projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$

$\Rightarrow \quad 4 = \frac{(\lambda\hat{i}+\hat{j}+4\hat{k}).(2\hat{i}+6\hat{j}+3\hat{k})}{\sqrt{2^2+6^2+3^2}}$

$\Rightarrow \quad 4 = \frac{2\lambda+6+12}{\sqrt{49}}$

$\Rightarrow \quad 28 = 2\lambda+18$

$\Rightarrow \quad 2\lambda = 10$

$\Rightarrow \quad \lambda = 5$

Hence, the value of λ is 5.

24. Find the value of the following expression:

$|\vec{a}\times\vec{b}|^2+(\vec{a}.\vec{b})^2$

25. If $\vec{a}$ is any non-zero vector, then find the value of $(\vec{a}.\hat{i})\hat{i}+(\vec{a}.\hat{j})\hat{j}+(\vec{a}.\hat{k})\hat{k}$.

Ans. Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$\therefore \quad \vec{a}.\hat{i} = a_1$

$\vec{a}.\hat{j} = a_2$

$\vec{a}.\hat{k} = a_3$

$\therefore \quad (\vec{a}.\hat{i})\hat{i}+(\vec{a}.\hat{j})\hat{j}+(\vec{a}.\hat{k})\hat{k}$

$= a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$= \vec{a}$

26. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 15$. [CBSE 2013]

Ans. Given, $\vec{a}$ is a unit vector.

Then, $|\vec{a}| = 1$

Now, we have $(\vec{x}-\vec{a}).(\vec{x}+\vec{a}) = 15$

$\Rightarrow \quad \vec{x}.\vec{x}-\vec{a}.\vec{x}+\vec{x}.\vec{a}-\vec{a}.\vec{a} = 15$

$\Rightarrow \quad \vec{x}.\vec{x}-\vec{a}.\vec{x}+\vec{a}.\vec{x}-\vec{a}.\vec{a} = 15$

[$\because$ Scalar product is commutative *i.e.* $\vec{a}.\vec{b} = \vec{b}.\vec{a}$]

$\Rightarrow \quad |\vec{x}|^2-|\vec{a}|^2 = 15$

$\Rightarrow \quad |\vec{x}|^2 - 1 = 15$

$\Rightarrow \quad |\vec{x}|^2 = 16$

$\Rightarrow \quad |\vec{x}| = 4$

[$\because$ length cannot be negative]

27. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, show that $\vec{a} - \vec{d}$ is parallel to $\vec{b} - \vec{c}$ where $\vec{a} \neq \vec{c}$ and $\vec{b} \neq \vec{c}$. **[DIKSHA]**

28. Write the projection of $(\vec{b} + \vec{c})$ on $\vec{a}$, where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. **[CBSE 2013]**

Ans. Here, $\vec{b} + \vec{c} = (\hat{i} + 2\hat{j} - 2\hat{k}) + (2\hat{i} - \hat{j} + 4\hat{k})$

$= 3\hat{i} + \hat{j} + 2\hat{k}$

Now, projection of $(\vec{b} + \vec{c})$ on $\vec{a}$

$$= \frac{(\vec{b} + \vec{c}).\vec{a}}{|\vec{a}|}$$

$$= \frac{(3\hat{i} + \hat{j} + 2\hat{k}).(2\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{2^2 + (-2)^2 + 1^2}}$$

$$= \frac{6 - 2 + 2}{\sqrt{9}}$$

$$= \frac{6}{3} = 2.$$

29. Dot product of a vector with vectors $\hat{i} - \hat{j} + \hat{k}, 2\hat{i} + \hat{j} - 3\hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively 4, 0 and 2. Find the vector. **[CBSE 2013]**

Ans. Let, the required vector be $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Also, let,

$\vec{a} = \hat{i} - \hat{j} + \hat{k}, \vec{b} = 2\hat{i} + \hat{j} - 3\hat{k}, \vec{c} = \hat{i} + \hat{j} + \hat{k}$.

Now, $\vec{r} \cdot \vec{a} = 4, \vec{r} \cdot \vec{b} = 0, \vec{r} \cdot \vec{c} = 2$ (Given)

$\Rightarrow \quad x - y + z = 4$ ...(i)

$2x + y - 3z = 0$ ...(ii)

$x + y + z = 2$ ...(iii)

Applying (iii)-(i), we get

$\Rightarrow (x + y + z) - (x - y + z) = 2 - 4$

$2y = -2$

$y = -1$ ...(iv)

Putting the value of y in (ii) and (iii), we get

$2x - 3z = 1$

And $\quad x + z - 3 = 0$

$\Rightarrow \quad x = 2, z = 1$

$\therefore$ Required vector is $\vec{r} = 2\hat{i} - \hat{j} + \hat{k}$.

30. Evaluate $(2\vec{a} + 3\vec{b}) \times (5\vec{a} + 7\vec{b})$.

31. You must have seen gardeners using a lawn mover to cut grass in their lawn. A lawn mover is a machine utilizing one or more revolving blades to cut a grass surface to an even height and the height of the cut grass may be fixed by the design of the mover. Let the displacement be represented by the vector $\vec{a}$ and the force applied be represented by the vector $\vec{b}$ as shown in figure below.

Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude, such that the angle between them is 60° and their scalar product is $\frac{9}{2}$.

[CBSE 2018]

Ans. Given that $|\vec{a}| = |\vec{b}|$, $\vec{a}.\vec{b} = \frac{9}{2}$ and angle between them is 60°.

We know, $\quad \vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$

$\Rightarrow \quad \frac{9}{2} = |\vec{a}||\vec{a}| \cos 60°$

$\Rightarrow \quad \frac{9}{2} = |\vec{a}|^2 \times \frac{1}{2}$

$\Rightarrow \quad |\vec{a}|^2 = 9$

$\Rightarrow \quad |\vec{a}| = 3$

[$\because$ magnitude cannot be negative]

Thus, $\quad |\vec{a}| = |\vec{b}| = 3$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

32. If the sum of two unit vectors $\vec{a}$ and $\vec{b}$ is a unit vector, show that the magnitude of their difference is $\sqrt{3}$. **[CBSE 2019, 12]**

Ans. Let $\vec{c} = \vec{a} + \vec{b}$. Then given $|\vec{c}| = 1$

To prove: $|\vec{a} - \vec{b}| = \sqrt{3}$

Proof: Since $\vec{c} = \vec{a} + \vec{b}$

$\Rightarrow \quad |\vec{c}| = |\vec{a} + \vec{b}|$

$\Rightarrow \quad 1 = |\vec{a} + \vec{b}|$

$\Rightarrow \quad |\vec{a} + \vec{b}|^2 = 1$

$\Rightarrow \quad (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = 1$

$\Rightarrow |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 = 1$

$\Rightarrow \quad 1 + 2\vec{a}.\vec{b} + 1 = 1$

$\Rightarrow \quad 2\vec{a}.\vec{b} = -1 \quad ...(i)$

Now, consider,

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}).(\vec{a} - \vec{b})$$

$$= |\vec{a}|^2 - 2\vec{a}.\vec{b} + |\vec{b}|^2$$

$= 1 - (-1) + 1$ [Using (i)]

$= 3$

$\Rightarrow \quad |\vec{a} - \vec{b}| = \sqrt{3}$

Hence, proved.

33. If $\vec{a} + \vec{b} + \vec{c} = 0$ and $|\vec{a}| = 5$, $|\vec{b}| = 6$ and $|\vec{c}| = 9$, then find the angle between $\vec{a}$ and $\vec{b}$. **[CBSE 2018]**

34. Find the sine of the angle between the vectors $\bar{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\bar{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$. **[NCERT Exemplar]**

Ans. Here, $\bar{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\bar{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$.

$\therefore \quad \cos\theta = \frac{\bar{a}.\bar{b}}{|\bar{a}||\bar{b}|}$

$$= \frac{(3\hat{i} + \hat{j} + 2\hat{k}).(2\hat{i} - 2\hat{j} + 4\hat{k})}{\sqrt{3^2 + 1^2 + 2^2}\sqrt{2^2 + (-2)^2 + 4^2}}$$

$$= \frac{6 - 2 + 8}{\sqrt{14}\sqrt{24}} = \frac{12}{\sqrt{2 \times 7 \times 3 \times 8}}$$

$$= \frac{12}{4\sqrt{21}} = \frac{3}{\sqrt{21}}$$

$\therefore \quad \sin\theta = \sqrt{1 - \cos^2\theta}$

$$= \sqrt{1 - \left(\frac{3}{\sqrt{21}}\right)^2} = \sqrt{1 - \frac{9}{21}}$$

$$= \sqrt{\frac{12}{21}} = \frac{2\sqrt{3}}{\sqrt{3}\sqrt{7}} = \frac{2}{\sqrt{7}}$$

Hence, the sine of angle between the given vectors is $\frac{2}{\sqrt{7}}$.

⚠ Caution

- *Alternatively, using $|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}| \sin\theta$, $\sin\theta$ can be calculated.*

35. Using vectors, prove that the parallelograms on the same base and between the same parallels are equal in area. **[NCERT Exemplar]**

36. If $\hat{a}$ and $\hat{b}$ are unit vectors, then prove that $|\hat{a} + \hat{b}| = 2\cos\frac{\theta}{2}$, where θ is the angle between them. **[CBSE Term-2 SQP 2022]**

Ans. $(\hat{a}+\hat{b}).(\hat{a}+\hat{b}) = |\hat{a}|^2+|\hat{b}|^2+2(\hat{a}.\hat{b})$

$$|\hat{a}+\hat{b}|^2 = 1+1+2\cos\theta$$

$$= 2(1+\cos\theta) = 4\cos^2\frac{\theta}{2}$$

$$\therefore \quad |\hat{a}+\hat{b}| = 2\cos\frac{\theta}{2}$$

[CBSE Marking Scheme Term-2 SQP 2022]

Explanation: Here, $\hat{a}$ and $\hat{b}$ are the unit vectors.

Dot product of $(\hat{a}+\hat{b}).(\hat{a}+\hat{b})$

$$= |\hat{a}|^2+|\hat{b}|^2+2(\hat{a}-\hat{b})$$

But $\hat{a}$ and $\hat{b}$ are unit vectors

$$\therefore \quad (\hat{a}+\hat{b})^2 = 1+1+2\ \hat{a}.\hat{b}\cos\theta$$

$$\Rightarrow \quad (\hat{a}+\hat{b})^2 = 2+2.1.1.\cos\theta$$

$$\Rightarrow \quad (\hat{a}+\hat{b})^2 = 2(1+\cos\theta)$$

$$\Rightarrow \quad (\hat{a}+\hat{b})^2 = 2\times 2\cos^2\frac{\theta}{2}$$

$$\Rightarrow \quad (\hat{a}+\hat{b}) = \sqrt{4\cos^2\frac{\theta}{2}}$$

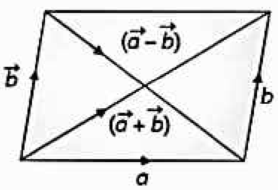
$$\Rightarrow \quad |\hat{a}+\hat{b}| = 2\cos\frac{\theta}{2}$$

Hence, proved.

37. The vectors $\vec{a} = 3\hat{i}-2\hat{j}+2\hat{k}$ and $\vec{b} = -\hat{i}-2\hat{k}$ are the adjacent sides of a parallelogram. Find the angle between its diagonals.

Ans. Given, $\vec{a} = 3\hat{i}-2\hat{j}+2\hat{k}$

$$\vec{b} = -\hat{i}-2\hat{k}$$



Clearly, $(\vec{a}+\vec{b})$ are $(\vec{a}-\vec{b})$ are the diagonals of the parallelogram

$$\therefore \quad \vec{a}+\vec{b} = (3\hat{i}-2\hat{j}+2\hat{k})+(-\hat{i}-2\hat{k})$$

$$= 2\hat{i}-2\hat{j}$$

$$\vec{a}-\vec{b} = (3\hat{i}-2\hat{j}+2\hat{k})-(-\hat{i}-2\hat{k})$$

$$= 4\hat{i}-2\hat{j}+4\hat{k}$$

Now, let θ be the acute angle between the diagonals.

$$\therefore \quad \cos\theta = \frac{(\vec{a}+\vec{b}).(\vec{a}-\vec{b})}{|\vec{a}+\vec{b}||\vec{a}-\vec{b}|}$$

$$= \frac{(2\hat{i}-2\hat{j}).(4\hat{i}-2\hat{j}+4\hat{k})}{\sqrt{4+4}\sqrt{16+4+16}}$$

$$= \frac{8+4}{2\sqrt{2}.6} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \theta = \frac{\pi}{4}$$

Hence, the angle between diagonals of the parallelogram is $\frac{\pi}{4}$.

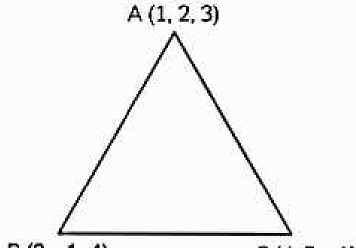
38. Using vectors, find the area of the triangle ABC with vertices A(1, 2, 3), B(2, –1, 4) and (4, 5, –1). [NCERT Exemplar]

Ans. Here, $\overline{AB} = (2-1)\hat{i}+(-1-2)\hat{j}+(4-3)\hat{k}$

$$= \hat{i}-3\hat{j}+\hat{k}$$

$$\overline{AC} = (4-1)\hat{i}+(5-2)\hat{j}+(-1-3)\hat{k}$$

$$= 3\hat{i}+3\hat{j}-4\hat{k}$$



So, $$\overline{AB}\times\overline{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$$

$$= \hat{i}(12-3)-\hat{j}(-4-3) + \hat{k}(3+9)$$

$$= 9\hat{i}+7\hat{j}+12\hat{k}$$

Now, Area of $\Delta ABC = \frac{1}{2}|\overline{AB}\times\overline{AC}|$

$$= \frac{1}{2}\sqrt{9^2+7^2+12^2}$$

$$= \frac{1}{2}\sqrt{274} \text{ sq. units}$$

Concept Applied

- *Remember that area of* $\Delta ABC = \frac{1}{2}|\overline{AB}\times\overline{AC}|$

39. **If θ is the angle between two vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$, find $\sin\theta$.**

[CBSE 2018]

40. **Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $3\hat{i}+7\hat{j}+\hat{k}$ and $5\hat{i}+6\hat{j}+2\hat{k}$ form the sides of a right angled triangle.** **[CBSE 2020]**

Ans. Let, $\vec{OA} = 2\hat{i}-\hat{j}+\hat{k}$

$\vec{OB} = 3\hat{i}+7\hat{j}+\hat{k}$

$\vec{OC} = 5\hat{i}+6\hat{j}+2\hat{k}$

Then, $\vec{AB} = (3\hat{i}+7\hat{j}+\hat{k})-(2\hat{i}-\hat{j}+\hat{k})$

$= (3-2)\hat{i}+(7+1)\hat{j}+(k-1)\hat{k}$

$= \hat{i}+8\hat{j}$

$\vec{AC} = (5\hat{i}+6\hat{j}+2\hat{k})-(2\hat{i}-\hat{j}+\hat{k})$

$= 3\hat{i}+7\hat{j}+\hat{k}$

$\vec{BC} = (5\hat{i}+6\hat{j}+2\hat{k})-(3\hat{i}+7\hat{j}+\hat{k})$

$= 2\hat{i}-\hat{j}+\hat{k}$

So, $|\vec{AB}| = \sqrt{1^2+8^2} = \sqrt{65}$

$|\vec{AC}| = \sqrt{3^2+7^2+1^2} = \sqrt{59}$

$|\vec{BC}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$

$\because |\vec{AC}|^2+|\vec{BC}|^2 = |\vec{AB}|^2$

So, be the converse of Pythagoras theorem, sides AB, BC and AC form a right triangle, right-angled at C.

Hence, proved

SHORT ANSWER Type-II Questions (SA-II)

[**3** marks]

41. **If $\vec{a}+\vec{b}+\vec{c}=0$, show that $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$. Interpret the result geometrically.**

[NCERT Exemplar]

Ans. Here, $\vec{a}+\vec{b}+\vec{c} = 0$

$\Rightarrow \quad \vec{b} = -\vec{c}-\vec{a}$

Now, $\vec{a}\times\vec{b} = \vec{a}\times(-\vec{c}-\vec{a})$

$= -\vec{a}\times\vec{c} - \vec{a}\times\vec{a}$

$= -\vec{a}\times\vec{c} \qquad [\because \vec{a}\times\vec{a}=0]$

$\therefore \quad \vec{a}\times\vec{b} = -\vec{a}\times\vec{c} \qquad$ ...(i)

And $\vec{b}\times\vec{c} = (-\vec{c}-\vec{a})\times\vec{c}$

$= -\vec{c}\times\vec{c}-\vec{a}\times c$

$= -\vec{a}\times\vec{c}$

$\therefore \quad \vec{b}\times\vec{c} = -\vec{a}\times\vec{c} \qquad$ ...(ii)

From (i) and (ii), we get

$\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a} \quad [\ -\vec{a}\times\vec{c}=\vec{c}\times\vec{a}]$

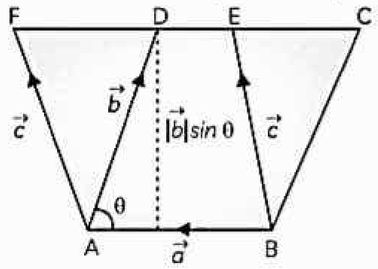
Hence, proved.

Geometrical interpretation:

If ABCD is a parallelogram, such that $\overline{AB} = \vec{a}$ and $\overline{AD} = \vec{b}$ and its adjacent sides are making an angle θ between each other, then

Area of parallelogram ABCD

$= |\vec{a}|\,|\vec{b}|\,\sin\theta = |\vec{a}\times\vec{b}|$



$\because$ Parallelograms on same base and between same parallels are equal in area

$\therefore \quad |\vec{a}\times\vec{b}| = |\vec{b}\times\vec{c}| = |\vec{a}\times\vec{c}|$

So, area of parallelogram formed by taking any two sides represented by $\vec{a}$, $\vec{b}$ and $\vec{c}$ as adjacents are equal.

42. **Three vertices A, B and D of a parallelogram ABCD are given by A(0, –3, 3), B(–5 –3, 0) and D(1, –3, 4). The area of the parallelogram ABCD is 6 sq. units.**

Using vector method find the value(s) of *m*.

Ans. We know,

Area of parallelogram ABCD = $|\vec{AB}\times\vec{AD}|$

Here, $\overrightarrow{AB}$ = P.V. of B – P.V. of A

$= (-5-0)\hat{i} + \{(m-3)-3\}\hat{j} + (0-3)\hat{k}$

$= -5\hat{i}+(m-6)\hat{j}-3\hat{k}$

and, $\overrightarrow{AD}$ = P.V. of D – P.V. of A

$= (1-0)\hat{i} + (-3+3)\hat{j} + (4-3)\hat{k}$

$= \hat{i}+\hat{k}$

So, $\overrightarrow{AB}\times\overrightarrow{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & m-6 & -3 \\ 1 & 0 & 1 \end{vmatrix}$

$= \hat{i}(m-6-0) - \hat{j}(-5+3) + \hat{k}(0-(m-6))$

$= (m-6)\hat{i} + 2\hat{j} - (m-6)\hat{k}$

$\Rightarrow |\overrightarrow{AB}\times\overrightarrow{AD}| = \sqrt{(m-6)^2+(2)^2+\{-(m-6)\}^2}$

$= \sqrt{2m^2-24m+76}$

$\because$ Area of parallelogram = 6 sq. units [Given]

$\Rightarrow \sqrt{2m^2-24m+76} = 6$

or, $2m^2 - 24m + 76 = 36$

$\Rightarrow 2m^2 - 24m + 40 = 0$

or, $m^2 - 12m + 20 = 0$

$\Rightarrow (m-10)(m-2) = 0$

$\Rightarrow m = 10, 2$

43. If $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $b = \hat{j}-\hat{k}$, find a vector $\vec{c}$ such that $\vec{a}\times\vec{c} = \vec{b}$ and $\vec{a}.\vec{c} = 3$. **[NCERT Exemplar]**

44. If $\vec{a} \neq \vec{0}$, $\vec{a}.\vec{b} = \vec{a}.\vec{c}$, $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$, then show that $\vec{b} = \vec{c}$.

Ans. We have, $\vec{a}.\vec{b} = \vec{a}.\vec{c}$

$\vec{a}.(\vec{b}-\vec{c}) = 0$

$\Rightarrow (\vec{b}-\vec{c}) = \vec{0}$

or $\vec{a}\perp(\vec{b}-\vec{c})$

$\Rightarrow \vec{b} = \vec{c}$

or $\vec{a}\perp(\vec{b}-\vec{c})$

Also, $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$

$\vec{a}\times(\vec{b}-\vec{c}) = \vec{0}$

$\Rightarrow (\vec{b}-\vec{c}) = \vec{0}$ or $\vec{a}\parallel(\vec{b}-\vec{c})$

$\Rightarrow \vec{b} = \vec{c}$ or $\vec{a}\parallel(\vec{b}-\vec{c})$

$\vec{a}$ cannot be both perpendicular to $(\vec{b}-\vec{c})$ and parallel to $(\vec{b}-\vec{c})$.

Hence, $\vec{b} = \vec{c}$

45. Let $\vec{a} = 4\hat{i}+5\hat{j}-\hat{k}$, $\vec{b} = \hat{i}-4\hat{j}+5\hat{k}$ and $\vec{c} = 3\hat{i}+\hat{j}-\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}.\vec{a} = 21$. **[CBSE 2018]**

Ans. Given, $\vec{a} = 4\hat{i}+5\hat{j}-\hat{k}$

$\vec{b} = \hat{i}-4\hat{j}+5\hat{k}$

$\vec{c} = 3\hat{i}+\hat{j}-\hat{k}$

Since, $\vec{d}$ is perpendicular to both $\vec{c}$ and $\vec{b}$

$\therefore \vec{d} = \lambda(\vec{c}\times\vec{b})$

$= \lambda\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -1 \\ 1 & -4 & 5 \end{vmatrix}$

$= \lambda[\hat{i}(5-4)-\hat{j}(15+1)+\hat{k}(-12-1)]$

$= \lambda[\hat{i}-16\hat{j}-13\hat{k}]$ ...(i)

Now, $\vec{d}.\vec{a} = 21$ (given)

$\therefore \lambda(\hat{i}-16\hat{j}-13\hat{k}).(4\hat{i}+5\hat{j}-\hat{k}) = 21$

$\Rightarrow \lambda(4-80+13) = 21$

$\Rightarrow \lambda(-63) = 21$

$\Rightarrow \lambda = \frac{-21}{63} = -\frac{1}{3}$

$\therefore \vec{d} = \frac{-1}{3}(\hat{i}-16\hat{j}-13\hat{k})$

46. $\vec{r}$ and $\vec{s}$ are unit vector. If $|\vec{r}+\vec{s}| = \sqrt{2}$, find:

(A) The value of $(4\vec{r}-\vec{s}).(2\vec{r}+\vec{s})$

(B) The angle between $\vec{r}$ and $\vec{s}$.

Ans. We have, $|\vec{r}+\vec{s}| = \sqrt{2}$

$\Rightarrow |\vec{r}+\vec{s}|^2 = 2$

$\Rightarrow |\vec{r}+\vec{s}||\vec{r}+\vec{s}| = 2$

$\Rightarrow \vec{r}.\vec{r}+\vec{r}.\vec{s}+\vec{s}.\vec{r}+\vec{s}.\vec{s} = 2$

$\Rightarrow |\vec{r}|^2+2\vec{r}.\vec{s}+|\vec{s}|^2 = 2 \quad [\because \vec{r}.\vec{s}=\vec{s}.\vec{r}]$

$\Rightarrow 1+2\vec{r}.\vec{s}+1 = 2$

[$\because \vec{r}$ and $\vec{s}$ are unit vectors]

$\Rightarrow \vec{r}.\vec{s} = 0$

(A) $(4\vec{r}-\vec{s}).(2\vec{r}+\vec{s})$

$= 8|\vec{r}|^2+4\vec{r}.\vec{s}-2\vec{s}.\vec{r}-|\vec{s}|^2$

$= 8\times 1 + 2\vec{r}.\vec{s}-1$

$= 7+2\vec{r}.\vec{s}$

$= 7+2(0)$

$= 7$

(B) Let θ be the angle between $\vec{r}$ and $\vec{s}$.

Then, $\cos\theta = \frac{\vec{r}.\vec{s}}{|\vec{r}||\vec{s}|} = \frac{0}{1\times 1} = 0$

$\Rightarrow \theta = \frac{\pi}{2}$.

47. **If $\vec{a}$ and $\vec{b}$ are unit vectors inclined at an angle θ, then prove that $\sin\frac{\theta}{2} = \frac{1}{2}|\hat{a}-\hat{b}|$.**

[CBSE 2011]

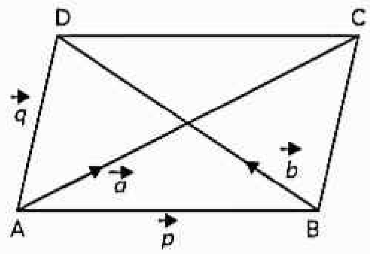
LONG ANSWER Type Questions (LA)

[4 & 5 marks]

48. Show that area of the parallelogram whose diagonals are given by $\vec{a}$ and $\vec{b}$ is $\frac{|\vec{a}\times\vec{b}|}{2}$. Also find the area of the parallelogram whose diagonals are $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+3\hat{j}-\hat{k}$.

[NCERT Exemplar]

Ans. Let, ABCD be a parallelogram such that $\overline{AB} = \vec{p}$ and $\overline{AD} = \vec{q}$.



$\Rightarrow \overline{BC} = \overline{AD} = \vec{q}$

By triangle law of vector addition, we have

$\overline{AC} = \overline{AB}+\overline{BC}$

$= \vec{p}+\vec{q} = \vec{a}$ (let) ...(i)

Similarly, $\overline{BD} = \overline{BA}+\overline{AD}$

$= -\vec{p}+\vec{q} = \vec{b}$ (let) ...(ii)

On adding (i) and (ii), we get

$\vec{a}+\vec{b} = 2\vec{q}$

$\Rightarrow \vec{q} = \frac{1}{2}(\vec{a}+\vec{b})$

On subtracting (ii) from (i), we get

$\vec{a}-\vec{b} = 2\vec{p}$

$\Rightarrow \vec{p} = \frac{1}{2}(\vec{a}-\vec{b})$

Now, $\vec{p}\times\vec{q} = \frac{1}{4}(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$

$= \frac{1}{4}(\vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\vec{b}\times\vec{b})$

$= \frac{1}{4}(\vec{a}\times\vec{b}+\vec{a}\times\vec{b})$

$= \frac{1}{2}(\vec{a}\times\vec{b})$

So, area of parallelogram ABCD

$= |\vec{p}\times\vec{q}| = \frac{1}{2}|\vec{a}\times\vec{b}|$

Hence, proved.

Now, area of parallelogram with diagonals $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+3\hat{j}-\hat{k}$

$= \frac{1}{2}|(2\hat{i}-\hat{j}+\hat{k})\times(\hat{i}+3\hat{j}-\hat{k})|$

$= \frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 1 & 3 & -1\end{vmatrix}$

$= \frac{1}{2}|\hat{i}(1-3)-\hat{j}(-2-1)+\hat{k}(6+1)|$

$$= \frac{1}{2}\left|-2\hat{i}+3\hat{j}+7\hat{k}\right|$$

$$= \frac{1}{2}\sqrt{(-2)^2+3^2+7^2}$$

$$= \frac{1}{2}\sqrt{4+9+49}$$

$$= \frac{1}{2}\sqrt{62} \text{ sq. units}$$

Concept Applied

- *If $\bar{p}$ and $\bar{q}$ are adjacent sides of a parallelogram, then the area of the parallelogram = $|\bar{p}\times\bar{q}|$*

49. Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\vec{p}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p}.\vec{c} = 18$. **[CBSE 2012, 10]**

Ans. Given, vectors are $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$

$$\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$$

$$\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$$

Let $\vec{p} = x\hat{i} + y\hat{j} + z\hat{k}$

We have, $\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

$\therefore \quad \vec{p} = \lambda(\vec{a}\times\vec{b})$, where λ is some scalar.

$$\because \quad \vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$$= \hat{i}(28+4) - \hat{j}(7-6) + \hat{k}(-2-12)$$

$$= 32\hat{i} - \hat{j} - 14\hat{k}$$

$$\therefore \quad \vec{p} = \lambda(32\hat{i} - \hat{j} - 14\hat{k}) \quad ...(i)$$

Also, $\vec{p}.\vec{c} = 18$

$\Rightarrow \lambda(32\hat{i} - \hat{j} - 14\hat{k}).(2\hat{i} - \hat{j} - 4\hat{k}) = 18$ [using (i)]

$\Rightarrow \quad \lambda(64 + 1 - 56) = 18$

$\Rightarrow \quad \lambda = \frac{18}{9} = 2$

Putting the value of λ in (i), we get

$$\vec{p} = 2(32\hat{i} - \hat{j} - 14\hat{k})$$

$$= 64\hat{i} - 2\hat{j} - 28\hat{k}$$

50. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three mutually perpendicular vectors of the same magnitude, then prove that $\vec{a}+\vec{b}+\vec{c}$ is equally inclined with the vectors $\vec{a}, \vec{b}$ and $\vec{c}$. **[CBSE 2017, 13, 11]**

51. The scalar product of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of vectors $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b} + \vec{c}$. **[CBSE 2014]**

Ans. Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$,

and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$

Now, $\vec{b} + \vec{c} = 2\hat{i} + 4\hat{j} - 5\hat{k} + \lambda\hat{i} + 2\hat{j} + 3\hat{k}$

$$= (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$$

$$\therefore \quad |\vec{b} + \vec{c}| = \sqrt{(2+\lambda)^2 + 6^2 + (-2)^2}$$

$$= \sqrt{4 + \lambda^2 + 4\lambda + 36 + 4}$$

$$= \sqrt{\lambda^2 + 4\lambda + 44}$$

Unit vector along $\vec{b} + \vec{c}$

$$= \frac{\vec{b} + \vec{c}}{|\vec{b} + \vec{c}|}$$

$$= \frac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{\lambda^2 + 4\lambda + 44}} \quad ...(i)$$

Given, scalar product of $(\hat{i} + \hat{j} + \hat{k})$ with above unit vector is 1.

i.e., $(\hat{i} + \hat{j} + \hat{k}) \cdot \frac{\vec{b} + \vec{c}}{|\vec{b} + \vec{c}|} = 1$

$\Rightarrow (\hat{i} + \hat{j} + \hat{k}) \cdot \frac{[(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}]}{\sqrt{\lambda^2 + 4\lambda + 44}} = 1$

$\Rightarrow 1(2+\lambda) + 1(6) - 1(-2) = \sqrt{\lambda^2 + 4\lambda + 44}$

$\Rightarrow \quad \lambda + 6 = \sqrt{\lambda^2 + 4\lambda + 44}$

$\Rightarrow \quad \lambda^2 + 36 + 12\lambda = \lambda^2 + 4\lambda + 44$

(Squaring both sides)

$\Rightarrow \qquad 8\lambda = 8$

$\Rightarrow \qquad \lambda = 1$

Now, put the value of $\lambda = 1$ in equation (i), we get Unit vector along $(\vec{b} + \vec{c})$

$$= \frac{(2+1)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{1^2 + 4(1) + 44}}$$

$$= \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{49}}$$

$$= \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{7}$$

$$= \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}$$

VERY SHORT ANSWER Type Questions (VSA)

[**1** mark]

1. If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} + 2\hat{k}$, then find $|\vec{a} \times \vec{b}|$.

Ans.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \hat{i}(-2-15) - \hat{j}(-4-9) + \hat{k}(10-3)$$

$$\vec{a} \times \vec{b} = -17\hat{i} + 13\hat{j} + 7\hat{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(-17)^2 + (13)^2 + (7)^2}$$

$$= \sqrt{289 + 169 + 49}$$

$$= \sqrt{507}$$

$$= \sqrt{3 \times 169}$$

$$|\vec{a} \times \vec{b}| = 13\sqrt{3}$$

[CBSE Topper 2015]

2. If $\bar{a}$ and $\bar{b}$ are unit vectors, then what is the angle between $\bar{a}$ and $\bar{b}$ for $\bar{a} - \sqrt{2}\,\bar{b}$ to be unit vector ?

Ans.

$\bar{a} - \sqrt{2}\bar{b}$ is a unit vector

$|\bar{a} - \sqrt{2}\bar{b}| = 1$

$|\bar{a} - \sqrt{2}\bar{b}|^2 = 1$

$|\bar{a}|^2 + |\sqrt{2}\bar{b}|^2 - 2\sqrt{2}\,\bar{a}.\bar{b} = 1$ $\quad (|\bar{a}| = |\bar{b}| = 1)$ $\quad$ (θ: Angle between vectors $\bar{a}$ & $\bar{b}$)

$1 + 2 - 2\sqrt{2}\,\bar{a}.\bar{b} = 1$ $\quad \bar{a}.\bar{b} = \frac{1}{\sqrt{2}}$ $\Rightarrow$ $|\bar{a}||\bar{b}|\cos\theta = \frac{1}{\sqrt{2}}$ $\quad$ $\boxed{\theta = 45^\circ}$ (Angle between $\bar{a}$ & $\bar{b}$ is 45°)

[CBSE Topper 2016]

3. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$.

Ans.

$$|\vec{a}| = |\vec{b}|$$

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$\frac{9}{2} = |\vec{a}||\vec{a}|\cos 60^\circ$$

$$\frac{9}{2} = |\vec{a}|^2 \times \frac{1}{2}$$

$$9 = |\vec{a}|^2$$

$$|\vec{a}| = 3.$$

Also $|\vec{a}| = |\vec{b}|$

$$|\vec{b}| = 3$$

$$|\vec{a}| = |\vec{b}| = 3 \quad \text{Ans.}$$

[CBSE Topper 2018]